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Leinert sets and complemented ideals in Fourier algebras

by

Michael Brannan (College Station, TX), Brian Forrest (Waterloo) and Cameron Zwarich (Cupertino, CA)

Abstract. We show how complemented ideals in the Fourier algebra $A(G)$ of G arise naturally from a class of thin sets known as Leinert sets. Moreover, we present an explicit example of a closed ideal in $A(\mathbb{F}_N)$, where $\mathbb{F}_N$ is the free group on $N \geq 2$ generators, that is complemented in $A(\mathbb{F}_N)$ but it is not completely complemented. Then by establishing an appropriate extension result for restriction algebras arising from Leinert sets, we show that any almost connected group G for which every complemented ideal in $A(G)$ is also completely complemented must be amenable.

1. Introduction. Throughout this paper, G will denote a locally compact group with a fixed left Haar measure dx. We equip $L^1(G)$ with convolution and involution given by $f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$, making it into an involutive Banach algebra which we call the *group algebra* of G. The *group C^*-algebra* of G, which we denote by $C^*(G)$, is simply the enveloping C^*-algebra of $L^1(G)$.

By a *representation* of G we will mean a homomorphism $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$, where $\mathcal{U}(\mathcal{H}_\pi)$ is the group of unitary operators on the Hilbert space $\mathcal{H}_\pi$. Given such a π, and $\xi, \eta \in \mathcal{H}$ we call the function

$$\pi_{\xi,\eta} : G \to \mathbb{C}, \qquad \pi_{\xi,\eta}(x) = \langle \pi(x)\xi \mid \eta \rangle_{\mathcal{H}_\pi},$$

a *coefficient function* of π. We say that π is continuous if each of its coefficient functions is continuous.

We let

$$B(G) = \{u = \pi_{\xi,\eta} : \pi \text{ is a continuous representation of } G \text{ and } \xi, \eta \in \mathcal{H}_\pi\}.$$

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Then $B(G)$ can be identified as the dual of $C^*(G)$ via the dual pairing

$$\langle u, f\rangle = \int_G u(x)f(x)\,dx$$

for every $f \in L^1(G)$. With respect to the dual norm and the usual pointwise operations, $B(G)$ becomes a commutative Banach algebra which we call the *Fourier–Sieltjes algebra* of G. The set of all elements $u = \pi_{\xi,\xi} \in B(G)$ is denoted by $P(G)$; it is precisely the cone of continuous positive definite functions on G and is identified with the space of all positive linear functionals on $C^*(G)$ under the identification $B(G) = C^*(G)^*$.

Amongst all continuous unitary representations of G, the most important would be the *left regular representation* $\lambda : G \to \mathcal{U}(L^2(G))$, defined by

$$\lambda(x)(f)(y) = f(x^{-1}y)$$

for each $f \in L^2(G)$. We then define

$$A(G) = \{u(x) = \lambda_{f,g}(x) : f, g \in L^2(G)\}.$$

This is a closed ideal of $B(G)$, which we call the *Fourier algebra* of G. Its dual space is the *group von Neumann algebra* $\mathrm{VN}(G)$, given by

$$\mathrm{VN}(G) = \overline{\mathrm{span}\{\lambda(x) : x \in G\}}^{\mathrm{WOT}} \subseteq \mathcal{B}(L^2(G)).$$

We note that as the predual of a von Neumann algebra the Fourier algebra inherits a natural operator space structure under which it becomes a completely contractive Banach algebra.

We refer the reader to [10] for the basic properties of $A(G)$ and $B(G)$.

The problem of identifying complemented ideals in the Fourier algebra $A(G)$ goes back to work of D. J. Newman [19]. He showed that if $\mathbb{T}$ is the circle group and if

$$H^1 = \{f \in L^1(\mathbb{T}) : \hat{f}(n) = 0 \text{ for every } n < 0\},$$

then H^1 is not complemented in $L^1(\mathbb{T})$. Later work of Rosenthall [26], Alspach and Matheson [2] and Alspach, Matheson and Rosenblatt [3], [4] showed that, for abelian groups, complemented ideals in $A(G)$ arise from certain elements of the closed coset ring $\mathcal{R}_c(G)$ of the group, but that a complete classification of such ideals is extremely difficult even for $\mathbb{R}^3$.

For nonabelian groups the problem of identifying complemented ideals in the Fourier algebra is still very much in its infancy. If we recognize the additional structure that $A(G)$ carries as an operator space, and we ask that our projection be completely bounded, then it is known from [12] that if G is amenable, then the ideal must be of the form $I = I_G(E)$ where $E \in \mathcal{R}_c(G)$. However, as we will also make clear later on, the necessity that $E \in \mathcal{R}_c(G)$ for $I = I_G(E)$ to be completely complemented does not hold for $\mathbb{F}_N$, the free group on $N \geq 2$ generators. The key here is to exhibit a set $E \notin \mathcal{R}_c(G)$

for which the characteristic function 1_E is a completely bounded multiplier of $A(G)$. Moreover, we will show that for many nonamenable almost connected groups G, there exists a set $E \notin \mathcal{R}_c(G)$ such that $I_G(E)$ is completely complemented in $A(G)$ (see Proposition 3.22).

Let G be a discrete group. We call a set $E \subseteq G$ a *Leinert set* if the restriction algebra

$$A_G(E) = \{\varphi : E \to \mathbb{C} : \varphi = u|_E \text{ for some } u \in A(G)\}$$

is isomorphic to $\ell^2(E)$ as Banach algebras.

We call E a *strong Leinert set* if $\ell^\infty(E) \subseteq M_{\mathrm{cb}}A(G)$, the algebra of completely bounded multipliers of $A(G)$.

Let G be a locally compact group, H a closed discrete subgroup of G, and let E be a Leinert set in H. Then we will show that:

(i) The ideal $I_G(E)$ of functions in $A(G)$ vanishing on E is always a complemented Banach subspace of $A(G)$ (Theorem 3.16).
(ii) $I_G(E)$ is always an *invariantly weakly complemented* ideal in $A(G)$. That is, $I_G(E)^\perp \subseteq \mathrm{VN}(G)$ is complemented by a projection which is also an $A(G)$-bimodule map (Proposition 3.19).
(iii) If H is a noncommutative free group, then there are Leinert sets $E \subseteq H$ for which the ideal $I_G(E)$ is (weakly) complemented in $A(G)$, but not (weakly) completely complemented in $A(G)$ (Corollary 3.17).
(iv) If E is a strong Leinert set in H, then under certain conditions (in particular, when $G \in [\mathrm{SIN}]_H$), $I_G(E)$ is complemented in $A(G)$ by a completely bounded projection (Proposition 3.22).

2. Preliminaries. In this section we will outline some basic facts, notation and definitions we will need. We begin with some well known results on operator spaces.

2.1. Operator spaces. If $\mathcal{H}$ is a Hilbert space, then we will denote by $\mathcal{H}_r$ and $\mathcal{H}_c$ the row and column operator Hilbert spaces over $\mathcal{H}$, respectively. Refer to [9] for the definitions and properties of these operator spaces.

Recall that if X is an operator space, then $\overline{X}$, the complex conjugate linear space, is naturally an operator space by defining, for each $n \in \mathbb{N}$, $\|[\overline{x_{ij}}]\|_{M_n(\overline{X})} = \|[x_{ij}]\|_{M_n(X)}$. If $\mathcal{H}$ is a Hilbert space, then by [9] we have the following completely isometric identifications:

$$(\mathcal{H}_c)^* = \overline{\mathcal{H}}_r, \qquad (\mathcal{H}_r)^* = \overline{\mathcal{H}}_c,$$

the dualities being given in both cases by the dual pairing

$$\langle \xi, \overline{\eta} \rangle = \langle \xi \,|\, \eta \rangle_{\mathcal{H}},$$

where we take inner products $\langle \cdot | \cdot \rangle_H$ to be conjugate-linear in the second variable.

Recall that if X and Y are two operator spaces, then the operator space projective tensor product of X and Y is denoted by $X \widehat{\otimes} Y$ and defined by the family of norms $\{\|\cdot\|_{n,\wedge} : M_n(X \otimes Y) \to \mathbb{R}_+\}_{n\in\mathbb{N}}$ where

$$\|u\|_{n,\wedge} = \inf\{\|\alpha\|\,\|x\|_p\|y\|_q\|\beta\| : u = \alpha(x \otimes y)\beta \in M_n(X \otimes Y)\},$$

with $x \in M_p(X),\ y \in M_q(Y),\ \alpha \in M_{n,pq}$, and $\beta \in M_{pq,n}$.

2.2. Complemented ideals in Fourier algebras. The Gelfand spectrum $\Delta(A(G))$ of $A(G)$ can be identified with G. If $E \subseteq G$ is closed, then we associate with E the closed ideal of $A(G)$ given by

$$I_G(E) = \{u \in A(G) : u(x) = 0 \text{ for every } x \in E\}.$$

When there is no possibility of confusion, we will omit the subscript G and write $I(E)$ instead of $I_G(E)$.

Definition 2.1. We say that a closed ideal I is *complemented* in $A(G)$ if there exists a bounded linear map P from $A(G)$ onto I such that $P^2 = P$. We say that I is *invariantly complemented* by P if

$$P(uv) = u \cdot P(v)$$

for every $u, v \in A(G)$.

We say that a closed ideal I is *weakly complemented* in $A(G)$ if there exists a bounded linear map P from $\mathrm{VN}(G)$ onto $I^\perp$ such that $P^2 = P$. We say that I is *invariantly weakly complemented* by P if

$$P(u \cdot T) = u \cdot P(T)$$

for every $u \in A(G)$ and $T \in \mathrm{VN}(G)$, where we define

$$\langle w, u \cdot T\rangle = \langle uw, T\rangle$$

for every $u, w \in A(G)$ and $T \in \mathrm{VN}(G)$.

We say that an ideal is (*invariantly*) *completely complemented* or (*invariantly*) *completely weakly complemented* if the implementing projection P can be chosen to be completely bounded.

2.3. Mutipliers and completely bounded multipliers of $A(G)$. A *multiplier* of $A(G)$ is a (necessarily bounded and continuous) function $v : G \to \mathbb{C}$ such that $vA(G) \subseteq A(G)$. For each such v of $A(G)$, the linear operator M_v on $A(G)$ defined by $M_v(u) = vu$ for each $u \in A(G)$ is bounded via the Closed Graph Theorem. The *multiplier algebra* of $A(G)$ is the closed subalgebra

$$MA(G) := \{M_v : v \text{ is a multiplier of } A(G)\}$$

of $\mathcal{B}(A(G))$, where $\mathcal{B}(A(G))$ denotes the algebra of all bounded linear operators from $A(G)$ to $A(G)$. Throughout this paper we will generally use v in place of the operator M_v and we will write $\|v\|_{MA(G)}$ to represent the norm of M_v in $\mathcal{B}(A(G))$.

Let $\mathbb{F}_N$ be the noncommutative free group on $2 \leq N \leq \infty$ generators $\{x_i\}_{i=1}^N$. For $g \in \mathbb{F}_N$ we denote by $|g|$ the usual word length of g with respect to the N generators. For the unit $e \in \mathbb{F}_N$, we have the convention that $|e| = 0$. The length function and its connection with harmonic analysis on $\mathbb{F}_N$ have been extensively studied by Haagerup [14]. In particular, the following result [14, Lemma 1.7] will be useful to us:

THEOREM 2.2. *Let $\phi : \mathbb{F}_N \to \mathbb{C}$ be a function for which*

$$\sup_{g\in\mathbb{F}_N} |\phi(g)|(1+|g|)^2 < \infty.$$

Then $\phi \in MA(\mathbb{F}_N)$ and

$$\|\phi\|_{MA(\mathbb{F}_N)} \leq 2 \sup_{g\in\mathbb{F}_N} |\phi(g)|(1+|g|)^2.$$

As was previously noted, since $A(G)$ is the predual of the von Neumann algebra $\mathrm{VN}(G)$ it carries a natural operator structure which makes $A(G)$ into a completely contractive Banach algebra. With this operator space structure we can define the *cb-multiplier algebra* of $A(G)$ to be

$$M_{\mathrm{cb}}A(G) := \mathrm{CB}(A(G)) \cap MA(G),$$

where $\mathrm{CB}(A(G))$ denotes the completely contractive Banach algebra of all completely bounded linear maps from $A(G)$ into itself. We let $\|v\|_{M_{\mathrm{cb}}A(G)}$ denote the *cb*-norm of the operator M_v. It is well known that $M_{\mathrm{cb}}A(G)$ is a closed subalgebra of $\mathrm{CB}(A(G))$ and is thus a (completely contractive) Banach algebra with respect to the norm $\|\cdot\|_{M_{\mathrm{cb}}A(G)}$.

It is known that in general,

$$A(G) \subseteq B(G) \subseteq M_{\mathrm{cb}}A(G) \subseteq MA(G),$$

and that for $v \in B(G)$,

$$\|v\|_{B(G)} \geq \|v\|_{M_{\mathrm{cb}}A(G)} \geq \|v\|_{MA(G)}.$$

Furthermore, in case G is an amenable group, we have

$$B(G) = M_{\mathrm{cb}}A(G) = MA(G)$$

and

$$\|v\|_{B(G)} = \|v\|_{M_{\mathrm{cb}}A(G)} = \|v\|_{MA(G)}$$

for any $v \in B(G)$.

The following characterization of the cb-multipliers of $A(G)$ was given by Jolissaint [16]:

THEOREM 2.3. *Let G be as above and let ϕ be a function on G. Then the following conditions are equivalent:*

(i) *ϕ belongs to $M_{\mathrm{cb}}A(G)$;*

(ii) *there exist a Hilbert space $\mathcal{H}$ and bounded continuous functions ξ, η from G to $\mathcal{H}$ such that $\phi(t^{-1}s) = \langle \xi(s) | \eta(t) \rangle_{\mathcal{H}}$ for all $s, t \in G$.*

Moreover, if these conditions are satisfied, then $\|\phi\|_{M_{\mathrm{cb}}A(G)} = \inf \|\xi\|_\infty \|\eta\|_\infty$ where the infimum is taken over all pairs as in condition (ii)*.*

We finish this section by recalling the link between cb-multipliers of $A(G)$ and Schur multipliers on $\mathcal{B}(L^2(G))$. In fact, since we will only require this correspondence in the case of discrete groups, we content ourselves with this special case.

Definition 2.4. Let G be a discrete group. A function $\sigma : G \times G \to \mathbb{C}$ is called a *Schur multiplier* on $\mathcal{B}(\ell^2(G))$ if the infinite matrix

$$S_\sigma T := [\sigma(g,h) T(g,h)]_{(g,h) \in G \times G}$$

belongs to $\mathcal{B}(\ell^2(G))$ for all $[T(g,h)]_{(g,h) \in G \times G} \in \mathcal{B}(\ell^2(G))$. We denote the vector space of all Schur multipliers of $\mathcal{B}(\ell^2(G))$ by $V^\infty(G)$. Moreover, $V^\infty(G)$ becomes a pointwise Banach algebra of functions on $G \times G$ when σ is given the norm of S_σ above as an operator on $\mathcal{B}(\ell^2(G))$.

In 1984, Bożejko and Fendler [8] established an isometric homomorphism between $M_{\mathrm{cb}}A(G)$ and $V^\infty(G)$. We state the following version of their result for discrete groups.

Theorem 2.5. *Let G be a discrete group, let $\mathcal{H}$ be a Hilbert space, and let $\xi, \eta : G \to \mathcal{H}$ be bounded functions. Then the function $\sigma_{\xi,\eta} : G \times G \to \mathbb{C}$ defined by*

$$\sigma_{\xi,\eta}(g,h) = \langle \xi(h) | \eta(g) \rangle_{\mathcal{H}} \qquad ((g,h) \in G \times G)$$

belongs to $V^\infty(G)$, and furthermore $\|\sigma_{\xi,\eta}\|_{V^\infty(G)} \le \|\xi\|_\infty \|\eta\|_\infty$.

Conversely, every $\sigma \in V^\infty(G)$ is of the above form and

$$\|\sigma\|_{V^\infty(G)} = \inf \|\xi\|_\infty \|\eta\|_\infty,$$

where the infimum is taken over all possible representations $\sigma = \sigma_{\xi,\eta}$ as above.

3. Uniformly discrete subsets, Leinert sets and complemented ideals. In this section, we will see how Leinert sets and strong Leinert sets generate complemented and completely complemented ideals respectively. In so doing we will exhibit a closed ideal in $A(\mathbb{F}_N)$ that is complemented but not even completely weakly complemented. We begin, though, by looking at the properties of uniformly discrete subsets of G.

3.1. Uniformly discrete sets

Definition 3.1. Let G be a locally compact group and let E be a subset of G. Then E is said to be (left) *uniformly discrete* in G if there exists some neighbourhood $\mathcal{U}$ of the identity $e \in G$ such that for all $g_1, g_2 \in E$,

$$g_1\mathcal{U} \cap g_2\mathcal{U} = \begin{cases} g_1\mathcal{U} & \text{if } g_1 = g_2, \\ \emptyset & \text{if } g_1 \neq g_2. \end{cases}$$

Observation. It is a relatively easy task to show that E is uniformly discrete if and only if there exists a neigbourhood $\mathcal{U}$ of $\{e\}$ such that if $g_1, g_2 \in E$ with $g_1 \neq g_2$, then $g_1^{-1} g_2 \notin \mathcal{U}$.

Examples. If G is a discrete group, then any subset $E \subseteq G$ is uniformly discrete in G. Indeed, just let $\mathcal{U} = \{e\}$ in the above definition. More generally, if H is a discrete closed subgroup of a locally compact group G, then any subset $E \subseteq H$ is a uniformly discrete subset of G. To see this, just let $\mathcal{U}$ be any open neighbourhood of e for which $\mathcal{U} \cap H = \{e\}$. Clearly any finite subset of a locally compact group G is uniformly discrete in G as well.

Remark. Of course there is also the parallel notion of right uniformly discrete sets, but we will not need to consider this right-sided version here. However, this does lead to the following natural question: *If $E \subset G$ is left uniformly discrete in G, is E automatically right uniformly discrete in G?* It is not hard to see that if G is discrete or abelian, then the answer is always yes. If $E \subseteq H$ for some closed discrete subgroup $H \leq G$, then the answer is yes as well. In general, E is left uniformly discrete if and only if E^{-1} is right uniformly discrete. Moreover, it was kindly pointed out to the authors by S. Knudby and H. Thiel [17] that the answer to the above question is no in general. Indeed, if $G = \mathbb{R} \rtimes \mathbb{R}_+$ is the $ax + b$ group, then $E = \{(m, 2^n) : m, n \in \mathbb{Z}\} \subset G$ is right uniformly discrete but fails to be left uniformly discrete.

We now show how the presence of uniformly discrete sets allows us to construct useful orthonormal families in $L^2(G)$. Let G be any locally compact group and let E be some fixed uniformly discrete subset of G. Let $\mathcal{U}$ be some open neighbourhood of e satisfying Definition 3.1 for E. Choose a symmetric neighbourhood $\mathcal{V}$ of e such that $\mathcal{V}^2 \subseteq \mathcal{U}$. Now define a unit vector $\xi \in L^2(G)$ by setting $\xi = \|1_{\mathcal{V}}\|_2^{-1} 1_{\mathcal{V}}$, and let $u \in P(G) \cap C_c(G)$ be defined by the equation

$$u = \overline{\xi} * \check{\xi} = \langle \lambda(\cdot)\xi \,|\, \xi \rangle = \lambda_{\xi,\xi}.$$

It is easy to see that for the function u we have $\operatorname{supp} u \subseteq \mathcal{U}$. Let us assume from now on that our function $u = \overline{\xi} * \check{\xi}$ has been fixed. We also note that u is a normalized positive definite function.

Lemma 3.2. *The family $\{\lambda(h)\xi\}_{h \in E}$ forms an orthonormal system in $L^2(G)$.*

Proof. For any $h \in E$ the left translation-invariance of Haar measure gives $\|\lambda(h)\xi\|_2 = \|\xi\|_2 = 1$. If $h_1, h_2 \in E$ are such that $h_2 \neq h_1$, then

$$\langle \lambda(h_1)\xi \mid \lambda(h_2)\xi\rangle = \int_G \xi(h_1^{-1}g)\overline{\xi(h_2^{-1}g)}\,dg = \frac{|h_1\mathcal{V}\cap h_2\mathcal{V}|}{|\mathcal{V}|}$$
$$\leq \frac{|h_1\mathcal{U}\cap h_2\mathcal{U}|}{|\mathcal{V}|} = \frac{|\emptyset|}{|\mathcal{V}|} = 0 \quad \text{(since } h_1 \neq h_2\text{)}. \blacksquare$$

We now define a linear map $\Gamma_u : \ell^2(E) \to C_0(G) \cap L^2(G)$ by

$$\Gamma_u\varphi = \sum_{h\in E} \varphi(h)(\delta_h * u),$$

where $(\delta_h * u)(x) = u(h^{-1}x)$ and $(u*\delta_h) = u(xh)$ for $h, x \in G$. The following theorem shows that this map has some interesting properties:

THEOREM 3.3. *Let $E \subseteq G$ and Γ_u be as above. Then:*

(i) $(\Gamma_u\varphi)|_E = \varphi$ *for all* $\varphi \in \ell^2(E)$.
(ii) $\Gamma_u(\ell^2(E)) \subseteq A(G)$ *and the map* $\Gamma_u : \ell^2(E) \to A(G)$ *is a contraction.*
(iii) $\Gamma_u : \overline{\ell^2(E)_r} \to A(G)$ *is a complete contraction.*

Proof. (i) By Lemma 3.2, given any $h_1 \neq h_2 \in E$ we have

$$u(h_2^{-1}h_1) = \langle \lambda(h_1)\xi \mid \lambda(h_2)\xi\rangle = 0.$$

Consequently, if $s \in E$ and $\varphi \in \ell^2(E)$, then

$$(\Gamma_u\varphi)(s) = \sum_{h\in E} \varphi(h)u(h^{-1}s) = \varphi(s)u(e) = \varphi(s).$$

That is, $\Gamma_u\varphi|_E = \varphi$ for all $\varphi \in \ell^2(E)$.

(ii) Let $\varphi \in \ell^2(E)$. Then we have

$$(\Gamma_u\varphi)(x) = \sum_{h\in E} \varphi(h)u(h^{-1}x) = \sum_{h\in E} \varphi(h)\langle \lambda(h^{-1}x)\xi|\xi\rangle$$
$$= \Big\langle \lambda(x)\xi \,\Big|\, \sum_{h\in E} \overline{\varphi(h)}\lambda(h)\xi \Big\rangle.$$

Thus

$$\Gamma_u\varphi = \lambda_{\xi, \sum_{h\in E}\overline{\varphi(h)}\lambda(h)\xi} \in A(G)$$

and

$$\|\Gamma_u\varphi\|_{A(G)} \leq \|\xi\|_2 \Big\| \sum_{h\in E} \overline{\varphi(h)}\lambda(h)\xi \Big\|_2 = \|\overline{\varphi}\|_2 = \|\varphi\|_2.$$

Therefore $\Gamma_u : \ell^2(E) \to A(G)$ is a contraction.

(iii) We will now show that $\Gamma_u : \overline{\ell^2(E)_r} \to A(G)$ is a complete contraction. Recall that the dual pairing

$$\langle \overline{\eta} * \check{\xi}, T\rangle = \langle T\xi \mid \eta\rangle \quad (\overline{\eta} * \check{\xi} \in A(G),\ T \in \mathrm{VN}(G))$$

identifies $A(G)^*$ with $\mathrm{VN}(G)$. Now define a linear map

$$q : \overline{L^2(G)_r} \widehat{\otimes} L^2(G)_c \to A(G) \quad \text{by} \quad q(\overline{\eta} \otimes \xi) = \overline{\eta} * \check{\xi}.$$

By [9] we have

$$(\overline{L^2(G)_r} \widehat{\otimes} L^2(G)_c)^* = \mathcal{CB}(L^2(G)_c) = \mathcal{B}(L^2(G))$$

completely isometrically. From this fact together with the above duality between $A(G)$ and $\mathrm{VN}(G)$, it follows that q is nothing other than the pre-adjoint of the canonical completely isometric inclusion $\mathrm{VN}(G) \hookrightarrow \mathcal{B}(L^2(G))$. Consequently,

$$q : \overline{L^2(G)_r} \widehat{\otimes} L^2(G)_c \to A(G)$$

is a complete quotient map.

Now let $n \in \mathbb{N}$ and $[\overline{\varphi_{ij}}] \in M_n(\overline{\ell^2(E)_r})$, and note that

$$\begin{aligned} \Gamma_u^{(n)}([\overline{\varphi_{ij}}]) &= \sum_{h\in E} [\overline{\varphi_{ij}(h)}] \otimes (\delta_h * u) = \sum_{h\in E} [\overline{\varphi_{ij}(h)}] \otimes (\overline{\lambda(h)\xi} * \check{\xi}) \\ &= (I_n \otimes q)\Big(\sum_{h\in E} [\overline{\varphi_{ij}(h)}] \otimes \overline{\lambda(h)\xi} \otimes \xi \Big) \\ &= (I_n \otimes q)\Big(\Big(\sum_{h\in E} [\overline{\varphi_{ij}(h)}] \otimes \overline{\lambda(h)\xi} \Big) \otimes \xi \Big). \end{aligned}$$

Therefore

$$\begin{aligned} \|\Gamma_u^{(n)}([\overline{\varphi_{ij}}])\|_{M_n(A(G))} &= \Big\| (I_n \otimes q)\Big(\Big(\sum_{h\in E} [\overline{\varphi_{ij}(h)}] \otimes \overline{\lambda(h)\xi} \Big) \otimes \xi \Big) \Big\|_{M_n(A(G))} \\ &\le \Big\| \Big(\sum_{h\in E} [\overline{\varphi_{ij}(h)}] \otimes \overline{\lambda(h)\xi} \Big) \otimes \xi \Big\|_{M_n \otimes \overline{L^2(G)_r} \widehat{\otimes} L^2(G)_c} \\ &\le \Big\| \sum_{h\in E} [\overline{\varphi_{ij}(h)}] \otimes \overline{\lambda(h)\xi} \Big\|_{M_n(\overline{L^2(G)_r})} \|\xi\|_{M_1(L^2(G)_c)} \\ &= \Big\| \sum_{h\in E} [\varphi_{ij}(h)] \otimes \lambda(h)\xi \Big\|_{M_n(L^2(G)_r)} \\ &= \Big\| \sum_{h\in E} [\varphi_{ij}(h)][\varphi_{ij}(h)]^* \Big\|^{1/2} \\ &= \|[\varphi_{ij}]\|_{M_n(\ell^2(E)_r)} = \|[\overline{\varphi_{ij}}]\|_{M_n(\overline{\ell^2(E)_r})}. \end{aligned}$$

Since $n \in \mathbb{N}$ and $[\overline{\varphi_{ij}}] \in M_n(\overline{\ell^2(E)_r})$ were arbitrary, $\Gamma_u : \overline{\ell^2(E)_r} \to A(G)$ is completely contractive. ∎

REMARK. Let $E \subseteq G$ and $u \in P(G) \cap C_c(G)$ be as above. In addition, suppose that the family $\{\lambda(h^{-1})\xi\}_{h\in E}$ forms an ONS in $L^2(G)$. In this case, define a new map $\check{\Gamma}_u : \ell^2(E) \to C_0(G) \cap L^2(G)$ by setting

$$\check{\Gamma}_u \varphi = \sum_{h\in E} \varphi(h)(u * \delta_{h^{-1}}).$$

Then we obtain the following analogue of Theorem 3.3:

THEOREM 3.4. *The map $\check{\Gamma}_u$ satisfies the following properties:*

(i) $(\check{\Gamma}_u\varphi)\big|_E = \varphi$ *for all* $\varphi \in \ell^2(E)$.
(ii) $\check{\Gamma}_u(\ell^2(E)) \subseteq A(G)$ *and the map* $\check{\Gamma}_u : \ell^2(E) \to A(G)$ *is a contraction.*
(iii) $\check{\Gamma}_u : \ell^2(E)_c \to A(G)$ *is a complete contraction.*

Proof. (i) and (ii) are proved in a similar way to Theorem 3.3(i), using the fact that $\{\lambda(h^{-1})\xi\}_{h\in E}$ forms an ONS in $L^2(G)$. For (iii), we also proceed as in the proof of Theorem 3.3. Let $n \in \mathbb{N}$ and let $[\varphi_{ij}] \in M_n(\ell^2(E)_c)$. Then

$$\begin{aligned}\|\check{\Gamma}_u^{(n)}[\varphi_{ij}]\|_{M_n(A(G))} &= \Big\|(I_n \otimes q)\Big(\overline{\xi} \otimes \Big(\sum_{h\in E}[\varphi_{ij}(h)] \otimes \lambda(h^{-1})\xi\Big)\Big)\Big\|_{M_n(A(G))} \\ &\le \Big\|(\overline{\xi} \otimes \Big(\sum_{h\in E}[\varphi_{ij}(h)] \otimes \lambda(h^{-1})\xi\Big)\Big\|_{M_n\otimes\overline{L^2(G)_r}\widehat{\otimes}L^2(G)_c} \\ &\le \|\overline{\xi}\|_{\overline{L^2(G)_r}}\Big\|\sum_{h\in E}[\varphi_{ij}(h)] \otimes \lambda(h^{-1})\xi\Big\|_{M_n\otimes L^2(G)_c} \\ &= \|[\varphi_{ij}]\|_{M_n(\ell^2(E)_c)}. \quad ■\end{aligned}$$

Given $T \in \mathrm{VN}(G)$ and $v \in A(G)$, recall (see [10]) that we can define a left $\mathrm{VN}(G)$-action on $A(G)$, denoted by Tv, by letting

$$Tv(x) = \langle \delta_x * \check{v}, T\rangle \qquad (x \in G),$$

where $\check{v}(x) = v(x^{-1})$ for any function $v : G \to \mathbb{C}$. (Note that from [10] we deduce that $\|\check{v}\|_{A(G)} = \|v\|_{A(G)}$.) It is not hard to see that when $v \in A(G) \cap L^2(G)$, Tv is nothing other than the image of the vector $v \in L^2(G)$ under the linear operator $T \in \mathrm{VN}(G) \subseteq \mathcal{B}(L^2(G))$.

Using the above module notation, we have the following corollary to Theorem 3.3:

COROLLARY 3.5. *Let $E \subseteq G$ be a uniformly discrete subset of G, let $u \in P(G)\cap C_c(G)$ be as in Theorem* 3.3, *and let $n \in \mathbb{N}$. Then for any matrix $[T_{ij}] \in M_n(\mathrm{VN}(G))$ we have*

$$\Big\|\sum_{h\in E}[T_{ij}\check{u}(h)]^*[T_{ij}\check{u}(h)]\Big\|^{1/2} \le \|[T_{ij}]\|_{M_n(\mathrm{VN}(G))}.$$

Suppose furthermore that the function u satisfies

$$u(h^{-1}xh) = u(x) \qquad (x \in G,\ h \in E).$$

Then

$$\max\Big\{\Big\|\sum_{h\in E}[T_{ij}\check{u}(h)]^*[T_{ij}\check{u}(h)]\Big\|^{1/2},\ \Big\|\sum_{h\in E}[T_{ij}\check{u}(h)][T_{ij}\check{u}(h)]^*\Big\|^{1/2}\Big\} \le \|[T_{ij}]\|.$$

Proof. Let $T = [T_{ij}] \in M_n(\mathrm{VN}(G))$, and consider the quantity

$$A(E,T) := \Big\| \sum_{h\in E} [T_{ij}\check{u}(h)]^*[T_{ij}\check{u}(h)] \Big\|^{1/2} = \|[T_{ij}\check{u}|_E]\|_{M_n(\ell^2(E)_c)}.$$

Since $\ell^2(E)_c^* = \overline{\ell^2(E)_r}$ completely isometrically, we have

$$\begin{aligned} A(E,T) &= \sup_{[\overline{\varphi_{kl}}]\in b_1(M_n(\overline{\ell^2(E)_r}))} \|[\langle T_{ij}\check{u}|_E, \overline{\varphi_{kl}}\rangle]\|_{M_{n^2}} \\ &= \sup_{[\overline{\varphi_{kl}}]\in b_1(M_n(\overline{\ell^2(E)_r}))} \Big\|\Big[\sum_{h\in E} T_{ij}\check{u}(h)\overline{\varphi_{kl}(h)}\Big]\Big\|_{M_{n^2}} \\ &= \sup_{[\overline{\varphi_{kl}}]\in b_1(M_n(\overline{\ell^2(E)_r}))} \Big\|\Big[\sum_{h\in E} \overline{\varphi_{kl}(h)}\langle \delta_h * u, T_{ij}\rangle\Big]\Big\|_{M_{n^2}} \\ &= \sup_{[\overline{\varphi_{kl}}]\in b_1(M_n(\overline{\ell^2(E)_r}))} \Big\| \Big\langle \sum_{h\in E} [\overline{\varphi_{kl}(h)}] \otimes (\delta_h * u), [T_{ij}] \Big\rangle \Big\|_{M_{n^2}} \\ &= \sup_{[\overline{\varphi_{kl}}]\in b_1(M_n(\overline{\ell^2(E)_r}))} \|\langle \Gamma_u^{(n)}[\overline{\varphi_{kl}}], [T_{ij}]\rangle\|_{M_{n^2}} \\ &\le \sup_{[\overline{\varphi_{kl}}]\in b_1(M_n(\overline{\ell^2(E)_r}))} \|\Gamma_u^{(n)}[\overline{\varphi_{kl}}]\|_{M_n(A(G))} \|[T_{ij}]\|_{M_n(\mathrm{VN}(G))} \\ &\le \|[T_{ij}]\|_{M_n(\mathrm{VN}(G))} \quad \text{(by Theorem 3.3(iii))}. \end{aligned}$$

Now suppose that $\delta_h * u * \delta_h = u$ for all $h \in E$. Let $[T_{ij}] \in M_n(\mathrm{VN}(G))$ and consider the quantity

$$B(E,T) := \Big\| \sum_{h\in E} [T_{ij}\check{u}(h)][T_{ij}\check{u}(h)]^* \Big\|^{1/2}.$$

We want to show that

$$\max\{A(E,T), B(E,T)\} \le \|[T_{ij}]\|_{M_n(\mathrm{VN}(G))}.$$

Observe that for any $h \in E$, the fact that $u * \delta_h = \delta_{h^{-1}} * u$ implies that

$$\begin{aligned} [T_{ij}\check{u}(h)]^* &= [\overline{T_{ji}\check{u}(h)}] = [\overline{\langle u, \lambda(h^{-1})T_{ji}\rangle}] = [\langle u, T_{ji}^*\lambda(h)\rangle] \\ &= [\langle u * \delta_h, T_{ji}^*\rangle] = [\langle \delta_{h^{-1}} * u, T_{ji}^*\rangle] = [T_{ji}^*\check{u}(h^{-1})], \end{aligned}$$

and consequently

$$B(E,T) = \Big\| \sum_{h\in E} [T_{ji}^*\check{u}(h^{-1})]^*[T_{ji}^*\check{u}(h^{-1})] \Big\| = A(E^{-1}, T^*).$$

Note that the condition on our function u forces E^{-1} to be uniformly discrete in G. Furthermore we can apply Theorem 3.3 to the set E^{-1} and the map $\tilde{\Gamma}_u : \overline{\ell^2(E^{-1})_r} \to A(G)$ given by $\tilde{\Gamma}_u\varphi = \sum_{h\in E} \varphi(h)(\delta_{h^{-1}} * u)$ to deduce that $\|\tilde{\Gamma}_u\|_{\mathrm{cb}} \le 1$. This allows us to use the same argument that was used to bound $A(E,T)$ to get $A(E^{-1},T) \le \|[T_{ij}]^*\|_{M_n(\mathrm{VN}(G))} = \|[T_{ij}]\|_{M_n(\mathrm{VN}(G))}$. ∎

If we restrict our attention to discrete groups, Corollary 3.5 can be viewed as a generalization of the well known fact that $\mathrm{VN}(G)$ embeds completely contractively into both $\ell^2(E)_c$ and $\ell^2(E)_r$. (See [23] for a proof of this in the discrete case.)

Corollary 3.6. *Let G be any discrete group and let $E \subseteq G$. Then for any Hilbert space $\mathcal{H}$ and any finitely supported function $a : E \to \mathcal{B}(\mathcal{H})$, we have*

$$\max\Big\{\Big\|\sum_{h\in E} a(h)^* a(h)\Big\|^{1/2}, \Big\|\sum_{h\in E} a(h)a(h)^*\Big\|^{1/2}\Big\} \leq \Big\|\sum_{h\in E} a(h)\otimes\lambda(h)\Big\|_{\mathcal{B}(\mathcal{H})\otimes_{\min}\mathrm{VN}(G)}.$$

Proof. Applying Corollary 3.5 to the set E and the function $u = \delta_e$, we obtain the above result for any finite-dimensional Hilbert space $\mathcal{H}$. The finite-dimensional case, however, is equivalent to the infinite-dimensional case (see [23, Chapter 2]). ∎

3.2. Leinert sets and strong Leinert sets in discrete groups. Recall again that for any discrete group G, $\mathrm{VN}(G)$ can be identified with a dense linear subspace of $\ell^2(G)$ (cf. Corollary 3.6). More concisely, the contractive embedding $\Lambda : \mathrm{VN}(G) \to \ell^2(G)$ is given by $\Lambda(T) = T\delta_e$, where $\delta_e \in \ell^2(G)$ is the basis vector associated to the unit $e \in G$. In the remainder, we shall identify $\mathrm{VN}(G)$ with $\Lambda(\mathrm{VN}(G))$ whenever it is convenient.

Now let $E \subseteq G$, and consider the annihilator $I(E)^\perp = (A(G)/I(E))^* = A(E)^* \subset \mathrm{VN}(G)$ of the ideal $I(E) \subseteq A(G)$. Since for any E, $\ell^2(E) \subseteq A(E)$ contractively, it follows by duality that $I(E)^\perp \subseteq \ell^2(E)$ contractively, with the equality $I(E)^\perp = \ell^2(E)$ if and only if E is a Leinert set (see [21]).

At this stage let us recall one sufficient condition for a set $E \subseteq G$ to be a Leinert set:

Definition 3.7. Let G be a discrete group. A set $E \subseteq G$ satisfies the *Leinert condition* if for all $n \in \mathbb{N}$ and for all $\{x_i\}_{i=1}^{2n} \subseteq E$ with $x_i \neq x_{i+1}$ we have $x_1x_2^{-1}x_3x_4^{-1}\cdots x_{2n-1}x_{2n}^{-1} \neq e$.

In [18], Leinert showed that every set $E \subseteq G$ satisfying the Leinert condition is in fact a Leinert set. Another useful characterization of Leinert sets in terms of multipliers of the Fourier algebra is due to Bożejko [7]:

Lemma 3.8. *Let G be a discrete group and $E \subseteq G$. Then the following are equivalent:*

(1) *E is a Leinert set.*
(2) *Every function in $\ell^\infty(E)$ belongs to $MA(G)$.*

It follows from the above lemma that for any Leinert set E in a discrete group G, we have
$$I(G \setminus E) = 1_E \cdot A(G) = \ell^2(E).$$
Note however that the condition $I(G\setminus E) = \ell^2(E)$ is not sufficient to ensure that E is a Leinert set. Indeed, it can be shown [21] that for *any* discrete group G, there exists an infinite set $E \subseteq G$ for which $I(G \setminus E) = \ell^2(E)$.

Clearly any finite subset of a discrete group is a Leinert set. A typical example of an infinite Leinert set in the countably generated free group $\mathbb{F}_\infty$ on the generators $\{x_i\}_{i\in\mathbb{N}}$ is the set E_n of all reduced words of length n in $\mathbb{F}_\infty$. For the Leinert set $E_1 = \{x_i, x_i^{-1} : i \in \mathbb{N}\}$, it can actually be shown [22] that $\ell^\infty(E_1) \subseteq M_{\mathrm{cb}}A(\mathbb{F}_\infty)$. This example leads us to the next definition.

Definition 3.9. Let E be a subset of a discrete group G. Then E is called a *strong Leinert* set if every function in $\ell^\infty(E)$ belongs to $M_{\mathrm{cb}}A(G)$.

The following result, due to Pisier [22, Proposition 3.2], characterizes strong Leinert sets in terms of the operator space structure of the subspace of $\mathrm{VN}(G)$ consisting of those operators supported on such sets.

Proposition 3.10. *Let E be a subset of a discrete group G. Then the following are equivalent:*

(1) *E is a strong Leinert set.*
(2) *There exists some $C > 0$ such that for any Hilbert space $\mathcal{H}$ and any finitely supported function $a : E \to \mathcal{B}(\mathcal{H})$ we have*
$$\Big\|\sum_{h\in E} a(h)\otimes\lambda(h)\Big\|_{\mathcal{B}(\mathcal{H})\otimes_{\min}\mathrm{VN}(G)} \le C\max\Big\{\Big\|\sum_{h\in E} a(h)^*a(h)\Big\|^{1/2}, \Big\|\sum_{h\in E} a(h)a(h)^*\Big\|^{1/2}\Big\}.$$

Remark. As mentioned above, the prototypical example of an infinite strong Leinert set is given by the set $E_1 = \{x_i, x_i^{-1}\}_{i\in\mathbb{N}} \subset \mathbb{F}_\infty$ consisting of the free generators and their inverses (see for example [22, Theorem 0.1]).

Next we will show that there exist Leinert sets E in $\mathbb{F}_N$ $(2 \le N \le \infty)$ which are not strong Leinert sets and that, in particular, $1_E \in MA(\mathbb{F}_N) \setminus M_{\mathrm{cb}}A(\mathbb{F}_N)$. The existence of Leinert sets in free groups which are not strong Leinert sets was first established by Bożejko [7]. The following result gives an explicit example of such a set in $\mathbb{F}_\infty$ with the above properties. The case $N < \infty$ then follows immediately because $\mathbb{F}_\infty$ can be realized as a subgroup of $\mathbb{F}_N$.

Proposition 3.11. *Let $E = \{x_i x_j^{-1} : 1 \le i \le j < \infty\}$ where $S = \{x_i\}$ denotes a countable set of free generators of $\mathbb{F}_\infty$. Then the function 1_E belongs to $MA(\mathbb{F}_\infty)$, but not to $M_{\mathrm{cb}}A(\mathbb{F}_\infty)$.*

Proof. Observe that

$$\sup_{g\in\mathbb{F}_\infty} |1_E(g)|(1+|g|)^2 = \sup_{g\in\mathbb{F}_\infty} (1+|g|)^2 = (1+2)^2 = 9,$$

so $1_E \in MA(\mathbb{F}_\infty)$ by Theorem 2.2. Now let $\phi := 1_E$ and suppose, to get a contradiction, that $\phi \in M_{\mathrm{cb}}A(\mathbb{F}_\infty)$. It then follows from this assumption and Theorem 2.5 that the function $\sigma_\phi : \mathbb{F}_\infty \times \mathbb{F}_\infty \to \mathbb{C}$ given by

$$\sigma_\phi(g,h) = \phi(gh^{-1})$$

for all $(g,h) \in \mathbb{F}_\infty \times \mathbb{F}_\infty$ belongs to $V^\infty(\mathbb{F}_\infty)$.

Let us now consider the associated Schur multiplier S_{σ_ϕ}. Let $\{\delta_g : g \in \mathbb{F}_\infty\}$ denote the canonical orthonormal basis for $\ell^2(\mathbb{F}_\infty)$ and identify $\mathcal{B}(\ell^2(S))$ with the corner $P\mathcal{B}(\ell^2(\mathbb{F}_\infty))P \subset \mathcal{B}(\ell^2(\mathbb{F}_\infty))$, where P is the orthogonal projection from $\ell^2(\mathbb{F}_\infty)$ onto the subspace $\ell^2(S)$. If $T = [T(x_i,x_j)]_{(i,j)\in\mathbb{N}\times\mathbb{N}} \in \mathcal{B}(\ell^2(S))$, then $S_{\sigma_\phi}T$ is given by the infinite matrix

$$S_{\sigma_\phi}T = [1_E(x_i x_j^{-1})T(x_i,x_j)]_{(i,j)\in\mathbb{N}\times\mathbb{N}}$$

where

$$1_E(x_i x_j^{-1}) = \begin{cases} 1 & \text{if } i \le j, \\ 0 & \text{if } i > j. \end{cases}$$

Thus the map $T \mapsto S_{\sigma_\phi}T$ is just the upper-triangular truncation map on $\mathcal{B}(\ell^2(S))$. Since $\ell^2(S)$ is not finite-dimensional, it follows that upper-triangular truncation is not bounded on $\mathcal{B}(\ell^2(S))$ (see for example [20, Problems 8.15 and 8.16]). The unboundedness of S_{σ_ϕ} contradicts the fact that $\sigma_\phi \in V^\infty(\mathbb{F}_\infty)$, and therefore we must have $\phi = 1_E \in MA(\mathbb{F}_\infty) \setminus M_{\mathrm{cb}}A(\mathbb{F}_\infty)$. ∎

Corollary 3.12. *There exists a Leinert set $E \subset \mathbb{F}_\infty$ such that $1_E \in MA(\mathbb{F}_\infty) \setminus M_{\mathrm{cb}}A(\mathbb{F}_\infty)$.*

Proof. Let $E = \{x_i x_j^{-1} : 1 \le i \le j < \infty\}$ be as in the previous proposition. Then it is a routine calculation to show that E satisfies the Leinert condition and is thus a Leinert set by [18]. The fact that $1_E \in MA(\mathbb{F}_\infty) \setminus M_{\mathrm{cb}}A(\mathbb{F}_\infty)$ is Proposition 3.11. ∎

Corollary 3.13. *Let $E \subseteq \mathbb{F}_\infty$ be a Leinert set such that $1_E \in MA(\mathbb{F}_\infty) \setminus M_{\mathrm{cb}}A(\mathbb{F}_\infty)$. Then the ideals $I(\mathbb{F}_\infty \setminus E)$ and $I(E)$ are complemented in $A(\mathbb{F}_\infty)$ but not completely complemented in $A(\mathbb{F}_\infty)$. Furthermore, the annihilators $I(\mathbb{F}_\infty \setminus E)^\perp$ and $I(E)^\perp$ are complemented in* VN$(\mathbb{F}_\infty)$ *but not completely complemented in* VN$(\mathbb{F}_\infty)$.

In particular, if $E = \{x_i x_j^{-1} : 1 \le i \le j < \infty\}$ where $S = \{x_i\}$ denotes the generators of $\mathbb{F}_\infty$, then $I(E)$ is complemented in $A(\mathbb{F}_\infty)$ but is not completely weakly complemented.

Proof. If we let $P : A(G) \to A(G)$ and $Q : A(G) \to A(G)$ be defined by $P(u) = 1_E u$ and $Q(u) = u - 1_E u$, then P and Q are projections onto $I(\mathbb{F}_\infty \setminus E)$ and $I(E)$ respectively.

Assume now that E is such that $I(E)^\perp$ is completely complemented in $\mathrm{VN}(\mathbb{F}_\infty)$. For a generic locally compact group G we let $A_{\mathrm{cb}}(G)$ be the closure of $A(G)$ when viewed as a subalgebra of $M_{\mathrm{cb}}A(G)$. Then we know that $A_{\mathrm{cb}}(\mathbb{F}_N)$ is operator amenable [13]. Since $\mathbb{F}_\infty$ can be realized as an open subgroup of $\mathbb{F}_N$, we see that $A(\mathbb{F}_\infty)$ is also operator amenable. It follows from [13, Theorem 3.4] that $1_E \in M_{\mathrm{cb}}A(\mathbb{F}_\infty)$, which contradicts our assumption that $1_E \in MA(\mathbb{F}_\infty) \setminus M_{\mathrm{cb}}A(\mathbb{F}_\infty)$.

A similar argument shows that $I(\mathbb{F}_\infty \setminus E)^\perp$ is not completely complemented in $\mathrm{VN}(\mathbb{F}_\infty)$. ■

3.3. Complemented ideals vanishing on Leinert sets in discrete subgroups of locally compact groups. Using our results on uniformly discrete subsets of locally compact groups, we will show that if H is a closed discrete subgroup of a locally compact group G and $E \subseteq H$ is a Leinert set, then $I_G(E)$ is always complemented in $A(G)$ and always invariantly weakly complemented in $A(G)$.

We begin with the following standard result:

PROPOSITION 3.14. *Let G be a locally compact group and let $E \subseteq G$ be a closed subset. Then the ideal $I(E) \subseteq A(G)$ has a Banach space complement in $A(G)$ if and only if there exists a bounded linear map $\Gamma : A(E) \to A(G)$ such that $\Gamma\varphi|_E = \varphi$ for all $\varphi \in A(E)$.*

Proof. First suppose that $P : A(G) \to I(E)$ is a bounded (surjective) projection. Fix $\epsilon > 0$. Given $\varphi \in A(E)$, let $u \in A(G)$ be chosen so that $u|_E = \varphi$ and $\|u\|_{A(G)} \le \|\varphi\|_{A(E)} + \epsilon$, and define

$$\Gamma\varphi = u - Pu \in A(G).$$

Note that we have $\|\Gamma\varphi\|_{A(G)} \le \|u\|_{A(G)} + \|P\|\,\|u\|_{A(G)} \le (1 + \|P\|)\|\varphi\|_{A(E)} + (1 + \|P\|)\epsilon$ and that $\Gamma\varphi|_E = u|_E - Pu|_E = \varphi$. Also note that if $u_1 \in A(G)$ is any other function such that $u_1|_E = \varphi$, then $u_1 - u \in I(E)$, which implies that $u_1 - Pu_1 - (u - Pu) = u_1 - u - P(u_1 - u) = 0$. Thus $\Gamma : A(E) \to A(G)$ is a well defined map. To see that Γ is linear, let $\varphi_1, \varphi_2 \in A(E)$ and let $\alpha \in \mathbb{C}$. Let $u_1, u_2 \in A(G)$ be extensions of φ_1 and φ_2 respectively. Then clearly $\alpha u_1 + u_2$ is an extension of $\alpha\varphi_1 + \varphi_2$, and consequently

$$\Gamma(\alpha\varphi_1 + \varphi_2) = \alpha u_1 + u_2 - P(\alpha u_1 + u_2) = \alpha\Gamma\varphi_1 + \Gamma\varphi_2.$$

Finally, since $\|\Gamma\| \le (1 + \|P\|)(1 + \epsilon) < \infty$, Γ is bounded. Therefore Γ is the required extension map.

Conversely, suppose $\Gamma : A(E) \to A(G)$ is a bounded linear map such that $\Gamma\varphi|_E = \varphi$ for all $\varphi \in A(E)$. For $u \in A(G)$ define $Pu = u - \Gamma(u|_E)$.

Then P is obviously linear, and $\|Pu\|_{A(G)} \leq (1 + \|\Gamma\|)\|u\|_{A(G)}$. Note that $Pu|_E = u|_E - \Gamma(u|_E)|_E = u|_E - u|_E = 0$, so $\operatorname{ran} P \subseteq I(E)$. Finally, if $u \in I(E)$, then $u|_E = 0$, so $Pu = u$. Therefore P is the required projection. ∎

LEMMA 3.15. *Let $H \leq G$ be a closed subgroup of a locally compact group G. If $E \subseteq H$ is any closed subset, then $A_G(E) \cong A_H(E)$ completely isometrically.*

Proof. It follows from Herz's restriction theorem [15] that $A(H) \cong A(G)/I_G(H)$ and $I_H(E) \cong I_G(E)/I_G(H)$ completely isometrically. Consequently,

$$\begin{aligned} A_H(E) &\cong A(H)/I_H(E) \cong (A(G)/I_G(H))/(I_G(E)/I_G(H)) \\ &\cong A(G)/I_G(E) \cong A_G(E) \end{aligned}$$

completely isometrically. ∎

We are now in a position to state the main theorem of this section.

THEOREM 3.16. *Let G be a locally compact group and let H be a discrete subgroup of G. If E is a Leinert set in H, then the ideal $I_G(E) \subseteq A(G)$ has a Banach space complement in $A(G)$.*

Proof. By Proposition 3.14, it suffices to find a bounded linear extension map $\Gamma : A_G(E) \to A(G)$ such that $\Gamma\varphi|_E = \varphi$ for all $\varphi \in A_G(E)$.

Since E is a Leinert set in H, Lemma 3.15 implies that there exists some $C > 0$ for which $\|\varphi\|_{\ell^2(E)} \leq C\|\varphi\|_{A_G(E)}$ for all $\varphi \in A_G(E)$. Since H is a discrete closed subgroup of G, E is uniformly discrete in G. Let $u \in P(G) \cap C_c(G)$ and $\Gamma_u : \ell^2(E) \to A(G)$ be given as in Theorem 3.3. Then $\Gamma_u : A_G(E) \to A(G)$ is bounded with norm $\leq C$ and is the required extension map. ∎

Theorem 3.16 can be used to show that the Fourier algebra of any locally compact group containing a noncommutative free group as a discrete subgroup has (weakly) complemented ideals which fail to be (weakly) completely complemented.

COROLLARY 3.17. *Let G be a locally compact group containing a noncommutative free group as a discrete subgroup. Then there are complemented ideals in $A(G)$ which are complemented as Banach subspaces of $A(G)$ but not as operator subspaces of $A(G)$.*

Proof. Since any noncommutative free group contains an isomorphic copy of $\mathbb{F}_\infty$ as a subgroup, G therefore contains a copy of $\mathbb{F}_\infty$ as a discrete subgroup. Let E be a Leinert set in $\mathbb{F}_\infty$ satisfying the properties of Corollary 3.12 and consider the closed ideal $I_G(E) \subseteq A(G)$.

By Theorem 3.16, $I_G(E)$ is complemented in $A(G)$ as a Banach subspace. To show that it is not complemented as an operator subspace, assume for contradiction that there exists a completely bounded projection $P : A(G) \to I_G(E)$. Then, by duality, the map $Q := \mathrm{id}_{\mathrm{VN}(G)} - P^*$ must be a completely bounded projection from $\mathrm{VN}(G)$ onto $I_G(E)^{\perp}$.

Now, since $I_G(\mathbb{F}_\infty)^{\perp}$ and $\mathrm{VN}(\mathbb{F}_\infty)$ are $*$-isomorphic as von Neumann algebras, and under this isomorphism $I_G(E)^{\perp}$ and $I_{\mathbb{F}_\infty}(E)^{\perp}$ are identified, we see that the restriction of Q to $I_G(\mathbb{F}_\infty) \cong \mathrm{VN}(\mathbb{F}_\infty)$ yields the existence of a completely bounded projection from $\mathrm{VN}(\mathbb{F}_\infty)$ onto $I_{\mathbb{F}_\infty}(E)^{\perp}$. This, however, contradicts Corollary 3.13. ∎

COROLLARY 3.18. *Let G be a locally compact group such that every complemented ideal in $A(G)$ is completely complemented. Then G has an open amenable subgroup. In particular, if G is almost connected, then G is amenable.*

Proof. Let G_e denote the connected component of the identity. If G_e is not amenable, then G_e contains $\mathbb{F}_2$ as discrete subgroup [25]. However, this is impossible by Corollary 3.17. Hence G_e is amenable. But G has an open almost connected subgroup H. Since H/G_e is compact and G_e is amenable, H is also amenable. ∎

REMARKS. (1) We do not know if for every nonamenable group G it is possible to find complemented ideals in $A(G)$ that are not completely complemented. Moreover, we do not know if there exists an amenable group G for which $A(G)$ contains a complemented ideals that is not also completely complemented. In fact, it would be desirable to show that no such amenable group exists. If we could establish this, then we could show, using operator amenability, that if G is amenable and $I(E)$ is complemented in $A(G)$, then $E \in \mathcal{R}_c(G)$, the closed coset ring of G.

(2) We note that if H is a closed subgroup of G and if $E \subset H$ is such that $I_H(E)$ is complemented in $A(H)$, then it does not follow that $I_G(E)$ is complemented in $A(G)$ even if E is uniformly discrete in G. To see an example of this, we let G be the $ax+b$ group. Then G has a normal subgroup H which is isomorphic to $\mathbb{R}$. In turn, H has a discrete subgroup H_1 which is isomorphic to $\mathbb{Z}$. Now since H is abelian, $I_H(H_1)$ is complemented in $A(H)$ (see [11, Proposition 3.4]). However, one can also follow the same reasoning as in [11, Example 3.13] to show that $I_G(H_1)$ is not complemented in $A(G)$. Consequently, the assumption in Theorem 3.16 that E be a Leinert set is crucial.

3.4. Leinert sets and invariantly weakly complemented ideals. Let G be a locally compact group and let H be a discrete subgroup of G. We will now show that whenever E is a Leinert set in H, then the complemented ideal $I_G(E) \subseteq A(G)$ is always invariantly weakly complemented in $A(G)$. This should be contrasted with the fact that for nondiscrete G, $I_G(E)$ is never invariantly complemented in $A(G)$.

We begin with a few preliminaries: Let X and Y be Banach spaces. Recall that $\mathcal{B}(X, Y^*)$ is isometrically isomorphic to $(X \otimes^\gamma Y)^*$, where $\otimes^\gamma$ denotes the Banach space projective tensor product. The duality is given by

$$\langle x \otimes y, \Gamma \rangle = \langle y, \Gamma x \rangle \quad (x \in X,\, y \in Y,\, \Gamma \in \mathcal{B}(X, Y^*)).$$

For a Banach space X and $C \geq 0$, we write $b_C(X) = \{x \in X : \|x\| \leq C\}$.

PROPOSITION 3.19. *Let G be a locally compact group and let H be a discrete subgroup of G. If E is a Leinert set in H, then the ideal $I_G(E) \subseteq A(G)$ is invariantly weakly complemented in $A(G)$.*

Proof. Given the Leinert set $E \subseteq H$, Lemma 3.15 tells us that there exists some $C > 0$ such that

$$\|u|_E\|_2 \leq C \|u|_E\|_{A_G(E)} \quad (u \in A(G)).$$

Since H is a discrete subgroup of G, we can find an open neighbourhood $\mathcal{V}$ of the identity $e \in G$ such that $\mathcal{V} \cap H = \{e\}$. Let $\{\mathcal{V}_\alpha\}$ be a neighbourhood basis at e such that $\mathcal{V}_\alpha \subseteq \mathcal{V}$ for all α. For each α, choose a function $u_\alpha \in P(G) \cap C_c(G)$ with $\operatorname{supp} u_\alpha \subseteq \mathcal{V}_\alpha$ and $u_\alpha(e) = 1$.

From Theorems 3.3 and 3.16, we know that for each α, the linear map $\Gamma_\alpha : A_G(E) \to A(G)$ given by

$$\Gamma_\alpha \varphi = \sum_{h \in E} \varphi(h)(\delta_h * u_\alpha) \quad (\varphi \in A_G(E))$$

is bounded with $\|\Gamma_\alpha\| \leq C$, and satisfies $\Gamma_\alpha \varphi|_E = \varphi$ for all $\varphi \in A_G(E)$.

Letting

$$P_\alpha := \Gamma_\alpha^* : \mathrm{VN}(G) \to I_G(E)^\perp \subseteq \mathrm{VN}(G),$$

we obtain a net of projections

$$\{P_\alpha\} \subseteq b_C\big(\mathcal{B}(\mathrm{VN}(G), \mathrm{VN}(G))\big) = b_C\big((\mathrm{VN}(G) \otimes^\gamma A(G))^*\big).$$

It is easy to see that for each α, P_α is defined by the following equation:

$$P_\alpha T = \sum_{h \in E} \langle u_\alpha, \lambda(h^{-1})T \rangle \lambda(h) \quad (T \in \mathrm{VN}(G)).$$

Note that the above sum converges in the ℓ^2-sense for all $T \in \mathrm{VN}(G)$.

Now consider the net $\{u_\alpha\} \subset A(G)$. By passing to a subnet if necessary, we may assume that $u_\alpha \to m \in \mathrm{VN}(G)^*$ weak*, where m is a topologically invariant mean on $\mathrm{VN}(G)$ (see [24, Theorem 4]). Since $\{P_\alpha\}$ is also a bounded net in $(\mathrm{VN}(G) \otimes^\gamma A(G))^*$, the Banach–Alaoglu theorem implies that by possibly passing to yet another weak*-convergent subnet, we may assume that $P = w^*\text{-}\lim_\alpha P_\alpha \in b_C(\mathcal{B}(\mathrm{VN}(G), \mathrm{VN}(G)))$ exists.

We will now show that P is an invariant projection onto $I_G(E)^\perp$. To do so, first observe that $P(\mathrm{VN}(G)) \subseteq I_G(E)^\perp$. Indeed, suppose that $v \in I_G(E)$

and $T \in \mathrm{VN}(G)$. Then

$$\begin{aligned}\langle v, PT\rangle &= \langle T\otimes v, P\rangle = \lim_\alpha \langle T\otimes v, P_\alpha\rangle \\ &= \lim_\alpha \langle v, P_\alpha T\rangle = \lim_\alpha \sum_{h\in E} v(h)\langle u_\alpha, \lambda(h^{-1})T\rangle = \lim_\alpha 0 = 0.\end{aligned}$$

Therefore $P(\mathrm{VN}(G)) \subseteq I_G(E)^\perp$. Next, we prove that $PT = T$ for all $T \in I_G(E)^\perp$. If $T \in I_G(E)^\perp$ then there exists some function $f \in \ell^2(E)$ such that $T = \sum_{h\in E} f(h)\lambda(h)$. But then for every α we have

$$\begin{aligned}P_\alpha T &= \sum_{h\in E}\langle u_\alpha, \lambda(h^{-1})T\rangle\lambda(h) = \sum_{h\in E}\sum_{s\in E} f(s)u_\alpha(h^{-1}s)\lambda(h) \\ &= \sum_{h\in E} f(h)\lambda(h) = T.\end{aligned}$$

From this it follows that $PT = T$. Therefore $P : \mathrm{VN}(G) \to I_G(E)^\perp$ is a bounded projection. It remains to show that P is invariant. For this, let $A_c(G) = A(G) \cap C_c(G)$, let $u \in A_c(G)$, $v \in A(G)$, and $T \in \mathrm{VN}(G)$. Then we compute

$$\begin{aligned}\langle u, P(v\cdot T)\rangle &= \lim_\alpha \langle u, P_\alpha(v\cdot T)\rangle = \lim_\alpha \sum_{h\in E} u(h)\langle u_\alpha, \lambda(h^{-1})(v\cdot T)\rangle \\ &= \lim_\alpha \sum_{h\in E} u(h)\langle v(\delta_h * u_\alpha), T\rangle \\ &= \lim_\alpha \sum_{h\in E} u(h)\langle \delta_h * ((\delta_{h^{-1}} * v)u_\alpha), T\rangle \\ &= \lim_\alpha \sum_{h\in E} u(h)\langle u_\alpha, (\delta_{h^{-1}} * v)\cdot(\lambda(h^{-1})T)\rangle \\ &= \sum_{h\in E} u(h)\langle(\delta_{h^{-1}} * v)\cdot(\lambda(h^{-1})T), m\rangle \\ &= \sum_{h\in E} u(h)v(h)\langle\lambda(h^{-1})T, m\rangle \qquad \text{(by the invariance of } m) \\ &= \lim_\alpha \sum_{h\in E} u(h)v(h)\langle u_\alpha, \lambda(h^{-1})T\rangle = \lim_\alpha \langle uv, P_\alpha T\rangle \\ &= \langle uv, PT\rangle = \langle u, v\cdot PT\rangle.\end{aligned}$$

That is, $\langle u, P(v\cdot T) = \langle u, v\cdot PT\rangle$ for all $u \in A_c(G)$. Since $A_c(G)$ is norm dense in $A(G)$, it follows that $P(v\cdot T) = v\cdot PT$. ∎

Remark. Observe that the above invariant projection is given by

$$PT = \sum_{h\in E}\langle\lambda(h^{-1})T, m\rangle\lambda(h) \qquad (T \in \mathrm{VN}(G)),$$

where $m \in \mathrm{VN}(G)^*$ denotes the invariant mean obtained above. Since for any nondiscrete group G, m must annihilate $C^*_\lambda(G)$ ([24]), it follows that

$$C^*_\lambda(G) \subseteq \ker P.$$

3.5. Strong Leinert sets and completely complemented ideals. Let $E \subseteq H \leq G$ be as in the previous section. A natural question that arises within the above framework is: *Under what conditions are the projections*

$$P, P_\alpha : \mathrm{VN}(G) \to I_G(E)^\perp$$

constructed above completely bounded? We know from Corollary 3.17 that it can happen that *none* of the projections $P, P_\alpha : \mathrm{VN}(G) \to I_G(E)^\perp$ are completely bounded. On the other hand, it seems natural to expect that if E is assumed to be a strong Leinert set in H, then $I_G(E)$ should be weakly completely complemented. In this final section, we present some evidence and partial results in this direction.

Let G_d denote the abstract group G equipped with its discrete topology, and let $\lambda_d : G \to \mathcal{U}(\ell^2(G))$ denote the left regular representation of G_d on $\ell^2(G)$. Let $C^*_{\lambda_d}(G_d)$ denote the reduced group C^*-algebra of G_d, and denote by $C^*_\delta(G)$ the norm closure of $\lambda(\ell^1(G))$ in $\mathcal{B}(L^2(G))$. We may identify $C^*_\delta(G)$ with the C^*-algebra $C^*_{\lambda_0} \subseteq \mathcal{B}(L^2(G))$ generated by the representation $\lambda_0 : G_d \to \mathcal{U}(L^2(G))$, where $\lambda_0(t) = \lambda(t)$ for all $t \in G$. Observe that $C^*_\delta(G)$ is a C^*-subalgebra of $\mathrm{VN}(G)$ and that $C^*_\delta(G)$ is σ-weakly dense in $\mathrm{VN}(G)$.

Our first goal is to show that if E is a strong Leinert set, then the restriction $P|_{C^*_\delta(G)} : C^*_\delta(G) \to I_G(E)^\perp$ of the invariant projection P constructed in Proposition 3.19 is completely bounded. To do so, we first need an elementary lemma.

Lemma 3.20. *Consider the algebras $\lambda(\ell^1(G)) \subseteq \mathcal{B}(L^2(G))$ and $\lambda_d(\ell^1(G)) \subseteq \mathcal{B}(\ell^2(G))$, and define $\pi : \lambda(\ell^1(G)) \to \lambda_d(\ell^2(G))$ by*

$$\pi(\lambda(f)) = \lambda_d(f) \quad (f \in \ell^1(G)).$$

Then π extends to a $*$-homomorphism from $C^*_\delta(G)$ onto $C^*_{\lambda_d}(G_d)$.

Proof. Clearly π is a $*$-homomorphism with dense range, therefore it suffices to show that π is continuous from $C^*_\delta(G)$ to $C^*_{\lambda_d}(G_d)$.

Recall that for any locally compact group G, the left regular representation λ_d is always weakly contained in λ_0 (see [5, Lemma 2]). Equivalently, this means that for any $u \in P(G_d) \cap A(G_d)$, there exists a net $\{u_\alpha\}$ of positive definite functions associated to λ_0 such that $u_\alpha \to u$ uniformly on compacta. Fix $u \in P(G_d) \cap A(G_d)$ and $f \in \ell^1(G)$, and let $\{u_\alpha\}$ be such a net converging uniformly on compacta to u. Then we have

$$\begin{aligned}\langle u, \lambda_d(f^* * f)\rangle &= \sum_{g\in G}(f^* * f)(g)u(g) = \lim_{\alpha}\sum_{g\in G}(f^* * f)(g)u_\alpha(g) \\ &= \lim_{\alpha}\langle u_\alpha, \lambda_0(f^* * f)\rangle \le \|\lambda_0(f^* * f)\| = \|\lambda_0(f)\|^2 = \|\lambda(f)\|^2.\end{aligned}$$

But this implies that

$$\|\pi(\lambda(f))\|^2 = \|\lambda_d(f)\|^2 = \sup\{\langle u, \lambda_d(f^* * f)\rangle : u\in P(G_d)\cap A(G_d)\} \le \|\lambda(f)\|^2,$$

i.e. π is continuous, and we are done. ■

PROPOSITION 3.21. *If E is a Leinert set in H such that $1_E \in M_{\rm cb}A(H)$ and $P : \mathrm{VN}(G) \to I_G(E)^\perp$ is the invariant projection constructed in Proposition* 3.19*, then $P|_{C^*_\delta(G)} : C^*_\delta(G) \to I_G(E)^\perp$ is completely bounded.*

Proof. First observe that since P is invariant, we have

$$P\lambda(t) = \begin{cases}\lambda(t) & \text{if } t \in E,\\ 0 & \text{if } t \notin E.\end{cases} \tag{3.1}$$

Indeed, if $t \in E$, then $\lambda(t) \in I(E)^\perp$, so $P\lambda(t) = \lambda(t)$. If $t \notin E$, then there exists some $u \in I_G(E)$ with $u(t) = 1$, and so for any $v \in A(G)$ we have

$$\langle v, P(\lambda(t))\rangle = \langle v, P(u\cdot\lambda(t))\rangle = \langle v, u\cdot P(\lambda(t))\rangle = \langle uv, P(\lambda(t))\rangle = 0.$$

That is, $P\lambda(t) = 0$.

In addition, define a map $\tilde P : \mathrm{VN}(G_d) \to I_{G_d}(E)^\perp$ by setting

$$\tilde P\lambda_d(t) = \begin{cases}\lambda_d(t) & \text{if } t \in E,\\ 0 & \text{if } t \notin E.\end{cases} \tag{3.2}$$

Since $1_E \in M_{\rm cb}A(H) \subseteq M_{\rm cb}A(G_d)$, $\tilde P$ is a well defined and completely bounded projection.

Now denote by $\mathrm{VN}_H(G_d)$ the w*-closure of $\lambda_d(\ell^1(H))$ in $\mathrm{VN}(G_d)$ and denote by $\mathrm{VN}_H(G)$ the w*-closure of $\lambda(\ell^1(H))$ in $\mathrm{VN}(G)$. Observe that both $\mathrm{VN}_H(G_d)$ and $\mathrm{VN}_H(G)$ are von Neumann subalgebras of $\mathrm{VN}(G_d)$ and $\mathrm{VN}(G)$, respectively, and since H is discrete, the map $\Phi : \mathrm{VN}_H(G_d) \to \mathrm{VN}_H(G)$ defined by

$$\Phi(\lambda_d(t)) = \lambda(t) \quad (t \in H) \tag{3.3}$$

is a $*$-isomorphism.

Let $\pi : C^*_\delta(G) \to C^*_{\lambda_d}(G_d)$ be the canonical $*$-homomorphism defined in Lemma 3.20. Then using formulas (3.1)–(3.3) it is easy to see that

$$P\lambda(t) = \Phi\big(\tilde P(\pi(\lambda(t)))\big) \quad (t \in G).$$

Extending by linearity and continuity, it follows that

$$P|_{C^*_\delta(G)} = \Phi\circ\tilde P\circ\pi.$$

Since Φ, $\tilde{P}$, and π are all completely bounded, $P|_{C^*_\delta(G)}$ must be completely bounded. ■

REMARK. The above result is somewhat unsatisfactory, since in most cases of interest, the projection $P : \mathrm{VN}(G) \to I_G(E)^\perp$ is not normal, and we therefore cannot infer complete boundedness of P from that of $P|_{C^*_\delta(G)}$ without further hypotheses. One such additional hypothesis is presented in the following proposition.

PROPOSITION 3.22. *Let E be a strong Leinert set contained in a discrete subgroup H of a locally compact group G. Let $u \in P(G) \cap C_c(G)$ and $\Gamma_u : A_G(E) \to A(G)$ be fixed as in Theorem 3.3 for E. If, in addition, $\delta_h * u * \delta_h = u$ for all $h \in E$, then $I_G(E)$ is completely complemented in $A(G)$. In particular, if G is a* [SIN]*-group, then $I_G(E)$ is always completely complemented in $A(G)$.*

Proof. It suffices to show that the bounded map $\Gamma_u : A_G(E) \to A(G)$ is completely bounded, or equivalently, that the projection

$$P_u = \Gamma_u^* : \mathrm{VN}(G) \to I_G(E)^\perp, \qquad P_u T = \sum_{h\in E} T\check{u}(h)\lambda(h) \quad (T \in \mathrm{VN}(G)),$$

is completely bounded.

Since $E \subseteq H$ is a strong Leinert set, Proposition 3.10 tells us that there is some $C > 0$ such that for any Hilbert space $\mathcal{H}$ and any finitely supported function $a : E \to \mathcal{B}(\mathcal{H})$ we have

$$\Big\| \sum_{h\in E} a(h) \otimes \lambda(h) \Big\|_{\mathcal{B}(\mathcal{H})\otimes_{\min}\mathrm{VN}(G)} \le C \max\Big\{ \Big\| \sum_{h\in E} a(h)^* a(h) \Big\|^{1/2}, \Big\| \sum_{h\in E} a(h) a(h)^* \Big\|^{1/2} \Big\}.$$

Let $n \in \mathbb{N}$ and $[T_{ij}] \in M_n(\mathrm{VN}(G))$ be arbitrary. Using the above inequality together with Corollary 3.5, we get

$$\begin{aligned} \|P_u^{(n)}[T_{ij}]\|_{M_n(\mathrm{VN}(G))} &= \Big\| \sum_{h\in E} [T_{ij}\check{u}(h)] \otimes \lambda(h) \Big\|_{M_n\otimes \mathrm{VN}(G)} \\ &\le C \max\Big\{ \Big\| \sum_{h\in E} [T_{ij}\check{u}(h)]^*[T_{ij}\check{u}(h)] \Big\|^{1/2}, \Big\| \sum_{h\in E} [T_{ij}\check{u}(h)][T_{ij}\check{u}(h)]^* \Big\|^{1/2} \Big\} \\ &\le C \|[T_{ij}]\|_{\mathrm{VN}(G)}. \end{aligned}$$

Thus P_u is completely bounded, and hence $I_G(E)$ is completely complemented.

Finally, if G is a [SIN]-group, note that for any neighbourhood $\mathcal{U}$ of the identity, one can find $u \in P(G) \cap C_c(G)$ such that $\delta_h * u * \delta_h = u$ for all

$h \in G$. Just choose such a u with small enough support and consider the associated extension map Γ_u. ■

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Michael Brannan
Department of Mathematics
Texas A&M University
Mailstop 3368
College Station, TX 77843-3368, U.S.A.
E-mail: mbrannan@math.tamu.edu

Brian Forrest
Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario, N2L 3G1 Canada
E-mail: beforrest@uwaterloo.ca

Cameron Zwarich
Cupertino, CA 95014, U.S.A.
E-mail: cwzwarich@me.com