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CONSTRUCTION OF A BRAIDED MONOIDAL CATEGORY FOR BRZEZIŃSKI CROSSED COPRODUCTS OF HOPF π-ALGEBRAS

BY

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Abstract. Let π be a group, C, H Hopf π-algebras, and $g_\alpha : C_\alpha \otimes H_\alpha \to H_\alpha \otimes H_\alpha$ and $T_\alpha : C_\alpha \otimes H_\alpha \to H_\alpha \otimes C_\alpha$ families of linear maps. We give necessary and sufficient conditions for the family of Brzeziński crossed coproduct coalgebras $\{C_\alpha \#^{g_\alpha}_{T_\alpha} H_\alpha\}_{\alpha\in\pi}$ to be a Hopf π-algebra. Moreover, necessary and sufficient conditions for the Brzeziński crossed coproduct Hopf π-algebra $C \natural^{g}_{T}{}^{\pi} H$ to be quasitriangular are derived, and in this case, the left π-module category ${}_{C \natural^{g}_{T}{}^{\pi} H}\mathcal{M}$ is a braided monoidal category.

1. Introduction. Let C be a coassociative and counitary coalgebra and H a vector space endowed with a distinguished element $\varepsilon \in H^*$. T. Brzeziński [1] gave necessary and sufficient conditions for the crossed coproduct $C \#^{g}_{T} H$ (with underlying vector space $C \otimes H$) to be a coalgebra (called Brzeziński's crossed coproduct coalgebra), which includes the Hopf crossed coproduct $C \rtimes_\alpha H$ [6], the twisted tensor coproduct [10, 6] and the Drinfel'd co-double.

Throughout we assume that π is a group. Hopf π-(co)algebras (or Hopf group-(co)algebras) appeared in the work of Turaev [14, 13] on homotopy quantum field theories. In the case where π is trivial, we recover the classical Hopf algebras. Hopf π-(co)algebras have been studied by many researchers from purely algebraic point of view [9, 8, 15–19]. In particular, Virelizier [15] generalized many properties of quasitriangular Hopf algebras to the setting of quasitriangular Hopf π-coalgebras (the (co)quasitriangular Hom-Hopf algebra structures can provide a solution of a corresponding Hom-Yang–Baxter equation, i.e., the Hom-version of (co)quasitriangular Hopf algebras, see [5, 7]). Ma et al. [9] studied the representation category of quasitriangular Hopf π-algebras, and gave necessary and sufficient conditions for the Hopf π-crossed coproduct algebra $C \rtimes^{\pi}_{\alpha} H$ to be a quasitriangular Hopf

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π-algebra. Ma et al. [8] gave necessary and sufficient conditions for a family of Brzeziński crossed product algebras with suitable comultiplication and counit to be a Hopf π-coalgebra, and also considered the coquasitriangular Brzeziński π-crossed product. Wang [17] considered the entwining structures between a group coalgebra, and a family of algebras indexed by the corresponding group. He also introduced group coalgebra Galois extensions (for the Hom-version of Hopf–Galois extensions see [2]). Wang [16] introduced the Doi–Hopf data and Doi–Hopf modules over Hopf group coalgebras (for the Hom-version of Doi–Hopf modules see [4]), and showed that the Yetter–Drinfeld modules introduced by Zunino are a special case. In [16], the smash product construction is generalized, and it is shown that Zunino's Drinfeld double can be written as such a smash product. Wang [18] also investigated properties of coquasitriangular Hopf π-algebras.

In contrast with classical Hopf algebras, the definition of a Hopf π-coalgebra is not selfdual. The dual of a Hopf π-coalgebra is a Hopf π-algebra, and the converse holds under some finiteness assumptions. But the representation category of quasitriangular Hopf π-(co)algebras gives rise to a braided monoidal category which is different from one coming from the corepresentation category of coquasitriangular Hopf π-(co)algebras.

In this paper, we mainly generalize the Brzeziński crossed coproduct to the case of Hopf π-algebras, and also give necessary and sufficient conditions for the Brzeziński crossed coproduct Hopf π-algebra to be quasitriangular. Finally, we apply the above result to the π-quantum codouble, $C \natural_{\theta}^{\pi} H$, and a concrete and complicated example is also provided.

2. Preliminaries. Throughout this paper we freely use the Hopf algebras and coalgebras terminology introduced in [1, 3, 9, 11, 12], and all algebraic systems are supposed to be over the field k. Given a k-space M, we write id_M for the identity map on M.

Next we recall a very general definition of crossed coproduct, introduced by T. Brzeziński [1].

Brzeziński crossed coproduct. Let $(C, \varDelta, \kappa)$ be a coalgebra, H a vector space and $\varepsilon \in H^*$ (the dual space of H). The vector space $C \otimes H$ is a coalgebra with counit $\kappa \otimes \varepsilon$, and the coproduct such that

$$c_1 \otimes c_2 \otimes \varepsilon(x_1)x_2 = c_1 \otimes c_2 \otimes x$$

if and only if there exist linear maps $g : C \otimes H \to H \otimes H$ (we write $g(c \otimes x) = c_g \otimes x^g$ for all $c \in C$ and $x \in H$) and $T : C \otimes H \to H \otimes C$ (we write $T(c \otimes x) = x_T \otimes c_T$ for all $x \in H$ and $c \in C$) which satisfy the following conditions:

$$
\begin{aligned}
&\text{(A1)} && \varepsilon(x_T)c_T = \varepsilon(x)c, \quad x_T\kappa(c_T) = x\kappa(c),\\
&\text{(A2)} && x_T \otimes c_{T1} \otimes c_{T2} = x_{Tt} \otimes c_{1t} \otimes c_{2T},\\
&\text{(A3)} && c_g\varepsilon(x^g) = \varepsilon(c_g)x^g = \kappa(c)x,\\
&\text{(A4)} && c_{2gT} \otimes c_{1TG} \otimes x^{gG} = c_{1G} \otimes c_{2g}{}^{G} \otimes x^g,\\
&\text{(A5)} && c_{2gT} \otimes x^g{}_t \otimes c_{1Tt} = c_{1g} \otimes x_T{}^g \otimes c_{2T},
\end{aligned}
$$

for all $c \in C$, $x \in H$ and $G = g$, $T = t$. The coproduct $\varDelta_{C\otimes H}$ in $C \otimes H$ explicitly reads

$$\varDelta_{C\otimes H}(c \otimes x) = c_1 \otimes c_{3gT} \otimes c_{2T} \otimes x^g$$

for all $c \in C$ and $x \in H$.

We call this coalgebra the *Brzeziński crossed coproduct* and denote it by $C \#^g_T H$.

Hopf π-algebra. A *π-algebra* consists of a set of k-spaces $H = \{H_\alpha\}_{\alpha\in\pi}$ together with maps

$$\mu_{\alpha,\beta} : H_\alpha \otimes H_\beta \to H_{\alpha\beta} \quad \text{and} \quad \eta : k \to H_1$$

such that

$$
\begin{aligned}
&\mu_{\alpha\beta,\gamma}(\mu_{\alpha,\beta} \otimes \mathrm{id}_{H_\gamma}) = \mu_{\alpha,\beta\gamma}(\mathrm{id}_{H_\alpha} \otimes \mu_{\beta,\gamma}),\\
&\mu_{\alpha,1}(\mathrm{id}_{H_\alpha} \otimes \eta) = \mathrm{id}_{H_\alpha} = \mu_{1,\alpha}(\eta \otimes \mathrm{id}_{H_\alpha}),
\end{aligned}
$$

for $\alpha, \beta, \gamma \in \pi$.

A π-algebra $H = \{H_\alpha\}_{\alpha\in\pi}$ is called a *semi-Hopf π-algebra* if every H_α is a coalgebra and $\mu_{\alpha,\beta}$ and η are coalgebra maps.

A *Hopf π-algebra* is a semi-Hopf π-algebra $H = \{H_\alpha\}_{\alpha\in\pi}$ together with maps $S_\alpha : H_\alpha \to H_{\alpha^{-1}}$ such that

$$\mu_{\alpha^{-1},\alpha}(S_\alpha \otimes \mathrm{id}_\alpha)\varDelta_\alpha = \mu_{\alpha,\alpha^{-1}}(\mathrm{id}_\alpha \otimes S_\alpha)\varDelta_\alpha = \eta\varepsilon_\alpha.$$

Note that $(H_1, \mu_{1,1}, \eta, \varDelta_1, \varepsilon_1, S_1)$ is a usual Hopf algebra.

Quasitriangular Hopf π-algebra. Let H be a Hopf π-algebra. Then H is *quasitriangular* if there exists $R \in H_1 \otimes H_1$ such that

$$
\begin{aligned}
&\text{(QT1)} && \varDelta_1(R^1) \otimes R^2 = R^1 \otimes \bar{R}^1 \otimes R^2\bar{R}^2,\\
&\text{(QT2)} && R^1 \otimes \varDelta_1(R^2) = R^1\bar{R}^1 \otimes \bar{R}^2 \otimes R^2,\\
&\text{(QT3)} && h_2R^1 \otimes h_1R^2 = R^1h_1 \otimes R^2h_2,\\
&\text{(QT4)} && \varepsilon_1(R^1)R^2 = R^1\varepsilon_1(R^2) = 1_1,
\end{aligned}
$$

for all $h \in H_\alpha$ and $R = \bar{R}$.

Remark. (1) In this case, R is called a *quasitriangular structure* over H.

(2) H_1 is a usual quasitriangular Hopf algebra if H is a quasitriangular Hopf π-algebra.

(3) By (QT1) and (QT4), R is invertible, and $R^{-1} = S_1(R^1) \otimes R^2$.

Let (H, R) be a quasitriangular Hopf π-algebra. Then the left H-π-module category ${}_H\mathcal{M}$ is a braided monoidal category.

Let C, H be Hopf π-algebras and $\theta = \theta^1 \otimes \theta^2 \in C_1 \otimes H_1$. If θ satisfies conditions (QT1), (QT2) and (QT4), then we say that (C, H, θ) is the *compatibility Hopf π-algebra triple* on $C \otimes H$.

We remark that if (H, R) is a quasitriangular Hopf π-algebra, then (H, H, R) is a compatibility Hopf π-algebra triple on $H \otimes H$.

From now on, we assume that $(H, \mu_H, 1_1, \varDelta_H, \varepsilon)$ and $(C, \mu_C, \bar{1}_1, \varDelta_C, \kappa)$ are Hopf π-algebras. Next, we introduce three new notions which will be used later.

Compatible (W, g, V)-Hopf π-algebra quadruple. Let C, H be Hopf π-algebras. Assume there exist a family of linear maps $g_\alpha : C_\alpha \otimes H_\alpha \to H_\alpha \otimes H_\alpha$ and two elements $V = V^1 \otimes V^2 \in C_1 \otimes C_1$ and $W = W^1 \otimes W^2 \in C_1 \otimes H_1$ such that

$$
\begin{aligned}
&\text{(CQ1)} \quad W^1{}_1 \otimes W^1{}_2 \otimes W_2 = W^1 \otimes \bar{W}^1 \otimes W^2 \bar{W}^2,\\
&\text{(CQ2)} \quad W^1 V^1 \otimes V^2{}_{g_1} \otimes W^{2 g_1} = W^1 \bar{W}^1 \otimes \bar{W}^2 \otimes W^2,\\
&\text{(CQ3)} \quad \kappa_1(W^1) W^2 = 1_1, \quad W^1 \varepsilon_1(W^2) = \bar{1}_1
\end{aligned}
$$

where $W = \bar{W}$. Then we call (C, H, W, g) a *compatible (W, g, V)-Hopf π-algebra quadruple.*

Anti-compatible (X, g, V)-Hopf π-algebra quadruple. Let C and H be Hopf π-algebras. Assume that there exist a family of linear maps $g_\alpha : C_\alpha \otimes H_\alpha \to H_\alpha \otimes H_\alpha$ and two elements $V = V^1 \otimes V^2 \in C_1 \otimes C_1$ and $X = X^1 \otimes X^2 \in H_1 \otimes C_1$. Then (H, C, X, g) is called an *anti-compatible (X, g, V)-Hopf π-algebra quadruple* if the following conditions hold $(X = \bar{X})$:

$$
\begin{aligned}
&\text{(ACQ1)} \quad V^1{}_{g_1} \otimes X^{1 g_1} \otimes V^2 X^2 = X^1 \otimes \bar{X}^1 \otimes X^2 \bar{X}^2,\\
&\text{(ACQ2)} \quad X^1 \otimes X^2{}_1 \otimes X^2{}_2 = X^1 \bar{X}^1 \otimes \bar{X}^2 \otimes X^2,\\
&\text{(ACQ3)} \quad \varepsilon_1(X^1) X^2 = \bar{1}_1, \quad X^1 \kappa_1(X^2) = 1_1.
\end{aligned}
$$

Quasitriangular-like Hopf π-algebra. Let C, H be Hopf π-algebras. Assume that there exist a family of maps $g_\alpha : C_\alpha \otimes H_\alpha \to H_\alpha \otimes H_\alpha$ and two elements $X = X^1 \otimes X^2 \in H_1 \otimes C_1$ and $W = W^1 \otimes W^2 \in C_1 \otimes H_1$. A *quasitriangular-like Hopf π-algebra* associated to (W, X, g) is a pair (H, U) where $U = U^1 \otimes U^2 \in H_1 \otimes H_1$ satisfying

$$
\begin{aligned}
&\text{(QTL1)} \quad W^1{}_{g_1} \otimes U^{1 g_1} \otimes W^2 U^2 = U^1 \otimes \bar{U}^1 \otimes U^2 \bar{U}^2,\\
&\text{(QTL2)} \quad U^1 X^1 \otimes X^2{}_{g_1} \otimes U^{2 g_1} = U^1 \bar{U}^1 \otimes \bar{U}^2 \otimes U^2,\\
&\text{(QTL3)} \quad x^{g_\alpha} U^1 \otimes c_{g_\alpha} U^2 = U^1 c_{g_\alpha} \otimes U^2 x^{g_\alpha},\\
&\text{(QTL4)} \quad \varepsilon_1(U^1) U^2 = U^1 \varepsilon_1(U^2) = 1_1,
\end{aligned}
$$

for all $x \in H_\alpha$.

Examples. (1) Let C, H be Hopf π-algebras. If the map g_α is trivial, i.e., $g_\alpha(c \otimes x) = c_{g_\alpha} \otimes x^{g_\alpha} = \kappa_\alpha(c)x_1 \otimes x_2$, and V satisfies (QT4), then a compatible (W, g, V)-Hopf π-algebra quadruple (C, H, W, g) is exactly a compatibility Hopf π-algebra triple on $C \otimes H$.

(2) Let C, H be Hopf π-algebras. If the map g_α is trivial and V satisfies (QT4), then an anti-compatible (X, g, V)-Hopf π-algebra quadruple (H, C, X, g) is exactly a compatibility Hopf π-algebra triple on $H \otimes C$.

(3) Let C, H be Hopf π-algebras. If the map g_α is trivial and V, W, X satisfy (QT4), (CQ3), (ACQ3), respectively, then a quasitriangular-like Hopf π-algebra (H, U) associated to (W, X, g) is exactly a quasitriangular Hopf π-algebra.

3. Brzeziński crossed coproduct Hopf π-algebra. In this section, we give the construction of the Brzeziński crossed coproduct Hopf π-algebra.

Let C, H be Hopf π-algebras. Then we have a π-algebra $C \#_T^{g}{}^{\pi} H = \{C_\alpha \#_{T_\alpha}^{g_\alpha} H_\alpha\}_{\alpha\in\pi}$ with the following multiplication and unit:

$$\overline{\mu_{\alpha,\beta}} : C_\alpha \#_{T_\alpha}^{g_\alpha} H_\alpha \otimes C_\beta \#_{T_\beta}^{g_\beta} H_\beta \to C_{\alpha\beta} \#_{T_{\alpha\beta}}^{g_{\alpha\beta}} H_{\alpha\beta}, \qquad \overline{\eta} : k \to C_1 \#_{T_1}^{g_1} H_1,$$
$$(c \otimes x) \otimes (d \otimes y) \mapsto cd \otimes xy, \qquad\qquad 1_k \mapsto \bar{1}_1 \otimes 1_1,$$

where $\alpha, \beta \in \pi$, $x \in H_\alpha$, $y \in H_\beta$ and $c \in C_\alpha$, $d \in C_\beta$.

Let $g_\alpha : C_\alpha \otimes H_\alpha \to H_\alpha \otimes H_\alpha$ (write $g_\alpha(c \otimes x) = c_{g_\alpha} \otimes x^{g_\alpha}$ for all $c \in C_\alpha$ and $x \in H_\alpha$) and $T_\alpha : C_\alpha \otimes H_\alpha \to H_\alpha \otimes C_\alpha$ (write $T_\alpha(c \otimes x) = x_{T_\alpha} \otimes c_{T_\alpha}$ for all $x \in H_\alpha$ and $c \in C_\alpha$) be two families of linear maps. Then $C_\alpha \#_{T_\alpha}^{g_\alpha} H_\alpha$ is a Brzeziński crossed coproduct with comultiplication

$$\varDelta_{C_\alpha \otimes H_\alpha}(c \otimes x) = c_1 \otimes c_{3g_\alpha T_\alpha} \otimes c_{2T_\alpha} \otimes x^{g_\alpha}$$

for all $c \in C_\alpha$ and $x \in H_\alpha$ if and only if conditions (A1)–(A5) hold for $G_\alpha = g_\alpha$, $T_\alpha = t_\alpha$ and $c \in C_\alpha$, $x \in H_\alpha$.

Theorem 3.1. *With the above notation. Then the family of Brzeziński crossed coproduct coalgebras $\{C_\alpha \#_{T_\alpha}^{g_\alpha} H_\alpha\}_{\alpha\in\pi}$ equipped with the multiplication $\overline{\mu_{\alpha,\beta}}$ and unit $\overline{\eta}$ is a semi-Hopf π-algebra if and only if, for all $c, d \in C$ and $x \in H_\alpha$, $y \in H_\beta$,*

$$\text{(B1)} \quad 1_{1T_1} \otimes \bar{1}_{1T_1} = 1_1 \otimes \bar{1}_1, \ \bar{1}_{1g_1} \otimes 1_1{}^{g_1} = 1_1 \otimes 1_1,$$
$$\text{(B2)} \quad c_{g_\alpha} d_{g_\beta} \otimes x^{g_\alpha} y^{g_\beta} = (cd)_{g_{\alpha\beta}} \otimes (xy)^{g_{\alpha\beta}},$$
$$\text{(B3)} \quad x_{T_\alpha} y_{T_\beta} \otimes c_{T_\alpha} d_{T_\beta} = (xy)_{T_{\alpha\beta}} \otimes (cd)_{T_{\alpha\beta}}.$$

We denote this semi-Hopf π-algebra by $C \natural_T^{g}{}^{\pi} H$ and call the *Brzeziński crossed coproduct semi-Hopf π-algebra.*

Proof. Sufficiency. We know that $\overline{\mu_{\alpha,\beta}}$ and $\overline{\eta}$ are coalgebra maps if and only if

$$\begin{aligned}
&\text{(C1)} \quad \bar{1}_1 \otimes 1_1 \otimes \bar{1}_1 \otimes 1_1 = \bar{1}_1 \otimes 1_{C g_1 T_1} \otimes 1_{C T_1} \otimes 1_1{}^{g_1},\\
&\text{(C2)} \quad c_1 d_1 \otimes c_{3 g_\alpha T_\alpha} d_{3 g_\beta T_\beta} \otimes c_{2 T_\alpha} d_{2 T_\beta} \otimes x^{g_\alpha} y^{g_\beta}\\
&\qquad\qquad = (cd)_1 \otimes (cd)_{3 g_{\alpha\beta} T_{\alpha\beta}} \otimes (cd)_{2 T_{\alpha\beta}} \otimes (xy)^{g_{\alpha\beta}}.
\end{aligned}$$

We prove that (C2) holds as follows:

$$\begin{aligned}
&(cd)_1 \otimes (cd)_{3 g_{\alpha\beta} T_{\alpha\beta}} \otimes (cd)_{2 T_{\alpha\beta}} \otimes (xy)^{g_{\alpha\beta}}\\
&\qquad = c_1 d_1 \otimes (c_3 d_3)_{g_{\alpha\beta} T_{\alpha\beta}} \otimes (c_2 d_2)_{T_{\alpha\beta}} \otimes (xy)^{g_{\alpha\beta}}\\
&\qquad = c_1 d_1 \otimes (c_{3 g_\alpha} d_{3 g_\beta})_{T_{\alpha\beta}} \otimes (c_2 d_2)_{T_{\alpha\beta}} \otimes x^{g_\alpha} y^{g_\beta} \quad \text{by (B3)}\\
&\qquad = c_1 d_1 \otimes c_{3 g_\alpha T_\alpha} d_{3 g_\beta T_\beta} \otimes c_{2 T_\alpha} d_{2 T_\beta} \otimes x^{g_\alpha} y^{g_\beta} \quad \text{by (B2).}
\end{aligned}$$

(C1) can be obtained from (B1).

The proof of the necessity is straightforward. ∎

Remarks. (1) Conditions (B1)–(B3) are equivalent to the maps $g = \{g_\alpha\}$ and $T = \{T_\alpha\}$ both being π-algebra maps.

(2) The structure of $C \natural_T^{g\,\pi} H$ is not dual to the one in [8] and is not a generalization of the Hopf π-crossed product in [9] since here C is also a Hopf π-algebra.

Example 3.2. (1) Let C, H be Hopf π-algebras. Let g_α be trivial, i.e., $g_\alpha(c \otimes x) = \kappa_\alpha(c) x_1 \otimes x_2$ in Theorem 3.1. Then a family $\{C \#_{T_\alpha} H_\alpha\}_{\alpha\in\pi}$ of twisted tensor coproduct coalgebras (see [10]) equipped with the multiplication $\overline{\mu_{\alpha,\beta}}$ and unit $\overline{\eta}$ is a semi-Hopf π-algebra if and only if $T = \{T_\alpha\}$ is a π-algebra map.

The resulting semi-Hopf π-algebra is called the *twisted tensor coproduct semi-Hopf π-algebra* and is denoted by $C \natural_T^{\pi} H$.

(2) Let C, H be Hopf π-algebras. Let T_α be trivial, i.e., $T_\alpha(c \otimes x) = x \otimes c$ in Theorem 3.1. Then a family $\{C_\alpha \#^{g_\alpha} H_\alpha\}_{\alpha\in\pi}$ of g-twisted coproduct coalgebras (see [6, 10]) equipped with the multiplication $\overline{\mu_{\alpha,\beta}}$ and unit $\overline{\eta}$ is a semi-Hopf π-algebra if and only if $g = \{g_\alpha\}$ is a π-algebra map.

We call this semi-Hopf π-algebra a *g-twisted coproduct semi-Hopf π-algebra* and denote it by $C \natural^{g\,\pi} H$.

Definition 3.3. Let C, H be Hopf π-algebras. Let $g = \{g_\alpha : C_\alpha \otimes H_\alpha \to H_\alpha \otimes H_\alpha\}$ and $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}$ be families of linear maps. Then S is called a *g-antipode* of H if for all $x \in H_\alpha$ and $c \in C$,

$$\begin{aligned}
&\text{(E1)} \quad c_{g_\alpha} S(x^{g_\alpha}) = \kappa_\alpha(c) \varepsilon_\alpha(x) 1_1,\\
&\text{(E2)} \quad S(c_{g_\alpha}) x^{g_\alpha} = \kappa_\alpha(c) \varepsilon_\alpha(x) 1_1.
\end{aligned}$$

In this case, we say H is a *g-Hopf algebra.*

Example 3.4. Let H be a Hopf π-algebra with antipode S. Assume that g_α is trivial. Then we can regard S as a g-antipode of H.

PROPOSITION 3.5. *Let* $(C, S_C), (H, S_H)$ *be Hopf* π*-algebras. Suppose that* $C \natural_T^{g\,\pi} H$ *is a Brzeziński crossed coproduct semi-Hopf* π*-algebra. Then it is a Hopf* π*-algebra with antipode* $\bar{S} = \{\bar{S}_\alpha\}$ *defined by*

$$\bar{S}_\alpha(c \otimes x) = S_{C\alpha}(c_{T_\alpha}) \otimes S_{H\alpha}(x_{T_\alpha})$$

if and only if H *is a* g*-Hopf* π*-algebra.*

Proof. Sufficiency. For all $c \in C_\alpha$, $x \in H_\alpha$ and $T_\alpha = t_\alpha$ (here $*$ denotes the convolution product), we have

$$\begin{aligned}(\bar{S}_\alpha * \mathrm{id}_{C_\alpha \#_{T_\alpha}^{g_\alpha} H_\alpha})(c \otimes x) &= S_{C\alpha}(c_{1T_\alpha})c_{2t_\alpha} \otimes S_{H\alpha}(c_{3g_\alpha t_\alpha T_\alpha})x^{g_\alpha} \\ &= S_{C\alpha}(c_{1T_\alpha 1})c_{1T_\alpha 2} \otimes S_{H\alpha}(c_{2g_\alpha T_\alpha})x^{g_\alpha} \quad \text{by (A2)} \\ &= \bar{1}_1 \otimes S_{H\alpha}(c_{g_\alpha})x^{g_\alpha} \quad \text{as } C \text{ is a Hopf } \pi\text{-algebra} \\ &= \kappa_\alpha(c)\varepsilon_\alpha(x)\bar{1}_1 \otimes 1_1 \quad \text{by (E2).}\end{aligned}$$

Similarly, $(\mathrm{id}_{C_\alpha \#_{T_\alpha}^{g_\alpha} H_\alpha} * \bar{S}_\alpha)(c \otimes x) = \kappa_\alpha(c)\varepsilon_\alpha(x)\bar{1}_1 \otimes 1_1$ can be obtained by (A5) and (E1).

So $\bar{S}$ is the antipode of $C \natural_T^{g\,\pi} H$.

Since the necessity can be easily verified, the proof is complete. ■

REMARK. If we add in Proposition 3.5 the assumptions of Example 3.2(1), (2), we get the antipodes in $C \natural_T^{\pi} H$ and $C \natural^{g\,\pi} H$, respectively.

4. Quasitriangular Brzeziński crossed coproduct Hopf π-algebra.

In this section, we give the construction of quasitriangular structures over a Brzeziński crossed coproduct Hopf π-algebra. Then by the result in [9], the left π-module category ${}_{C \natural_T^{g\,\pi} H}\mathcal{M}$ is braided.

The following is obvious:

PROPOSITION 4.1. *Let* $C \natural_T^{g\,\pi} H$ *be a Brzeziński crossed coproduct Hopf* π*-algebra. Define*

$$\psi_\alpha : C_\alpha \#_{T_\alpha}^{g_\alpha} H_\alpha \to C_\alpha, \qquad \psi_\alpha(c \otimes x) = c\varepsilon_\alpha(x),$$
$$\varphi_\alpha : C_\alpha \#_{T_\alpha}^{g_\alpha} H_\alpha \to H_\alpha, \qquad \varphi_\alpha(c \otimes x) = \kappa_\alpha(c)x,$$

for all $c \in C_\alpha$ *and* $x \in H_\alpha$*. Then* ψ_1 *is a bialgebra map.*

Let $C \natural_T^{g\,\pi} H$ be a Brzeziński crossed coproduct Hopf π-algebra and consider $R = R^1 \otimes R^2 \otimes R^3 \otimes R^4 \in C_1 \#_{T_1}^{g_1} H_1 \otimes C_1 \#_{T_1}^{g_1} H_1$. Define

$$U = U^1 \otimes U^2 = (\varphi_1 \otimes \varphi_1)(R) \in H_1 \otimes H_1,$$
$$V = V^1 \otimes V^2 = (\psi_1 \otimes \psi_1)(R) \in C_1 \otimes C_1,$$
$$W = W^1 \otimes W^2 = (\psi_1 \otimes \varphi_1)(R) \in C_1 \otimes H_1,$$
$$X = X^1 \otimes X^2 = (\varphi_1 \otimes \psi_1)(R) \in H_1 \otimes C_1.$$

The following result is straightforward.

Proposition 4.2. *With the above notation, if R satisfies* (QT4)*, then*

$$\begin{aligned}
&(1)\quad \varepsilon_1(U^1)U^2 = U^1\varepsilon_1(U^2) = 1_1,\\
&(2)\quad \kappa(W^1)W^2 = 1_1, \quad W^1\varepsilon_1(W^2) = \bar{1}_1,\\
&(3)\quad \varepsilon_1(X^1)X^2 = \bar{1}_1, \quad X^1\kappa(X^2) = 1_1,\\
&(4)\quad \kappa(V^1)V^2 = V^1\kappa(V^2) = \bar{1}_1.
\end{aligned}$$

Proposition 4.3. *With the above notation, if $(C \natural^{g\ \pi}_T H, R)$ is a quasitriangular Hopf π-algebra, then*

$$R = W^1V^1 \otimes U^1X^1 \otimes V^2X^2 \otimes W^2U^2.$$

Proof. By (QT1) and (QT2), for $R = r = \bar{R} = \bar{r}$, we have

$$\begin{aligned}
(4.1)\qquad & R^1_1 \otimes R^2_1 \otimes R^1_2 \otimes R^2_2 \otimes R^3_1 \otimes R^4_1 \otimes R^3_2 \otimes R^4_2\\
&= R^1r^1 \otimes R^2r^2 \otimes \bar{r}^1\bar{R}^1 \otimes \bar{r}^2\bar{R}^2 \otimes r^3\bar{R}^3 \otimes r^4\bar{R}^4 \otimes R^3\bar{r}^3 \otimes R^4\bar{r}^4
\end{aligned}$$

Applying $\psi_1 \otimes \varphi_1 \otimes \psi_1 \otimes \varphi_1$ to (4.1), by Proposition 4.2, we have

$$R = W^1V^1 \otimes U^1X^1 \otimes V^2X^2 \otimes W^2U^2. \blacksquare$$

Proposition 4.4. *Let $C \natural^{g\ \pi}_T H$ be a Brzeziński crossed coproduct Hopf π-algebra. Assume that $R = W^1V^1 \otimes U^1X^1 \otimes V^2X^2 \otimes W^2U^2$ is a quasitriangular structure over $C \natural^{g\ \pi}_T H$. Then*

$$\begin{aligned}
&(\mathrm{F1})\quad c_{T_\alpha}W^1 \otimes x_{T_\alpha}W^2 = W^1c \otimes W^2x,\\
&(\mathrm{F2})\quad xX^1 \otimes cX^2 = X^1x_{T_\alpha} \otimes X^2c_{T_\alpha},\\
&(\mathrm{F3})\quad X^1{}_{T_1} \otimes V^1{}_{T_1} \otimes V^2X^2 = X^1 \otimes V^1 \otimes X^2V^2,\\
&(\mathrm{F4})\quad U^1{}_{T_1} \otimes W^1{}_{T_1} \otimes W^2U^2 = U^1 \otimes W^1 \otimes U^2W^2,\\
&(\mathrm{F5})\quad W^1V^1 \otimes W^2{}_{T_1} \otimes V^2{}_{T_1} = V^1W^1 \otimes W^2 \otimes V^2,\\
&(\mathrm{F6})\quad U^1X^1 \otimes U^2{}_{T_1} \otimes X^2{}_{T_1} = X^1U^1 \otimes U^2 \otimes X^2,
\end{aligned}$$

where $c \in C_\alpha$ and $x \in H_\alpha$.

Proof. In the following, we freely use the properties of the Hopf π-algebra H. By (QT1),

$$(4.2)\qquad R^1{}_1 \otimes R^1{}_{3g_1T_1} \otimes R^1{}_{2T_1} \otimes R^{2g_1} \otimes R^3 \otimes R^4 = R^1 \otimes R^2 \otimes r^1 \otimes r^2 \otimes R^3r^3 \otimes R^4r^4.$$

By (QT2),

$$(4.3)\qquad R^1 \otimes R^2 \otimes R^3{}_1 \otimes R^3{}_{3g_1T_1} \otimes R^3{}_{2T_1} \otimes R^{4g_1} = R^1r^1 \otimes R^2r^2 \otimes r^3 \otimes r^4 \otimes R^3 \otimes R^4,$$

and by $(QT3)$ one knows that

$$(4.4)\qquad c_{2T_\alpha}R^1 \otimes x^{g_\alpha}R^2 \otimes c_1R^3 \otimes c_{3g_\alpha T_\alpha}R^4 = R^1c_1 \otimes R^2c_{3g_\alpha T_\alpha} \otimes R^3c_{2T_\alpha} \otimes R^4x^{g_\alpha}.$$

Applying $\psi_\alpha \otimes \varphi_\alpha$ to (4.4) and using Proposition 4.2, we get (F1). Similarly, applying $\varphi_\alpha \otimes \psi_\alpha$ to (4.4) and using Proposition 4.2 we obtain (F2).

Applying $\varphi_1 \otimes \psi_1 \otimes \psi_1$ to (4.2), we get (F3) by Propositions 4.2 and 4.3. Similarly, applying $\varphi_1 \otimes \psi_1 \otimes \varphi_1$ to (4.2), we see that (F4) holds.

(F5) can be obtained by applying $\psi_1 \otimes \varphi_1 \otimes \psi_1$ to (4.3) and by invoking Propositions 4.2 and 4.3. Likewise, we get (F6) by applying $\varphi_1 \otimes \varphi_1 \otimes \psi_1$ to (4.3) and making use of Propositions 4.2 and 4.3. ∎

PROPOSITION 4.5. *Let* $C \natural^{g\,\pi}_{T} H$ *be a Brzeziński crossed coproduct Hopf* π*-algebra. Assume that* $R = W^1V^1 \otimes U^1X^1 \otimes V^2X^2 \otimes W^2U^2$ *is a quasitriangular structure on* $C \natural^{g\,\pi}_{T} H$*. Then:*

(1) (C, V) *is a quasitriangular Hopf* π*-algebra.*
(2) (C, H, W, g) *is a compatible* (W, g, V)*-Hopf* π*-algebra quadruple.*
(3) (H, C, X, g) *is an anti-compatible* (X, g, V)*-Hopf* π*-algebra quadruple.*
(4) (H, U) *is a quasitriangular-like Hopf* π*-algebra associated to* (W, X, g)*.*

Proof. It follows from Proposition 4.1 that V, W, X and U respectively satisfy (QT4), (CQ3), (ACQ3) and (QTL4).

(1) Since $\psi_1 : C_1 \natural^{g_1}_{T_1} H_1 \to C_1$ is a bialgebra map, and $(C \natural^{g\,\pi}_{T} H, R)$ is a quasitriangular Hopf π-algebra, so is (C, V).

(2) By Propositions 4.2 and 4.3, (2) is obvious by applying $\psi_1 \otimes \psi_1 \otimes \varphi_1$ to (4.2) and applying $\psi_1 \otimes \varphi_1 \otimes \varphi_1$ to (4.3).

(3) This is easy to see by applying $\varphi_1 \otimes \varphi_1 \otimes \pi_1$ to (4.2) and applying $\varphi_1 \otimes \pi_1 \otimes \pi_1$ to (4.3), in view of Propositions 4.2 and 4.3.

(4) Applying $\varphi_1 \otimes \varphi_1 \otimes \varphi_1$ to (4.2), one gets (QTL1) by Propositions 4.2 and 4.3; and by applying $\varphi_1 \otimes \varphi_1 \otimes \varphi_1$ to (4.3) and invoking Propositions 4.2 and 4.3 one has (QTL2). We apply $\varphi_\alpha \otimes \varphi_\alpha$ to (4.4) to complete the proof ∎

THEOREM 4.6. *Let* $C \natural^{g\,\pi}_{T} H$ *be a Brzeziński crossed coproduct Hopf* π*-algebra. Suppose there exist* $V \in C_1 \otimes C_1$, $W \in C_1 \otimes H_1$, $X \in H_1 \otimes C_1$ *and* $U \in H_1 \otimes H_1$ *such that:*

(1) (C, V) *is a quasitriangular Hopf* π*-algebra.*
(2) (C, H, W, g) *is a compatible* (W, g, V)*-Hopf* π*-algebra quadruple.*
(3) (H, C, X, g) *is an anti-compatible* (X, g, V)*-Hopf* π*-algebra quadruple.*
(4) (H, U) *is a quasitriangular-like Hopf* π*-algebra associated to* (W, X, g)*.*
(5) *Conditions* (F1)–(F6) *in Proposition* 4.4 *hold.*

Then $(C \natural^{g\,\pi}_{T} H, R)$ *is a quasitriangular Hopf* π*-algebra with the quasitriangular structure given by*

$$R = W^1V^1 \otimes U^1X^1 \otimes V^2X^2 \otimes W^2U^2.$$

Proof. It is obvious that R satisfies (QT4). Now we check that (QT1) holds:

$$\begin{aligned}
&(\Delta_{C_1\#^{g_1}_{T_1}H_1}\otimes \mathrm{id}_{C_1\#^{g_1}_{T_1}H_1})(R)\\
&= (W^1V^1)_1\otimes(W^1V^1)_{3g_1T_1}\otimes(W^1V^1)_{2T_1}\otimes(U^1X^1)^{g_1}\otimes V^2X^2\otimes W^2U^2\\
&= W^1{}_1V^1{}_1\otimes(W^1{}_3V^1{}_3)_{g_1T_1}\otimes(W^1{}_2V^1{}_2)_{T_1}\otimes(U^1X^1)^{g_1}\otimes V^2X^2\otimes W^2U^2\\
&\overset{\text{(B2)}}{=} W^1{}_1V^1{}_1\otimes(W^1{}_{3g_1}V^1{}_{3G_1})_{T_1}\otimes(W^1{}_2V^1{}_2)_{T_1}\otimes U^{1g_1}X^{1G_1}\otimes V^2X^2\otimes W^2U^2\\
&\overset{\text{(B3)}}{=} W^1{}_1V^1{}_1\otimes W^1{}_{3g_1T_1}V^1{}_{3G_1t_1}\otimes W^1{}_{2T_1}V^1{}_{2t_1}\otimes U^{1g_1}X^{1G_1}\otimes V^2X^2\otimes W^2U^2\\
&\overset{\text{(CQ1)}}{=} W^1V^1{}_1\otimes\bar{\bar{W}}^1{}_{g_1T_1}V^1{}_{3G_1t_1}\otimes\bar{W}^1_{T_1}V^1{}_{2t_1}\\
&\qquad\otimes U^{1g_1}X^{1G_1}\otimes V^2X^2\otimes W^2\bar{W}^2\bar{\bar{W}}^2U^2\\
&\overset{\text{(QT1)}}{=} W^1V^1\otimes\bar{\bar{W}}^1{}_{g_1T_1}\bar{\bar{V}}^1{}_{G_1t_1}\otimes\bar{W}^1_{T_1}\bar{V}^1_{t_1}\\
&\qquad\otimes U^{1g_1}X^{1G_1}\otimes V^2\bar{V}^2\bar{\bar{V}}^2X^2\otimes W^2\bar{W}^2\bar{\bar{W}}^2U^2\\
&\overset{\text{(ACQ1)}}{=} W^1V^1\otimes\bar{\bar{W}}^1{}_{g_1T_1}X^1{}_{t_1}\otimes\bar{W}^1_{T_1}\bar{V}^1_{t_1}\otimes U^{1g_1}\bar{X}^1\otimes V^2\bar{V}^2X^2\bar{X}^2\otimes W^2\bar{W}^2\bar{\bar{W}}^2U^2\\
&\overset{\text{(QTL1)}}{=} W^1V^1\otimes U^1{}_{T_1}X^1{}_{t_1}\otimes\bar{W}^1_{T_1}\bar{V}^1_{t_1}\otimes\bar{U}^1\bar{X}^1\otimes V^2\bar{V}^2X^2\bar{X}^2\otimes W^2\bar{W}^2U^2\bar{U}^2\\
&\overset{\text{(F4)}}{=} W^1V^1\otimes U^1X^1{}_{t_1}\otimes\bar{W}^1\bar{V}^1_{t_1}\otimes\bar{U}^1\bar{X}^1\otimes V^2\bar{V}^2X^2\bar{X}^2\otimes W^2U^2\bar{W}^2\bar{U}^2\\
&\overset{\text{(F3)}}{=} W^1V^1\otimes U^1X^1\otimes\bar{W}^1\bar{V}^1\otimes\bar{U}^1\bar{X}^1\otimes V^2X^2\bar{V}^2\bar{X}^2\otimes W^2U^2\bar{W}^2\bar{U}^2\\
&= R^1\otimes R^2\otimes r^1\otimes r^2\otimes R^3r^3\otimes R^4r^4.
\end{aligned}$$

Likewise, (QT2) holds for R. Finally, we show that (QT3) holds:

$$\begin{aligned}
&R^1c_1\otimes R^2c_{3g_\alpha T_\alpha}\otimes R^3c_{2T_\alpha}\otimes R^4x^{g_\alpha}\\
&\qquad= W^1V^1c_1\otimes U^1X^1c_{3g_\alpha T_\alpha}\otimes V^2X^2c_{2T_\alpha}\otimes W^2U^2x^{g_\alpha}\\
&\qquad\overset{\text{(F2)}}{=} W^1V^1c_1\otimes U^1c_{3g_\alpha}X^1\otimes V^2c_2X^2\otimes W^2U^2x^{g_\alpha}\\
&\qquad\overset{\text{(QTL3)}}{=} W^1V^1c_1\otimes x^{g_\alpha}U^1X^1\otimes V^2c_2X^2\otimes W^2c_{3g_\alpha}U^2\\
&\qquad\overset{\text{(QT3)}}{=} W^1c_2V^1\otimes x^{g_\alpha}U^1X^1\otimes c_1V^2X^2\otimes W^2c_{3g_\alpha}U^2\\
&\qquad\overset{\text{(F1)}}{=} c_{2T_\alpha}W^1V^1\otimes x^{g_\alpha}U^1X^1\otimes c_1V^2X^2\otimes c_{3g_\alpha T_\alpha}W^2U^2\\
&\qquad= c_{2T_\alpha}R^1\otimes x^{g_\alpha}R^2\otimes c_1R^3\otimes c_{3g_\alpha T_\alpha}R^4. \ \blacksquare
\end{aligned}$$

The following result is an immediate consequence of Propositions 4.3–4.5 and Theorem 4.6.

THEOREM 4.7. *A Brzeziński crossed coproduct Hopf π-algebra $C\natural^{g}_{T}{}^{\pi} H$ is quasitriangular if and only if there exist $V\in C_1\otimes C_1$, $W\in C_1\otimes H_1$, $X\in H_1\otimes C_1$ and $U\in H_1\otimes H_1$ such that (C,V) is a quasitriangular Hopf π-algebra, (C,H,W,g) is a compatible (W,g,V)-Hopf π-algebra quadruple, (H,C,X,g) is an anti-compatible (X,g,V)-Hopf π-algebra quadruple, (H,U) is a quasitriangular-like Hopf π-algebra associated to (W,X,g) and conditions* (F1)–(F6) *in Proposition* 4.4 *are satisfied. Moreover, the quasitriangular structure R on $C\natural^{g}_{T}{}^{\pi} H$ has a decomposition*

$$R=W^1V^1\otimes U^1X^1\otimes V^2X^2\otimes W^2U^2.$$

Remarks. (1) Theorem 4.7 shows that if C is not a quasitriangular Hopf π-algebra then the Brzeziński crossed coproduct Hopf π-algebra $C \natural_T^{g\,\pi} H$ is not quasitriangular either.

(2) When π is trivial, i.e., $\pi = \{1\}$, we get the main result of [6].

Corollary 4.8. *If there exist* $V \in C_1 \otimes C_1$, $W \in C_1 \otimes H_1$, $X \in H_1 \otimes C_1$ *and* $U \in H_1 \otimes H_1$ *such that* (C, V) *is a quasitriangular Hopf* π*-algebra,* (C, H, W, g) *is a compatible* (W, g, V)*-Hopf* π*-algebra quadruple,* (H, C, X, g) *is an anti-compatible* (X, g, V)*-Hopf* π*-algebra quadruple,* (H, U) *is a quasitriangular-like Hopf* π*-algebra associated to* (W, X, g) *and conditions* (F1)–(F6) *in Proposition* 4.4 *are satisfied, then the left* π*-module category* ${}_{C\natural_T^{g\,\pi}H}\mathcal{M}$ *is a braided monoidal category.*

5. Applications. In this section, we give applications of the results proved in Sections 2–4.

Proposition 5.1. *A twisted tensor coproduct Hopf* π*-algebra* $C \natural_T^\pi H$ *is quasitriangular if and only if there exist* $V \in C_1 \otimes C_1$, $W \in C_1 \otimes H_1$, $X \in H_1 \otimes C_1$ *and* $U \in H_1 \otimes H_1$ *such that* (C, V) *and* (H, U) *are quasitriangular Hopf* π*-algebras,* (C, H, W) *is a compatibility Hopf* π*-algebra triple on* $C \otimes H$, (H, C, X) *is a compatibility Hopf* π*-algebra triple on* $H \otimes C$, *and conditions* (F1)–(F6) *in Proposition* 4.4 *are satisfied. Moreover, the quasitriangular structure* R *on* $C \natural_T^\pi H$ *has a decomposition*

$$R = W^1V^1 \otimes U^1X^1 \otimes V^2X^2 \otimes W^2U^2.$$

Proof. Let g_α be trivial, i.e., $g_\alpha(c \otimes x) = \kappa_\alpha(c)x_1 \otimes x_2$ in Theorem 4.7. ∎

Remark. Let π be trivial. Then we can get a quasitriangular structure on the twisted tensor coproduct Hopf algebra $C \natural_T H$ ([6]).

Given a compatibility Hopf π-algebra triple (C, H, θ), one may construct a new Hopf π-algebra $C \natural_\theta^\pi H = \{C_\alpha \otimes_\theta H_\alpha\}_{\alpha\in\pi}$. For each $\alpha \in \pi$, $C_\alpha \otimes_\theta H_\alpha$ is a coalgebra with the coproduct given by

$$\overline{\Delta}_\alpha(c \otimes x) = c_1 \otimes \theta^2 x_1(\theta^{-1})^2 \otimes \theta^1 c_2(\theta^{-1})^1 \otimes x_2$$

for all $c \in C_\alpha$ and $x \in H_\alpha$, and counit $\kappa_\alpha \otimes \varepsilon_\alpha$. The product and the unit are given by $\overline{\mu_{\alpha,\beta}}$ and $\overline{\eta}$ of Section 3.

We call the Hopf π-algebra $C \natural_\theta^\pi H$ a π*-quantum codouble.*

Theorem 5.2. *Let* C, H *be Hopf* π*-algebras,* (C, H, θ) *a compatibility Hopf* π*-algebra triple and* $C \natural_\theta^\pi H$ *the* π*-quantum codouble. Then* $C \natural_\theta^\pi H$ *is a quasitriangular Hopf* π*-algebra if and only if* C *and* H *are quasitriangular Hopf* π*-algebras. Moreover, if* (C, V) *and* (H, U) *are quasitriangular Hopf* π*-algebras, then the quasitriangular structure on* $C \natural_\theta^\pi H$ *is given by*

$$R = \theta^1V^1 \otimes U^1\theta^{-12} \otimes V^2\theta^{-11} \otimes \theta^2U^2.$$

Proof. The necessity is straightforward since $\psi_1 : C_1 \otimes_\theta H_1 \to C_1$, $\psi_1(c \otimes x) = c\varepsilon_1(x)$, and $\varphi_1 : C_1 \otimes_\theta H_1 \to H_1$, $\varphi_1(c \otimes x) = \kappa_1(c)x$, are surjective Hopf algebra maps.

Sufficiency. If (C, H, θ) is a compatibility Hopf π-algebra triple on $C \otimes H$, then $(H, C, \sigma \circ \theta^{-1})$ is a compatibility Hopf π-algebra triple on $H \otimes C$ where σ is the flip map.

Set $W = \theta$ and $X = \sigma \circ \theta^{-1}$. It remains to verify that (F1)–(F6) of Proposition 4.4 hold.

(F1) and (F2) are obvious.

For (F3), we argue as follows:

$$\begin{aligned} {X^1}_{T_1} \otimes {V^1}_{T_1} \otimes V^2 X^2 &= \theta^2 \bar{\theta}^{-12} \theta^{-12} \otimes \theta^1 V^1 \theta^{-11} \otimes V^2 \bar{\theta}^{-11} \\ &\overset{\text{(QT1)}}{=} \theta^2 \theta^{-12} \otimes \theta^1 V^1 {\theta^{-11}}_1 \otimes V^2 {\theta^{-11}}_2 \\ &\overset{\text{(QT3)}}{=} \theta^2 \theta^{-12} \otimes \theta^1 {\theta^{-11}}_2 V^1 \otimes {\theta^{-11}}_1 V^2 \\ &\overset{\text{(QT1)}}{=} \theta^2 \theta^{-12} \bar{\theta}^{-12} \otimes \theta^1 \theta^{-11} V^1 \otimes \bar{\theta}^{-11} V^2 \\ &= X^1 \otimes V^1 \otimes X^2 V^2, \end{aligned}$$

where $\theta = \bar{\theta}$.

Similarly, (F4)–(F6) are satisfied. ■

Corollary 5.3. *If (C, V), (H, U) are quasitriangular Hopf π-algebras, then the left π-module category ${}_{C \natural_\theta^\pi H}\mathcal{M}$ is a braided monoidal category.*

In the following, we give a concrete example.

Step 1. Let $A = k\{1, a \mid a^2 = 1\}$ be a group algebra (see [11]). Let π be a group. Then we have a π-algebra $H = A[\pi] = \{A\alpha\}_{\alpha \in \pi}$ with $H_\alpha = A\alpha$ for $\alpha \in \pi$. Multiplication in $A[\pi]$ is given by

$$(a\alpha)(b\beta) = (ab)(\alpha\beta)$$

where $a, b \in A$ and $\alpha, \beta \in \pi$ (see [13]).

Define

$$\begin{aligned} \Delta_{H_\alpha} &: H_\alpha \to H_\alpha \otimes H_\alpha, \quad \Delta_{H_\alpha}(a\alpha) = a\alpha \otimes a\alpha, \\ \varepsilon_{H_\alpha} &: H_\alpha \to k, \quad \varepsilon_{H_\alpha}(a\alpha) = 1, \quad \forall a\alpha \in H_\alpha. \end{aligned}$$

Then H_α is a coalgebra for all $\alpha \in \pi$. It is obvious that H is a semi-Hopf π-algebra.

Define

$$S_\alpha : H_\alpha \to H_{\alpha^{-1}}, \quad S_\alpha(a\alpha) = a\alpha^{-1}, \quad \forall \alpha \in \pi.$$

Then $(H, S = \{S_\alpha\})$ is a Hopf π-algebra. By a direct computation we can see that (H, U) is a quasitriangular Hopf π-algebra, where

$$U = \tfrac{1}{2}(1_A 1_\pi \otimes 1_A 1_\pi + a1_\pi \otimes 1_A 1_\pi + 1_A 1_\pi \otimes a1_\pi - a1_\pi \otimes a1_\pi) \in H_1 \otimes H_1.$$

Step 2. Let $H_4 = k\{1, g, x, gx \mid g^2 = 1, x^2 = 0, xg = -gx\}$ be the Sweedler Hopf algebra (see [11]). Its coalgebra structure and antipode are given by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes g + 1 \otimes x, \quad \Delta(gx) = gx \otimes 1 + g \otimes gx,$$
$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0, \quad \varepsilon(gx) = 0,$$

and

$$S_{H_4}(g) = g, \quad S_{H_4}(x) = gx, \quad S_{H_4}(gx) = -x.$$

Let π be a group. Then we have a π-algebra $C = H_4[\pi] = \{H_4\alpha\}_{\alpha\in\pi}$ with $C_\alpha = H_4\alpha$ for $\alpha \in \pi$. Multiplication in $H_4[\pi]$ is given by

$$(c\alpha)(d\beta) = (cd)(\alpha\beta)$$

for $c, d \in H_4$ and $\alpha, \beta \in \pi$ (see [13]).

Define

$$\Delta_{C_\alpha} : C_\alpha \to C_\alpha \otimes C_\alpha, \quad \Delta_{C_\alpha}(c\alpha) = c_1\alpha \otimes c_2\alpha,$$
$$\varepsilon_{C_\alpha} : C_\alpha \to k, \;\; \varepsilon_{C_\alpha}(c\alpha) = \varepsilon(c), \quad \forall c\alpha \in C_\alpha.$$

Then C_α is a coalgebra for all $\alpha \in \pi$. It is obvious that C is a semi-Hopf π-algebra.

Define $S'_\alpha : C_\alpha \to C_{\alpha^{-1}}$ for $\alpha \in \pi$ by

$$1\alpha \mapsto 1\alpha^{-1}, \quad g\alpha \mapsto g\alpha^{-1}, \quad x\alpha \mapsto (gx)\alpha^{-1}, \quad (gx)\alpha \mapsto (-x)\alpha^{-1}.$$

Then $(C, S' = \{S'_\alpha\})$ is a Hopf π-algebra. By a tedious computation we can check that (C, V) is a quasitriangular Hopf π-algebra, where

$$V = \tfrac{1}{2}(1_{H_4}1_\pi \otimes 1_{H_4}1_\pi + g1_\pi \otimes 1_{H_4}1_\pi + 1_{H_4}1_\pi \otimes g1_\pi - g1_\pi \otimes g1_\pi)$$
$$+ \tfrac{l}{2}l\big(x1_\pi \otimes x1_\pi + x1_\pi \otimes (gx)1_\pi + (-gx)1_\pi \otimes x1_\pi + (gx)1_\pi \otimes (gx)1_\pi\big),$$

where $l \in k$.

Step 3. Under the notation introduced above, we define linear maps $T_\alpha : C_\alpha \otimes H_\alpha \to H_\alpha \otimes C_\alpha$, $\alpha \in \pi$, by

$$\begin{aligned} &1_{H_4}\alpha \otimes 1_A\alpha \mapsto 1_A\alpha \otimes 1_{H_4}\alpha, && x\alpha \otimes 1_A\alpha \mapsto a\alpha \otimes x\alpha,\\ &1_{H_4}\alpha \otimes a\alpha \mapsto a\alpha \otimes 1_{H_4}\alpha, && x\alpha \otimes a\alpha \mapsto 1_A\alpha \otimes x\alpha,\\ &g\alpha \otimes 1_A\alpha \mapsto 1_A\alpha \otimes g\alpha, && (gx)\alpha \otimes 1_A\alpha \mapsto a\alpha \otimes (gx)\alpha,\\ &g\alpha \otimes a\alpha \mapsto a\alpha \otimes g\alpha, && (gx)\alpha \otimes a\alpha \mapsto 1_A\alpha \otimes (gx)\alpha. \end{aligned}$$

Then by a routine computation we get a family of twisted tensor coproduct coalgebras $\{C_\alpha \#_{T_\alpha} H_\alpha\}_{\alpha\in\pi}$.

Furthermore, $C \#^\pi_T H = \{C_\alpha \#_{T_\alpha} H_\alpha\}_{\alpha\in\pi}$ equipped with the following multiplication and unit:

$$\overline{\mu_{\alpha,\beta}} : C_\alpha \#_{T_\alpha} H_\alpha \otimes C_\beta \#_{T_\beta} H_\beta \to C_{\alpha\beta} \#_{T_{\alpha\beta}} H_{\alpha\beta}, \qquad \overline{\eta} : k \to T_4 \times^\pi H_1,$$
$$(c\alpha \otimes h\alpha) \otimes (d\beta \otimes g\beta) \mapsto (cd)(\alpha\beta) \otimes (hg)(\alpha\beta), \quad 1_k \mapsto 1_{H_4}1_\pi \otimes 1_A 1_\pi,$$

for $\alpha, \beta \in \pi$, $c\alpha \in H_\alpha$, $d\beta \in H_\beta$ and $h\alpha \in H_\alpha$, $g\beta \in H_\beta$ is a semi-Hopf π-algebra, denoted by $C \natural_T^\pi H$.

Moreover, $(C \natural_T^\pi H, \bar{S} = \{\bar{S}_\alpha\})$ is a Hopf π-algebra, where the antipode $\bar{S} = \{\bar{S}_\alpha\}$ is given by

$$\begin{aligned}
\bar{S}_\alpha(1_{H_4}\alpha \otimes 1_A\alpha) &= 1_{H_4}\alpha^{-1} \otimes 1_A\alpha^{-1}, & \bar{S}_\alpha(1_{H_4}\alpha \otimes a\alpha) &= 1_{H_4}\alpha^{-1} \otimes a\alpha^{-1},\\
\bar{S}_\alpha(g\alpha \otimes 1_A\alpha) &= g\alpha^{-1} \otimes 1_A\alpha^{-1}, & \bar{S}_\alpha(g\alpha \otimes a\alpha) &= g\alpha^{-1} \otimes a\alpha^{-1},\\
\bar{S}_\alpha(x\alpha \otimes 1_A\alpha) &= (gx)\alpha^{-1} \otimes a\alpha^{-1}, & \bar{S}_\alpha(x\alpha \otimes a\alpha) &= (gx)\alpha^{-1} \otimes 1_A\alpha^{-1},\\
\bar{S}_\alpha((gx)\alpha \otimes 1_A\alpha) &= (-x)\alpha^{-1} \otimes a\alpha^{-1}, & \bar{S}_\alpha((gx)\alpha \otimes a\alpha) &= (-x)\alpha^{-1} \otimes 1_A\alpha^{-1}.
\end{aligned}$$

STEP 4. Under the notation introduced above, we define

$$\begin{aligned}
W &= \tfrac{1}{2}(1_{H_4}1_\pi \otimes 1_A 1_\pi + 1_{H_4}1_\pi \otimes a1_\pi + g1_\pi \otimes 1_A1_\pi - g1_\pi \otimes a1_\pi) \in C_1 \otimes H_1,\\
X &= \tfrac{1}{2}(1_A1_\pi \otimes 1_{H_4}1_\pi + a1_\pi \otimes 1_{H_4}1_\pi + 1_A1_\pi \otimes g1_\pi - a1_\pi \otimes g1_\pi) \in H_1 \otimes C_1.
\end{aligned}$$

Then it is straightforward to prove that (C, H, W) and (H, C, X) are compatibility Hopf π-algebra triples.

EXAMPLE 5.4. Under the notation introduced above, the twisted tensor coproduct Hopf π-algebra $(C \natural_T^\pi H, R)$ is a quasitriangular Hopf π-algebra, where

$$\begin{aligned}
R = \tfrac{1}{2}(&1_{H_4}1_\pi \otimes 1_A1_\pi \otimes 1_{H_4}1_\pi \otimes 1_A1_\pi + g1_\pi \otimes a1_\pi \otimes 1_{H_4}1_\pi \otimes 1_A1_\pi\\
&+1_{H_4}1_\pi \otimes 1_A1_\pi \otimes g1_\pi \otimes a1_\pi - g1_\pi \otimes a1_\pi \otimes g1_\pi \otimes a1_\pi)\\
&+ \tfrac{1}{2}l\big(x1_\pi \otimes a1_\pi \otimes x1_\pi \otimes a1_\pi + x1_\pi \otimes a1_\pi \otimes (gx)1_\pi \otimes 1_A1_\pi\\
&+ (-gx)1_\pi \otimes 1_A1_\pi \otimes x1_\pi \otimes a1_\pi + (gx)1_\pi \otimes 1_A1_\pi \otimes (gx)1_\pi \otimes 1_A1_\pi\big).
\end{aligned}$$

Indeed, in view of Steps 1–4, it remains to prove that conditions (F1)–(F6) in Proposition 4.4 hold. But these are easily checked.

REMARK. When $\pi = \{1\}$, we get the usual case of Hopf algebras.

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