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Atomic decompositions for Hardy spaces related to Schrödinger operators

by

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Abstract. Let $\mathbf{L}^U = -\mathbf{\Delta} + U$ be a Schrödinger operator on $\mathbb{R}^d$, where $U \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a non-negative potential and $d \geq 3$. The Hardy space $H^1(\mathbf{L}^U)$ is defined in terms of the maximal function of the semigroup $\mathbf{K}_t^U = \exp(-t\mathbf{L}^U)$, namely

$$H^1(\mathbf{L}^U) = \Big\{ f \in L^1(\mathbb{R}^d) : \|f\|_{H^1(\mathbf{L}^U)} := \Big\| \sup_{t>0} |\mathbf{K}_t^U f| \Big\|_{L^1(\mathbb{R}^d)} < \infty \Big\}.$$

Assume that $U = V + W$, where $V \geq 0$ satisfies the global Kato condition

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} V(y)|x-y|^{2-d}\, dy < \infty.$$

We prove that, under certain assumptions on $W \geq 0$, the space $H^1(\mathbf{L}^U)$ admits an atomic decomposition of local type. An atom a for $H^1(\mathbf{L}^U)$ either is of the form $a(x) = |Q|^{-1}\chi_Q(x)$, where Q are special cubes determined by W, or satisfies the cancellation condition $\int_{\mathbb{R}^d} a(x)\omega(x)\, dx = 0$, where ω is given by $\omega(x) = \lim_{t\to\infty} \mathbf{K}_t^V \mathbf{1}(x)$. Furthermore, we show that, in some cases, the above cancellation condition can be replaced by $\int_{\mathbb{R}^d} a(x)\, dx = 0$. However, we construct an example where the atomic spaces with these two cancellation conditions are not equivalent as Banach spaces.

1. Background and statement of results

1.1. Introduction. Let U be a non-negative, locally integrable function on $\mathbb{R}^d$. In this article we consider the Schrödinger operator given by $-\mathbf{\Delta}+U$, where $\mathbf{\Delta}$ is the standard Laplacian on $\mathbb{R}^d$ and U is called the *potential*. Throughout the whole paper we assume that $d \geq 3$.

To be more precise, let us recall what we mean by the Schrödinger operator. First, define the quadratic form

$$\mathbf{Q}^U(f,g) = \int_{\mathbb{R}^d} \nabla f(x)\overline{\nabla g(x)}\, dx + \int_{\mathbb{R}^d} U(x) f(x)\overline{g(x)}\, dx$$

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with domain $\mathrm{Dom}(\mathbf{Q}^U) = \{f \in L^2(\mathbb{R}^d) : \nabla f, \sqrt{U} f \in L^2(\mathbb{R}^d)\}$. This quadratic form is closed, thus it defines a self-adjoint operator $\mathbf{L}^U : \mathrm{Dom}(\mathbf{L}^U) \to L^2(\mathbb{R}^d)$. In particular, $\mathrm{Dom}(\mathbf{L}^U)$ is equal to

$$\Big\{f \in \mathrm{Dom}(\mathbf{Q}^U) : \exists h \in L^2(\mathbb{R}^d)\ \forall g \in \mathrm{Dom}(\mathbf{Q}^U)\ \ \mathbf{Q}^U(f,g) = \int_{\mathbb{R}^d} h\overline{g}\, dx\Big\},$$

and $\mathbf{L}^U f := h$ when f and h are as above. Formally, we write

$$\mathbf{L}^U = -\boldsymbol{\Delta} + U.$$

Let $(\mathbf{K}_t^U)_{t>0}$ be the semigroup generated by $\mathbf{L}^U$ on $L^2(\mathbb{R}^d)$. The Feynman–Kac formula (see, e.g., [17, Chap. V]) states that

$$\mathbf{K}_t^U f(x) = E^x\Big(\exp\Big(-\int_0^t U(X_s)\, ds\Big) f(X_t)\Big), \tag{1.1}$$

where X_s is the Brownian motion on $\mathbb{R}^d$. From (1.1) one finds that $\mathbf{K}_t^U$ has an integral kernel $K_t^U(x,y)$ and it is clear that $V \geq 0$ implies Gaussian upper bounds for $K_t^U(x,y)$, i.e.

$$0 \leq K_t^U(x,y) \leq (4\pi t)^{-d/2} \exp\Big(-\frac{|x-y|^2}{4t}\Big) =: P_t(x-y). \tag{1.2}$$

The Hardy space $H^1(\mathbf{L}^U)$ associated with $\mathbf{L}^U$ is defined as follows. Let

$$\mathbf{M}^U f(x) = \sup_{t>0} |\mathbf{K}_t^U f(x)|$$

be the maximal operator associated with $(\mathbf{K}_t^U)_{t>0}$. We say that a function $f \in L^1(\mathbb{R}^d)$ belongs to the maximal Hardy space $H^1(\mathbf{L}^U)$ if

$$\|f\|_{H^1(\mathbf{L}^U)} := \|\mathbf{M}^U f(x)\|_{L^1(\mathbb{R}^d)} < \infty. \tag{1.3}$$

In this paper, atomic Hardy spaces play a special role. A general definition is as follows. Assume that a family of functions $\mathcal{A} \subseteq L^1(\mathbb{R}^d)$ is given. A function $a \in \mathcal{A}$ will be called an *atom* and we always assume that $\|a\|_{L^1(\mathbb{R}^d)} \leq 1$. We say that a function f belongs to the atomic Hardy space $H^1_{\mathrm{at}}(\mathcal{A})$ if

$$f(x) = \sum_{j=1}^\infty \lambda_j a_j(x), \tag{1.4}$$

where $a_j \in \mathcal{A}$ and $\lambda_j \in \mathbb{C}$ with $\sum_{j=1}^\infty |\lambda_j| < \infty$. Obviously, the series (1.4) is absolutely convergent in $L^1(\mathbb{R}^d)$. Whenever $f \in H^1_{\mathrm{at}}(\mathcal{A})$ we set

$$\|f\|_{H^1_{\mathrm{at}}(\mathcal{A})} = \inf\Big\{\sum_{j=1}^\infty |\lambda_j| : f \text{ as in } (1.4)\Big\}. \tag{1.5}$$

It is not difficult to check that $H^1_{\mathrm{at}}(\mathcal{A}) \subseteq L^1(\mathbb{R}^d)$ and $H^1_{\mathrm{at}}(\mathcal{A})$ is a Banach space.

An important result in the classical theory of Hardy spaces is the atomic decomposition theorem [1], [14]. It asserts that $H^1(-\mathbf{\Delta}) = H^1_{\mathrm{at}}(\mathcal{A}_{\mathrm{class}})$ and the corresponding norms are equivalent. Here $\mathcal{A}_{\mathrm{class}}$ is the set of classical atoms, that is, $a \in \mathcal{A}_{\mathrm{class}}$ if there exist a cube Q such that $\operatorname{supp} a \subseteq Q$ (localization condition), $\|a\|_\infty \le |Q|^{-1}$ (size condition), and $\int_Q a(x)\,dx = 0$ (cancellation condition). By $|S|$ we denote the Lebesgue measure of a set S, and

$$Q = Q(c_Q, r_Q) = \Big\{ y = (y_1, \ldots, y_d) \in \mathbb{R}^d : \max_{i=1,\ldots,d} |(c_Q)_i - y_i| < r_Q \Big\},$$

where c_Q and r_Q are the center and the radius of Q, respectively. Denote

$$d_Q = \operatorname{diam}(Q) = 2\sqrt{d}\, r_Q.$$

The question we shall be concerned with is: does $H^1(\mathbf{L}^U)$ coincide with $H^1_{\mathrm{at}}(\mathcal{A})$ for a potential U and a family $\mathcal{A}$? If so, are the norms (1.3) and (1.5) comparable?

There are partial answers to the question above. A general result of Hofmann et al. [13] gives an atomic and molecular characterization of $H^1(\mathbf{L}^U)$ for any positive potential $U \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Also, using [13], Dziubański and Zienkiewicz [10] proved another general atomic characterization of $H^1(\mathbf{L}^U)$. The atoms in [13] are of the form $a = (\mathbf{L}^U)^M b$, where M is a fixed positive integer and b satisfies some localization and size conditions [13, Theorem 7.1]. Likewise, the atoms in [10] are given by $a = \mathbf{K}^U_t b - b$ for similar b.

Although the approaches just mentioned are useful in many situations, they also have some weaknesses. One of them is that the atoms are images of some functions under the operator $\mathbf{L}^U$ (or its semigroup), and they no more satisfy simple geometric conditions (localization, size, cancellation). One would also like to better understand the nature of $H^1(\mathbf{L}^U)$ by describing it in terms of simpler, "geometric" atoms. In the 1990's Dziubański and Zienkiewicz started studying atomic decompositions of Hardy spaces for Schrödinger operators. In this paper we continue this approach. For more results of this type see [2–11]. Let us finally mention that this approach was successfully used e.g. for proving Riesz transform characterizations of $H^1(\mathbf{L}^U)$ for some U, while such characterization is not known in general.

Before proceeding to our main results, we present results of [11] and [8], which are the starting point for our considerations.

1.2. The space $H^1(\mathbf{L}^V)$. Assume that a potential $V \ge 0$ satisfies the following *global Kato condition*

$$\text{(S)} \qquad \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |x-y|^{2-d} V(y)\,dy < \infty.$$

In other words, $\mathbf{\Delta}^{-1}V \in L^\infty(\mathbb{R}^d)$. Let $\omega = \omega(V)$ be the function defined by

$$\omega(x) = \lim_{t\to\infty} \int_{\mathbb{R}^d} K_t^V(x,y)\,dy. \tag{1.6}$$

The function ω is $\mathbf{L}^V$-harmonic and satisfies

$$0 < \delta < \omega(x) \le 1 \tag{1.7}$$

with some δ for all $x \in \mathbb{R}^d$ [11, Lemma 2.1]. It is well-known [16] that the integral kernel $K_t^V(x,y)$ has not only Gaussian upper bounds, but also Gaussian lower bounds, that is, there are $\kappa_1, \kappa_2 > 0$ such that

$$K_t^V(x,y) \ge \kappa_1 t^{-d/2} \exp\left(-\frac{|x-y|^2}{\kappa_2 t}\right). \tag{1.8}$$

By definition, a function a is an ω-*atom* if there exists a cube Q such that

$$\operatorname{supp} a \subseteq Q, \qquad \|a\|_\infty \le |Q|^{-1}, \qquad \int_Q a(x)\omega(x)\,dx = 0.$$

Let $\mathcal{A}_\omega$ be the set of ω-atoms. Corollary 1.2 of [11] states that $H^1(\mathbf{L}^V) = H^1_{\mathrm{at}}(\mathcal{A}_\omega)$ and

$$\|f\|_{H^1(\mathbf{L}^V)} \simeq \|f\|_{H^1_{\mathrm{at}}(\mathcal{A}_\omega)}. \tag{1.9}$$

Let us mention that condition (S) above is satisfied for example when V is compactly supported and $V \in L^p(\mathbb{R}^d)$ for some $p > d/2$. For more general examples, see [6] and [11].

1.3. The space $H^1(\mathbf{L}^W)$. For $\theta > 0$ (small) and a cube $Q = Q(c_Q, r_Q)$ denote $Q^* = Q(c_Q, (1+\theta)r_Q)$. Assume a family $\mathcal{Q}$ of cubes is given and there exist $C, \theta > 0$ such that for $Q_1, Q_2 \in \mathcal{Q}$, $Q_1 \ne Q_2$, we have

$$\bigcup_{Q\in\mathcal{Q}} \mathrm{cl}(Q) = \mathbb{R}^d, \tag{G$_1$}$$

$$|Q_1 \cap Q_2| = 0, \tag{G$_2$}$$

$$\text{if } Q_1^{****} \cap Q_2^{****} \ne \emptyset, \text{ then } C^{-1} d_{Q_1} \le d_{Q_2} \le C d_{Q_1}. \tag{G$_3$}$$

Observe that, under these assumptions, the family $\{Q^{****} : Q \in \mathcal{Q}\}$ is a finite covering of $\mathbb{R}^d$. In the following, we briefly say that $\mathcal{Q}$ satisfies (G) if it satisfies (G$_1$)–(G$_3$).

Suppose that for a potential $W \ge 0$ and a family $\mathcal{Q}$ as above there exist $\varepsilon, \delta, C > 0$ such that

$$\sup_{y\in Q^{**}} \int_{\mathbb{R}^d} K^W_{2^n d_Q^2}(x,y)\,dx \le C n^{-1-\varepsilon} \qquad (Q \in \mathcal{Q}, n \in \mathbb{N}), \tag{D}$$

$$\int_0^{2t} (\mathbf{1}_{Q^{***}} W) * P_s(x)\,ds \le C(t/d_Q^2)^\delta \qquad (x \in \mathbb{R}^d,\ Q \in \mathcal{Q},\ t \le d_Q^2), \tag{K}$$

where $P_t(x-y) = K_t^0(x,y)$ is the kernel of the classical heat semigroup (see (1.2)). By definition, a $\mathcal{Q}$-*atom* is a function a such that one of the following holds:

- there exist $Q \in \mathcal{Q}$ and a cube $K \subset Q^{**}$ such that

$$\operatorname{supp} a \subseteq K, \qquad \|a\|_\infty \le |K|^{-1}, \qquad \int_K a(x)\,dx = 0;$$

- $a(x) = |Q|^{-1}\chi_Q(x)$ for some $Q \in \mathcal{Q}$.

Let $\mathcal{A}_\mathcal{Q}$ be the set of $\mathcal{Q}$-atoms. By [8, Theorem 2.2] we have $H^1(\mathbf{L}^W) = H^1_{\mathrm{at}}(\mathcal{A}_\mathcal{Q})$ and

$$\|f\|_{H^1(\mathbf{L}^W)} \simeq \|f\|_{H^1_{\mathrm{at}}(\mathcal{A}_\mathcal{Q})}.$$

A list of examples of potentials W and related families $\mathcal{Q}$ can be found in [8]. Here we only mention one simple example, which we shall use later. Let $t > 0$ and denote by $\mathcal{Q}^{[t]}$ a family of cubes of radius t that satisfies (G). If $W^{[t]}(x) = t^{-2}$, then the pair $(W^{[t]}, \mathcal{Q}^{[t]})$ satisfies (D), (K), (G) with constants independent of t.

1.4. Main results. In this paper, V always denotes a potential satisfying (S), and ω is related to V by (1.6). Similarly, the pair $(W, \mathcal{Q})$ always satisfies (D), (K), and (G). Notice that in $H^1_{\mathrm{at}}(\mathcal{A}_\omega)$ and $H^1_{\mathrm{at}}(\mathcal{A}_\mathcal{Q})$ two different phenomena appear. Every atom $a \in \mathcal{A}_\omega$ (an atom for $\mathbf{L}^V$) satisfies the weighted cancellation condition with respect to the weight ω. On the other hand, for $a \in \mathcal{A}_\mathcal{Q}$, there are "local" atoms, i.e. atoms of the type $|Q|^{-1}\chi_Q(x)$ that do not satisfy any cancellation condition.

The goal of this paper is to study $\mathbf{L}^{V+W}$ and its Hardy space $H^1(\mathbf{L}^{V+W})$. We shall prove that in atomic decompositions for this space both phenomena described above appear simultaneously. Define $\mathcal{A}_{\omega,\mathcal{Q}}$ to be the set of $(\omega, \mathcal{Q})$-*atoms*, that is, functions such that one of the following holds:

- there exist $Q \in \mathcal{Q}$ and a cube $K \subset Q^{**}$ such that

$$\operatorname{supp} a \subseteq K, \qquad \|a\|_\infty \le |K|^{-1}, \qquad \int_K a(x)\omega(x)\,dx = 0;$$

- $a(x) = |Q|^{-1}\chi_Q(x)$ for some $Q \in \mathcal{Q}$.

The following theorem gives an atomic characterization of $H^1(\mathbf{L}^{V+W})$ in the spirit of [8] and [11].

Theorem A. *Assume that $d \ge 3$, $V \ge 0$ satisfies* (S), *and $W \ge 0$ with a family $\mathcal{Q}$ satisfy* (D), (K), (G). *Then*

$$C^{-1}\|f\|_{H^1(\mathbf{L}^{V+W})} \le \|f\|_{H^1_{\mathrm{at}}(\mathcal{A}_{\omega,\mathcal{Q}})} \le C\|f\|_{H^1(\mathbf{L}^{V+W})}. \tag{1.10}$$

In particular, $H^1(\mathbf{L}^{V+W}) = H^1_{\mathrm{at}}(\mathcal{A}_{\omega,\mathcal{Q}})$.

In Theorem A atoms are localized to cubes $Q \in \mathcal{Q}$ and they satisfy the weighted cancellation condition with weight ω. However, it is not hard to see that every $(\omega, \mathcal{Q})$-atom can be written as a linear combination of $\mathcal{Q}$-atoms. Indeed, if a is such that $\operatorname{supp} a \subseteq K \subseteq Q^{**}$, $\|a\|_\infty \le |K|^{-1}$, and $\int_K a(x)\omega(x)\,dx = 0$ for $Q \in \mathcal{Q}$, then

$$a(x) = \big(a(x) - \kappa |Q|^{-1}\mathbb{1}_Q(x)\big) + \kappa|Q|^{-1}\mathbb{1}_Q(x) = b_1(x) + b_2(x),$$

where $\kappa = \int_K a(x)\,dx$, $|\kappa| \le 1$. Observe that $\int_{Q^{**}} b_1(x)\,dx = 0$ and $\operatorname{supp} b_1 \subseteq Q^{**}$. Thus both b_1 and b_2 are multiples of $\mathcal{Q}$-atoms. What we have just shown is that for every $(\omega, \mathcal{Q})$-atom a we have $a \in H^1_{\mathrm{at}}(\mathcal{A}_\mathcal{Q})$ and

$$\|a\|_{H^1_{\mathrm{at}}(\mathcal{A}_\mathcal{Q})} \le T. \tag{1.11}$$

The constant T in (1.11) possibly depends on a. This leads us to the following question: do $H^1_{\mathrm{at}}(\mathcal{A}_{\omega,\mathcal{Q}})$ and $H^1_{\mathrm{at}}(\mathcal{A}_\mathcal{Q})$ coincide as Banach spaces? In Theorem B we prove that, under a certain Lipschitz assumption, the answer to this question is affirmative. However, a more difficult task is to find an example such that $\|f\|_{H^1_{\mathrm{at}}(\mathcal{A}_\mathcal{Q})} \not\simeq \|f\|_{H^1_{\mathrm{at}}(\mathcal{A}_{\omega,\mathcal{Q}})}$. This is done in Example C.

THEOREM B. *Assume that* $0 < \delta \le \omega \le 1$, $\mathcal{Q}$ *satisfies* (G), *and there exists* $\lambda > 0$ *such that*

$$|\omega(x) - \omega(y)| \le C(|x-y|/d_Q)^\lambda \quad (Q \in \mathcal{Q},\ x, y \in Q^{**}). \tag{1.12}$$

Then

$$\|f\|_{H^1_{\mathrm{at}}(\mathcal{A}_\mathcal{Q})} \simeq \|f\|_{H^1_{\mathrm{at}}(\mathcal{A}_{\omega,\mathcal{Q}})}. \tag{1.13}$$

As an example that fulfills the assumptions of Theorem B one could take $W^{[1]}$, $\mathcal{Q}^{[1]}$ (see the end of Subsection 1.3) and $\omega = \omega(V)$, with V such that $\operatorname{supp} V \subseteq Q(0,1)$ and $V \in L^p(\mathbb{R}^d)$ for $p > d/2$ (for details see [9]). In this case ω satisfies a global Hölder condition.

EXAMPLE C. Let $\mathcal{Q}^{[1]}$ be as above, and $\omega = \omega(\mathcal{V})$, where $\mathcal{V}$ is the potential given in (6.1) below. There exists a sequence of $(\omega, \mathcal{Q}^{[1]})$-atoms a_j such that

$$\lim_{j\to\infty} \|a_j\|_{H^1_{\mathrm{at}}(\mathcal{A}_\mathcal{Q})} = \infty. \tag{1.14}$$

In other words, $\|f\|_{H^1_{\mathrm{at}}(\mathcal{A}_{\omega,\mathcal{Q}})} \not\simeq \|f\|_{H^1_{\mathrm{at}}(\mathcal{A}_\mathcal{Q})}$.

The paper is organized as follows. Local Hardy spaces are investigated in Section 2, in particular we prove an atomic decomposition theorem for a local version of $H^1(\mathbf{L}^V)$. In Section 3 we state some auxiliary estimates, similar to those from [8], that will be used in the proof of Theorem A, which is given in Section 4. The proof of Theorem B is presented in Section 5. In Section 6 we provide details of Example C and prove (1.14). Finally, in the Appendix we give a proof of the estimate $\|f\|_{L^1(\mathbb{R}^d)} \le \big\|\sup_{t\le\tau} |\mathbf{K}_t^U f|\big\|_{L^1(\mathbb{R}^d)}$, where $U \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

To end this section, let us make a short remark. In some papers local atomic spaces are defined in a slightly different manner. The remark below clarifies that different definitions lead to the same atomic Hardy space, that is, they are equivalent as Banach spaces.

REMARK 1.15. Let us consider $\mathcal{Q}$ and ω as above and a function $\mathfrak{a}$ that satisfies:

(1.16) there exist $Q \in \mathcal{Q}$ and a cube $K \subset Q^{**}$ such that

$$\operatorname{supp} \mathfrak{a} \subseteq K, \qquad 4d_K \geq d_Q, \qquad \|\mathfrak{a}\|_\infty \leq |K|^{-1}.$$

For each $\mathfrak{a}$ satisfying (1.16), we have $\|\mathfrak{a}\|_{H^1_{\mathrm{at}}(\mathcal{A}_{\omega,\mathcal{Q}})} \leq C$, with a universal C. To see this, observe that $\mathfrak{a}$ is a linear combination of two atoms: an atom satisfying the cancellation condition, and $|Q|^{-1}\chi_Q(x)$, namely

$$\mathfrak{a}(x) = \left(\mathfrak{a}(x) - \frac{\int_K \mathfrak{a}}{|Q|}\chi_Q(x)\right) + \frac{\int_K \mathfrak{a}}{|Q|}\chi_Q(x).$$

On the other hand, atoms of the form $|Q|^{-1}\chi_Q$ satisfy (1.16). Therefore, the functions $\mathfrak{a}$ as above can be substitutes for the atoms of the form $|Q|^{-1}\chi_Q(x)$ in the definition of $\mathcal{A}_{\omega,\mathcal{Q}}$.

2. Local Hardy spaces. In this section we consider a potential $V \geq 0$ which satisfies the global Kato condition (S). For $f \in L^1(\mathbb{R}^d)$ define a local version of $\mathbf{M}^V$ at scale $\tau > 0$ by

$$\mathbf{M}^V_\tau f(x) = \sup_{t \leq \tau^2} |\mathbf{K}^V_t f(x)|.$$

By definition, a function $f \in L^1(\mathbb{R}^d)$ is in the *local Hardy space* $h^1_\tau(\mathbf{L}^V)$ when $\mathbf{M}^V_\tau f$ is in $L^1(\mathbb{R}^d)$. We set

$$\|f\|_{h^1_\tau(\mathbf{L}^V)} := \|\mathbf{M}^V_\tau f\|_{L^1(\mathbb{R}^d)}.$$

In the special case $V \equiv 0$, the space $h^1_\tau(-\boldsymbol{\Delta})$ is the classical local Hardy space introduced by Goldberg [12]. It follows from [12] that

$$C^{-1}\|f\|_{H^1_{\mathrm{at}}(\mathcal{A}_{\mathcal{Q}[\tau]})} \leq \|f\|_{h^1_\tau(-\boldsymbol{\Delta})} \leq C\|f\|_{H^1_{\mathrm{at}}(\mathcal{A}_{\mathcal{Q}[\tau]})}, \tag{2.1}$$

where C does not depend on τ. The following proposition is a generalization of (2.1) for $h^1_\tau(\mathbf{L}^V)$ localized to a cube of diameter comparable to τ. It will play a crucial role in the proof of Theorem A.

PROPOSITION 2.2. *Assume that Q is a cube.*

(a) *Let a be an ω-atom such that* $\operatorname{supp} a \subseteq Q^{**}$ ***or*** $a(x) = |Q|^{-1}\chi_Q(x)$. *Then*

$$\|\mathbf{M}^V_{d_Q} a\|_{L^1(\mathbb{R}^d)} \leq C. \tag{2.3}$$

(b) *Assume that* $f \in L^1(\mathbb{R}^d)$, $\operatorname{supp} f \subseteq Q^*$, *and* $\mathbf{M}^V_{d_Q} f(x) \in L^1(\mathbb{R}^d)$. *There exist* λ_j *and* a_j, *the latter being either* ω*-atoms* ***or*** *of the form* $|Q|^{-1}\chi_Q(x)$, *such that*

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x), \qquad \sum_{j=1}^{\infty} |\lambda_j| \le C\|\mathbf{M}^V_{d_Q} f\|_{L^1(\mathbb{R}^d)}.$$

The constant C *above depends only on* d *and* θ *from the definition of* Q^*.

Proof. Consider first an ω-atom a. Obviously, $\mathbf{M}^V_{d_Q} a(x) \le \mathbf{M}^V a(x)$, so (2.3) holds by (1.9). In the case when $a = |Q|^{-1}\chi_Q$ we use (1.2) and (2.1) to get

$$\|\mathbf{M}^V_{d_Q} a\|_{L^1(\mathbb{R}^d)} \le \|\mathbf{M}^0_{d_Q} a\|_{L^1(\mathbb{R}^d)} \le C.$$

Now, let f be as in the assumptions of (b). Set

$$g(x) = f(x) - \mathbf{K}^V_{d_Q^2/2} f(x),$$

so that

$$f(x)\omega(x) = g(x)\omega(x) + \mathbf{K}^V_{d_Q^2/2} f(x)\omega(x) =: h_1(x) + h_2(x).$$

We claim that $h_1 \in H^1(-\mathbf{\Delta})$ and $h_2 \in h^1_{d_Q}(-\mathbf{\Delta})$ with

$$\|h_1\|_{H^1(-\mathbf{\Delta})} \le C\|\mathbf{M}^V_{d_Q} f\|_{L^1(\mathbb{R}^d)}, \tag{2.4}$$

$$\|h_2\|_{h^1_{d_Q}(-\mathbf{\Delta})} \le C\|\mathbf{M}^V_{d_Q} f\|_{L^1(\mathbb{R}^d)}. \tag{2.5}$$

To prove (2.4), observe that

$$\Big\| \sup_{t \le d_Q^2/2} |\mathbf{K}^V_t g| \Big\|_{L^1(\mathbb{R}^d)} \le 2\|\mathbf{M}^V_{d_Q} f\|_{L^1(\mathbb{R}^d)} < \infty.$$

Likewise,

$$\Big\| \sup_{t > d_Q^2/2} |\mathbf{K}^V_t g(x)| \Big\|_{L^1(\mathbb{R}^d)} \le C\|f\|_{L^1(\mathbb{R}^d)}$$

by the argument identical as in the proof of [10, Proposition 6.1]. By Corollary 7.8 below, we have $\|f\|_{L^1(\mathbb{R}^d)} \le \|\mathbf{M}^V_{d_Q} f\|_{L^1(\mathbb{R}^d)}$. Thus $g \in H^1(\mathbf{L}^V)$ and, by (1.9), $h_1 = g \cdot \omega \in H^1(-\mathbf{\Delta})$, so (2.4) is proved.

Now, we turn to (2.5). It is clear that

$$h_2(x) = \sum_{K \in \mathcal{Q}^{[d_Q]}} \mathbf{K}^V_{d_Q^2/2} f(x)\omega(x)\chi_K(x) = \sum_{K \in \mathcal{Q}^{[d_Q]}} h_K(x),$$

and using Corollary 7.8 once again, we see that

$$\begin{aligned}\|h_K\|_\infty &\le C \int_{Q^*} d_Q^{-d} \exp\Big(-\frac{|x-y|^2}{2d_Q^2}\Big)|f(y)|\,dy\\ &\le C|Q|^{-1}\exp\Big(-\frac{d(Q^*,K)^2}{2d_Q^2}\Big)\|\mathbf{M}^V_{d_Q} f\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

Clearly, $\operatorname{supp} h_K \subseteq K$, so by using the classical atomic characterization of $h^1_{d_Q}(-\boldsymbol{\Delta})$ we find that $\|h_K\|_{h^1_{d_Q}(-\boldsymbol{\Delta})} \le C\exp\big(-\frac{d(Q^*,K)^2}{2d_Q^2}\big)\|\mathbf{M}^V_{d_Q} f\|_{L^1(\mathbb{R}^d)}$. Adding up gives

$$\begin{aligned}\|h_2\|_{h^1_{d_Q}(-\boldsymbol{\Delta})} &\le C\|\mathbf{M}^V_{d_Q} f\|_{L^1(\mathbb{R}^d)} \sum_{K\in\mathcal{Q}^{[d_Q]}} \exp\Big(-\frac{d(Q^*,K)^2}{2d_Q^2}\Big)\\ &\le C\|\mathbf{M}^V_{d_Q} f\|_{L^1(\mathbb{R}^d)},\end{aligned}$$

where the last inequality is a simple geometric observation.

Having proved (2.4) and (2.5), we finish the proof by the following argument. The function $f\cdot\omega$ is supported in Q^* and $f\cdot\omega \in h^1_{d_Q}(-\boldsymbol{\Delta})$ with $\|f\cdot\omega\|_{h^1_{d_Q}(-\boldsymbol{\Delta})} \le C\|\mathbf{M}^V_{d_Q} f\|_{L^1(\mathbb{R}^d)}$. By the classical local characterization of $h^1_{d_Q}(-\boldsymbol{\Delta})$, we have $f\cdot\omega = \sum_j \lambda_j a_j$, where a_j are either classical atoms, or of the form $|Q|^{-1}\chi_Q(x)$. Moreover, $\sum_j|\lambda_j| \le C\|\mathbf{M}^V_{d_Q} f\|_{L^1(\mathbb{R}^d)}$. Then $f = \sum_j \lambda_j b_j$ where $b_j = a_j/\omega$ are either ω-atoms or $b_j = \omega^{-1}|Q|^{-1}\chi_Q$. In the latter case, b_j can be decomposed into a linear combination of a $|Q|^{-1}\chi_Q$-atom and an ω-atom, exactly as in Remark 1.15. ■

The following corollary is a "global" version of Proposition 2.2 and can be proved by standard techniques.

COROLLARY 2.6. *There exists a constant C, independent of $\tau > 0$, such that*

$$\|f\|_{h^1_\tau(\mathbf{L}^V)} \simeq \|f\|_{H^1_{\mathrm{at}}(\mathcal{A}_{\omega,\mathcal{Q}^{[\tau]}})}.$$

In particular, $h^1_\tau(\mathbf{L}^V) = H^1_{\mathrm{at}}(\mathcal{A}_{\omega,\mathcal{Q}^\tau})$.

3. Auxiliary estimates. In this section we present tools that will be used in the proof of Theorem A. The proofs of Lemmas 3.4, 3.5, 3.7, 3.8 are very similar to their analogues in [8]. Therefore we do not give all the details here, but we just sketch how to adapt the proofs from [8] to our setting.

Let $U_1, U_2 \ge 0$ be two potentials. A well-known perturbation formula (see, e.g., [15, Chap. 3]) states that

$$\mathbf{K}^{U_1}_t - \mathbf{K}^{U_1+U_2}_t = \int_0^t \mathbf{K}^{U_1}_{t-s}\, U_2\, \mathbf{K}^{U_1+U_2}_s\, ds. \tag{3.1}$$

For the kernels this reads

$$\tag{3.2} K_t^{U_1}(x,y) - K_t^{U_1+U_2}(x,y) = \int_0^t \int_{\mathbb{R}^d} K_{t-s}^{U_1}(x,z) U_2(z) K_s^{U_1+U_2}(z,y)\, dz\, ds.$$

With a family $\mathcal{Q}$ satisfying (G) we associate a partition of unity $\Phi = \{\phi_Q\}_{Q\in\mathcal{Q}}$ such that

$$\tag{3.3} 0 \le \phi_Q \in C_c^\infty(Q^*), \quad \mathbb{1}_{\mathbb{R}^d} = \sum_{Q\in\mathcal{Q}} \phi_Q, \quad \|\nabla\phi_Q\|_\infty \le C d_Q^{-1}.$$

LEMMA 3.4. *Let $U \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ be a positive potential. For $f \in L^1(\mathbb{R}^d)$ and $Q \in \mathcal{Q}$,*

$$\Big\| \sup_{t\le d_Q^2} |\mathbf{K}_t^U(\phi_Q f)| \Big\|_{L^1((Q^{**})^c)} \le \|\phi_Q f\|_{L^1(\mathbb{R}^d)}.$$

Proof. Let c_Q be the center of Q. For $t \le d_Q^2$, $y \in Q^*$ and $x \notin Q^{**}$ we have

$$\sup_{t\le d_Q^2} K_t^U(x,y) \le \sup_{t\le d_Q^2} C t^{-d/2} \exp\Big(-\frac{|x-c_Q|^2}{ct}\Big) \le C d_Q^{-d} \exp\Big(-\frac{|x-c_Q|^2}{c d_Q^2}\Big).$$

The lemma follows by integrating the last expression on $(Q^{**})^c$. ∎

LEMMA 3.5. *Assume* (K). *For $f \in L^1(\mathbb{R}^d)$ and $Q \in \mathcal{Q}$,*

$$\Big\| \sup_{t\le d_Q^2} |(\mathbf{K}_t^V - \mathbf{K}_t^{V+W})(\phi_Q f)| \Big\|_{L^1(\mathbb{R}^d)} \le C\|\phi_Q f\|_{L^1(\mathbb{R}^d)}.$$

Sketch of the proof. Using (3.1) we write

$$\begin{aligned}(\mathbf{K}_t^V - \mathbf{K}_t^{V+W})(\phi_Q f) &= \int_0^t \mathbf{K}_{t-s}^V (W\cdot \mathbb{1}_{(Q^{***})^c}) \mathbf{K}_s^{V+W}(\phi_Q f)\, ds \\ &\quad + \int_0^t \mathbf{K}_{t-s}^V (W\cdot \mathbb{1}_{Q^{***}}) \mathbf{K}_s^{V+W}(\phi_Q f)\, ds.\end{aligned}$$

Both summands can be estimated as in [8, Lemma 3.11]. In order to repeat the arguments of [8], one should have in mind that, by (3.2),

$$\tag{3.6} K_t^{V+W}(x,y) \le K_t^U(x,y) \le P_t(x-y),$$

where U is either V or W. The details are omitted. ∎

For each $Q \in \mathcal{Q}$ we set

$$\mathcal{Q}_{\mathrm{loc},Q} = \{Q' \in \mathcal{Q} : Q^{***} \cap Q'^{***} \ne \emptyset\},$$
$$\mathcal{Q}_{\mathrm{glob},Q} = \{Q'' \in \mathcal{Q} : Q^{***} \cap Q''^{***} = \emptyset\}.$$

Roughly speaking, for each Q, the set $\mathcal{Q}_{\mathrm{loc},Q}$ is the set of cubes $Q' \in \mathcal{Q}$ that are "close" to Q. For a function f denote

$$f_{\mathrm{loc},Q} = \sum_{Q' \in \mathcal{Q}_{\mathrm{loc},Q}} \phi_{Q'} f, \qquad f_{\mathrm{glob},Q} = f - f_{\mathrm{loc},Q}.$$

The following two lemmas and their proofs are almost identical to those of [8, Lemmas 3.7 and 3.8]. To see this, one only has to use (3.6). The details are left to the reader.

LEMMA 3.7. *For $f \in L^1(\mathbb{R}^d)$ and $Q \in \mathcal{Q}$,*

$$\Big\| \sup_{t>0} |\mathbf{K}_t^{V+W}(\phi_Q \cdot f_{\mathrm{loc},Q}) - \phi_Q \cdot \mathbf{K}_t^{V+W}(f_{\mathrm{loc},Q})| \Big\|_{L^1(Q^{**})} \le C \|f_{\mathrm{loc},Q}\|_{L^1(\mathbb{R}^d)}.$$

LEMMA 3.8. *Assume* (D). *For $f \in L^1(\mathbb{R}^d)$ and $Q \in \mathcal{Q}$,*

$$\sum_{Q \in \mathcal{Q}} \Big\| \sup_{t \le d_Q^2} |\mathbf{K}_t^{V+W}(f_{\mathrm{glob},Q})| \Big\|_{L^1(Q^*)} \le C \|f\|_{L^1(\mathbb{R}^d)}.$$

4. Proof of Theorem A. In the proof below, we shall often use the fact that, for $0 \le U \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\tau > 0$, we have

$$\|f\|_{L^1(\mathbb{R}^d)} \le \|\mathbf{M}_\tau^U f\|_{L^1(\mathbb{R}^d)}. \tag{4.1}$$

This is a consequence of the semigroup property and the Gaussian estimates. A detailed proof is given in the Appendix (see Proposition 7.5 and Corollary 7.8).

First implication. We start by proving the second inequality of (1.10), that is, for a function f such that $\|f\|_{H^1(\mathbf{L}^{V+W})} < \infty$ we will find $(\omega, \mathcal{Q})$-atoms a_i such that

$$f(x) = \sum_{i=1}^\infty \lambda_i a_i(x) \quad \text{and} \quad \sum_{i=1}^\infty |\lambda_i| \le C \|f\|_{H^1(\mathbf{L}^{V+W})}.$$

Let ϕ_Q be as in (3.3), in particular $f = \sum_{Q \in \mathcal{Q}} \phi_Q f$. The key estimate, which we now prove, is the following:

$$\sum_{Q \in \mathcal{Q}} \Big\| \sup_{t \le d_Q^2} |\mathbf{K}_t^V(\phi_Q f)(x)| \Big\|_{L^1(\mathbb{R}^d)} \le C \|f\|_{H^1(\mathbf{L}^{V+W})}. \tag{4.2}$$

By Lemma 3.4 we get $\sum_{Q \in \mathcal{Q}} \|\ldots\|_{L^1((Q^{**})^c)} \le C \|f\|_{L^1(\mathbb{R}^d)}$. Now we concentrate our attention on Q^{**}. Notice that

$$\begin{aligned} \mathbf{K}_t^V(\phi_Q f) &= [(\mathbf{K}_t^V - \mathbf{K}_t^{V+W})(\phi_Q f)] + [\mathbf{K}_t^{V+W}(\phi_Q f) - \phi_Q \cdot \mathbf{K}_t^{V+W}(f_{\mathrm{loc},Q})] \\ &\quad + [-\phi_Q \cdot \mathbf{K}_t^{V+W}(f_{\mathrm{glob},Q})] + [\phi_Q \cdot \mathbf{K}_t^{V+W}(f)] \\ &=: A_1 + A_2 + A_3 + A_4. \end{aligned}$$

Notice that $\phi_Q f_{\mathrm{loc},Q} = \phi_Q f$. Lemmas 3.5, 3.7 and 3.8 lead to

$$\begin{aligned}\sum_{k=1}^{3} \sum_{Q\in\mathcal{Q}} \Big\| \sup_{t\le d_Q^2} |A_k| \Big\|_{L^1(Q^{**})} &\le C \sum_{Q\in\mathcal{Q}} (\|\phi_Q \cdot f\|_{L^1(\mathbb{R}^d)} + \|f_{\mathrm{loc},Q}\|_{L^1(\mathbb{R}^d)}) \\ &\quad + \|f\|_{L^1(\mathbb{R}^d)} \\ &\le C\|f\|_{L^1(\mathbb{R}^d)} \le C\|f\|_{H^1(\mathbf{L}^{V+W})},\end{aligned}$$

where we have used (4.1) and

$$\begin{aligned}\sum_{Q\in\mathcal{Q}} \|f_{\mathrm{loc},Q}\|_{L^1(\mathbb{R}^d)} &\le \sum_{Q\in\mathcal{Q}} \sum_{Q'\in\mathcal{Q}_{\mathrm{loc},Q}} \|\phi_{Q'} f\|_{L^1(\mathbb{R}^d)} \\ &= \sum_{Q'\in\mathcal{Q}} \sum_{Q\in\mathcal{Q}_{\mathrm{loc},Q'}} \|\phi_{Q'} f\|_{L^1(\mathbb{R}^d)} \\ &\le C \sum_{Q'\in\mathcal{Q}} \|\phi_{Q'} f\|_{L^1(\mathbb{R}^d)} \le C\|f\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

The proof of (4.2) is finished by noticing that

$$\sum_{Q\in\mathcal{Q}} \Big\| \sup_{t\le d_Q^2} |A_4| \Big\|_{L^1(Q^{**})} \le C\|f\|_{H^1(\mathbf{L}^{V+W})}.$$

Having proved (4.2), we apply Proposition 2.2(b) to $\phi_Q f$, obtaining $\lambda_{j,Q}$ and $(\omega, \mathcal{Q})$-atoms $a_{j,Q}$ such that

$$\phi_Q(x) f(x) = \sum_{j=1}^{\infty} \lambda_{j,Q} a_{j,Q}(x)$$

with

$$\sum_{j=1}^{\infty} |\lambda_{j,Q}| \le C \Big\| \sup_{t\le d_Q^2} |\mathbf{K}_t^V(\phi_Q f)(x)| \Big\|_{L^1(\mathbb{R}^d)}.$$

Therefore,

$$f(x) = \sum_{j,Q} \lambda_{j,Q} a_{j,Q}(x) \quad \text{with} \quad \sum_{j,Q} |\lambda_{j,Q}| \le C\|f\|_{H^1(\mathbf{L}^{V+W})},$$

and the proof of the first part is finished.

Second implication. By the sublinearity of $\mathbf{M}^{V+W}$ it is enough to prove that

$$\Big\| \sup_{t>0} |\mathbf{K}_t^{V+W} a| \Big\|_{L^1(\mathbb{R}^d)} \le C$$

for $a \in \mathcal{A}_{\omega,\mathcal{Q}}$. Assume then that $\operatorname{supp} a \subseteq Q^{**}$, where $Q \in \mathcal{Q}$. By the definition of $\mathcal{Q}_{\mathrm{loc},Q}$ and ϕ_Q it is clear that $a = a_{\mathrm{loc},Q}$. From (G_3) there exists a universal constant $m \in \mathbb{N}$ such that $d_{Q'}^2 \ge 2^{-m} d_Q^2$ whenever $Q' \in \mathcal{Q}_{\mathrm{loc},Q}$.

We have

$$\Big\| \sup_{t\le 2^{-m}d_Q^2} |\mathbf{K}_t^{V+W} a| \Big\|_{L^1(\mathbb{R}^d)} \le \sum_{Q'\in \mathcal{Q}_{\mathrm{loc},Q}} \Big\| \sup_{t\le d_{Q'}^2} |(\mathbf{K}_t^{V+W} - \mathbf{K}_t^{V})(\phi_{Q'} a)| \Big\|_{L^1(\mathbb{R}^d)} + \Big\| \sup_{t\le d_Q^2} |\mathbf{K}_t^{V} a| \Big\|_{L^1(\mathbb{R}^d)}.$$

By Lemma 3.5, the sum is bounded by $C\|a\|_{L^1(\mathbb{R}^d)} \le C$. The second summand is bounded by Proposition 2.2(a).

What is left is to consider $t \ge 2^{-m}d_Q^2$. Denote

$$I_j = [2^j d_Q^2, 2^{j+1} d_Q^2], \qquad I_j^{\Diamond} = [2^{j-1} d_Q^2, 3\cdot 2^{j-1} d_Q^2].$$

Note that $I_j = \{x + 2^{j-1}d_Q^2 : x \in I_j^{\Diamond}\}$. By (1.2) it is not hard to check that for $g\in L^1(\mathbb{R}^d)$ we have

$$\Big\| \sup_{t\in I_j^{\Diamond}\cup I_j} |\mathbf{K}_t^{V+W} g| \Big\|_{L^1(\mathbb{R}^d)} \le C\|g\|_{L^1(\mathbb{R}^d)},$$

where C does not depend on j or g. Therefore, for $j\ge 2$,

$$\Big\| \sup_{t\in I_j} |\mathbf{K}_t^{V+W} a| \Big\|_{L^1(\mathbb{R}^d)} \le \Big\| \sup_{t\in I_j^{\Diamond}} \mathbf{K}_t^{V+W} (\mathbf{K}^W_{2^{j-1}d_Q^2} |a|) \Big\|_{L^1(\mathbb{R}^d)} \le C \big\| \mathbf{K}^W_{2^{j-1}d_Q^2} |a| \big\|_{L^1(\mathbb{R}^d)} \le C j^{-1-\varepsilon},$$

where in the last inequality we have used (D). The proof is finished by noticing that

$$\Big\| \sup_{t\ge 2^{-m}d_Q^2} |\mathbf{K}_t^{V+W} a| \Big\|_{L^1(\mathbb{R}^d)} \le \sum_{j=-m}^{\infty} \Big\| \sup_{t\in I_j} |\mathbf{K}_t^{V+W} a| \Big\|_{L^1(\mathbb{R}^d)} \le C\Big(m + 2 + \sum_{j=2}^{\infty} j^{-1-\varepsilon}\Big) \le C.$$

5. Proof of Theorem B. The proof follows by a known procedure that uses atomic decompositions. Assume that $W, V, \mathcal{Q}, \omega$ are given and ω satisfies (1.12).

To prove one of the inequalities of (1.13) it is enough to show that

$$\|a\|_{H^1_{\mathrm{at}}(\mathcal{A}_{\mathcal{Q}})} \le C \tag{5.1}$$

for every $a\in \mathcal{A}_{\omega,\mathcal{Q}}$. Obviously, if a is an atom of the form $a(x) = |Q|^{-1}\chi_Q(x)$, the inequality (5.1) holds with $C=1$. Assume then that a is such that $\operatorname{supp} a \subseteq K \subseteq Q^{**}$, $Q\in\mathcal{Q}$, $\|a\|_\infty \le |K|^{-1}$, and $\int_K a(x)\omega(x)\,dx = 0$. Take a sequence of cubes G_n such that

$$K = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_N \subseteq Q^{**}, \qquad d_{G_{n+1}} = 2d_{G_n} \quad (n = 0,\dots,N-1),$$

and $d_Q \le 2d_{G_N}$. Observe that $N \le C(\log_2(d_Q/d_K) + 1)$ and $a(x) = \sum_{n=0}^{N+2} b_n(x)$, where

$$\begin{aligned} b_0(x) &= a(x) - t_0\chi_{G_0}(x), \\ b_n(x) &= t_{n-1}\chi_{G_{n-1}}(x) - t_n\chi_{G_n}(x) \quad (n = 1, \dots, N), \\ b_{N+1} &= t_N\chi_{G_N}(x) - t_{N+1}|Q|^{-1}\chi_Q(x), \\ b_{N+2} &= t_{N+1}|Q|^{-1}\chi_Q(x). \end{aligned}$$

The constants t_n are chosen so that $\int b_n(x)\,dx = 0$ for $n = 0, \dots, N+1$, i.e.

$$\begin{aligned} t_0 &= |G_0|^{-1} \int_{G_0} a(x)\,dx, \\ t_n &= 2^{-d} t_{n-1} \quad (n = 1, \dots, N), \\ t_{N+1} &= t_N|G_N|. \end{aligned}$$

The key estimate, which uses (1.12) and the cancellation property, is the following:

$$\begin{aligned} |t_0| &= |K|^{-1}\omega(c_K)^{-1}\Big|\int_K a(x)(\omega(c_K) - \omega(x))\,dx\Big| \\ &\le C|K|^{-2}\int_K \left(\frac{|x - c_K|}{d_Q}\right)^{\lambda} dx \le C|K|^{-1}\left(\frac{d_K}{d_Q}\right)^{\lambda} \le C2^{-cN}|K|^{-1}. \end{aligned}$$

Thus $|t_n| \le C2^{-cN}|G_n|^{-1}$ for $n = 1, \dots, N$, and $|t_{N+1}| \le C$.

Obviously, $\operatorname{supp} b_n \subseteq G_n$ for $n = 0, \dots, N$, and $\operatorname{supp} b_{N+1} \subseteq Q^{**}$. Moreover,

$$\begin{aligned} \|b_0\|_\infty &\le |K|^{-1} + |t_0| \le C|K|^{-1}, \\ \|b_n\|_\infty &\le C|t_{n-1}| \le C2^{-cN}|G_n|^{-1} \quad (n = 1, \dots, N), \\ \|b_{N+1}\|_\infty &\le C|Q^{**}|^{-1}. \end{aligned}$$

As a consequence, all b_n are multiples of $H^1_{\rm at}(\mathcal{A}_{\mathcal{Q}})$-atoms and (5.1) is proved, since

$$\|a\|_{H^1_{\rm at}(\mathcal{A}_{\mathcal{Q}})} \le \sum_{n=0}^{N+2} \|b_n\|_{H^1_{\rm at}(\mathcal{A}_{\mathcal{Q}})} \le CN2^{-cN} + 3C \le C.$$

For the second inequality one should consider $a \in H^1_{\rm at}(\mathcal{A}_{\mathcal{Q}})$ and prove that

$$\|a\|_{H^1_{\rm at}(\mathcal{A}_{\omega,\mathcal{Q}})} \le C.$$

This can be done in a similar fashion. The details are omitted.

6. Example C. Denote $c_n = 2^n\mathbf{e}_1$ and $C_n = Q(c_n, 1/(2n))$, where $\mathbf{e}_1$ denotes the vector $(1, 0, \dots, 0)$ in $\mathbb{R}^d$. The potential $\mathcal{V}$ that is needed for

Example C is

$$\mathcal{V}(x)=\sum_{k=2}^{\infty}k^2\chi_{C_k}(x). \tag{6.1}$$

LEMMA 6.2. *$\mathcal{V}$ satisfies* (S).

Proof. Fix $x\in\mathbb{R}^d$. Write

$$\int_{\mathbb{R}^d}\mathcal{V}(y)|x-y|^{2-d}\,dy=\sum_{k=2}^{\infty}k^2\int_{C_k}|x-y|^{2-d}\,dy=:\sum_{k=2}^{\infty}I_k.$$

We have

$$I_k\le k^2\int_{C_k}|y-c_k|^{2-d}\,dy\le C\qquad(x\in\mathbb{R}^d), \tag{6.3}$$

$$I_k\le Ck^2\int_{C_k}|x-c_k|^{2-d}\,dy\le C(k|x-c_k|)^{2-d}\qquad(x\notin 2C_k). \tag{6.4}$$

Consider $x=(x_1,\dots,x_d)$ and let $N\ge 2$ be such that $2^N<x_1\le 2^{N+1}$ ($N=2$ when $x_1\le 8$). Then

$$\sum_{k=2}^{\infty}I_k=\sum_{k=2}^{N-1}I_k+(I_N+I_{N+1})+\sum_{k=N+2}^{\infty}I_k(x)=A_1+A_2+A_3,$$

with obvious modification when $N=2$. Obviously, $A_2\le C$ by (6.3). Moreover, for $k\ne N$ and $k\ne N+1$, we have $|x-c_k|\ge c2^{\max(N,k)}$, so using (6.4) we obtain

$$A_1\le C\sum_{k=2}^{N-2}(k2^N)^{2-d}\le C,\qquad A_3\le C\sum_{k=N+1}^{\infty}(k2^k)^{2-d}\le C. \ \blacksquare$$

For the rest of this section, by ω we mean $\omega(\mathcal{V})$ for $\mathcal{V}$ given by (6.1). The following lemma gives essential information about local oscillations of ω.

PROPOSITION 6.5. *Let c_n and C_n be as above, and*

$$d_n=c_n+(\tau/n)\mathbf{e}_1,\qquad D_n=Q(d_n,1/(2n)).$$

There exist $\tau>3$, $c_0>0$, and $N\in\mathbb{N}$ such that for $n\ge N$ we have

$$\inf_{x\in D_n,\ y\in C_n}(\omega(x)-\omega(y))\ge c_0. \tag{6.6}$$

Let us remark that ω satisfying (6.6) cannot fulfill the global Hölder condition. To see this, just observe that $|c_n-d_n|\to 0$ and $\omega(d_n)-\omega(c_n)\ge c_0$.

Proof of Proposition 6.5. Recall that $K_t^{\mathcal{V}}(x,y)$ always satisfies Gaussian upper bounds (see (1.2)). By Lemma 6.2, there are also Gaussian lower bounds. Let κ_1,κ_2 be as in (1.8) and set $\kappa=\min(\kappa_1,\kappa_2)$. Set $U_1=0,\ U_2=\mathcal{V}$

in (3.2), integrate with respect to x, and let t tend to infinity. We obtain

$$1-\omega(p)=\int_0^\infty\int_{\mathbb{R}^d}\mathcal{V}(z)K_s^{\mathcal{V}}(z,p)\,dz\,ds.$$

It is enough to show that, for properly chosen τ and c_0, the following estimates hold for $x\in D_n$, $y\in C_n$, and n large enough:

$$1-\omega(y)=\int_0^\infty\int_{\mathbb{R}^d}\mathcal{V}(z)K_s^{\mathcal{V}}(z,y)\,dz\,ds\ge 2c_0, \tag{6.7}$$

$$1-\omega(x)=\int_0^\infty\int_{\mathbb{R}^d}\mathcal{V}(z)K_s^{\mathcal{V}}(z,x)\,dz\,ds\le c_0. \tag{6.8}$$

Fix $n\ge 2$ and $y\in C_n$. By (1.8) and (6.1),

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}^d}\mathcal{V}(z)K_s^{\mathcal{V}}(z,y)\,dz\,ds &\ge c\int_0^\infty\int_{C_n}\kappa n^2 s^{-d/2}\exp\Big(-\frac{|z-y|^2}{\kappa s}\Big)\,dz\,ds\\ &= c\kappa n^2\int_{C_n}|z-y|^{2-d}dz\cdot\int_0^\infty s^{-d/2}e^{-1/(\kappa s)}\,ds\\ &\ge c(d,\kappa)=:2c_0.\end{aligned}$$

Thus (6.7) is proved. For $x\in D_n$,

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}^d}\mathcal{V}(z)K_s^{\mathcal{V}}(z,x)\,dy\,ds &\le C\sum_{k=2}^\infty k^2\int_0^\infty\int_{C_k}s^{-d/2}\exp\Big(-\frac{|z-x|^2}{4s}\Big)\,dz\,ds\\ &\le Cn^2\int_{C_n}|z-x|^{2-d}\,dz+C\sum_{2\le k\ne n}k^2\int_{C_k}|z-x|^{2-d}\,dz\\ &=: A_1+A_2.\end{aligned}$$

Observe that if $x\in D_n$ and $z\in C_n$, then $|x-z|\ge(\tau-2)/(2n)$. Therefore,

$$A_1\le Cn^2(\tau/n)^{2-d}n^{-d}=C\tau^{2-d}\le c_0/2,$$

where the last inequality holds for τ large enough. Fix such a τ. In what follows we consider only $n\ge N_1$ such that $d(c_n,d_n)\le 1/2$. For such n and $k\ne n$ we have $|z-x|\ge c2^{\max(n,k)}$ for $z\in C_k$ and $x\in D_n$. Thus,

$$\begin{aligned}A_2=\sum_{2\le k<n}\cdots+\sum_{k>n}\cdots &\le C\sum_{2\le k<n}k^2 2^{n(2-d)}k^{-d}+C\sum_{k>n}k^2 2^{k(2-d)}k^{-d}\\ &\le Cn2^{n(2-d)}+C2^{n(2-d)}\le c_0/2,\end{aligned}$$

where the last estimate holds for $n\ge N_2$. The proof of (6.8) is finished by taking $N=\max(N_1,N_2)$. ∎

Recall that $\mathcal{Q}^{[1]}$ consists of the cubes of radii equal to 1 that satisfy (G). We are now in a position to prove that the spaces $H^1_{\mathrm{at}}(\mathcal{A}_{\mathcal{Q}^{[1]}})$ and $H^1_{\mathrm{at}}(\mathcal{A}_{\mathcal{Q}^{[1]},\omega})$ are not equivalent as Banach spaces.

PROPOSITION 6.9. *There exists a sequence a_n of $(\mathcal{Q}^{[1]},\omega)$-atoms such that*

$$\|a_n\|_{H^1_{\mathrm{at}}(\mathcal{A}_{\mathcal{Q}^{[1]}})} \geq c \ln n. \tag{6.10}$$

Proof. In this proof we use the notation introduced in this section. For a cube R set $\omega(R) = \int_R \omega(x)\,dx$ and $\mu_n = \omega(D_n)\omega(C_n)^{-1}$. The atoms we are looking for are

$$a_n(x) = \zeta n^d(\mu_n \chi_{C_n}(x) - \chi_{D_n}(x)),$$

where $\zeta > 0$ is a constant that will be fixed in a moment.

Let us check that the a_n are $(\mathcal{Q}^{[1]},\omega)$-atoms. Obviously $\operatorname{supp} a_n \subseteq K_n := Q(c_n, (\tau+1)/n)$. By the definition of μ_n, $\int_{K_n} a_n(x)\omega(x)\,dx = 0$. Recall that $|C_n| = |D_n|$, so by (1.7) we get $\mu_n \leq \delta^{-1}$. Moreover, by using Proposition 6.5, for $n \geq N$,

$$\begin{aligned}\mu_n &\geq \frac{\inf\{\omega(x) : x \in D_n\}}{\sup\{\omega(y) : y \in C_n\}} \\ &= 1 + \frac{\inf\{\omega(x) - \omega(y) : x \in D_n,\, y \in C_n\}}{\sup\{\omega(y) : y \in C_n\}} \geq 1 + c_0.\end{aligned} \tag{6.11}$$

What is left is to check the size condition. By choosing a proper $\zeta > 0$ we can write

$$\|a_n\|_\infty \leq \zeta n^d \delta^{-1} \leq |K_n|^{-1},$$

so the a_n are indeed $(\omega, \mathcal{Q})$-atoms.

Now we prove (6.10). For the collection $\mathcal{Q}^{[1]}$, the space $H^1_{\mathrm{at}}(\mathcal{A}_{\mathcal{Q}^{[1]}})$ is the classical local Hardy space. Equivalently, the norm can be given by a local maximal operator (see (2.1)),

$$\|f\|_{H^1_{\mathrm{at}}(\mathcal{Q}^{[1]})} \simeq \Big\|\sup_{t\leq 1} |\mathbf{K}^0_t f|\Big\|_{L^1(\mathbb{R}^d)}.$$

Denote

$$S_n = \{x \in \mathbb{R}^d : \sqrt{d}/n < |x - c_n| < 1,\ (x)_1 < (c_n)_1\}, \tag{6.12}$$

where $(x)_1$ is the first coordinate of $x \in \mathbb{R}^d$ (see Figure 1). Obviously, $|S_n| \simeq C$. Assume now that $x \in S_n$ for some n. By (6.11),

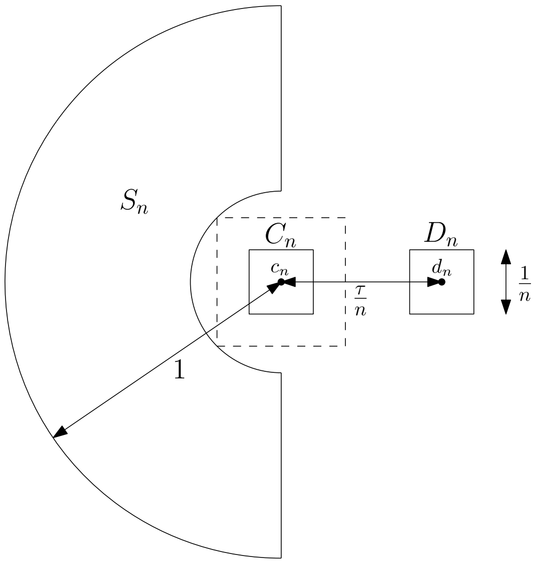

Fig. 1. The sets C_n, D_n, S_n

$$\begin{aligned}\mathbf{K}_t^0 a_n(x) &= \zeta n^d \int_{\mathbb{R}^d} (4\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right)(\mu_n \chi_{C_n}(y) - \chi_{D_n}(y))\, dy \\ &\geq C n^d t^{-d/2} \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{4t}\right)(\chi_{C_n}(y) - \chi_{D_n}(y))\, dy \\ &\quad + C n^d t^{-d/2} c_0 \int_{C_n} \exp\left(-\frac{|x-y|^2}{4t}\right) dy \\ &=: A_1 + A_2.\end{aligned}$$

We claim that $A_1 \geq 0$. Indeed, $D_n = C_n + (\tau/n)\mathbf{e}_1$ and for $x \in S_n$, $y_1 \in C_n$ and $y_2 = y_1 + (\tau/n)\mathbf{e}_1$ we have $|y_1 - x| < |y_2 - x|$ (cf. (6.12)). We obtain

$$A_1 = C n^d t^{-d/2} \int_{C_n} \left(\exp\left(-\frac{|x-y|^2}{4t}\right) - \exp\left(-\frac{|x-y-\frac{\tau}{n}\mathbf{e}_1|^2}{4t}\right)\right) dy \geq 0.$$

Now we deal with A_2. For $x \in S_n$ and $y \in C_n$ we have $|x-y| \leq 2|x-c_n|$. Thus,

$$A_2 \geq C t^{-d/2} \exp\left(-\frac{|x-c_n|^2}{t}\right).$$

Taking $t = |x-c_n|^2 \leq 1$ we find that $\sup_{t\leq 1} A_2 \geq C|x-c_n|^{-d}$. The proof is finished by noticing that

$$\Big\| \sup_{t\leq 1} |\mathbf{K}_t^0 a_n(x)| \Big\|_{L^1(S_n)} \geq C \int_{S_n} |x-c_n|^{-d}\, dx \geq C \ln n,$$

where the last inequality is easily obtained by integrating in spherical coordinates. ∎

7. Appendix. In this Appendix we consider a semigroup $(\mathbf{T}_t)_{t>0}$, strongly continuous on $L^2(\mathbb{R}^d)$, that has positive integral kernel $T_t(x,y)$ satisfying (1.2). Obviously, all Schrödinger semigroups $\mathbf{K}_t^U$ with $0 \le U \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfy these assumptions. Our goal is to give a precise proof of a natural estimate given in Corollary 7.8.

LEMMA 7.1. *Let $r > 0$. For a.e. $x \in \mathbb{R}^d$,*

$$\lim_{t\to 0} \int_{|x-y|>r} T_t(x,y)\,dy = 0, \tag{7.2}$$

$$\lim_{t\to 0} \int_{|x-y|<r} T_t(x,y)\,dy = 1. \tag{7.3}$$

Proof. The limit (7.2) is a simple consequence of (1.2). To prove (7.3) we shall use the fact that $\lim_{t\to 0} \mathbf{T}_t f = f$, where the convergence is in $L^2(\mathbb{R}^d)$.

From L^2 convergence we have a.e. convergence for a subsequence. Applying this to $f_n(x) = \chi_{Q(0,n)}(x)$, by a diagonal argument we obtain a sequence $t_k > 0$ that tends to zero and such that for a.e. $x \in \mathbb{R}^d$ we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} T_{t_k}(x,y)\,dy = 1. \tag{7.4}$$

Now, we are going to prove (7.4) for an arbitrary sequence s_j such that $\lim_j s_j = 0$. Without loss of generality we can assume that t_k is decreasing. For $j \in \mathbb{N}$, let k_j be such that $t_{k_j-1} < s_j \le t_{k_j}$ ($k_j = 1$ when $s_j > t_{k_1}$). Then $t_{k_j} = s_j + r_j$, where $\lim_{j\to\infty} t_{k_j} = \lim_{j\to\infty} r_j = 0$. By (1.2) and the semigroup property,

$$\int_{\mathbb{R}^d} T_{t_{k_j}}(x,y)\,dy = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} T_{s_j}(x,z) T_{r_j}(z,y)\,dz\,dy \le \int_{\mathbb{R}^d} T_{s_j}(x,z) \le 1.$$

Letting $j \to \infty$, by (7.4) we see that $\lim_{j\to\infty} \int_{\mathbb{R}^d} T_{s_j}(x,z)\,dz = 1$. ∎

PROPOSITION 7.5. *Assume that $f \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$. For almost every $x \in \mathbb{R}^d$,*

$$\lim_{t\to 0} \mathbf{T}_t f(x) = f(x).$$

Proof. Since $f \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) \subseteq L^1_{\mathrm{loc}}(\mathbb{R}^d)$, by the Lebesgue differentiation theorem we have

$$\lim_{s\to 0} |Q(x,s)|^{-1} \int_{Q(x,s)} |f(y) - f(x)|\,dy = 0 \tag{7.6}$$

for a.e. $x \in \mathbb{R}^d$. Assume that $x \in \mathbb{R}^d$ is such that (7.6), (7.2) and (7.3) are satisfied for all rational $r > 0$. The set of such points has full measure. For

$\varepsilon > 0$ fixed, we shall show that $|\mathbf{T}_t f(x) - f(x)| < C\varepsilon$ for t small enough. Let $r > 0$ be a fixed rational number such that for $s < r$ we have

$$\int_{Q(x,s)} |f(y) - f(x)|\, dy \le \varepsilon |Q(x,s)|. \tag{7.7}$$

We can assume that $\sqrt{t} < r$. For such t, write

$$\begin{aligned}\mathbf{T}_t f(x) - f(x) &= f(x)\Big(\int_{|x-y|<r} T_t(x,y)\, dy - 1\Big) + \int_{|x-y|>r} T_t(x,y) f(y)\, dy \\ &\quad + \int_{|x-y|<\sqrt{t}} T_t(x,y)(f(y) - f(x))\, dy \\ &\quad + \int_{\sqrt{t}<|x-y|<r} T_t(x,y)(f(y) - f(x))\, dy \\ &=: A_1 + A_2 + A_3 + A_4.\end{aligned}$$

By using (7.3), we find that $A_1 < \varepsilon$ for t small enough. For the summand A_2 we consider two cases:

- if $f \in L^\infty(\mathbb{R}^d)$, then $|A_2| < \varepsilon$ for t small enough by (7.2),
- if $f \in L^1(\mathbb{R}^d)$, then $|A_2| \le Ct^{-d/2} \exp(-r^2/t) \|f\|_{L^1(\mathbb{R}^d)} < \varepsilon$ for t small enough.

By (1.2) and (7.7), for t small enough,

$$A_3 \le Ct^{-d/2} \int_{|x-y|<\sqrt{t}} |f(y) - f(x)|\, dy \le C\varepsilon.$$

To estimate A_4 denote $N = \lceil \log_2(r/\sqrt{t}) \rceil$, so that $r \le \sqrt{t}\, 2^N \le 2r$. Let

$$R_n = \{x \in \mathbb{R}^d : r2^{-n} < |x-y| < r2^{-n+1}\}$$

for $n = 1, \ldots, N$. By (1.2) and (7.7),

$$\begin{aligned}A_4 &\le Ct^{-d/2} \sum_{n=1}^{N} \int_{R_n} \exp\Big(-\frac{|x-y|^2}{4t}\Big) |f(y) - f(x)|\, dy \\ &\le Ct^{-d/2} \sum_{n=1}^{N} \exp\Big(-\frac{r2^{-n}}{ct}\Big) \int_{R_n} |f(y) - f(x)|\, dy \\ &\le C\varepsilon \sum_{n=1}^{N} \Big(\frac{r2^{-n}}{\sqrt{t}}\Big)^d \exp\Big(-\frac{r2^{-n}}{c\sqrt{t}}\Big) \le C\varepsilon \frac{\sqrt{t}\, 2^N}{r} \le C\varepsilon. \ \blacksquare\end{aligned}$$

COROLLARY 7.8. *Let $0 \le U \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\tau > 0$. Then*

$$\|f\|_{L^1(\mathbb{R}^d)} \le \|\mathbf{M}^U_\tau f\|_{L^1(\mathbb{R}^d)}.$$

REMARK 7.9. Let us point out that Corollary 7.8 could be proved without using Proposition 7.5, as was pointed out by the referee. One can truncate a given $f \in L^1(\mathbb{R}^d)$ at a level M to get an L^2 function, and then use the L^2-continuity of $\mathbf{T}_t$ together with the contractivity of $\mathbf{T}_t$ on $L^1(\mathbb{R}^d)$. The details are omitted.

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