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Calderón–Zygmund decompositions in tent spaces and weak-type endpoint bounds for two quadratic functionals of Stein and Fefferman–Stein

by

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Abstract. We prove some Calderón–Zygmund type decompositions for functions in the tent spaces introduced by Coifman, Meyer and Stein. These decompositions will be useful in the operator theory on tent spaces developed in the author's 2015 thesis. As an application of these decompositions to the study of quadratic functionals on tent spaces, we give a unified proof for tent space generalizations of C. Fefferman's endpoint weak-type estimates for grand square functions and of C. Fefferman and Stein's endpoint weak-type estimates for box maximal functions.

1. Introduction. Let $\mathbb{R}_+^{1+n} = \mathbb{R}_+ \times \mathbb{R}^n$ with $\mathbb{R}_+ = (0,\infty)$. For $t > 0$ and $y \in \mathbb{R}^n$, (t,y) denotes a point in $\mathbb{R}_+^{1+n}$. Let $B(x,t) = \{y \in \mathbb{R}^n \mid |x-y| < t\}$ be the open ball centered at $x \in \mathbb{R}^n$ and of radius $t > 0$.

For $0 < p \le \infty$, let $\|\cdot\|_p$ be the $L^p(\mathbb{R}^n)$ quasi-norm. Denote by $L^2_{\mathrm{loc}}(\mathbb{R}_+^{1+n})$ the collection of all locally square integrable functions in $\mathbb{R}_+^{1+n}$. For $0 < p < \infty$ and $\alpha > 0$, we say an $L^2_{\mathrm{loc}}(\mathbb{R}_+^{1+n})$ function f belongs to the *α-aperture tent space* ${}^{\alpha}T^p_2$ if

$$\|f\|_{{}^{\alpha}T^p_2} := \|\mathcal{A}^{(\alpha)}(f)\|_p < \infty,$$

where

$$\mathcal{A}^{(\alpha)}(f)(x) := \Bigg(\iint\limits_{\mathbb{R}_+^{1+n}} \frac{1_{B(x,\alpha t)}(y)}{t^n} |f(t,y)|^2 \, \frac{dt\,dy}{t}\Bigg)^{1/2}, \qquad x \in \mathbb{R}^n.$$

Note that the scale of α-aperture tent spaces ${}^{\alpha}T^p_2$ has equivalent quasi-norms for different α, that is, for any $L^2_{\mathrm{loc}}(\mathbb{R}_+^{1+n})$ function f we have

$$\|f\|_{{}^{\alpha}T^p_2} \simeq \|f\|_{{}^{\beta}T^p_2}, \qquad 0 < p < \infty,\ 0 < \alpha, \beta < \infty. \tag{1.1}$$

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This change of aperture result can be found, for example, in [CMS, Tb]. We omit the aperture parameter α in $\mathcal{A}^{(\alpha)}$ and ${}^{\alpha}T_2^p$ if $\alpha = 1$. Let

$$\widehat{\Omega} := \{(t,y) \in \mathbb{R}_+^{1+n} \mid B(y,t) \subset \Omega\}$$

be the tent over the set $\Omega \subset \mathbb{R}^n$. Let $|E|$ be the volume of the set E in $\mathbb{R}^n$. Then we say an $L^2_{\mathrm{loc}}(\mathbb{R}_+^{1+n})$ function f is in the tent space T_2^∞ if

$$\|f\|_{T_2^\infty} := \|\mathcal{C}(f)\|_\infty < \infty,$$

where

$$\mathcal{C}(f)(x) := \sup_{B \ni x} \bigg(\frac{1}{|B|} \iint\limits_{\widehat{B}} |f(t,y)|^2 \, \frac{dt\,dy}{t} \bigg)^{1/2}, \quad x \in \mathbb{R}^n.$$

Here the supremum is taken over all balls in $\mathbb{R}^n$ which contain x. Note that T_2^p, $0 < p \le \infty$, is the scale of tent spaces introduced in [CMS].

In this paper we give some Calderón–Zygmund type decompositions for functions in T_2^p, $0 < p < \infty$. Let $\mathbb{N}^*$ be the set of integers ≥ 1.

Theorem 1.1. *For any* $f \in T_2^p$, $0 < p < \infty$, *and any* $l > 0$, *there exists* $C = C(n,p) > 0$ *such that we can always find a family* $\{B_i\}_{i \in \mathbb{N}^*}$ *of balls in* $\mathbb{R}^n$ *and a Calderón–Zygmund type decomposition*

$$f = g + \sum_{i \in \mathbb{N}^*} b_i,$$

with $\operatorname{supp} b_i \subset \widehat{B}_i$, *such that*

$$\|g\|_{T_2^\infty} \le Cl, \tag{1.2}$$

$$\|b_i\|_{T_2^p}^p \le Cl^p |B_i|, \tag{1.3}$$

$$\sum_{i \in \mathbb{N}^*} |B_i| \le Cl^{-p} \|f\|_{T_2^p}^p. \tag{1.4}$$

Moreover, the supports of b_i *are mutually disjoint.*

The number $l > 0$ involved in the above theorem is often called the *height* for the corresponding Calderón–Zygmund decomposition.

Let $\lambda > 1$. Define the *grand square function* of Stein type as

$$\mathcal{S}_\lambda^*(f)(x) := \bigg(\iint\limits_{\mathbb{R}_+^{1+n}} \Big(\frac{t}{|x-y| + t} \Big)^{\lambda n} |f(t,y)|^2 \, \frac{dt\,dy}{t^{n+1}} \bigg)^{1/2}, \quad x \in \mathbb{R}^n.$$

It can be easily verified that $\mathcal{S}_\lambda^*$ is $T_2^2 \to L^2$ bounded as $\lambda > 1$. For f_0 in the Hardy space $H^p(\mathbb{R}^n)$, $0 < p < \infty$, and $P_t(x)$ being the Poisson kernel $t/(|x|^2 + t^2)^{(n+1)/2}$, one has $f = t\nabla_{t,y}(f_0 * P_t) \in T_2^p$. Moreover, the grand square function

$$g_\lambda^*(f_0) := \mathcal{S}_\lambda^*(f),$$

first studied by A. Zygmund and E. Stein, satisfies the strong type estimate

$$\|g_\lambda^*(f_0)\|_p = \|\mathcal{S}_\lambda^*(t\nabla_{t,y}(f_0 * P_t))\|_p \lesssim \|\mathcal{A}(t\nabla_{t,y}(f_0 * P_t))\|_p \tag{1.5}$$

if $\lambda > \max(1, 2/p)$ (see [S, Theorem 2, p. 91]). This result has a natural tent space generalization: for $0 < p < \infty$ and f locally square integrable in $\mathbb{R}_+^{1+n}$, one has

$$\|\mathcal{S}_\lambda^*(f)\|_p \lesssim \|\mathcal{A}(f)\|_p \tag{1.6}$$

if $\lambda > \max(1, 2/p)$ (see for example [A]). As suggested in [A], the endpoint case of (1.6) for $\lambda > 1$ and $p = 2/\lambda$ corresponds to a tent space generalization of the endpoint case of (1.5) for g_λ^*, the latter proved in C. Fefferman's thesis [Fa] (see [Fb, Theorem 1]).

Let $L^{p,\infty}$, $0 < p < \infty$, be the Lorentz space in $\mathbb{R}^n$. As application of Theorem 1.1, our first result is the following weak-type estimate for the quadratic functional $\mathcal{S}_\lambda^*$.

Corollary 1.2. *Let* $\lambda > 1$. *Then* $\mathcal{S}_\lambda^*$ *is* $T_2^{2/\lambda} \to L^{2/\lambda,\infty}$ *bounded.*

Remark 1.3. It would also be interesting to know if one can obtain for $\lambda > 1$ the Lorentz space estimate $\mathcal{S}_\lambda^* : T_2^{2/\lambda} \to L^{2/\lambda,2}$. The motivation of this question comes from [ST].

Corollary 1.2 has a natural companion. Let $\lambda > 1$. Define the *box maximal function* of Fefferman–Stein type by setting, for $x \in \mathbb{R}^n$,

$$\mathcal{C}_\lambda^*(f)(x) := \sup_{r>0} \biggl(\frac{1}{|B(x,r)|^\lambda} \iint\limits_{(0,r)\times B(x,r)} t^{\lambda n - n} |f(t,y)|^2 \,\frac{dt\,dy}{t} \biggr)^{1/2}.$$

See [FS, Lemmas 8 and 9] for more information.

Our second result is the following weak-type estimate for $\mathcal{C}_\lambda^*$.

Corollary 1.4. *Let* $\lambda > 1$. *Then* $\mathcal{C}_\lambda^*$ *is* $T_2^{2/\lambda} \to L^{2/\lambda,\infty}$ *bounded.*

It can be verified that $\mathcal{C}_\lambda^*$ is $T_2^2 \to L^{2,\infty}$ bounded for $\lambda > 1$. This uses the pointwise relation

$$\mathcal{C}_\lambda^*(f)(x) \lesssim \mathcal{C}(f)(x), \qquad \forall x \in \mathbb{R}^n,$$

together with the weak-type estimate $\mathcal{C} : T_2^2 \to L^{2,\infty}$ which was given in [CMS, proof of Theorem 3(b)] as a consequence of the maximal theorem.

Remark 1.5. Calderón–Zygmund type decompositions as in Theorem 1.1 are motivated by the operator theory on tent spaces developed in [H], where we established weak-type estimates for certain singular integral operators (including maximal regularity operators) on tent spaces.

Remark 1.6. The weak-type estimates in Corollaries 1.2 and 1.4 are not new. Corollary 1.2 and the corresponding weighted norm inequalities were first proved in [AS] (1) by using an integration lemma over the cones as in [CMS, Lemma 2], which requires certain nice geometric properties of $\mathbb{R}^n$. Our proof is direct and can possibly be extended to rough geometry. Corollary 1.4 was implied by some pointwise estimates in [CW] proved via Carleson measures. See also [B] and [Ta].

In proving the above corollaries, we shall also need the following L^2 estimate on the bad functions $\{b_i\}$ which arise from the Calderón–Zygmund decomposition.

Lemma 1.7. *Given any* $f \in T_2^p$ *with* $0 < p < \infty$, *let* $f = g + \sum_{i\in\mathbb{N}^*} b_i$ *be the Calderón–Zygmund decomposition associated to the height* $l > 0$ *as in Theorem* 1.1. *If* $0 < p < 2$ *and* $\lambda = 2/p > 1$, *we have the fractional integral estimate*

$$\iint\limits_{\widehat{B_i}} t^{\lambda n-n}|b_i(t,y)|^2\,\frac{dt\,dy}{t} \lesssim \|b_i\|_{T_2^p}^2 \lesssim l^2|B_i|^{2/p}. \tag{1.7}$$

Here $\{B_i\}_{i\in\mathbb{N}^*}$ *is the family of balls found in Theorem* 1.1.

Observe that the second inequality in (1.7) just rewrites (1.3) in the Calderón–Zygmund decompositions. It suffices to prove the first inequality.

The proof is essentially a general embedding estimate $T_2^p(\mathbb{R}_+^{1+n}) \hookrightarrow L^2(\mathbb{R}_+^{1+n}, t^{\lambda n-n-1}dtdy)$, with $0 < p < 2$. Thus the statement of this lemma can be more general, but this information will not be used in this paper. See [H, Chapter 2] for related embedding results.

2. Proof of Theorem 1.1.

Let $\mathbf{M}$ be the maximal function in $\mathbb{R}^n$, that is,

$$\mathbf{M}(h)(x) := \sup_{B\ni x}\frac{1}{|B|}\int\limits_B |h|, \qquad x\in\mathbb{R}^n.$$

Fix $\alpha > 7$. Let

$$\Omega_l = \{x\in\mathbb{R}^n \mid \mathbf{M}(\mathcal{A}^{(\alpha)}(f)^p)(x) > l^p\}.$$

By the maximal theorem and the lower semicontinuity of $\mathbf{M}(\mathcal{A}^{(\alpha)}(f)^p)$, we know that Ω_l is an open set of finite measure. Let $\Omega_l = \bigcup_{i\in\mathbb{N}^*} Q_i$ be a Whitney decomposition of Ω_l into cubes, and denote by $B_i = cQ_i$ the ball with the same center as Q_i and radius c times the diameter of Q_i. We choose c sufficiently large such that

$$\widehat{B}_i \supset \Delta_i := \widehat{\Omega}_l \cap (Q_i\times(0,\infty))$$

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(1) We thank Professor Pascal Auscher for pointing out this reference.

uniformly in $i \in \mathbb{N}^*$. Then we let

$$b_i = f|_{\Delta_i} \quad \text{and} \quad g = f - \sum_{i\in\mathbb{N}^*} b_i.$$

We claim that this yields the desired decomposition.

Since $\alpha > 7$, we claim that for any $x \in Q_i$, there exists $x_i \in \complement(\Omega_l)$, the complement of Ω_l in $\mathbb{R}^n$, such that

$$\mathcal{A}(g)(x) \le \mathcal{A}^{(\alpha)}(g)(x_i).$$

In fact, we can simply select x_i to be an arbitrary point in $5Q_i \cap \complement(\Omega_l)$. Recall that in the general Whitney decomposition arguments (see [S, Theorem 1, p. 167] for example) one has

$$\operatorname{diam}(Q_i) \le \operatorname{dist}(Q_i, \complement(\Omega_l)) \le 4 \operatorname{diam}(Q_i).$$

Hence $5Q_i \cap \complement(\Omega_l)$ is non-empty. With $r_i = \operatorname{diam}(Q_i)/2$, we also have

$$\operatorname{dist}(x_i, Q_i) \le 4r_i \quad \text{and} \quad \operatorname{dist}(Q_i, \complement(\Omega_l)) \ge 2r_i.$$

From geometrical observations, to meet

$$\sup_{x\in Q_i} \mathcal{A}(g)(x) \le \inf_{x_i \in 4Q_i \cap \complement(\Omega_l)} \mathcal{A}^{(\alpha)}(g)(x_i),$$

it suffices to take

$$\alpha > \frac{\operatorname{dist}(x_i, Q_i) + \operatorname{diam}(Q_i) + \frac{1}{2}\operatorname{dist}(Q_i, \complement(\Omega_l))}{\frac{1}{2}\operatorname{dist}(Q_i, \complement(\Omega_l))} = \frac{4r_i + 2r_i}{r_i} + 1 = 7. \tag{2.1}$$

This proves the claim. Note that this claim is uniform in $i \in \mathbb{N}^*$.

Now for the "good" part g, with $C = C(n,p)$ different at each step, we have

$$\begin{aligned}\|\mathcal{C}(g)\|_{L^\infty} &\le C\|\mathcal{A}(g)\|_{L^\infty} \le C\|\mathcal{A}(g)|_{\Omega_l}\|_{L^\infty} + C\|\mathcal{A}(g)|_{\complement(\Omega_l)}\|_{L^\infty}\\ &\le C\|\mathcal{A}^{(\alpha)}(f)|_{\complement(\Omega_l)}\|_{L^\infty} + C\|\mathcal{A}(f)|_{\complement(\Omega_l)}\|_{L^\infty}\\ &\le C\|\mathcal{A}^{(\alpha)}(f)|_{\complement(\Omega_l)}\|_{L^\infty} \le Cl.\end{aligned}$$

In the above estimates we mainly used, in order: in the first inequality the endpoint comparison of $\mathcal{A}$ and $\mathcal{C}$ at $p = \infty$ (see [CMS, Theorem 3(b)]); in the third inequality the claim just proved

$$\mathcal{A}(g)(x) \le \mathcal{A}^{(\alpha)}(g)(x_i) \le \mathcal{A}^{(\alpha)}(f)(x_i), \qquad \forall x \in Q_i,$$

and the construction $f|_{\complement(\widehat{\Omega_l})} = g$; in the fourth inequality the geometrical fact $\alpha > 1$; and in the fifth inequality the Lebesgue differentiation theorem applied to $\mathcal{A}^{(\alpha)}(f)^p$.

By similar geometrical observations to those in (2.1), there exists $k = k(n,c) \ge 5$ such that for any $i \in \mathbb{N}^*$, we have the α-aperture tent $\widehat{(kQ_i)}^\alpha \supset \widehat{B}_i$, where

$$\widehat{\Omega}^\alpha := \{(t,y) \in \mathbb{R}^{1+n}_+ \mid B(y,\alpha t) \subset \Omega\}.$$

Now for the "bad" part b, first we know that $\operatorname{supp} b_i \subset \widehat{B}_i$. With $C = C(n,p,c)$ different at each step, we can estimate

$$\|b_i\|_{T^p_2} \le \|f|_{\widehat{B}_i}\|_{T^p_2} \le C\|\mathcal{A}^{(\alpha)}(f|_{\widehat{B}_i})\|_p \le C\|\mathcal{A}^{(\alpha)}(f)|_{kQ_i}\|_p \le C\lambda|B_i|^{1/p}.$$

Here we mainly used, in order, $\alpha > 1$, $\widehat{(kQ_i)}^\alpha \supset \widehat{B}_i$ for $k \ge 5$ and the existence of $x_i \in kQ_i \cap \complement(\Omega_l)$, and the construction of Ω_l from the maximal function.

Moreover, by the maximal theorem

$$\sum_{i\in\mathbb{N}^*} |B_i| \le C \sum_{i\in\mathbb{N}^*} |Q_i| = C|\Omega_l| \le Cl^{-p}\|f\|^p_{{}^\alpha T^p_2} \le Cl^{-p}\|f\|^p_{T^p_2},$$

where $C = C(n,p,c)$. The last estimate used (1.1) (2) on the change of apertures.

Finally, using the fact that the Whitney cubes $\{Q_i\}$ are mutually disjoint (see [S, Theorem 1, p. 167]), we see that the supports of b_i are mutually disjoint.

3. Proofs of Corollaries 1.2 and 1.4

Proof of Corollary 1.2. Recall that $\mathcal{S}^*_\lambda$ is $T^2_2 \to L^2$ bounded when $\lambda > 1$. By density of $T^{2/\lambda}_2 \cap T^2_2$ in $T^{2/\lambda}_2$ it suffices to show, for any $f \in T^{2/\lambda}_2 \cap T^2_2$,

$$|\{x \in \mathbb{R}^n \mid \mathcal{S}^*_\lambda(f)(x) > l\}| \lesssim \frac{1}{l^{2/\lambda}} \int\limits_{\mathbb{R}^n} \mathcal{S}^*_\lambda(f)^{2/\lambda}(x)\,dx, \qquad \forall l > 0.$$

Let $f = g + \sum_{i\in\mathbb{N}^*} b_i$ be the Calderón–Zygmund decomposition associated to height $l > 0$, the Whitney cubes $\{Q_i\}_{i\in\mathbb{N}^*}$ and the balls $\{B_i\}_{i\in\mathbb{N}^*}$ as in Theorem 1.1, such that $g = f|_{\complement(\widehat{\Omega})}$ with $\Omega = \bigcup_i Q_i$, and $b_i = f|_{\triangle_i}$ with $\triangle_i = (Q_i \times (0,\infty)) \cap \widehat{\Omega} \subset \widehat{B}_i$.

By sublinearity of the quadratic functional $\mathcal{S}^*_\lambda$,

$$\mathcal{S}^*_\lambda(f)(x) \le \mathcal{S}^*_\lambda(g)(x) + \mathcal{S}^*_\lambda(f-g)(x) := G_1(x) + G_2(x), \qquad \forall x \in \mathbb{R}^n,$$

and it remains to check that G_k $(k=1,2)$ is in $L^{2/\lambda,\infty}$.

By $T^2_2 \to L^2$ boundedness of $\mathcal{S}^*_\lambda$, we have

$$|\{x \in \mathbb{R}^n \mid G_1(x) > l/2\}| \lesssim l^{-2}\|\mathcal{S}^*_\lambda(g)\|^2_{T^2_2} \lesssim l^{-2}\|g\|^2_{T^2_2}.$$

By the interpolation control of g from (1.2),

$$l^{-2}\|g\|^2_{T^2_2} \lesssim l^{-2/\lambda}\|f\|^{2/\lambda}_{T^{2/\lambda}_2}.$$

This shows that $G_1 \in L^{2/\lambda,\infty}$.

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(2) We remark that (1.1) can be proved by the vector-valued approach proposed in [HTV]. This allows us to obtain (1.1) without using [CMS, Lemma 2], the latter (as seen in [Tb]) actually leading to (1.1) with sharp dependence on the aperture.

By the properties of the Calderón–Zygmund decomposition,

$$|\{x \in \mathbb{R}^n \mid G_2(x) > l/2\}| \lesssim l^{-2/\lambda} \|f\|_{T_2^{2/\lambda}}^{2/\lambda} + \Big|\Big\{x \in \mathbb{R}^n \setminus \bigcup_i 4B_i \;\Big|\; G_2(x) > l/2\Big\}\Big|.$$

Let $\Omega^* = \bigcup_i 4B_i$. Then for any $x \in \mathbb{R}^n \setminus \Omega^*$, we have $|x - y_i| \sim |x - y|$, where y_i denotes the center of Q_i and B_i, and y is any point in Q_i. Therefore, for any $x \in \mathbb{R}^n \setminus \Omega^*$,

$$\begin{aligned} G_2^2(x) &= \iint\limits_{\widehat{\Omega}} \Big(\frac{t}{|x-y|+t}\Big)^{\lambda n} \Big|\sum_{i \in \mathbb{N}^*} b_i(t,y)\Big|^2 \frac{dt\,dy}{t^{n+1}} \\ &= \sum_{i \in \mathbb{N}^*} \iint\limits_{\triangle_i} \Big(\frac{t}{|x-y|+t}\Big)^{\lambda n} |b_i(t,y)|^2 \frac{dt\,dy}{t^{n+1}} \\ &\lesssim \sum_{i \in \mathbb{N}^*} \frac{1}{|x-y_i|^{\lambda n}} \iint\limits_{\triangle_i} t^{\lambda n - n} |b_i(t,y)|^2 \frac{dt\,dy}{t} \\ &\lesssim \sum_{i \in \mathbb{N}^*} \frac{\|b_i\|_{T_2^{2/\lambda}}^2}{|x-y_i|^{\lambda n}} \lesssim \sum_{i \in \mathbb{N}^*} \frac{l^2 |B_i|^\lambda}{|x-y_i|^{\lambda n}}, \end{aligned}$$

where we have used Lemma 1.7 in the last two estimates. In the second equality above, we have also used the fact that the supports of the bad functions $\{b_i\}$ are mutually disjoint, which is guaranteed by the Calderón–Zygmund decomposition.

Then it remains to show

$$|\{x \in \mathbb{R}^n \setminus \Omega^* \mid H(x) > 1\}| \lesssim l^{-2/\lambda} \|f\|_{T_2^{2/\lambda}}^{2/\lambda}$$

with

$$H(x) := \sum_{i \in \mathbb{N}^*} \frac{|B_i|^\lambda}{|x-y_i|^{\lambda n}}.$$

However, by the Chebyshev inequality for $H(x)$ restricted to $\mathbb{R}^n \setminus \Omega^*$, we have

$$\begin{aligned} |\{x \in \mathbb{R}^n \setminus \Omega^* \mid H(x) > 1\}| &\le \int\limits_{\mathbb{R}^n \setminus \Omega^*} H(x)\,dx = \sum_{i \in \mathbb{N}^*} |B_i|^\lambda \int\limits_{\mathbb{R}^n \setminus \Omega^*} \frac{dx}{|x-y_i|^{\lambda n}} \\ &\le \sum_{i \in \mathbb{N}^*} |B_i|^\lambda \int\limits_{\mathbb{R}^n \setminus B_i} \frac{dx}{|x-y_i|^{\lambda n}} \lesssim \sum_{i \in \mathbb{N}^*} |B_i|. \end{aligned}$$

Thus, with the property (1.4) from the Calderón–Zygmund decomposition, we finish the proof that $G_2 \in L^{2/\lambda,\infty}$.

The proof of Corollary 1.2 can be concluded by combining the estimates just proved for G_1 and G_2. ∎

Proof of Corollary 1.4. Recall that we have the $T_2^2 \to L^{2,\infty}$ boundedness of $\mathcal{C}_\lambda^*$ when $\lambda > 1$. We examine the preceding arguments carried out for $\mathcal{S}_\lambda^*$, and we notice that in estimating the new version of G_2^2, which we can write as

$$G_2^2(x) = \sum_{i\in\mathbb{N}^*} \sup_{r>0} \frac{1}{r^{\lambda n}} \iint\limits_{\triangle_i\cap\{B(x,r)\times(0,r)\}} t^{\lambda n-n}|b_i(t,y)|^2\,\frac{dt\,dy}{t},$$

we always have $r \ge C|x-y_i|$ in each summand since for $x \in \mathbb{R}^n \setminus \Omega^*$ and $y \in B(x,r)$, $|x-y_i| \sim |x-y|$ and $|x-y| \le r$. Hence for $x \in \mathbb{R}^n \setminus \Omega^*$,

$$G_2^2(x) \lesssim \sum_{i\in\mathbb{N}^*} \sup_{r>0} \frac{1}{|x-y_i|^{\lambda n}} \iint\limits_{\triangle_i\cap\{B(x,r)\times(0,r)\}} t^{\lambda n-n}|b_i(t,y)|^2\,\frac{dt\,dy}{t}.$$

Furthermore, removing the restriction to $B(x,r)\times(0,r)$ in the integrals, we get

$$\begin{aligned} G_2^2(x) &\lesssim \sum_{i\in\mathbb{N}^*} \frac{1}{|x-y_i|^{\lambda n}} \iint\limits_{\triangle_i} t^{\lambda n-n}|b_i(t,y)|^2\,\frac{dt\,dy}{t} \\ &\lesssim \sum_{i\in\mathbb{N}^*} \frac{l^2|B_i|^\lambda}{|x-y_i|^{\lambda n}}, \qquad x \in \mathbb{R}^n \setminus \Omega^*. \end{aligned}$$

Note that this goes back to the step in the above proof for $\mathcal{S}_\lambda^*$.

The other arguments remain unchanged. ∎

Remark 3.1. In the classical setting for harmonic extensions, namely for $f = t\nabla_{t,y}(f_0 * P_t)$, both corollaries were proved in [MW] exploiting the pointwise relation between $\mathcal{S}_\lambda^*$ and $\mathcal{C}_\lambda^*$. Such relations are not true for general tent space functions. Here we give a unified proof of Corollaries 1.2 and 1.4 through our Calderón–Zygmund type decompositions in tent spaces, and this approach is close to the original spirit of [Fb] and [FS]. More precisely, see [Fb, pp. 20–21] and [FS, pp. 181–182]. Our arguments on G_2 in the above proofs reveal that, modulo the good function $g \in T_2^\infty$ in the Calderón–Zygmund type decomposition $f = g + b = g + \sum_{i\in\mathbb{N}^*} b_i$, the quadratic functionals $\mathcal{S}_\lambda^*(b)(x)$ and $\mathcal{C}_\lambda^*(b)(x)$ have comparable upper bounds when the cone with vertex x does not intersect the support of the bad function b.

4. Proof of Lemma 1.7. First, by a straightforward extension (3) of [CMS, Theorem 1(c)], the bad function $b_i \in T_2^p$, $0 < p < 2$, admits an atomic decomposition, say

$$b_i = \sum_j \lambda_{ij} b_{ij},$$

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(3) See for example [V, p. 52] for such a precise statement. Be aware that the setup of [V] is quite different from that of [CMS].

where the atom b_{ij} is supported in the tent $\widehat{B}_{ij}$ over some ball $B_{ij} \subset \mathbb{R}^n$ and b_{ij} also satisfies the size requirement

$$\|b_{ij}\|_{L^2(t^{-1}dtdy)} \le |B_{ij}|^{1-2/p}.$$

Moreover, the coefficients $\{\lambda_{ij}\}_j \in l^p$ satisfy

$$\|\{\lambda_{ij}\}_j\|_{l^p} \lesssim \|b_i\|_{T_2^p}. \tag{4.1}$$

We point out that the reverse $\|b_i\|_{T_2^p} \lesssim \|\{\lambda_{ij}\}_j\|_{l^p}$ is valid only for $p \le 1$.

Then we note that the decomposition equality $b_i = \sum_j \lambda_{ij} b_{ij}$ holds in pointwise sense. This follows by inspection of the proof of [CMS, Theorem 1(c)] (see also [R] for more precise arguments).

Next it suffices to prove that the fractional integral estimate as in (1.7) holds uniformly on the atoms b_{ij}. Now we explain this in detail.

Note that $t \le r_{B_{ij}}$, the radius of the ball B_{ij}, and $\lambda n - n > 0$. The verification of the fractional integral estimate on atoms is as follows:

$$\iint\limits_{\widehat{B}_{ij}} t^{\lambda n-n}|b_{ij}(t,y)|^2 \,\frac{dt\,dy}{t} \lesssim (r_{B_{ij}})^{\lambda n-n}|B_{ij}|^{1-2/p} \lesssim 1.$$

Hence, as b_{ij} have disjoint supports (again by inspection of the proof of [CMS, Theorem 1(c)]), we get

$$\begin{aligned}\iint\limits_{\triangle_i} t^{\lambda n-n}|b_i(t,y)|^2 \,\frac{dt\,dy}{t} &= \iint\limits_{\triangle_i} t^{\lambda n-n}\Big|\sum_j \lambda_{ij} b_{ij}(t,y)\Big|^2 \,\frac{dt\,dy}{t}\\ &\le \sum_j \lambda_{ij}^2 \iint\limits_{\widehat{B}_{ij}} t^{\lambda n-n}|b_{ij}(t,y)|^2\,\frac{dt\,dy}{t} \lesssim \|\{\lambda_{ij}^2\}_j\|_{l^1}.\end{aligned}$$

Since $p/2 = 1/\lambda < 1$, we have

$$\|\{\lambda_{ij}^2\}_j\|_{l^1} \le \|\{\lambda_{ij}^2\}_j\|_{l^{p/2}} = \|\{\lambda_{ij}\}_j\|_{l^p}^2.$$

Combining this with (4.1), we finish the proof of the lemma.

Remark 4.1. It may come as a surprise that in the proof of Lemma 1.7, which is used to prove the weak-type bounds in Corollaries 1.2 and 1.4 via Calderón–Zygmund decompositions in tent spaces, we have in addition to resort to atomic decompositions in tent spaces. This is a very interesting function space property of tent spaces. In this respect, see also [H, Chapter 2] for related results concerning singular integral operators on tent spaces.

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