STUDIA MATHEMATICA 239 (1) (2017)

Estimates for Bellman functions related to dyadic-like maximal operators on weighted spaces

by

ANTONIOS D. MELAS, ELEFTHERIOS N. NIKOLIDAKIS, and DIMITRIOS CHELIOTIS (Athens)

Abstract. We provide some new estimates for Bellman type functions for the dyadic maximal operator on $\mathbb{R}^n$ and for maximal operators on martingales related to weighted spaces. Using a type of symmetrization principle, introduced for the dyadic maximal operator in earlier works of the authors, we introduce certain conditions on the weight that imply an estimate for the maximal operator on the corresponding weighted space. Also using a well known estimate for the maximal operator by a double maximal operator with different measures related to the weight, we give new estimates for the above Bellman type functions.

1. Introduction. The dyadic maximal operator on $\mathbb{R}^n$ is defined by

$$M_d\phi(x) = \sup\left\{ \frac{1}{|S|} \int_S |\phi(u)|\, du : x \in S, S \subseteq \mathbb{R}^n \text{ is a dyadic cube} \right\} \tag{1.1}$$

for every $\phi \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ where the dyadic cubes are the cubes formed by the grids $2^{-N}\mathbb{Z}^n$ for $N = 0, 1, 2, \ldots$.

As is well known, M_d satisfies the following L^p inequality (for martingales known as *Doob's inequality*):

$$\|M_d\phi\|_p \le \frac{p}{p-1}\|\phi\|_p \tag{1.2}$$

for every $p > 1$ and every $\phi \in L^p(\mathbb{R}^n)$, which is best possible (see [2], [3] for the general martingales and [36] for dyadic ones).

For the study of the behavior of this maximal operator in more depth, the so called Bellman functions [12] have been introduced. Such functions related to the L^p inequality (1.2) have been precisely evaluated in [6]. Define $\mathrm{Av}_E(\psi) = |E|^{-1} \int_E |\psi|$, where $E \subseteq \mathbb{R}^n$ is measurable of positive measure and

2010 *Mathematics Subject Classification*: Primary 42B25.
Key words and phrases: Bellman, dyadic, maximal, Lorentz.
Received 28 January 2016; revised 23 March 2017.
Published online 7 July 2017.

DOI: 10.4064/sm8524-3-2017

ψ is measurable on E. Fixing a dyadic cube Q define the localized maximal operator $M'_d\phi$ as in (1.1) but with the dyadic cubes S being assumed to be contained in Q. Then for any $p > 1$ we let

$$\mathcal{B}_p(F,f) = \sup\left\{ \frac{1}{|Q|} \int_Q (M'_d\phi)^p : \mathrm{Av}_Q(\phi^p) = F,\ \mathrm{Av}_Q(\phi) = f \right\} \tag{1.3}$$

where ϕ is nonnegative in $L^p(Q)$ and the variables F, f satisfy $0 < f, f^p \le F$. By a scaling argument it is easy to see that the above is independent of the choice of Q (so we have just written $\mathcal{B}_p(F,f)$ and we may take $Q = [0,1]^n$). It has been shown in [6, Theorem 3, p. 327] that

$$\mathcal{B}_p(F,f) = F\omega_p\left(\frac{f^p}{F}\right)^p \tag{1.4}$$

where $\omega_p : [0,1] \to [1, p/(p-1)]$ is the inverse function of $H_p(z) = -(p-1)z^p + pz^{p-1}$. Actually [6], the Bellman function can be defined with respect to the maximal operator $M_{\mathcal{T}}$ on a nonatomic probability space (X,μ) equipped with a tree $\mathcal{T}$ (see Section 2), and the resulting Bellman function is always the same. The fact that ω_p is strictly decreasing with range $[1, p/(p-1)]$ shows that the constant in Doob's inequality can be approached by considering functions whose integral is very small compared to their p-norm (in the limit $f \to 0$ keeping F fixed). For example for $p = 2$ we get the following sharp improvement of Doob's inequality:

$$\|M_{\mathcal{T}}\phi\|_2 \le \|\phi\|_2 + (\|\phi\|_2^2 - \|\phi\|_1^2)^{1/2} < 2\|\phi\|_2 \tag{1.5}$$

which aside from the L^2 norm of ϕ also involves in a sharp way the *variance* of ϕ.

The introduction of the more general setting of a probability space (X,μ) equipped with a tree $\mathcal{T}$, the dyadic subcubes of $[0,1]^n$ being a special case, allows us to include maximal operators similar to the dyadic ones but related to parabolic rectangles or more generally to nonhomogeneous dilations, but also to include structures that have no uniformity at all and are related to general uniformly integrable martingales. It is interesting that these Bellman functions are independent of the type of the tree $\mathcal{T}$, but depend only on its combinatorial structure. In sharp contrast to this are the corresponding results on infimum-type Bellman functions [10], where no uniform, over all trees, lower estimate exists and strong dependence on the homogeneity of the tree appears.

Here we will be concerned with the behavior of these maximal operators on weighted spaces. It is well known that for any positive locally integrable function w on Q, the estimate

$$\int_Q (M'_d\phi)^p w \le C \int_Q \phi^p w \tag{1.6}$$

holds for all ϕ if and only if w is a dyadic A_p weight in the sense that

$$\sup\Big\{|I|^{-p}\Big(\int_I w\Big)\Big(\int_I w^{-1/(p-1)}\Big)^{p-1} : I \text{ a dyadic subcube of } Q\Big\} = [w]_p < \infty.$$

Also it is known that the best possible C is of the order of $[w]_p^{p/(p-1)}$, the exponent being best possible. Related to this, one may try to derive estimates for the following Bellman function related to weights:

$$\begin{aligned}(1.7)\qquad \mathcal{B}_{p,r}(F,f,u,v) &= \sup\Big\{\frac{1}{|Q|}\int_Q (M_d'\phi)^p w : \phi \ge 0 \text{ measurable with } \operatorname{Av}_Q(\phi^p w) = F,\\ &\qquad \operatorname{Av}_Q(\phi) = f,\ w \text{ a dyadic } A_p \text{ weight such that } [w]_p \le r,\\ &\qquad \operatorname{Av}_Q(w) = u, \operatorname{Av}_Q(w^{-1/(p-1)}) = v\Big\}.\end{aligned}$$

The estimates here will be proved in the general setting of tree-like families on probability spaces and the related maximal operator, as described in the next section. Also we remark that we use the condition $\operatorname{Av}_Q(\phi) = f$ instead of $\operatorname{Av}_Q(\phi w) = f$ because the variable f is related to the maximal operator $M_d'\phi$, which is defined with respect to the Lebesgue measure, and not to the weight [6].

We will derive two types of estimates related to the above problems. In the first we will use a related condition on some symmetrization of the weight to find the exact form of the corresponding weighted Bellman function. This condition is related to the behavior of a certain equimeasurable rearrangement w^{**} of the weight w on $(0,1)$, and depends on this rearrangement. The motivation of this is our attempt to generalize and relate to weights our recent results on sharp Lorentz norm estimates of the corresponding maximal operators on trees (see also [15], [9]) in which the multiplying factor is not necessarily decreasing, and the result in [16] on the effect of the decreasing symmetrization on an A_p weight. This related Bellman type function is exactly evaluated in Section 2.

Then in Section 3 we obtain certain new estimates for the Bellman function (1.7) related to A_p but with respect to a general tree, and to the corresponding maximal operator, by using an estimate of the maximal operator via two applications of maximal operators on the same tree but with different measures; this is described in Section 3. We get the upper bound

$$(1.8)\qquad \mathcal{B}_{p,r}(F,f,u,v) \le p^{p'} r^{p'} F\omega_p\Big(\frac{f^p}{v^{p-1}F}\Big)^p.$$

This estimate is of the right order with respect to r but the appearance of the factor $p^{p'}$ makes it not sharp in view of estimate (1.4), this factor coming from the weighted maximal operator with respect to $w^{-1/(p-1)}$ on the space $L^{p'}$. It would be desirable to improve on that.

On the other hand, looking for generalizations of (1.7) on arbitrary trees (see (3.1) and Section 2) we get a lower bound of the form

$$\inf_{\mathcal{T}} \mathcal{B}_{p,r}^{\mathcal{T}}(F, f, u, v) \geq F\omega_p\left(\frac{uf^p}{F}\omega_{p'}(r^{-1/(p-1)})\right)^p,$$

which is sharp for r and u close to 1 in view of (1.4).

There are several other problems in harmonic analysis where Bellman functions naturally arise. Such problems (including the dyadic Carleson imbedding and weighted inequalities) are described in [14] (see also [12], [13]), where also connections to stochastic optimal control are provided, from which it follows that the corresponding Bellman functions satisfy certain nonlinear second order PDEs.

The exact computation of a Bellman function is a difficult task which is connected with the deeper structure of the corresponding harmonic analysis problem. Thus far, several Bellman functions have been computed (see [2], [3], [6], [24], [25], [31], [32], [33]). L. Slavin, A. Stokolos and V. Vasyunin [29] linked the Bellman function computation to solving certain PDEs of the Monge–Ampère type, and in this way they obtained an alternative proof for the Bellman functions related to the dyadic maximal operator in [6]. Also in [33] using the Monge–Ampère equation approach a more general Bellman function than the one related to the dyadic Carleson imbedding theorem has been precisely evaluated, thus generalizing the corresponding result of [6]. For more recent developments and results related to the Bellman function technique we refer to [26], [27], [30], [5], [37], [18], [19], [20], [21], [22], [1], [23], [11].

In this paper, as in our previous ones, we use Bellman functions as a means to gain deeper understanding of the corresponding maximal operators and we are not using the standard techniques like Bellman dynamics and induction, corresponding PDEs, obstacle conditions etc. Instead, we rely on the combinatorial structure of these operators combined with the symmetrization principle introduced in [7], [15] for this type of extremal problems.

2. Trees, maximal operators and symmetrization. As in [6], we let (X, μ) be a nonatomic probability space (i.e. $\mu(X) = 1$). Two measurable subsets A, B of X will be called *almost disjoint* if $\mu(A \cap B) = 0$.

DEFINITION 1. A set $\mathcal{T}$ of measurable subsets of X will be called a *tree* if:

(i) $X \in \mathcal{T}$ and $\mu(I) > 0$ for every $I \in \mathcal{T}$.

(ii) To every $I \in \mathcal{T}$ there corresponds a finite subset $\mathcal{C}(I) \subseteq \mathcal{T}$ containing at least two elements such that:

(a) the elements of $\mathcal{C}(I)$ are pairwise almost disjoint subsets of I,

(b) $I = \bigcup_{J \in \mathcal{C}(I)} J$.

(iii) $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}_{(m)}$ where $\mathcal{T}_{(0)} = \{X\}$ and $\mathcal{T}_{(m+1)} = \bigcup_{I \in \mathcal{T}_{(m)}} \mathcal{C}(I)$.

(iv) $\lim_{m\to\infty} \sup_{I \in \mathcal{T}_{(m)}} \mu(I) = 0$ and $\mathcal{T}$ differentiates L^1.

The second condition in (iv) means that for every ψ in $L^1(X,\mu)$ there is a set D of μ-measure 1 such that for every x in D there is a sequence $I_m \in \mathcal{T}_{(m)}$ with $x \in I_m$ for all m and $\mu(I_m)^{-1} \int_{I_m} \psi \, d\mu \to \psi(x)$ as $m \to \infty$. By removing the measure zero exceptional set $E(\mathcal{T}) = \bigcup_{I \in \mathcal{T}} \bigcup_{J_1, J_2 \in \mathcal{C}(I),\, J_1 \neq J_2} (J_1 \cap J_2)$ we may replace the almost disjointness above by disjointness.

Now given any tree $\mathcal{T}$ we define the associated maximal operator as follows:

$$M_{\mathcal{T}}\phi(x) = \sup\left\{ \frac{1}{\mu(I)} \int_I |\phi| \, d\mu : x \in I \in \mathcal{T} \right\} \tag{2.1}$$

for every $\phi \in L^1(X,\mu)$.

The above setting can be used not only for the dyadic maximal operator but also for the maximal operator on martingales, hence many of the results here can be viewed as generalizations and refinements of the classical Doob inequality.

Also for any locally integrable positive function w on X, which will be called a *weight*, we denote $\sigma = w^{-1/(p-1)}$, and for any $I \in \mathcal{T}$ we write $w(I) = \int_I w \, d\mu$ and $\sigma(I) = \int_I \sigma \, d\mu$.

Definition 2. A weight w on X will be called A_p *with respect to* $\mathcal{T}$ if

$$[w]_{\mathcal{T},p} = [w]_p = \sup_{I \in \mathcal{T}} \frac{w(I)\sigma(I)^{p-1}}{\mu(I)^p} < \infty.$$

Estimates for the above maximal operator can be studied through the symmetrization of ϕ as introduced in [7] and [15] and used in [9] to evaluate Bellman functions related to Lorentz norms. In order to apply this in the context of weights, we introduce the following condition on a weight w on X.

Definition 3. Let $p > 1$. A weight w on X will be called A_p^* if for some equimeasurable rearrangement w^{**} of w on $(0,1)$ (not necessarily decreasing) there exist constants $c, a > 0$ such that for every t in $(0,1]$,

$$\int_t^1 \frac{w^{**}(s)}{s^p} \, ds + c \leq a \frac{w^{**}(t)}{t^{p-1}} \tag{2.2}$$

and also

$$\lim_{t\to 0^+} t^p \int_t^1 \frac{w^{**}(s)}{s^p}\, ds = 0. \tag{2.3}$$

By equimeasurable in the above definition we mean that w^{**} is measurable on $(0,1)$ and $|\{w^{**} > \lambda\}| = \mu(\{w > \lambda\})$ for every $\lambda > 0$. We do not require w^{**} to be decreasing, being motivated by our study of Lorentz norms in [9], where also increasing weights on $(0,1)$ appear. Note that if we write $r(t) = w^{**}(t)/t^{p-1}$, the first condition implies that $r(t) > c/a > 0$ for all t, hence $\lim_{t\to 0^+} \int_t^1 (w^{**}(s)/s^p)\, ds = \infty$, and so $\lim_{t\to 0^+} r(t) = \infty$. Hence we conclude that there is a best possible pair (a,c) for each such weight and its corresponding rearrangement w^{**}, namely

$$a = \sup_t r(t)^{-1} \int_t^1 \frac{r(s)}{s}\, ds \quad \text{and} \quad c = \inf_t \left(a r(t) - \int_t^1 \frac{r(s)}{s}\, ds \right).$$

We will refer to this pair as the *constants of the A_p^* weight w with respect to w^{**}*.

Example. Suppose that $w^{**}(t) = kt^b$ with $k,b \in \mathbb{R}$. Then the above conditions hold if and only if $-1 < b < p-1$, which is exactly the range making w^{**} an A_p weight on $(0,1)$. Moreover the corresponding constants a,c can be easily seen to be $a = 1/(p-1-b)$, $c = k/(p-1-b)$.

Now we take into consideration the following theorem proved in [15] and [7].

Theorem 1. *Let $G : [0,\infty) \to [0,\infty)$ be nondecreasing and $h : (0,1] \to \mathbb{R}^+$ be locally integrable. Then for any nonatomic probability space (X,μ), equipped with any tree-like family $\mathcal{T}$, for any nonincreasing right continuous integrable function $g : (0,1] \to \mathbb{R}^+$ and any $k \in (0,1]$, the following equality holds (where ψ^* denotes the decreasing equimeasurable rearrangement of ψ):*

$$\sup\left\{ \int_0^k G[(M_{\mathcal{T}}\phi)^*(t)]h(t)\, dt : \phi \text{ measurable on } X \text{ with } \phi^* = g \right\}$$
$$= \int_0^k G\left(\frac{1}{t}\int_0^t g(u)\, du \right) h(t)\, dt.$$

Given $p > 1$, we define the following variant of the Bellman function (1.7):

$$\mathcal{B}_p^*(F,f,a,c)=\sup\Big\{\int_0^1((M_{\mathcal{T}}\phi)^*)^p w^{**}:\int_0^1(\phi^*)^p w^{**}=F,\ \int_X\phi=f,\ w \text{ is an } A_p^* \text{ weight on } X, \text{ and } w^{**} \text{ an equimeasurable rearrangement of } w \text{ with constants } a,c\Big\}. \tag{2.4}$$

Here ϕ^* and $(M_{\mathcal{T}}\phi)^*$ denote the equimeasurable **decreasing** rearrangements of ϕ and $M_{\mathcal{T}}\phi$. Note that in case w^{**} is decreasing, we have $\int_0^1((M_{\mathcal{T}}\phi)^*)^p w^{**} \ge \int_X (M_{\mathcal{T}}\phi)^p w$ by the classical rearrangement inequality, and similarly $\int_0^1(\phi^*)^p w^{**}\ge\int_X\phi^p w$, whereas when w^{**} is increasing the opposite relations hold. Now we can prove the following.

THEOREM 2. *Given $p>1$ we have*

$$\mathcal{B}_p^*(F,f,a,c)=(p-1)^p a^p F\omega_p\bigg(\frac{cf^p}{(p-1)^{p-1}a^pF}\bigg)^p, \tag{2.5}$$

the domain of this function being all (F,f,a,c) such that $0<cf^p\le(p-1)^{p-1}a^pF$.

Proof. First fix an A_p^* weight w on X with an equimeasurable rearrangement w^{**} having constants a,c. In view of the above mentioned result it suffices to consider the expression $\Delta_w(g)=\int_0^1(t^{-1}\int_0^t g(u)\,du)^p w^{**}(t)\,dt$ when g runs over all nonnegative decreasing right continuous functions on $(0,1]$ satisfying $\int_0^1 g(t)\,dt=f$ and $\int_0^1 g(t)^p w^{**}(t)\,dt=F$. We next define the following function on $(0,1)$:

$$u(t)=\int_t^1\frac{w^{**}(s)}{s^p}\,ds+c,$$

so that $u'(t)=-t^{-p}w^{**}(t)$. Considering first any bounded such function g we compute by integration by parts

$$\int_0^1 u(t)\Big(\int_0^t g(u)\,du\Big)^{p-1}g(t)\,dt=\frac1p\int_0^1 u(t)\Big[\Big(\int_0^t g(u)\,du\Big)^p\Big]'\,dt$$
$$=\frac1p\Big(\int_0^1 g(u)\,du\Big)^p u(1)+\frac1p\int_0^1\Big(t^{-1}\int_0^t g(u)\,du\Big)^p w^{**}(t)\,dt=c\frac{f^p}{p}+\frac1p\Delta(g),$$

the integration by parts term $\lim_{t\to0+}u(t)(\int_0^t g(u)\,du)^p$ being zero because of condition (2.3) since g is assumed bounded. Now using Young's inequality $xy\le x^p/p+y^{p'}/p'$ (where $p'=p/(p-1)$) in the first integral (with $\lambda>0$ to be determined later) combined with the condition $u(t)t^{p-1}/w^{**}(t)\le a$ from the above definition we get

$$\begin{aligned}\int_0^1 u(t)\Big(\int_0^t g(u)\,du\Big)^{p-1} g(t)\,dt \\ &= \int_0^1 (\lambda g(t)w^{**}(t)^{1/p})\bigg(\frac{w^{**}(t)^{1/p}}{\lambda^{1/(p-1)}t}\int_0^t g(u)\,du\bigg)^{p-1}\frac{u(t)t^{p-1}}{w^{**}(t)}\,dt \\ &\le \frac{a}{p}\int_0^1 \lambda^p g(t)^p w^{**}(t)\,dt + \frac{a}{p'}\int_0^1 \lambda^{-p'}\bigg(\frac{1}{t}\int_0^t g(u)\,du\bigg)^p w^{**}(t)\,dt \\ &= \frac{a\lambda^p}{p}\int_0^1 g(t)^p w^{**}(t)\,dt + a\frac{\lambda^{-p'}}{p'}\int_0^1\bigg(\frac{1}{t}\int_0^t g(u)\,du\bigg)^p w^{**}(t)\,dt \\ &= a\frac{\lambda^p}{p}F + a\frac{\lambda^{-p'}}{p'}\Delta_w(g).\end{aligned}$$

Therefore by writing $\lambda^{p'} = (p-1)a(\beta+1)$ with $\beta > 0$ and using the above inequalities we get

$$\Delta_w(g) \le \Big(1+\frac{1}{\beta}\Big)\frac{(\beta+1)^{p-1}(p-1)^p a^p F - (p-1)cf^p}{p-1}. \tag{2.6}$$

Next, given an arbitrary g, the above estimate can be used for the truncations $g_M = \min(g, M)$ and F, f replaced by the corresponding quantities for g_M; then we take $M \to \infty$ and use monotone convergence to infer that (2.6) holds for the general nonnegative decreasing right continuous function g on $(0,1]$ satisfying $\int_0^1 g(t)\,dt = f$ and $\int_0^1 g(t)^p w^{**}(t)\,dt = F$. Moreover since $\Delta_w(g) > 0$, the inequality (2.6) implies that $(\beta+1)^{p-1}(p-1)^p a^p F - (p-1)cf^p > 0$ for every $\beta > 0$, and so letting $\beta \to 0^+$ we conclude that (F, f) must satisfy the inequality $cf^p \le (p-1)^{p-1}a^p F$ given in the statement of the theorem.

Writing $A = (p-1)^p a^p F$ and $B = (p-1)cf^p$ it is easy to compute (see for example [6, p. 326]) that the minimum possible value of the right hand side of (2.6) is equal to $A\omega_p(B/A)^p$. This proves the inequality

$$\mathcal{B}_p^*(F, f, a, c) \le (p-1)^p a^p F\omega_p\bigg(\frac{cf^p}{(p-1)^{p-1}a^p F}\bigg)^p. \tag{2.7}$$

Now we consider the continuous positive decreasing function

$$g_\delta(t) = f(1-\delta)t^{-\delta} \tag{2.8}$$

where $0 \le \delta < 1$, and any A_p^* weight w that is equimeasurable to

$$w^{**}(t) = kt^b, \qquad k > 0,\ -1 < b < p-1, \tag{2.9}$$

where δ is chosen small enough to satisfy $\delta < (1+b)/p$. Clearly then $\int_0^1 g_\delta(t)\,dt = f$ and

$$\int_0^1 g_\delta(t)^p w^{**}(t)\,dt = \frac{kf^p(1-\delta)^p}{1+b-\delta p}.$$

Next note that

$$\frac{1}{t}\int_0^t g_\delta(u)\,du = \frac{g_\delta(t)}{1-\delta} \qquad \text{for all } t \in (0,1]$$

and so

$$\Delta_w(g_\delta) = \left(\frac{1}{1-\delta}\right)^p \int_0^1 g_\delta(t)^p w^{**}(t)\,dt.$$

The condition $\int_0^1 g_\delta(t)^p w^{**}(t)\,dt = F$ is then equivalent to the following equation in δ:

$$\frac{(1-\delta)^p}{1+b-\delta p} = \frac{F}{kf^p}. \tag{2.10}$$

To study this equation we write

$$z = \frac{p-1-b}{p-1}\,\frac{1}{1-\delta}, \tag{2.11}$$

and note that (2.10) is then equivalent to

$$-(p-1)z^p + pz^{p-1} = \frac{kf^p}{(\frac{p-1}{p-1-b})^{p-1}F}, \tag{2.12}$$

thus

$$z = \omega_p\left(\frac{kf^p}{(\frac{p-1}{p-1-b})^{p-1}F}\right), \tag{2.13}$$

so the corresponding δ clearly satisfies $\delta < (1+b)/p$, and using (2.11) we get

$$\Delta_w(g_\delta) = \left(\frac{p-1}{p-1-b}\right)^p F\omega_p\left(\frac{kf^p}{(\frac{p-1}{p-1-b})^{p-1}F}\right). \tag{2.14}$$

But now note that the constants a, c of the weight w are $a = 1/(p-1-b)$ and $c = k/(p-1-b)$, and so

$$\Delta_w(g_\delta) = (p-1)^p a^p F\omega_p\left(\frac{cf^p}{(p-1)^{p-1}a^pF}\right)^p; \tag{2.15}$$

moreover, taking into consideration the mapping properties of ω_p we conclude that by varying k, b with $-1 < b < p-1$ we can achieve all possible F, f, a, c satisfying $0 < cf^p \le (p-1)^{p-1}a^pF$. This completes the proof. ∎

3. Estimation via double maximal operators. Here we will use an inequality introduced by A. Lerner [4] for the nondyadic case. We fix $p > 1$, we let w be an A_p weight with respect to the tree $\mathcal{T}$ and we denote, for any I in $\mathcal{T}$, $w(I) = \int_I w\,d\mu$, $\sigma = w^{-1/(p-1)}$, $\sigma(I) = \int_I \sigma\,d\mu$. Also we denote by $M_{\mathcal{T},w}$ the maximal operator with respect to the tree $\mathcal{T}$ but when X is equipped with the measure $w\mu$ instead of μ, and similarly for $M_{\mathcal{T},\sigma}$.

PROPOSITION 1. *Let w be an A_p weight with respect to the tree $\mathcal{T}$ and with $\mathcal{T}$-constant*

$$[w]_p = \sup_{I\in\mathcal{T}} \frac{w(I)\sigma(I)^{p-1}}{\mu(I)^p}.$$

Then for any ϕ we have the following pointwise estimate:

$$(M_{\mathcal{T}}\phi)^{p-1} \le [w]_p M_{\mathcal{T},w}[M_{\mathcal{T},\sigma}(\phi\sigma^{-1})^{p-1}w^{-1}].$$

Proof. This follows from the following inequalities valid for any $I \in \mathcal{T}$:

$$\begin{aligned}\Big(\frac{1}{\mu(I)}\int_I \phi\,d\mu\Big)^{p-1} &= \frac{w(I)\sigma(I)^{p-1}}{\mu(I)^p}\bigg(\frac{\mu(I)}{w(I)}\Big(\frac{1}{\sigma(I)}\int_I \phi\sigma^{-1}\sigma\,d\mu\Big)^{p-1}\bigg)\\ &\le [w]_p\frac{1}{w(I)}\int_I M_{\mathcal{T},\sigma}(\phi\sigma^{-1})^{p-1}w^{-1}w\,d\mu,\end{aligned}$$

since $M_{\mathcal{T},\sigma}(\phi\sigma^{-1})(x) \ge \sigma(I)^{-1}\int_I \phi\sigma^{-1}\sigma\,d\mu$ for every x in I. ■

As a first application, fixing a tree $\mathcal{T}$ on a probability space (X,μ) and given an A_p weight w in the sense of Definition 2, we define the following generalization of the Bellman function (1.7), where $p > 1$:

$$\begin{aligned}(3.1)\quad &\mathcal{B}^{\mathcal{T}}_{p,r}(F,f,u,v)\\ &= \sup\Big\{\int_X (M_{\mathcal{T}}\phi)^p w\,d\mu : \phi \ge 0 \text{ measurable with } \int_X \phi^p w\,d\mu = F,\\ &\int_X \phi\,d\mu = f,\ w \text{ an } A_p\text{-weight with respect to } \mathcal{T} \text{ such that } [w]_p \le r,\\ &\int_X w\,d\mu = u,\ \int_X w^{-1/(p-1)}\,d\mu = v\Big\}.\end{aligned}$$

This expression, depending on the weight w, will now be estimated from above in terms of the $\mathcal{T}$-constant $[w]_p$ of the weight and $\sigma(X)$.

THEOREM 3. *For any tree $\mathcal{T}$ on a probability space (X,μ) and any A_p weight w, and for any ϕ with $\int_X \phi^p w\,d\mu = F$ and $\int_X \phi\,d\mu = f$, we have*

$$\begin{aligned}(3.2)\quad &\int_X (M_{\mathcal{T}}\phi)^p w\,d\mu\\ &\le [w]_p^{p'} F\omega_p\Big(\frac{f^p}{\sigma(X)^{p-1}F}\Big)^p \omega_{p'}\bigg(\frac{(\int_X \phi^{p-1}w\,d\mu)^{p'}}{w(X)^{p'-1}F\omega_p\big(\frac{f^p}{\sigma(X)^{p-1}F}\big)^p}\bigg)^{p'}.\end{aligned}$$

In particular

$$(3.3)\qquad \mathcal{B}^{\mathcal{T}}_{p,r}(F,f,u,v) \le p^{p'}r^{p'}F\omega_p\Big(\frac{f^p}{v^{p-1}F}\Big)^p.$$

Proof. By applying [6, (1.12)] for the exponent $p' = p/(p-1)$ to the function $\rho = (M_{\mathcal{T},\sigma}(\phi\sigma^{-1}))^{p-1}w^{-1}$ and with respect to the tree $\mathcal{T}$ but on the probability space $(X, w(X)^{-1}w\,d\mu)$ (where as usual $w(X) = \int_X w\,d\mu$), we get

$$\begin{aligned}(3.4)\qquad \frac{1}{[w]_p^{p'}}\int_X (M_{\mathcal{T}}\phi)^p w\,d\mu &\le w(X)\int_X (M_{\mathcal{T},w}\rho)^{p'} w\frac{d\mu}{w(X)}\\ &\le w(X)\int_X \rho^{p'} w\frac{d\mu}{w(X)}\cdot\omega_{p'}\bigg(\frac{\big(\int_X \rho w\frac{d\mu}{w(X)}\big)^{p'}}{\int_X \rho^{p'} w\frac{d\mu}{w(X)}}\bigg)^{p'}.\end{aligned}$$

Note that (as proved in [6]) the function $x\omega_{p'}(y^{p'}/x)^{p'}$ is increasing in x and decreasing in y. Now we have

$$\begin{aligned}\int_X \rho w\,\frac{d\mu}{w(X)} &= \int_X (M_{\mathcal{T},\sigma}(\phi\sigma^{-1}))^{p-1}\,\frac{d\mu}{w(X)}\\ &\ge \int_X (\phi\sigma^{-1})^{p-1}\,\frac{d\mu}{w(X)} = \int_X \phi^{p-1}w\,\frac{d\mu}{w(X)},\end{aligned}$$

and applying [6, (1.12)] for the exponent p to the function $\rho = \phi\sigma^{-1}$ and with respect to the tree $\mathcal{T}$ but on the probability space $(X, \sigma(X)^{-1}\sigma\,d\mu)$, we get (since $\sigma^{-(p-1)} = w$)

$$\begin{aligned}(3.5)\qquad \int_X \rho^{p'} w\,d\mu &= \int_X (M_{\mathcal{T},\sigma}(\phi\sigma^{-1}))^p w^{-p'} w\,d\mu = \int_X (M_{\mathcal{T},\sigma}(\phi\sigma^{-1}))^p \sigma\,d\mu\\ &\le \sigma(X)\int_X (\phi\sigma^{-1})^p\sigma\frac{d\mu}{\sigma(X)}\cdot\ \omega_p\bigg(\frac{(\int_X \phi\sigma^{-1}\sigma\frac{d\mu}{\sigma(X)})^p}{\int_X (\phi\sigma^{-1})^p\sigma\frac{d\mu}{\sigma(X)}}\bigg)^p\\ &= \int_X \phi^p w\,d\mu\cdot\omega_p\bigg(\frac{(\int_X \phi\,d\mu)^p}{\sigma(X)^{p-1}\int_X \phi^p w\,d\mu}\bigg)^p = F\omega_p\bigg(\frac{f^p}{\sigma(X)^{p-1}F}\bigg)^p.\end{aligned}$$

Now combining the above estimates we get

$$\begin{aligned}(3.6)\qquad &\frac{1}{[w]_p^{p'}}\int_X (M_{\mathcal{T}}\phi)^p w\,d\mu\\ &\le F\omega_p\bigg(\frac{f^p}{\sigma(X)^{p-1}F}\bigg)^p\omega_{p'}\bigg(\frac{(\int_X \phi^{p-1}w\,d\mu)^{p'}}{w(X)^{p'-1}F\omega_p\big(\frac{f^p}{\sigma(X)^{p-1}F}\big)^p}\bigg)^{p'},\end{aligned}$$

which proves (3.2). Since $\omega_{p'}(x) \le \frac{p'}{p'-1} = p$, the estimate (3.3) follows too. ■

To get lower bounds for the Bellman function we invoke the following construction.

Fixing α with $0 < \alpha < 1$ and using [6, Lemma 1], we fix now a tree $\mathcal{T}$, for example the dyadic subintervals of $[0,1]$, and choose for every $I \in \mathcal{T}$ a family $\mathcal{F}(I) \subseteq \mathcal{T}$ of pairwise almost disjoint subsets of I such that

$$\sum_{J\in\mathcal{F}(I)} \mu(J) = (1-\alpha)\mu(I). \tag{3.7}$$

Then we define $\mathcal{S} = \mathcal{S}_\alpha$ to be the intersection of all subsets $\mathcal{U}$ of $\mathcal{T}$ such that $X \in \mathcal{U}$ and $\mathcal{F}(I) \subseteq \mathcal{U}$ for every $I \in \mathcal{U}$. Clearly $\mathcal{S} = \{X\} \cup \mathcal{F}(X) \cup \bigcup_{I\in\mathcal{F}(X)} \mathcal{F}(I) \cup \cdots$. Next for every $I \in \mathcal{S}$ we define

$$A_I = I \setminus \bigcup_{J\in\mathcal{F}(I)} J \tag{3.8}$$

and note that the sets A_I are pairwise disjoint, $\mu(A_I) = \alpha\mu(I)$ and $I = \bigcup_{J\in\mathcal{S},\, J\subseteq I} A_J$ for every $I \in \mathcal{S}$. Also since $\mathcal{S} = \bigcup_{m\geq 0} \mathcal{S}_{(m)}$ where $\mathcal{S}_{(0)} = \{X\}$ and $\mathcal{S}_{(m+1)} = \bigcup_{I\in\mathcal{S}_{(m)}} \mathcal{F}(I)$, we can define $\operatorname{rank}(I) = r(I)$ for $I \in \mathcal{S}$ to be the unique integer m such that $I \in \mathcal{S}_{(m)}$ and remark that

$$\sum_{\substack{\mathcal{S}\ni J\subseteq I\\ r(J)=r(I)+m}} \mu(J) = (1-\alpha)^m \mu(I) \quad \text{for every } I \in \mathcal{S}.$$

Next for any $\lambda, \gamma > 0$ with $\gamma(1-\alpha) < 1$ we define

$$\psi = \sum_{I\in\mathcal{S}} \lambda\gamma^{r(I)} \chi_{A_I}. \tag{3.9}$$

Then for any $I \in \mathcal{S}$,

$$\frac{1}{\mu(I)} \int_I \psi \, d\mu = \frac{\lambda\alpha}{1-\gamma(1-\alpha)} \gamma^{r(I)}. \tag{3.10}$$

Hence taking

$$\phi_\alpha = \sum_{I\in\mathcal{S}} \lambda_1 \gamma_1^{r(I)} \chi_{A_I}, \qquad w_\alpha = \sum_{I\in\mathcal{S}} \lambda_2 \gamma_2^{r(I)} \chi_{A_I} \tag{3.11}$$

with γ_1, γ_2 such that $\gamma_1(1-\alpha) < 1$, $\gamma_1^p\gamma_2(1-\alpha) < 1$, $\gamma_2(1-\alpha) < 1$, $\gamma_2^{-1/(p-1)}(1-\alpha) < 1$ we have

$$\int_X \phi_\alpha \, d\mu = \frac{\lambda_1\alpha}{1-\gamma_1(1-\alpha)}, \qquad \int_X \phi_\alpha^p w \, d\mu = \frac{\lambda_1^p \lambda_2 \alpha}{1-\gamma_1^p\gamma_2(1-\alpha)}, \tag{3.12}$$

(3.13)

$$\int_X w_\alpha \, d\mu = \frac{\lambda_2\alpha}{1-\gamma_2(1-\alpha)}, \qquad \int_X w_\alpha^{-1/(p-1)} \, d\mu = \frac{\lambda_2^{-1/(p-1)}\alpha}{1-\gamma_2^{-1/(p-1)}(1-\alpha)},$$

and for any $I \in \mathcal{S}$,

$$\frac{w_\alpha(I)[w_\alpha^{-1/(p-1)}(I)]^{p-1}}{\mu(I)^p} = \frac{\alpha^p}{[1-\gamma_2(1-\alpha)][1-\gamma_2^{-1/(p-1)}(1-\alpha)]^{p-1}} = \Big(\int_X w\,d\mu\Big)\Big(\int_X w^{-1/(p-1)}\,d\mu\Big)^{p-1}. \tag{3.14}$$

Thus w_α is an A_p weight but with respect to the *tree* $\mathcal{S}_\alpha$ on (X,μ) and with $[w_\alpha]_p$ equal to the right hand side of the above relation. Moreover

$$M_{\mathcal{S}}\phi_\alpha \geq \sum_{I\in\mathcal{S}} \frac{1}{\mu(I)} \int_I \phi_a\,d\mu\,\chi_{A_I} = \frac{\alpha}{1-\gamma_1(1-\alpha)}\phi_\alpha, \tag{3.15}$$

hence

$$\int_X (M_{\mathcal{S}}\phi)^p w\,d\mu \geq \Big(\frac{\alpha}{1-\gamma_1(1-\alpha)}\Big)^p \int_X \phi_\alpha^p w\,d\mu. \tag{3.16}$$

Now fix appropriate variables F,f,r,u (and v with $r=uv^{p-1}$) and fix $\alpha>0$ small. Then with $z=z(\alpha)>0$ and $\zeta=\zeta(\alpha)>0$ to be defined later we define

$$\lambda_1=\frac{f}{z},\qquad \lambda_2=\frac{u}{\zeta},\qquad \gamma_1=\frac{z-\alpha}{z(1-\alpha)},\qquad \gamma_2=\frac{\zeta-\alpha}{\zeta(1-a)} \tag{3.17}$$

so that the constraints $\int_X \phi_\alpha\,d\mu=f$, $\int_X \phi_\alpha^p w\,d\mu=F$, $\int_X w_\alpha\,d\mu=u$ and $\int_X w_\alpha^{-1/(p-1)}\,d\mu=v=(r/u)^{1/(p-1)}$ are satisfied for the functions ϕ_a, w_α in (3.11), which, as is easy to see, amounts to the following equations:

$$\frac{1}{\zeta^{1/(p-1)}}-\frac{(1-\alpha)^{p'}}{(\zeta-\alpha)^{1/(p-1)}} = r^{-1/(p-1)}a, \tag{3.18}$$

$$z^p\zeta(1-\alpha)^p-(z-\alpha)^p(\zeta-\alpha)=\frac{uf^p}{F}(1-\alpha)^p\alpha. \tag{3.19}$$

Now using [6, Lemma 3 (and its proof)] we first infer from (3.11) that $\zeta(\alpha)$ is uniquely defined and

$$\zeta(\alpha)\to\omega_{p'}(r^{-1/(p-1)})^{-1}\qquad \text{as } \alpha\to 0^+.$$

Then using the proof of the same lemma we find that $z(\alpha)$ is also uniquely determined and converges as $a\to 0^+$ to the solution Z of the equation

$$-pZ^p+Z^p\omega_{p'}(r^{-1/(p-1)})+pZ^{p-1}=\frac{uf^p}{F}\omega_{p'}(r^{-1/(p-1)}). \tag{3.20}$$

Moreover the inequalities following (3.11) automatically hold for $z(\alpha),\zeta(\alpha)$ as $\alpha\to 0^+$. Since $\omega_{p'}(r^{-1/(p-1)})\geq 1$ we get $H_p(Z)\leq \frac{uf^p}{F}\omega_{p'}(r^{-1/(p-1)})$, and since ω_p is decreasing we get

$$Z\geq \omega_p\Big(\frac{uf^p}{F}\omega_{p'}(r^{-1/(p-1)})\Big). \tag{3.21}$$

Now using (3.16) and noting that $\frac{\alpha}{1-\gamma_1(1-\alpha)}$ is equal to $z(\alpha)$ with the values of the parameters defined above, and taking a sequence $\alpha_m \to 0^+$, the corresponding trees $\mathcal{T}_m = \mathcal{S}_{a_m}$ and the functions from (3.11), we obtain the following.

PROPOSITION 2. *Given appropriate F, f, r, u there exists a sequence of trees $\mathcal{T}_m$ on (X, μ) and two sequences (ϕ_m) and (w_m) of positive measurable functions on (X, μ) such that $\int_X \phi_m \, d\mu = f$, $\int_X \phi_m^p w_m \, d\mu = F$, each w_m is an A_p weight with respect to the tree $\mathcal{T}_m$ with $[w_m]_p = r$ and $\int_X w_m \, d\mu = u$, and*

$$\lim_{m\to\infty} \int_X (M_{\mathcal{T}_m}\phi_m)^p w_m \, d\mu \geq F\omega_p\bigg(\frac{uf^p}{F}\omega_{p'}(r^{-1/(p-1)})\bigg)^p. \tag{3.22}$$

The above proposition implies a lower bound on the class of functions $\mathcal{B}_{p,r}^{\mathcal{T}}(F, f, u, v)$ when viewed over all trees $\mathcal{T}$.

Acknowledgements. This research has been co-financed by the European Union and Greek national funds through the Operational Program “Education and Lifelong Learning” of the National Strategic Reference Framework (NSRF), ARISTEIA I, MAXBELLMAN 2760, research number 70/3/11913.

The authors would like to thank the referee for his thorough suggestions and remarks.

References

[1] T. N. Bekjan, Z. Chen and A. Osękowski, *Noncommutative maximal inequalities associated with convex functions*, Trans. Amer. Math. Soc. 369 (2017), 409–427.

[2] D. L. Burkholder, *Martingales and Fourier analysis in Banach spaces*, in: Probability and Analysis (Varenna, 1985), Lecture Notes in Math. 1206, Springer, 1986, 61–108.

[3] D. L. Burkholder, *Boundary value problems and sharp inequalities for martingale transforms*, Ann. Probab. 12 (1984), 647–702.

[4] A. K. Lerner, *An elementary approach to several results on the Hardy–Littlewood maximal operator*, Proc. Amer. Math. Soc. 136 (2008), 2829–2833.

[5] A. A. Logunov, L. Slavin, D. M. Stolyarov, V. Vasyunin and P. B. Zatitskiy, *Weak integral conditions for BMO* Proc. Amer. Math. Soc. 143 (2015), 2913–2926.

[6] A. D. Melas, *The Bellman functions of dyadic-like maximal operators and related inequalities*, Adv. Math. 192 (2005), 310–340.

[7] A. D. Melas, *Sharp general local estimates for dyadic-like maximal operators and related Bellman functions*, Adv. Math. 220 (2009), 1631–1654.

[8] A. D. Melas, *Dyadic-like maximal operators on $L\log L$ functions*, J. Funct. Anal. 257 (2009), 367–426.

[9] A. D. Melas and E. N. Nikolidakis, *Sharp Lorentz estimates for dyadic-like maximal operators and related Bellman functions*, submitted.

[10] A. D. Melas, E. N. Nikolidakis and Th. Stavropoulos, *Sharp local lower L^p-bounds for dyadic-like maximal operators*, Proc. Amer. Math. Soc. 141 (2013), 3171–3181.

[11] P. Ivanisvili, N. N. Osipov, D. M. Stolyarov, V. I. Vasyunin and P. B. Zatitskiy, *Bellman function for extremal problems in BMO*, Trans. Amer. Math. Soc. 368 (2016), 3415–3468.

[12] F. Nazarov and S. Treil, *The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis*, Algebra i Analiz 8 (1996), no. 5, 32–162 (in Russian).

[13] F. Nazarov, S. Treil and A. Volberg, *The Bellman functions and two-weight inequalities for Haar multipliers* J. Amer. Math. Soc. 12 (1999), 909–928.

[14] F. Nazarov, S. Treil and A. Volberg, *Bellman function in stochastic control and harmonic analysis (how our Bellman function got its name)*, in: Oper. Theory Adv. Appl. 129, Birkhäuser, 2001, 393–424.

[15] E. N. Nikolidakis, *The geometry of the dyadic maximal operator*, Rev. Mat. Iberoamer. 30 (2014), 1397–1411.

[16] E. N. Nikolidakis and A. D. Melas, *Dyadic weights on $\mathbb{R}^n$ and reverse Hölder inequalities*, Studia Math. 234 (2016), 281–290.

[17] E. N. Nikolidakis and A. D. Melas, *A sharp integral rearrangement inequality for the dyadic maximal operator and applications*, Appl. Comput. Harmon. Anal. 38 (2009), 242–261.

[18] A. Osękowski, *Sharp $L^{p,\infty} \to L^q$ estimates for the dyadic-like maximal operators* J. Fourier Anal. Appl. 20 (2014), 911–933.

[19] A. Osękowski, *Sharp estimates for Lipschitz class*, J. Geom. Anal. 26 (2016), 1346–1369.

[20] A. Osękowski, *Sharp weak-type estimates for the dyadic-like maximal operators*, Taiwanese J. Math. 19 (2015), 1031–1050.

[21] A. Osękowski, *Weighted inequalities for the martingale square and maximal functions*, Statist. Probab. Lett. 120 (2017), 95–100.

[22] S. Petermichl, L. Slavin and B. D. Wick, *New estimates for the Beurling–Ahlfors operator on differential forms*, J. Operator Theory 65 (2011), 307–324.

[23] G. Rey and A. Reznikov, *Extremizers and sharp weak-type estimates for positive dyadic shifts*, Adv. Math. 254 (2014), 664–681.

[24] L. Slavin and V. Vasyunin, *Sharp results in the integral-form John–Nirenberg inequality*, Trans. Amer. Math. Soc. 363, (2011), 4135–4169.

[25] L. Slavin and A. Volberg, *The explicit BF for a dyadic Chang–Wilson–Wolff theorem. The s-function and the exponential integral*, in: Contemp. Math. 444, Amer. Math. Soc., 2007, 215–228.

[26] L. Slavin, *Best constants for a family of Carleson sequences*, Adv. Math. 289 (2016), 685–724.

[27] L. Slavin and V. Vasyunin, *Sharp L^p estimates on BMO*, Indiana Univ. Math. J. 61 (2012), 1051–1110.

[28] E. M. Stein, *Harmonic Analysis*, Princeton Math. Ser. 43, Princeton Univ. Press, 1993.

[29] L. Slavin, A. Stokolos and V. Vasyunin, *Monge–Ampère equations and Bellman functions: The dyadic maximal operator*, C. R. Math. Acad. Sci. Paris 346 (2008), 585–588.

[30] D. M. Stolyarov, V. I. Vasyunin and P. B. Zatitskiy, *Monotonic rearrangements of functions with small mean oscillation*, Studia Math. 231 (2015), 257–267.

[31] V. Vasyunin, *The sharp constant in the reverse Hölder inequality for Muckenhoupt weights*, Algebra i Analiz 15 (2003), no. 1, 73–117 (in Russian).

[32] V. Vasyunin and A. Volberg, *The Bellman functions for a certain two weight inequality: a case study*, Algebra i Analiz 18 (2006), no. 2, 24–56 (in Russian).

[33] V. Vasyunin and A. Volberg, *Monge–Ampère equation and Bellman optimization of Carleson embedding theorems*, Amer. Math. Soc. Transl. (2) 226 (2009), 195–238.

[34] V. Vasyunin and A. Volberg, *Sharp constants in the classical weak form of the John–Nirenberg inequality*, Proc. London Math. Soc. (3) 108 (2014), 1417–1434.

[35] A. Volberg, *Bellman approach to some problems in harmonic analysis*, in: Séminaire des équations aux derivées partielles, École Polytéchnique, 2002, exp. 20, 14 pp.

[36] G. Wang, *Sharp maximal inequalities for conditionally symmetric martingales and Brownian motion*, Proc. Amer. Math. Soc. 112 (1991), 579–586.

[37] P. B. Zatitskiy, P. Ivanishvili and D. M. Stolyarov, *Bellman vs. Beurling: sharp estimates of uniform convexity for L^p spaces*, St. Petersburg Math. J. 27 (2016), 333–343.

Antonios D. Melas, Eleftherios N. Nikolidakis, Dimitrios Cheliotis
Department of Mathematics
University of Athens
Panepistimiopolis 15784, Athens, Greece
E-mail: amelas@math.uoa.gr
lefteris@math.uoc.gr
dcheliotis@math.uoa.gr