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A TIGHT NEIGHBORHOOD UNION CONDITION ON FRACTIONAL (g,f,n',m)-CRITICAL DELETED GRAPHS

BY

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Abstract. A graph G is called a fractional (g,f,n',m)-critical deleted graph if it remains a fractional (g,f,m)-deleted graph after deleting any n' vertices. We prove that if G is a graph of order n, $1 \le a \le g(x) \le f(x) \le b$ for any $x \in V(G)$, $\delta(G) \ge b^2/a + n' + 2m$, $n > ((a+b)(2(a+b)+2m-1)+bn')/a$, and $|N_G(x_1) \cup N_G(x_2)| \ge b(n+n')/(a+b)$ for any nonadjacent vertices x_1 an x_2, then G is a fractional (g,f,n',m)-critical deleted graph. The result is tight on the neighborhood union condition in some sense.

1. Introduction. All graphs considered in this paper are finite, loopless, and have no multiple edges. The notation and terminology used but undefined in this paper can be found in [2]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $n = |V(G)|$. For a vertex $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. Let $\Delta(G)$ and $\delta(G)$ denote the minimum degree and the maximum degree of G, respectively. For $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and let $G - S = G[V(G) \setminus S]$. For two disjoint subsets S and T of $V(G)$, we use $e_G(S,T)$ to denote the number of edges with one end in S and the other in T. Given a vertex $x \in V(G)$, let $E(x) = \{e \in E(G) \mid e \text{ is incident to } x\}$.

Suppose that g and f are two integer-valued functions on $V(G)$ such that $0 \le g(x) \le f(x)$ for all $x \in V(G)$. A *fractional (g,f)-factor* is a function h that assigns to each edge of G a number in [0,1] so that for each vertex x we have $g(x) \le d_G^h(x) \le f(x)$, where $d_G^h(x) = \sum_{e \in E(x)} h(e)$ is called the *fractional degree* of x in G. If $g(x) = f(x) = k$ ($k \ge 1$ is an integer) for all $x \in V(G)$, then a fractional (g,f)-factor is just a *fractional k-factor*.

A graph G is called a *fractional (g,f,m)-deleted graph* if for each edge subset $H \subseteq E(G)$ with $|H| = m$ there exists a fractional (g,f)-factor h such that $h(e) = 0$ for all $e \in H$. That is, after any m edges are removed,

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the resulting graph still has a fractional (g, f)-factor. A graph G is called a *fractional* (g, f, n')*-critical graph* if after deleting any n' vertices from G, the resulting graph still has a fractional (g, f)-factor.

The first author of this paper first introduced the concept of a fractional (g, f, n', m)-critical deleted graph [3]. A graph G is called a *fractional* (g, f, n', m)*-critical deleted graph* if it remains a fractional (g, f, m)-deleted graph after deleting any n' vertices. If $g(x) = f(x)$ for all $x \in V(G)$, then a fractional (g, f, m)-deleted graph, fractional (g, f, n')-critical graph, and fractional (g, f, n', m)-critical deleted graph are called a fractional (f, m)-deleted graph, fractional (f, n')-critical graph, and fractional (f, n', m)-critical deleted graph, respectively. Furthermore, if $g(x) = f(x) = k$ ($k \geq 1$ is an integer) for all $x \in V(G)$, then a fractional (g, f, m)-deleted graph, fractional (g, f, n')-critical graph, and fractional (g, f, n', m)-critical deleted graph are called just a fractional (k, m)-deleted graph, fractional (k, n')-critical graph, and fractional (k, n', m)-critical deleted graph, respectively.

Yu et al. [8] proved that if G is a connected graph with $\delta(G) \geq k \geq 1$, $n \geq 4k - 3$, and $\max\{d_G(x), d_G(y)\} \geq n/2$ for each pair of nonadjacent vertices x and y, then G has a fractional k-factor. Liu and Zhang [7] revealed a relation between the fractional k-factor and the toughness of a graph by showing that for an integer $k \geq 2$, if $n \geq k + 1$ and $t(G) \geq k - 1/k$, then G admits a fractional k-factor. Anstee [1] gave a necessary and sufficient condition for a graph to have a fractional (g, f)-factor as follows:

Theorem 1.1 (Anstee [1]). *Suppose that f and g are integer-valued functions defined on the vertex set of a graph G such that $0 \leq g(x) \leq f(x)$ for each $x \in V(G)$. Then G has a fractional (g, f)-factor if and only if for every subset S of $V(G)$, $g(T) - d_{G-S}(T) \leq f(S)$, where $T = \{x \in V(G) \backslash S \mid d_{G-S}(x) \leq g(x)\}$.*

Given integers $k, m \geq 1$, a graph G is called a *fractional* (k, m)*-deleted graph* if after removing any m edges from G, the resulting graph has a fractional k-factor. Zhou [9] showed that if G is a graph with $n \geq 4k - 5 + 2(2k + 1)m$ and $\delta(G) \geq n/2$, then G is a fractional (k, m)-deleted graph. Moreover, Zhou [10] established a neighborhood condition for a fractional (k, m)-deleted graph.

Theorem 1.2 (Zhou [10]). *Let $k \geq 2$ and $m \geq 0$ be integers. If G is a connected graph with $n \geq 9k - 1 - \sqrt{2(k-1)^2 + 2} + 2(2k + 1)m$, $\delta(G) \geq k + m + (m + 1)^2 - 1/(4k)$, and*

$$|N_G(x) \cup N_G(y)| \geq \tfrac{1}{2}(n + k - 2)$$

for each pair of nonadjacent vertices x and y, then G is a fractional (k, m)-deleted graph.

Zhou and Bian [13] determined an independent set degree condition for fractional (g, f)-deleted graphs.

THEOREM 1.3 (Zhou and Bian [13]). *Let $r \geq 2$, $\varDelta \geq 0$ and $2 \leq a \leq b-\varDelta$ be integers, and let G be a graph of order n with*

$$n \geq \frac{(a+b)(a+b+1+(r-2)(b-\varDelta))}{a+\varDelta}$$

and

$$\delta(G) \geq \frac{(r-1)(b-\varDelta)(b+1)}{a+\varDelta}.$$

Let g and f be integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) - \varDelta \leq b - \varDelta$ for each $x \in V(G)$. If

$$\max\{d(x_1), \dots, d(x_r)\} \geq \frac{(b-\varDelta)n}{a+b}$$

for any independent subset $\{x_1, \dots, x_r\}$ in G, then G is a fractional (g, f)-deleted graph.

They also show that when $a = b = k$, Theorem 1.3 can be strengthened.

THEOREM 1.4 (Zhou and Bian [13]). *Let k and r be integers with $r \geq 2$ and $k \geq 2$, and let G be a graph of order n with $n \geq 2rk + 1$ and $\delta(G) \geq (r-1)(k+1)$. If*

$$\max\{d(x_1), \dots, d(x_r)\} \geq n/2$$

for any independent subset $\{x_1, \dots, x_r\}$ in G, then G is a fractional k-deleted graph.

Other results on fractional deleted graphs and fractional critical graphs can be found in Gao and Gao [4], Gao et al. [5], Gao and Wang [6], and Zhou [11], [12].

In this paper, we determine a neighborhood union condition for fractional (g, f, n', m)-critical deleted graphs, and also establish the sharpness of the bound. Our main conclusion is stated as follows; it will be proved in the next section.

THEOREM 1.5. *Let G be a graph of order n. Let a, b, n', m be integers with $1 \leq a \leq b$ and $n', m \geq 0$. Let g, f be integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If*

$$\delta(G) \geq \frac{b^2}{a} + n' + 2m, \qquad n > \frac{(a+b)(2(a+b)+2m-1)+bn'}{a},$$

and

$$|N_G(x_1) \cup N_G(x_2)| \geq \frac{b(n+n')}{a+b}$$

for any nonadjacent vertices $x_1, x_2 \in V(G)$, then G is a fractional (g, f, n', m)-critical deleted graph.

Clearly, the neighborhood union condition in Theorem 1.5 is better than any previous bound, and it also improves the result of Yu et al. [8]. Setting $n' = 0$ in Theorem 1.5, we get the following corollary.

Corollary 1.6. *Let* G *be a graph of order* n*. Let* a, b, m *be integers with* $1 \leq a \leq b$ *and* $m \geq 0$*. Let* g, f *be integer-valued functions defined on* $V(G)$ *such that* $a \leq g(x) \leq f(x) \leq b$ *for each* $x \in V(G)$*. If* $\delta(G) \geq b^2/a + 2m$*,* $n > (a+b)(2(a+b)+2m-1)/a$*, and*

$$|N_G(x_1) \cup N_G(x_2)| \geq \frac{bn}{a+b}$$

for any nonadjacent vertices $x_1, x_2 \in V(G)$*, then* G *is a fractional* (g, f, m)*-deleted graph.*

By setting $m = 0$, we obtain the following neighborhood union condition.

Corollary 1.7. *Let* G *be a graph of order* n*. Let* a, b, n' *be integers with* $1 \leq a \leq b$ *and* $n' \geq 0$*. Let* g, f *be integer-valued functions defined on* $V(G)$ *such that* $a \leq g(x) \leq f(x) \leq b$ *for each* $x \in V(G)$*. If* $\delta(G) \geq b^2/a + n'$*,* $n > (a+b)((2(a+b)-1)+bn')/a$*, and*

$$|N_G(x_1) \cup N_G(x_2)| \geq \frac{b(n+n')}{a+b}$$

for any nonadjacent vertices $x_1, x_2 \in V(G)$*, then* G *is a fractional* (g, f, n')*-critical graph.*

Note that $e_H(S,T)$ denotes the number of edges in a subgraph H with one end in S and the other in T. Set $d_H(x) = |\{y \in H \mid xy \in E(G)\}|$. The proof of our main results is based on the following lemma.

Lemma 1.8 (Gao [3]). *Let* G *be a graph, and* g, f *be integer-valued functions defined on* $V(G)$ *such that* $g(x) \leq f(x)$ *for each* $x \in V(G)$*. Let* n'*,* m *be nonnegative integers. Then* G *is a fractional* (g, f, n', m)*-critical deleted graph if and only if*

$$\begin{aligned} &f(S) - g(T) + d_{G-S}(T) \\ &\geq \max_{U \subseteq S,\, |U|=n',\, H \subseteq E(G-U),\, |H|=m} \Big\{ f(U) + \sum_{x \in T} d_H(x) - e_H(T,S) \Big\} \end{aligned} \tag{1.1}$$

for all disjoint subsets S, T *of* $V(G)$ *with* $|S| \geq n'$*.*

2. Proof of Theorem 1.5. Assume to the contrary that G satisfies the conditions of the theorem, but is not a fractional (g, f, n', m)-critical deleted graph. By Corollary 1.8, noting that $\sum_{x \in T} d_H(x) - e_H(T,S) \leq 2m$, we see that there exist disjoint subsets S and T of $V(G)$ with $|S| \geq n'$ such that

$$f(S \setminus U) + \sum_{x \in T} d_{G-S}(x) - g(T) \leq 2m - 1. \tag{2.1}$$

This can be formulated as

$$f(S)+\sum_{x\in T} d_{G-S}(x)-g(T)\le bn'+2m-1. \tag{2.2}$$

We choose subsets S and T such that $|T|$ is minimal. Obviously, $T\neq\emptyset$.

LEMMA 2.1. *$d_{G-S}(x)\le g(x)-1\le b-1$ for any $x\in T$.*

Proof. If $d_{G-S}(x)\ge g(x)$ for some $x\in T$, then the subsets S and $T\setminus\{x\}$ satisfy (2.1). This contradicts the choice of S and T. ■

Let $d_1=\min\{d_{G-S}(x)\mid x\in T\}$ and choose $x_1\in T$ such that $d_{G-S}(x_1)=d_1$. If $T-N_T[x_1]\neq\emptyset$, let $d_2=\min\{d_{G-S}(x)\mid x\in T-N_T[x_1]\}$ and choose $x_2\in T-N_T[x_1]$ such that $d_{G-S}(x_2)=d_2$. Hence, $0\le d_1\le d_2\le b-1$ and $x_1x_2\neq E(G)$.

LEMMA 2.2. $|T|\ge b+1$.

Proof. Assume that $|T|\le b$. Then $|S|+d_1\ge d_G(x_1)\ge\delta(G)\ge b^2/a+n'+2m$. By (2.1) and $0\le d_1\le b-1$, we have

$$\begin{aligned}2m-1&\ge f(S-U)-g(T)+d_{G-S}(T)\\&\ge a(|S|-n')+d_1|T|-b|T|\\&=a(|S|-n')+(d_1-b)|T|\\&\ge a(b^2/a-d_1+2m)+(d_1-b)b\\&=b^2+d_1(b-a)-b^2+2am\ge 2m,\end{aligned}$$

a contradiction. ■

Since $d_{G-S}(x)\le b-1$ and $|T|\ge b+1$, we can choose nonadjacent vertices x_1 and x_2 in T.

By the neighborhood union condition of the theorem, we infer that

$$\frac{b(n+n')}{a+b}\le|N_G(x_1)\cup N_G(x_2)|\le|S|+d_1+d_2$$

and

$$|S|\ge\frac{b(n+n')}{a+b}-(d_1+d_2). \tag{2.3}$$

Using $n-|S|-|T|\ge0$, $b-d_2>0$ and (2.2), we get

$$\begin{aligned}&(n-|S|-|T|)(b-d_2)\\&\quad\ge a|S|+\sum_{x\in T}(d_{G-S}(x)-b)-bn'-2m+1\\&\quad\ge a|S|+(d_1-b)|N_T[x_1]|+(d_2-b)(|T|-|N_T[x_1]|)-bn'-2m+1\\&\quad=a|S|+(d_1-d_2)|N_T[x_1]|+(d_2-b)|T|-bn'-2m+1\\&\quad\ge a|S|+(d_1-d_2)(d_1+1)+(d_2-b)|T|-bn'-2m+1.\end{aligned}$$

It follows that

$$0 \le n(b-d_2) - (a+b-d_2)|S| + (d_2-d_1)(d_1+1) + bn' + 2m - 1. \tag{2.4}$$

In view of (2.3), (2.4), $d_1 \le d_2 \le b-1$ and

$$n > (a+b)\big((2(a+b)+2m-1)+bn'\big)/a,$$

we have

$$\begin{aligned} 0 &\le n(b-d_2) - (a+b-d_2)\left(\frac{b(n+n')}{a+b} - (d_1+d_2)\right) \\ &\quad + (d_2-d_1)(d_1+1) + bn' + 2m - 1 \\ &= -nd_2\frac{a}{a+b} + d_2\frac{bn'}{a+b} + (a+b)(d_1+d_2) - d_1^2 - d_2^2 + d_2 - d_1 + 2m - 1 \\ &< (a+b)(d_1-d_2) - d_1^2 - d_2^2 + 2d_2 - d_1 + 2m(1-d_2) - 1 \\ &\le -d_1^2 - d_2^2 + 2d_2 - d_1 + 2m(1-d_2) - 1. \end{aligned}$$

If $d_2 = 0$, then $d_1 = d_2 = 0$. By (2.3), we get $|S| \ge b(n+n')/(a+b)$ and $|T| \le n - |S| \le (an-bn')/(a+b)$. Since $d_{G-S}(T) \ge \sum_{x\in T} d_H(x) - e_H(T,S)$, we obtain

$$\begin{aligned} &f(S) + d_{G-S}(T) - g(T) - bn' - \Big(\sum_{x\in T} d_H(x) - e_H(T,S)\Big) \\ &\quad \ge a\frac{b(n+n')}{a+b} - b\frac{an-bn'}{a+b} - bn' + \Big(d_{G-S}(T) - \sum_{x\in T} d_H(x) + e_H(T,S)\Big) \\ &\quad \ge 0, \end{aligned}$$

a contradiction.

If $d_2 \ge 1$, then

$$\begin{aligned} 0 &< -d_1^2 - d_2^2 + 2d_2 - d_1 + 2m(1-d_2) - 1 \\ &\le -d_2^2 + 2d_2 - d_1^2 - d_1 - 1. \end{aligned}$$

Let

$$h_1(d_2) = -d_2^2 + 2d_2 - d_1^2 - d_1 - 1.$$

Hence,

$$\max\{h_1(d_2)\} = h_1(1) = -d_1^2 - d_1 \le 0,$$

a contradiction. This completes the proof of Theorem 1.5. ■

3. Sharpness. Theorem 1.5 is best possible in some sense. Namely, we can construct graphs such that the neighborhood union condition in Theorem 1.5 cannot be replaced by $|N_G(x_1) \cup N_G(x_2)| \ge b(n+n')/(a+b) - 1$.

Let $G_1 = K_{bt+n'}$ be a complete graph, $G_2 = (at+1)K_1$ be a graph consisting of $at+1$ isolated vertices, and $G = G_1 \vee G_2$, where t is sufficiently large (such that $\delta(G) \geq b^2/a + n' + 2m$, and $n > (a+b)((2(a+b)+2m-1)+bn')/a)$ and $0 \leq n' < a/(b-a)$. Then $n = |G_1| + |G_2| = (a+b)t + n' + 1$, and for any nonadjacent vertices $x_1, x_2 \in V(G_2)$, we deduce

$$\frac{b(n+n')}{a+b} > |N_G(x_1) \cup N_G(x_2)| = bt + n' > \frac{b(n+n')}{a+b} - 1.$$

Let $S = V(G_1)$, and $g(x) = f(x) = a$ for any $x \in V(G_1)$; and set $T = V(G_2)$, and $g(x) = f(x) = b$ for any $x \in V(G_2)$. Then

$$\begin{aligned} f(S \setminus U) - g(T) + d_{G-S}(T) - \Big(\sum_{x \in T} d_H(x) - e_H(T,S)\Big) & \\ = a(|S| - n') - b|T| = a(bt) - b(at+1) = -b < 0 & \end{aligned}$$

for any $U \subseteq S$ and $|U| = n'$. Hence, G is not a fractional (g, f, n', m)-critical deleted graph.

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