ANNALES
POLONICI MATHEMATICI
119.2 (2017)

Noncompact complete manifolds with cyclic parallel Ricci curvature

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Abstract. Let (M^n, g) be a noncompact complete n-dimensional Riemannian manifold with cyclic parallel Ricci curvature and positive Yamabe constant. When the scalar curvature R is negative, assuming that the L^β-norms (see Theorem 1.1 for the range of β) of the Weyl curvature are finite, we show that (M^n, g) is a space form if $n \geq 7$ and the $L^{n/2}$-norms of the traceless Ricci curvature and Weyl curvature are small enough. When $R = 0$, the same rigidity result is also obtained for all dimensions $n \geq 3$ without the assumption on the L^β-norms of the Weyl curvature.

1. Introduction. In 1978, Gray [G] introduced two classes of Riemannian manifolds, $\mathscr{A}$ and $\mathscr{B}$, which are defined by covariant derivatives of the Ricci tensor:

Class $\mathscr{A}$: a Riemannian manifold (M, g) belongs to $\mathscr{A}$ if and only if its Ricci tensor Rc is *cyclic-parallel*, that is,

$$\nabla_X Rc(Y, Z) + \nabla_Z Rc(X, Y) + \nabla_Y Rc(Z, X) = 0 \tag{1.1}$$

for all vector fields X, Y, Z tangent to M. (1.1) is equivalent to requiring that Rc is a Killing tensor, namely,

$$\nabla_X Rc(X, X) = 0, \quad \forall X \in \mathfrak{X}(M).$$

Class $\mathscr{B}$: (M, g) belongs to $\mathscr{B}$ if and only if its Ricci tensor Rc is a *Codazzi tensor*, that is,

$$\nabla_X Rc(Y, Z) = \nabla_Y Rc(X, Z), \tag{1.2}$$

which is equivalent to (M, g) having *harmonic curvature* (see [Be]).

Obviously, all manifolds belonging to $\mathscr{A}$ or $\mathscr{B}$, which are known as *Einstein-like*, have constant scalar curvature. Moreover, any manifold that

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2010 *Mathematics Subject Classification*: Primary 53C24; Secondary 53C20.

Key words and phrases: noncompact manifolds, cyclic parallel Ricci curvature, Yamabe constant, space form.
Received 28 October 2016; revised 4 March 2017.
Published online 11 July 2017.

belongs to $\mathscr{A} \cap \mathscr{B}$ must have parallel Ricci curvature, which is locally the Riemannian product of Einstein manifold according to the de Rham decomposition theorem. More interesting examples which are Einstein-like but not Einstein can be found in [G].

Einstein-like manifolds have been extensively studied. Note that conditions (1.1) and (1.2) yield different consequences in dimension three. In fact, any 3-dimensional Riemannian manifold (M^3, g) belonging to class $\mathscr{A}$ is locally homogeneous (see [PD]). On the other hand, 3-dimensional metrics in class $\mathscr{B}$ may not be locally homogeneous. Derdziński [Der] gave the first examples of Riemannian metrics in class $\mathscr{B}$ having a nonparallel Ricci tensor, and provided a complete description of the ones having at most two distinct Ricci eigenvalues, in terms of suitable warped product metrics. See [BV], [Ca], [KK], [LY], [MDD], [PQ], [TY] and the references therein for more examples of Riemannian manifolds in class $\mathscr{A}$.

The aim of this paper is to investigate the $L^{n/2}$-type rigidity for noncompact manifolds in class $\mathscr{A}$. Given an n-dimensional complete Riemannian manifold (M^n, g) $(n \geq 3)$, it is well known that the Riemannian curvature tensor Rm can be decomposed into three orthogonal components which have the same symmetries as Rm (see for instance [H]):

$$Rm = U + V + W, \tag{1.3}$$

where U, V and W denote the scalar curvature part, the traceless Ricci part and the Weyl curvature tensor, respectively.

In 2011, Kim [K] studied the $L^{n/2}$-type rigidity for noncompact complete manifolds with harmonic curvature (i.e., belonging to class $\mathscr{B}$) and nonpositive scalar curvature, and proved

Theorem A. *Let (M^n, g) be a complete noncompact Riemannian n-manifold with harmonic curvature and Yamabe constant $Q(M^n, g) > 0$. Assume that $\int_{M^n} (|W|^2 + |\mathring{Rc}|^2)\, dV_g$ is finite, and* (1) *or* (2) *below holds:*

(1) *The scalar curvature R is* 0 *and $n \geq 3$.*
(2) *$R < 0$ and $n \geq 8$.*

Then there exists a small number $C_1 > 0$ depending on n and $Q(M^n, g)$ such that if

$$\int_{M^n} (|\mathring{Rc}|^{n/2} + |W|^{n/2})\, dV_g \leq C_1,$$

then (M^n, g) is a space form. Here $\mathring{Rc} \equiv Rc - (R/n)g$ is the traceless Ricci tensor.

The corresponding rigidity result for complete noncompact n-manifolds (M^n, g) with of $R > 0$ was established in [Ch] for all dimensions $n \geq 3$.

When (M^n, g) is in class $\mathscr{A}$, Deng [Den] studied its rigidity and showed that

THEOREM B. *Let (M^n, g) be a complete noncompact Riemannian n-manifold with cyclic parallel Ricci curvature and $Q(M^n, g) > 0$. If $R \leq 0$, then there exists a small number $C_2 > 0$ depending on the dimension n and $Q(M^n, g)$ such that if $\int_{M^n} (|\mathring{Rc}|^{n/2} + |W|^{n/2})\, dV_g \leq C_2$, then (M^n, g) is an Einstein manifold.*

Comparing with Theorem A, it is to be expected that the noncompact manifold (M^n, g) in Theorem B is a space form under some curvature assumptions. In fact, it is shown that

THEOREM 1.1. *Let (M^n, g) be a complete noncompact Riemannian n-manifold with cyclic parallel Ricci curvature and $Q(M^n, g) > 0$.*

(i) *When $R < 0$, assume that $\int_{M^n} |W|^\beta \, dV_g < \infty$ for some constant β in*

$$\left(\frac{2(n-3)\sqrt{n(n-2)}}{(n-1)(\sqrt{n(n-2)}+\sqrt{n^2-10n+24})}, \frac{2(n-3)\sqrt{n(n-2)}}{(n-1)(\sqrt{n(n-2)}-\sqrt{n^2-10n+24})}\right) \triangleq I(n).$$

Then there exists a small number $C_3 > 0$ depending on the dimension $n \geq 7$ and $Q(M^n, g)$ such that if $\int_{M^n} (|\mathring{Rc}|^{n/2} + |W|^{n/2})\, dV_g \leq C_3$, then M^n is a space form.

(ii) *When $R = 0$, then there exists a small number $C_4 > 0$ depending on the dimension $n \geq 3$ and $Q(M^n, g)$ such that if $\int_{M^n} (|\mathring{Rc}|^{n/2} + |W|^{n/2})\, dV_g \leq C_4$, then M^n is a space form.*

REMARK 1.2. It is easy to check that $2 \in I(n)$ for $n \geq 8$. By taking $\beta = 2$ in Theorem 1.1 we establish a similar rigidity result to Theorem A. Thus, when $R \leq 0$, we see that noncompact manifolds with positive Yamabe constant in class $\mathscr{A}$ have $L^{n/2}$-type rigidity similar to those of manifolds in class $\mathscr{B}$. Moreover, when $R = 0$, Theorem 1.1 generalizes the corresponding result in Theorem B.

2. Notation and preliminaries. By the assumptions of Theorems 1.1 and B, we know that there exists a small number $C_2 > 0$ such that if $\int_{M^n} (|\mathring{Rc}|^{n/2} + |W|^{n/2})\, dV_g \leq C_2$, then (M^n, g) is an Einstein manifold.

Let (M^n, g) be an n-dimensional ($n \geq 3$) smooth complete Riemannian manifold. In local coordinates, we denote by $g = (g_{ij})$ the Riemannian metric on M^n with coefficients g_{ij}, and denote the inverse matrix by $(g^{ij}) = (g_{ij})^{-1}$. Correspondingly, the decompositions (1.3) and the equation (1.1) can be rewritten respectively as

$$R_{ijkl} = W_{ijkl} + \frac{R}{n(n-1)}\,(g_{il}g_{jk} - g_{ik}g_{jl}) + \frac{1}{n-2}(\mathring{R}_{il}g_{jk} + \mathring{R}_{jk}g_{il} - \mathring{R}_{ik}g_{jl} - \mathring{R}_{jl}g_{ik})$$

and

$$\nabla_i R_{jk} + \nabla_k R_{ij} + \nabla_j R_{ki} = 0, \tag{2.1}$$

where R_{ijkl}, W_{ijkl}, R_{ij} and $\mathring{R}_{jk} = R_{jk} - (R/n)g_{jk}$ denote respectively the components of the Riemannian curvature tensor, the Weyl curvature tensor, the Ricci tensor and the traceless Ricci tensor.

In what follows, we always adopt the Einstein summation convention. On an Einstein manifold, the following equalities can be easily obtained:

$$\nabla R = 0, \tag{2.2}$$

$$\nabla_p W_{ijkl} + \nabla_l W_{ijpk} + \nabla_k W_{ijlp} = 0, \tag{2.3}$$

$$\nabla^l W_{ijkl} = 0. \tag{2.4}$$

Moreover, the first Bianchi identity for the Weyl tensor for any Riemannian manifold is

$$W_{ijkl} + W_{iljk} + W_{iklj} = 0. \tag{2.5}$$

We now recall the definition of the Yamabe constant. Given a complete noncompact Riemannian n-manifold (M^n, g) of dimension $n \geq 3$, the *Yamabe constant* $Q(M^n, g)$ is defined by

$$Q(M^n, g) \equiv \inf_{0\neq u\in C_0^\infty(M^n)} \frac{\int_{M^n}\big(|\nabla u|^2 + \frac{n-2}{4(n-1)}Ru^2\big)\,dV_g}{(\int_{M^n}|u|^{2n/(n-2)}\,dV_g)^{(n-2)/n}}. \tag{2.6}$$

The important works of Schoen [Sc], Trudinger [T] and Yamabe [Y] for the Yamabe problem showed that the infimum in (2.6) is always achieved.

In order to prove Theorem 1.1, we need the following formula.

LEMMA 2.1. *Let (M^n, g) $(n \geq 3)$ be an n-dimensional Einstein manifold. Then*

$$\begin{aligned}\tfrac{1}{2}\Delta|W|^2 &= |\nabla W|^2 - 4W^{ijkl}(W^p{}_{li}{}^h W_{pkjh} + W^p{}_{lk}{}^h W_{pijh}) + \frac{2R}{n}|W|^2 \\ &= |\nabla W|^2 - 2W^{ijkl}\big(2W^p{}_{li}{}^h W_{pkjh} + \tfrac{1}{2}W^{ph}{}_{kl}W_{phji}\big) + \frac{2R}{n}|W|^2,\end{aligned} \tag{2.7}$$

where Δf denotes the Laplacian of f given by the trace of Hess f.

Proof. By the definition of $|W|^2$ and (2.3), we have

$$\begin{aligned} \tfrac{1}{2}\Delta|W_{ijkl}|^2 &= |\nabla W|^2 + W^{ijkl}\nabla^p\nabla_p W_{ijkl} \\ &= |\nabla W|^2 + W^{ijkl}\nabla^p(\nabla_l W_{ijkp} - \nabla_k W_{ijlp}) \\ &= |\nabla W|^2 + 2W^{ijkl}\nabla^p\nabla_l W_{ijkp}. \end{aligned} \tag{2.8}$$

Using the Ricci identities, (2.2) and (2.4), we get

$$\begin{aligned} &2W^{ijkl}\nabla^p\nabla_l W_{ijkp} \\ &= 2W^{ijkl}(\nabla_l\nabla^p W_{ijkp} - R^p{}_{li}{}^h W_{hjkp} - R^p{}_{lj}{}^h W_{ihkp} \\ &\quad - R^p{}_{lk}{}^h W_{ijhp} - R^p{}_{lp}{}^h W_{ijkh}) \\ &= 2W^{ijkl}(-2R^p{}_{li}{}^h W_{hjkp} - R^p{}_{lk}{}^h W_{ijhp} + R_l{}^h W_{ijkh}) \\ &= -2W^{ijkl}(2W^p{}_{li}{}^h W_{pkjh} + W^p{}_{lk}{}^h W_{phji}) + \frac{2R}{n}|W|^2 \\ &\quad - \frac{2R}{n(n-1)}W^{ijkl}\big(2W_{pkjh}(g^{ph}g_{li} - g^p_i g^h_l) + W_{phji}(g^{ph}g_{lk} - g^p_k g^h_l)\big) \\ &= \frac{2R}{n}|W|^2 - 2W^{ijkl}(2W^p{}_{li}{}^h W_{pkjh} + W^p{}_{lk}{}^h W_{phji}) \\ &\quad + \frac{2R}{n(n-1)}W^{ijkl}(2W_{ikjl} + W_{ijlk}). \end{aligned} \tag{2.9}$$

By (2.5), we see that

$$W^{ijkl}(2W_{ikjl} + W_{ijlk}) = W^{ijkl}(W_{ikjl} + W_{ijlk} - W_{iljk}) = 0 \tag{2.10}$$

and

$$\begin{aligned} W^{ijkl}W^p{}_{lk}{}^h W_{phji} &= W^{ijkl}W^p{}_{lk}{}^h(W_{pijh} - W_{pjih}) = 2W^{ijkl}W^p{}_{lk}{}^h W_{pijh} \\ &= W^{ijkl}(W^p{}_{kl}{}^h - W^{ph}{}_{lk})W_{phji} \\ &= -W^{ijkl}(W^p{}_{lk}{}^h - W^{ph}{}_{kl})W_{phji} \\ &= \tfrac{1}{2}W^{ijkl}W^{ph}{}_{kl}W_{phji}, \end{aligned}$$

which together with (2.8)–(2.10) yields (2.7). ∎

Remark 2.2. Formula (2.7) can also be derived from [Si] where Singer computed the Laplacian of the Weyl curvature tensor. Here we give its proof for completeness and because it seems to be of independent interest.

Making use of the $*$-notation, we can rewrite (2.7) as

$$W^{ijkl}\Delta W_{ijkl} = \frac{1}{2}\Delta|W_{ijkl}|^2 - |\nabla W|^2 = -W * W * W + \frac{2R}{n}|W|^2, \tag{2.11}$$

where $W * W * W$ denotes the cubic terms of W, which is bounded by $c|W|^3$ for a positive constant c that depends on n. This together with (2.11) gives

$$W^{ijkl}\Delta W_{ijkl} \geq \frac{2R}{n}|W|^2 - c|W|^3. \tag{2.12}$$

3. Proof of Theorem 1.1. We need to prove $|W| = 0$. For simplicity of notation, we write $u = |W|$. By the assumption of cyclic parallel Ricci curvature, the refined Kato inequality $|\nabla W|^2 \geq \frac{n+1}{n-1}\big|\nabla|W|\big|^2$ for any Einstein manifold (see [Br]), and (2.12), we have

$$\begin{aligned} u\Delta u = |W|\Delta|W| &= W^{ijkl}\Delta W_{ijkl} + |\nabla W|^2 - \big|\nabla|W|\big|^2 \\ &\geq W^{ijkl}\Delta W_{ijkl} + \frac{2}{n-1}\big|\nabla|W|\big|^2 \\ &\geq \frac{2R}{n}u^2 - cu^3 + \frac{2}{n-1}|\nabla u|^2. \end{aligned} \tag{3.1}$$

For any $a > \frac{n-3}{2(n-1)}$, it follows from (3.1) and the formula $|\nabla u|^2 = \frac{|\nabla(u^a)|^2}{a^2u^{2a-2}}$ that

$$\begin{aligned} u^a\Delta(u^a) &= a(a-1)u^{2a-2}|\nabla u|^2 + au^{2a-1}\Delta u \\ &= \frac{a-1}{a}|\nabla(u^a)|^2 + au^{2a-2}u\Delta u \\ &\geq \left(1 - \frac{1}{a} + \frac{2}{(n-1)a}\right)|\nabla(u^a)|^2 - acu^{2a+1} + \frac{2aR}{n}u^{2a}. \end{aligned} \tag{3.2}$$

Assume that ϕ is a smooth compactly supported function on M^n. On the one hand, multiplying (3.2) by ϕ^2 and integrating over M^n gives

$$\begin{aligned} &\left(1 - \frac{1}{a} + \frac{2}{(n-1)a}\right)\int_{M^n}\phi^2|\nabla(u^a)|^2\,dV_g \\ &\quad\leq ac\int_{M^n}\phi^2u^{2a+1}\,dV_g + \int_{M^n}\phi^2u^a\Delta(u^a)\,dV_g - \frac{2a}{n}\int_{M^n}\phi^2Ru^{2a}\,dV_g \\ &\quad\leq ac\int_{M^n}\phi^2u^{2a+1}\,dV_g - 2\int_{M^n}\phi u^a\nabla\phi\cdot\nabla(u^a)\,dV_g \\ &\qquad - \int_{M^n}\phi^2|\nabla(u^a)|^2\,dV_g - \frac{2a}{n}\int_{M^n}\phi^2Ru^{2a}\,dV_g. \end{aligned}$$

Employing the Young inequality $\xi a^2 + b^2/\xi \geq 2ab$ for all positive constants a, b and ξ, we obtain

$$\begin{aligned} &\left(1 - \frac{1}{a} + \frac{2}{(n-1)a}\right)\int_{M^n}\phi^2|\nabla(u^a)|^2\,dV_g \\ &\qquad\leq ac\int_{M^n}\phi^2u^{2a+1}\,dV_g + (\xi - 1)\int_{M^n}\phi^2|\nabla(u^a)|^2\,dV_g \\ &\qquad\quad + \frac{1}{\xi}\int_{M^n}|\nabla\phi|^2u^{2a}\,dV_g - \frac{2a}{n}\int_{M^n}\phi^2Ru^{2a}\,dV_g. \end{aligned}$$

Therefore,

$$\left(2-\frac{n-3}{(n-1)a}-\xi\right)\int\limits_{M^n}\phi^2|\nabla(u^a)|^2\,dV_g \le ac\int\limits_{M^n}\phi^2u^{2a+1}\,dV_g+\frac{1}{\xi}\int\limits_{M^n}|\nabla\phi|^2u^{2a}\,dV_g-\frac{2a}{n}\int\limits_{M^n}\phi^2Ru^{2a}\,dV_g. \tag{3.3}$$

In the rest of the proof we choose $0<\xi<2-\frac{n-3}{(n-1)a}$ so that $2-\frac{n-3}{(n-1)a}-\xi>0$.

On the other hand,

$$\begin{aligned}\int\limits_{M^n}&\left(|\nabla(\phi u^a)|^2+\frac{n-2}{4(n-1)}R\phi^2u^{2a}\right)dV_g\\ &\le\int\limits_{M^n}\bigg(\phi^2|\nabla(u^a)|^2+u^{2a}|\nabla\phi|^2+2\phi u^a\nabla\phi\cdot\nabla(u^a)\\ &\qquad\qquad+\frac{n-2}{4(n-1)}R\phi^2u^{2a}\bigg)\,dV_g\\ &\le\int\limits_{M^n}\bigg((1+\eta)\phi^2|\nabla(u^a)|^2+(1+1/\eta)u^{2a}|\nabla\phi|^2\\ &\qquad\qquad+\frac{n-2}{4(n-1)}R\phi^2u^{2a}\bigg)\,dV_g,\end{aligned} \tag{3.4}$$

where $\eta>0$ is a constant.

Substituting (3.3) into (3.4), we get

$$\begin{aligned}\int\limits_{M^n}&\left(|\nabla(\phi u^a)|^2+\frac{n-2}{4(n-1)}R\phi^2u^{2a}\right)dV_g\\ &\le\frac{(1+\eta)ac}{2-\frac{n-3}{(n-1)a}-\xi}\int\limits_{M^n}\phi^2u^{2a+1}\,dV_g\\ &\quad+\left(1+\frac{1}{\eta}+\frac{1+\eta}{\big(2-\frac{n-3}{(n-1)a}-\xi\big)\xi}\right)\int\limits_{M^n}|\nabla\phi|^2u^{2a}\,dV_g\\ &\quad+\left(\frac{n-2}{4(n-1)}-\frac{2(1+\eta)a}{n\big(2-\frac{n-3}{(n-1)a}-\xi\big)}\right)\int\limits_{M^n}R\phi^2u^{2a}\,dV_g\\ &\triangleq c_2\int\limits_{M^n}\phi^2u^{2a+1}\,dV_g+c_3\int\limits_{M^n}|\nabla\phi|^2u^{2a}\,dV_g+c_4\int\limits_{M^n}R\phi^2u^{2a}\,dV_g,\end{aligned} \tag{3.5}$$

where

$$c_1=2-\frac{n-3}{(n-1)a}-\xi>0,\qquad c_2=\frac{(1+\eta)ac}{c_1}>0,$$
$$c_3=1+\frac{1}{\eta}+\frac{1+\eta}{c_1\xi}>0,\qquad c_4=\frac{n-2}{4(n-1)}-\frac{2a(1+\eta)}{nc_1}.$$

We now divide our proof into the following two cases:

CASE (i): $R < 0$. It is easily seen that

$$c_4 = \frac{n-2}{4(n-1)} - \frac{2a(1+\eta)}{nc_1} = \frac{n(n-2)c_1 - 8(n-1)(1+\eta)a}{4n(n-1)c_1} > 0$$

if and only if

$$8(n-1)(1+\eta)a^2 - n(n-2)(2-\xi)a + \frac{n(n-2)(n-3)}{n-1} < 0,$$

which is equivalent to

$$\begin{aligned} a \in &\Big(\frac{n(n-2) - \sqrt{n(n-2)(n^2-10n+24)}}{8(n-1)}, \\ &\qquad \frac{n(n-2) + \sqrt{n(n-2)(n^2-10n+24)}}{8(n-1)}\Big) \\ = &\Big(\frac{(n-3)\sqrt{n(n-2)}}{(n-1)(\sqrt{n(n-2)} + \sqrt{n^2-10n+24})}, \\ &\qquad \frac{(n-3)\sqrt{n(n-2)}}{(n-1)(\sqrt{n(n-2)} - \sqrt{n^2-10n+24})}\Big) \\ = &\tfrac{1}{2}I(n) \end{aligned}$$

for sufficiently small positive constants η and ξ. Under the assumption that $n \geq 7$, we check at once that

$$\frac{(n-3)\sqrt{n(n-2)}}{(n-1)(\sqrt{n(n-2)} + \sqrt{n^2-10n+24})} > \frac{n-3}{2(n-1)}.$$

By choosing η and ξ are sufficiently small, we can choose $2a \in I(n)$ such that $c_4 > 0$. This together with (3.5), the definition of $Q(M^n, g)$ and $R < 0$ yields

$$\begin{aligned} (3.6) \qquad &Q(M^n, g)\Big(\int_{M^n} (\phi u^a)^{\frac{2n}{n-2}}\, dV_g\Big)^{\frac{n-2}{n}} \\ &\leq \int_{M^n} \Big(|\nabla(\phi u^a)|^2 + \frac{n-2}{4(n-1)} R\phi^2 u^{2a}\Big)\, dV_g \\ &\leq c_2 \int_{M^n} \phi^2 u^{2a+1}\, dV_g + c_3 \int_{M^n} |\nabla\phi|^2 u^{2a}\, dV_g \\ &\leq c_2 \Big(\int_{M^n} (\phi u^a)^{\frac{2n}{n-2}}\, dV_g\Big)^{\frac{n-2}{n}} \Big(\int_{M^n} u^{n/2}\, dV_g\Big)^{2/n} + c_3 \int_{M^n} |\nabla\phi|^2 u^{2a}\, dV_g. \end{aligned}$$

The assumption that $\int_{M^n} (|\mathring{Rc}|^{n/2} + |W|^{n/2})\, dV_g \leq C_1$ now leads to the inequality $\int_{M^n} u^{n/2}\, dV_g \leq C_1$, hence the first term on the right-hand side of

(3.6) can be absorbed in the left-hand side. Therefore, there exists a constant $C > 0$, depending on n and $Q(M^n, g)$, such that

$$C\Big(\int_{M^n} (\phi u^a)^{\frac{2n}{n-2}}\, dV_g\Big)^{\frac{n-2}{n}} \le \int_{M^n} |\nabla\phi|^2 u^{2a}\, dV_g. \tag{3.7}$$

Let $B_r = \{x \in M^n : d(x, x_0) \le r\}$ for some fixed $x_0 \in M^n$ and choose ϕ with

$$\phi = \begin{cases} 1 & \text{on } B_r, \\ 0 & \text{on } M^n - B_{2r}, \end{cases} \qquad |\nabla\phi| \le 2/r \quad \text{on } B_{2r} - B_r,$$

and $0 \le \phi \le 1$. From (3.7), we get

$$\begin{aligned} C\Big(\int_{M^n} (\phi u^a)^{\frac{2n}{n-2}}\, dV_g\Big)^{\frac{n-2}{n}} &\le \int_{B_{2r}-B_r} |\nabla\phi|^2 u^{2a}\, dV_g \triangleq \int_{B_{2r}-B_r} |\nabla\phi|^2 u^{\beta}\, dV_g \\ &\le \frac{4}{r^2}\int_{M^n} u^{\beta}\, dV_g. \end{aligned} \tag{3.8}$$

Letting $r \to \infty$, by the assumption that $\int_{M^n} u^\beta\, dV_g$ is finite and (3.8), we obtain $u = |W| = 0$.

Case (ii): $R = 0$. By the assumption of $n \ge 3$, for any positive constant η, we can choose $a = n/4 > 1/2$, $\xi = 1/(n(n-1)) > 0$ such that c_1 is positive. This together with (3.5) and $R = 0$ yields

$$\begin{aligned} Q(M^n, g)&\Big(\int_{M^n} (\phi u^a)^{\frac{2n}{n-2}}\, dV_g\Big)^{\frac{n-2}{n}} \\ &= Q(M^n, g)\Big(\int_{M^n} (\phi u^{n/4})^{\frac{2n}{n-2}}\, dV_g\Big)^{\frac{n-2}{n}} \\ &\le \int_{M^n}\Big(|\nabla(\phi u^{n/4})|^2 + \frac{n-2}{4(n-1)} R\phi^2 u^{n/2}\Big)\, dV_g \\ &\le c_2\Big(\int_{M^n} (\phi u^{n/4})^{\frac{2n}{n-2}}\, dV_g\Big)^{\frac{n-2}{n}}\Big(\int_{M^n} u^{n/2}\, dV_g\Big)^{2/n} + c_3\int_{M^n} u^{n/2}|\nabla\phi|^2\, dV_g. \end{aligned} \tag{3.9}$$

Since $Q(M^n, g) > 0$ and $\int_{M^n} u^{n/2}\, dV_g$ is sufficiently small due to the assumption that $\int_{M^n}(|\mathring{Rc}|^{n/2} + |W|^{n/2})\, dV_g \le C_2$, the first term on the right-hand side of (3.9) can be absorbed in the left-hand side. Therefore, there exists a constant $c_5 > 0$, depending on n and $Q(M^n, g)$, such that

$$c_5\Big(\int_{M^n} (\phi u^{n/4})^{\frac{2n}{n-2}}\, dV_g\Big)^{\frac{n-2}{n}} \le \int_{M^n} u^{n/2}|\nabla\phi|^2\, dV_g.$$

The rest of the proof runs as before. ∎

Acknowledgements. The author would like to express his sincere thanks to the referees for their helpful comments and suggestions during the preparation of the paper, and is greatly indebted to Professor Yi Hua Deng for sharing his paper. This work was supported by grants of National Nature Science Foundation of China (No. 11371330) and Natural Science Foundation of Anhui Provincial Education Department (No. KJ2014A196).

References

[Be] A. L. Besse, *Einstein Manifolds*, Springer, Berlin, 2008.

[Br] T. Branson, *Kato constants in Riemannian geometry*, Math. Res. Lett. 7 (2000), 245–261.

[BV] P. Bueken and L. Vanhecke, *Three- and four-dimensional Einstein-like manifolds and homogeneity*, Geom. Dedicata 75 (1999), 123–136.

[Ca] G. Calvaruso, *Riemannian 3-metrics with a generic Codazzi Ricci tensor*, Geom. Dedicata 151 (2011), 259–267.

[Ch] Y. W. Chu, *Complete noncompact manifolds with harmonic curvature*, Front. Math. China 7 (2012), 19–27.

[Den] Y. H. Deng, *Rigidity of noncompact manifolds with cyclic parallel Ricci curvature*, Ann. Polon. Math. 112 (2014), 101–108.

[Der] A. Derdziński, *Classification of certain compact Riemannian manifolds with harmonic curvature and non-parallel Ricci tensor*, Math. Z. 172 (1980), 273–280.

[G] A. Gray, *Einstein-like manifolds which are not Einstein*, Geom. Dedicata 7 (1978), 259–280.

[H] G. Huisken, *Ricci deformation of the metric on a Riemannian manifold*, J. Differential Geom. 21 (1985), 47–62.

[KK] U. H. Ki and Y. H. Kim, *Complete and non-compact conformally flat manifolds with constant scalar curvature*, Geom. Dedicata 40 (1991), 45–52.

[K] S. Kim, *Rigidity of noncompact complete manifolds with harmonic curvature*, Manuscripta Math. 135 (2011), 107–116.

[LY] Q. C. Li and W. J. Yan, *On Ricci tensor of focal submanifolds of isoparametric hypersurfaces*, Sci. China Math. 58 (2015), 1723–1736.

[MDD] S. Mallick, A. De and U. C. De, *On generalized Ricci recurrent manifolds with applications to relativity*, Proc. Nat. Acad. Sci. India Sect. A Phys. Sci. 83 (2013), 143–152.

[PD] H. Pedersen and P. Tod, *The Ledger curvature conditions and D'Atri geometry*, Differential Geom. Appl. 11 (1999), 155–162.

[PQ] C. K. Peng and C. Qian, *Homogeneous Einstein-like metrics on spheres and projective spaces*, Differential Geom. Appl. 44 (2016), 63–76.

[Sc] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. 20 (1984), 479–495.

[Si] M. Singer, *Positive Einstein metrics with small $L^{n/2}$-norm of the Weyl tensor*, Differential Geom. Appl. 2 (1992), 269–274.

[TY] Z. Z. Tang and W. J. Yan, *Isoparametric foliation and a problem of Besse on generalizations of Einstein condition*, Adv. Math. 285 (2015), 1970–2000.

[T] N. S. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa 22 (1968), 265–274.

[Y] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. 12 (1960), 21–37.

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