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Metric characterizations of super weakly compact operators

by

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Abstract. We define the notion of factorization of a family of metric spaces through a bounded, linear operator between Banach spaces. This notion serves as the analogue of uniform bi-Lipschitz embeddings of this family of metric spaces into a given Banach space. We prove operator versions of well-known non-linear characterizations of superreflexivity due to Bourgain, Johnson–Schechtman, and Baudier. More precisely, we give a non-linear characterization of non-super weakly compact operators as those through which the binary tree, diamond, and Laakso graphs may be factored with uniform distortion.

1. Introduction. In [12] Ribe showed that any two uniformly homeomorphic Banach spaces must be finitely representable in each other. This result suggests the principle at the heart of the Ribe program, which is that a given local property of Banach spaces can be formulated as a purely metric property. One important result in this direction is Bourgain's characterization of superreflexive Banach spaces as those into which one can embed the binary trees of arbitrary depth with uniform distortion [4]. Using other metric spaces besides binary trees, Johnson and Schechtman [8] and Baudier [1] produced further metric characterizations of superreflexive spaces. Johnson and Schechtman used the class of diamond graphs and the class of Laakso graphs. Baudier used a single binary tree of infinite depth. Chávez-Domínguez [6] initiated the extension of the Ribe program to the classification of operators rather than Banach spaces, and extended a remarkable number of non-linear results to operators.

The main result of this work is to continue the Ribe program for maps by generalizing known characterizations of superreflexive spaces to characterize super weakly compact operators. To that end, rather than studying bi-Lipschitz maps of metric spaces into Banach spaces and estimating the

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distortion, we define a quantification of how well a given metric space can be preserved through a linear operator. When the operator in question is the identity of a Banach space, this quantification recovers the notion of Lipschitz distortion, so that our results do indeed generalize the works of Bourgain, Johnson–Schechtman, and Baudier. We say that a class $\mathcal{M}$ of metric spaces *factors through* the linear operator $A : X \to Y$ if the members of $\mathcal{M}$ can be uniformly preserved through A, in a sense which is formally defined in Section 2. The main result of this work is the following.

THEOREM 1. *Let $A : X \to Y$ be an operator and let $\mathcal{M}$ be the class of binary trees, diamond graphs, Laakso graphs, or the binary tree of infinite depth. Then $\mathcal{M}$ factors through A if and only if A fails to be super weakly compact.*

The idea of a superproperty of an operator is due to Pietsch [11]. Recall that for a property P, an operator $A : X \to Y$ has *super P* if for every ultrafilter $\mathcal{U}$, the induced operator $A_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ has property P. Thus $A : X \to Y$ is super weakly compact if for every ultrafilter $\mathcal{U}$, $A_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ is weakly compact. In the case that A is the identity of X, this recovers superreflexivity of X.

2. Notation and terminology. Given Banach spaces $Y \subset X$, we write $Y \leq X$ to mean that Y is a closed subspace of X.

For any set Λ, we let $\Lambda^{<\mathbb{N}}$ denote the finite sequences in Λ. For any $s \in \Lambda^{<\mathbb{N}}$, we let $|s|$ denote the length of s. For any $s \in \Lambda^{<\mathbb{N}}$ and any integer i with $0 \leq i \leq |s|$, we let $s|_i$ denote the initial segment of s having length i. We write $s \prec t$ if s is a proper initial segment of t. If $s, t \in \Lambda^{<\mathbb{N}}$, we let $s^\frown t$ denote the concatenation of s with t. If s is a non-empty sequence, we let s^- denote the maximal proper initial segment of s.

Given Banach spaces X, Y and an operator $A : X \to Y$, and a number $D \geq 1$, we say that a family $\mathcal{M}$ of metric spaces *D-factors through* A if for each $(M, d) \in \mathcal{M}$, there exists a function $f : (M, d) \to X$ such that for all $s, t \in M$,

$$D^{-1} d(s,t) \leq \|Af(s) - Af(t)\| \quad \text{and} \quad \|f(s) - f(t)\| \leq d(s,t).$$

We say $\mathcal{M}$ *factors through* A if it D-factors through A for some $D \geq 1$. If $\mathcal{M} = \{(M, d)\}$, we will say (M, d) factors through A if $\mathcal{M}$ factors through A.

Of course, $\mathcal{M}$ factors through an operator A if and only if $\mathcal{M}$ factors through any non-zero multiple of A, so if we are not concerned with the constant D, we may assume $\|A\| \leq 1$. In this case, in order to see that $\mathcal{M}$ factors through A, it is sufficient to prove that there exists $D \geq 1$ such that for each $(M, d) \in \mathcal{M}$, there exists $f : M \to X$ such that for all $s, t \in M$, $\|f(s) - f(t)\| \leq d(s, t)$ and $D^{-1} d(s, t) \leq \|Af(s) - Af(t)\|$. It is easy to see that if A is the identity operator on X, then $\mathcal{M}$ factors through A if and only if the members of $\mathcal{M}$ admit uniformly bi-Lipschitz embeddings into X.

3. Metric characterization of super weakly compact operators

3.1. Basics of super weakly compact operators. Recall that an operator $A : X \to Y$ is *super weakly compact* if and only if for any ultrafilter $\mathcal{U}$, the ultrapower $A_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ is weakly compact. Beauzamy introduced super weakly compact operators in [3] where they are called *opérateurs uniformément convexifiants.*

We will use the following facts, due to James. The first part of the following lemma is well-known, while the quantifications in the latter part are implicit in James's result. For the convenience of the reader, we sketch the proof.

LEMMA 2 ([7]). *Let $A : X \to Y$ be an operator. Let*

$$R = R(A) = \sup\{\|y^{**}\|_{Y^{**}/Y} : y^{**} \in \overline{AB_X}^{w^*}\},$$

where Y is identified with a subspace of Y^{**} and the w^*-closure is taken in Y^{**}. Then A fails to be weakly compact if and only if $R(A) > 0$, and in this case there exist $\theta = \theta(R) > 0$, $b = b(R) \geq 1$, and $(x_n) \subset B_X$ such that (Ax_n) is b-basic and every convex combination of (Ax_n) has norm at least θ.

Sketch of proof. Let $j_X : X \to X^{**}$ and $j_Y : Y \to Y^{**}$ denote the canonical inclusions. Note that $j_Y A = A^{**} j_X$. Then $A^{**}B_{X^{**}} = \overline{j_Y AB_X}^{w^*}$. Indeed, since $j_Y AB_X \subset A^{**}B_{X^{**}}$ and $A^{**}B_{X^{**}}$ is w^*-closed, we have $\overline{j_Y AB_X}^{w^*} \subset A^{**}B_{X^{**}}$. Now if $x^{**} \in B_{X^{**}}$, by Goldstine's theorem there exists a net $(x_\lambda) \subset B_X$ such that $j_X x_\lambda \underset{w^*}{\longrightarrow} x^{**}$. Then

$$j_Y AB_X \ni j_Y Ax_\lambda = A^{**} j_X x_\lambda \underset{w^*}{\longrightarrow} A^{**}x^{**},$$

yielding

$$A^{**}B_{X^{**}} \subset \overline{j_Y AB_X}^{w^*}.$$

If A is weakly compact, then the weak-w^*-continuity of j_Y implies that $j_Y(\overline{AB_X})$ is w^*-compact, and therefore closed, so that $\overline{j_Y AB_X}^{w^*} \subset j_Y(\overline{AB_X}) \subset j_Y Y$, and $R(A) = 0$. Conversely, if $R(A) = 0$, then $A^{**}B_{X^{**}} \subset j_Y Y$. This means $\overline{j_Y AB_X}^{w^*} \subset j_Y Y$, and $j_Y^{-1} : \overline{j_Y AB_X}^{w^*} \to Y$ is a w^*-weak homeomorphism onto $\overline{AB_X}$. By w^*-compactness of $\overline{j_Y AB_X}^{w^*}$, we see that $\overline{AB_X}$ is weakly compact.

Since no confusion will arise, we dispense with writing j_X, j_Y in the remainder of the proof and identify X, Y with subspaces of X^{**}, Y^{**}, respectively. Now suppose A is not weakly compact. Fix $x^{**} \in B_{X^{**}}$ such that $\|A^{**}x^{**}\|_{Y^{**}/Y} > 0$ and fix a number ψ between 0 and $\|A^{**}x^{**}\|_{Y^{**}/Y}$. By replacing x^{**} with $(1-\delta)x^{**}$ for a suitably small $\delta > 0$ and relabeling, we may assume $\|x^{**}\| < 1$. By the Hahn–Banach theorem, there exists $y^{***} \in Y^{***}$ such that $y^{***}(A^{**}x^{**}) = 1$, $y^{***}|_Y \equiv 0$, and $\|y^{***}\| \le 1/\psi$. We may also fix $y^* \in Y^*$ such that $A^{**}x^{**}(y^*) = 1$ and $\|y^*\| \le 1/\psi$. Fix $\varepsilon > 0$. We may recursively choose $x_1, x_2, \ldots \in B_X$ and finite subsets $F_1, F_2, \ldots \subset B_{Y^*}$ such that for every $n \in \mathbb{N}$,

(i) for every $y \in \operatorname{span}\{Ax_1 - A^{**}x^{**}, \ldots, Ax_n - A^{**}x^{**}\}$,
$$\|y\| \le (1+\varepsilon) \max_{y_1^* \in F_n} |y_1^*(y)|,$$

(ii) for every $m > n$ and every $y_1^* \in F_n \cup \{y^*\}$, $y_1^*(Ax_m) = A^{**}x^{**}(y_1^*)$.

This recursion can be completed because, assuming that $x_1, \ldots, x_{n-1}$ and $F_1, \ldots, F_{n-1}$ have been chosen, we may apply Helly's theorem to choose x_n in B_X and then use compactness of the unit ball of $\operatorname{span}\{Ax_1 - A^{**}x^{**}, \ldots, Ax_n - A^{**}x^{**}\}$ to choose F_n. Now for any scalar sequence $(a_i)_{i=1}^n$ and any $1 \le m < n$,

$$\begin{aligned}\Big\|\sum_{i=1}^m a_i(Ax_i - A^{**}x^{**})\Big\| &\le (1+\varepsilon) \max_{y_1^* \in F_m} \Big|y_1^*\Big(\sum_{i=1}^m a_i(Ax_i - A^{**}x^{**})\Big)\Big| \\ &= (1+\varepsilon) \max_{y_1^* \in F_m} \Big|y_1^*\Big(\sum_{i=1}^n a_i(Ax_i - A^{**}x^{**})\Big)\Big| \\ &\le (1+\varepsilon)\Big\|\sum_{i=1}^n a_i(Ax_i - A^{**}x^{**})\Big\|.\end{aligned}$$

We see that $(Ax_n - A^{**}x^{**})_{n=1}^\infty$ is $(1+\varepsilon)$-basic. Moreover, if $B = I_{Y^{**}} - A^{**}x^{**} \otimes y^{***}$ and $C = I_{Y^{**}} - A^{**}x^{**} \otimes y^*$, then $\max\{\|B\|, \|C\|\} \le 1 + 1/\psi$ and for any $n \in \mathbb{N}$,

$$B(Ax_n - A^{**}x^{**}) = (Ax_n - A^{**}x^{**}) - y^{***}(Ax_n - A^{**}x^{**})A^{**}x^{**} = Ax_n$$

and

$$C(Ax_n) = Ax_n - y^*(Ax_n)A^{**}x^{**} = Ax_n - A^{**}x^{**}(y^*)A^{**}x^{**} = Ax_n - A^{**}x^{**}.$$

Here, $A^{**}x^{**} \otimes y^{***}$ is the operator from Y^{**} into itself given by

$$A^{**}x^{**} \otimes y^{***}(y^{**}) = y^{***}(y^{**})A^{**}x^{**},$$

and $A^{**}x^{**} \otimes y^* : Y^{**} \to Y^{**}$ is defined similarly.

These facts together show that (Ax_n) is $(1+\varepsilon)(1+1/\psi)^2$-basic. Finally, for any $n \in \mathbb{N}$ and any $y = \sum_{i=1}^n a_i Ax_i \in \operatorname{co}(Ax_i : i \le n)$,

$$\|y\| \ge \sum_{i=1}^n a_i y^*(Ax_i) = \sum_{i=1}^n a_i A^{**}x^{**}(y^*) > \psi.$$

Therefore for any $b > (1 + 1/R(A))^2$ and any $\theta < 1/R(A)$, we may choose $x^{**} \in B_{X^{**}}$, ψ, and ε to satisfy the conclusion with this b and θ. ∎

Lemma 3. *Let X be a Banach space, Z a subspace of X, $A : X \to Y$ an operator, and $\mathcal{U}$ an ultrafilter.*

(i) *The closed subspace $X_{\mathcal{U}}(Z) := \{(x_n) + \mathcal{N}_X \in X_{\mathcal{U}} : \lim_{\mathcal{U}} \|x_n\|_{X/Z} = 0\}$ of $X_{\mathcal{U}}$ is isometrically isomorphic to $Z_{\mathcal{U}}$.*

(ii) *If $\dim(X/Z) < \infty$, $X_{\mathcal{U}}/X_{\mathcal{U}}(Z)$ is isometrically isomorphic to X/Z.*

(iii) *$(A|_Z)_{\mathcal{U}} : Z_{\mathcal{U}} \to Y_{\mathcal{U}}$ is isometrically identifiable with $A_{\mathcal{U}}|_{X_{\mathcal{U}}(Z)} : X_{\mathcal{U}}(Z) \to Y_{\mathcal{U}}$.*

Here, $\mathcal{N}_X = \{(x_n) \in \ell_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\}$, and $\mathcal{N}_Y, \mathcal{N}_Z$ are defined similarly.

Proof. The first statement follows from the fact that the map $Z_{\mathcal{U}} \ni (z_n) + \mathcal{N}_Z \mapsto (z_n) + \mathcal{N}_X \in X_{\mathcal{U}}(Z)$ is an isometric isomorphism. The second follows from the fact that since bounded sequences in X/Z must converge through $\mathcal{U}$, for any sequence $(x_n) \in \ell_\infty(X)$, $\lim_{\mathcal{U}}(x_n + Z) \in X/Z$ exists. One then checks that the map $(x_n) + X_{\mathcal{U}}(Z) \mapsto \lim_{\mathcal{U}}(x_n + Z)$ is a well-defined isometric isomorphism. The third fact follows from the first together with the equality $(A|_Z)_{\mathcal{U}}((z_n) + \mathcal{N}_Z) = (Az_n) + \mathcal{N}_Y = A_{\mathcal{U}}((z_n) + \mathcal{N}_X)$. ∎

Proposition 4. *If $A : X \to Y$ fails to be weakly compact, then*

$$\inf_{\dim(X/Z)<\infty} R(A|_Z) > 0,$$

where R is defined in Lemma 2.

Proof. We will show that for any $Z \le X$ with $\dim(X/Z) < \infty$, $R(A|_Z) \ge R(A)/3$. Fix $0 < \theta < R(A)$. There exists $y^{**} \in \overline{AB_X}^{w^*}$ with $\|y^{**}\|_{Y^{**}/Y} > \theta$ by the definition of $R(A)$. Fix a net $(x_\lambda) \subset B_X$ such that $Ax_\lambda \xrightarrow[w^*]{} y^{**}$. By passing to a subnet of (x_λ), we may fix $x \in X$ such that $x_\lambda + Z \to x + Z$ in norm in X/Z. Of course, $x + Z \in B_{X/Z}$. By replacing x with a different member of the equivalence class $x + Z$, we may assume $\|x\| < 2$. For each λ, we fix $z_\lambda \in Z$ such that $\|x_\lambda - x - z_\lambda\| \le 2\|x_\lambda - x\|_{X/Z} \xrightarrow[\lambda]{} 0$. Thus $\limsup_\lambda \|z_\lambda\| \le \limsup_\lambda \|x_\lambda - x\| < 3$. By passing to a subnet once more, we may assume $z_\lambda/3 \in B_Z$, and note that $Az_\lambda/3 \xrightarrow[w^*]{} (y^{**} - Ax)/3$ and $\|(y^{**} - Ax)/3\|_{Y^{**}/Y} \ge \theta/3$. ∎

Corollary 5. *If $A : X \to Y$ fails to be super weakly compact, then there exist $\psi > 0$ and $c \ge 1$ such that for every natural number $n \in \mathbb{N}$, and every $Z \le X$ such that $\dim(X/Z) < \infty$, there exists $(z_i)_{i=1}^n \subset B_Z$ such that $(Az_i)_{i=1}^n$ is c-basic and every convex combination of $(Az_i)_{i=1}^n$ has norm at least ψ.*

Proof. If A fails to be super weakly compact, we may fix an ultrafilter $\mathcal{U}$ such that $A_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ fails to be weakly compact. If $Z \le X$ is finite-codimensional, then $R(A_{\mathcal{U}}|_{X_{\mathcal{U}}(Z)}) \ge R(A_{\mathcal{U}})/3 > 0$, since $A_{\mathcal{U}}$ fails to be weakly compact and since $\dim(X_{\mathcal{U}}/X_{\mathcal{U}}(Z)) = \dim(X/Z) < \infty$. Thus we deduce the existence of $\theta > 0$ and $b \geq 1$ depending only on $R(A_{\mathcal{U}})$ such that there exists an infinite sequence $(\chi_i) \subset B_{X_{\mathcal{U}}(Z)}$ such that $(A_{\mathcal{U}}\chi_i)$ is b-basic and each convex combination of $(A_{\mathcal{U}}\chi_i)$ has norm at least θ. We identify $(A|_Z)_{\mathcal{U}} : Z_{\mathcal{U}} \to Y_{\mathcal{U}}$ with $A_{\mathcal{U}}|_{X_{\mathcal{U}}(Z)} : X_{\mathcal{U}}(Z) \to Y_{\mathcal{U}}$ and assume that this sequence (χ_i) is contained in $Z_{\mathcal{U}}$ and replace $A_{\mathcal{U}}|_{X_{\mathcal{U}}(Z)}$ with $(A|_Z)_{\mathcal{U}}$. Since $(A|_Z)_{\mathcal{U}}$ is finitely representable in $A|_Z$, for each $n \in \mathbb{N}$ and $\varepsilon > 0$ we may fix $(z_i)_{i=1}^n \in B_Z$ such that $(Az_i)_{i=1}^n$ is $(1+\varepsilon)$-equivalent to $((A|_Z)_{\mathcal{U}}\chi_i)_{i=1}^n$. Thus we deduce the result for any constant $\psi \in (0,\theta)$ and any $c > b$. ∎

3.2. Binary trees, diamond graphs, and Laakso graphs. We let $B_n = \bigcup_{i=0}^n \{0,1\}^i$, including the empty sequence $\emptyset$. We let $B = \bigcup_{n\in\mathbb{N}} B_n$. We treat B and B_n as graphs, where t is adjacent to $t^\frown 0$ and $t^\frown 1$. We endow B_n and B with the graph distances, noting that each B_n is isometrically a subset of B. We next let $r_n = 2^n - 1$ and $\Lambda_n = \{t \in B : r_n \leq |t| < r_{n+1}\}$ for $n = 0, 1, 2, \ldots$. We let $\ell(s)$ denote the n such that $r_n \leq |s| < r_{n+1}$.

Our construction of the diamond and Laakso graphs follows the presentation given in [8]. We next recall the definitions of the Hamming and diamond graphs. The vertex set of the *Hamming graph* is $\{0,1\}^{2^n}$, and the edge set E_n consists of all pairs of vertices which differ at exactly one coordinate. We let D_n denote the vertex set of the *diamond graph*, and it will be a subset of $\{0,1\}^{2^n}$. The edge set of the diamond graph will be the restriction $\{(s,t) \in E_n : s,t \in D_n\}$ of E_n to the vertex set D_n. For any $n \in \mathbb{N}$, let $d : \{0,1\}^n \to \{0,1\}^{2n}$ be the doubling function $d(k_1,\ldots,k_n) = (k_1,k_1,k_2,k_2,\ldots,k_n,k_n)$. We let $D_0 = \{(0),(1)\}$. If D_n has been defined, we let $D'_n = \{d(t) : t \in D_n\}$ and $D''_n = \{s \in \{0,1\}^{2^{n+1}} : (\exists t \in D'_n)((s,t) \in E_{n+1})\}$. We then let $D_{n+1} = D'_n \cup D''_n$.

We also recall the definitions of the Laakso graphs. The *Laakso graph* L_n will be a subset of $\{0,1\}^{4^n}$. Given any $n \in \mathbb{N}$ and $s = (a_i)_{i=1}^n \in \{0,1\}^n$, let

$$q(s) = (a_1,a_1,a_1,a_1,a_2,a_2,a_2,a_2,\ldots,a_n,a_n,a_n,a_n) \in \{0,1\}^{4n}.$$

Let $L_0 = \{(0),(1)\}$ with a single edge. Suppose that L_{n-1} has been defined. Let $L'_n = \{q(s) : s \in L_{n-1}\}$. Suppose that $s,t \in L_{n-1}$ are equal except at their jth coordinate. Suppose also that $s = u^\frown 0^\frown v$ and $t = u^\frown 1^\frown v$. Let $L_n^{s,t}$ consist of the following sequences:

$$\begin{array}{lll} q(u)^\frown(1,1,1,1)^\frown q(v), & q(u)^\frown(1,1,0,1)^\frown q(v), & q(u)^\frown(1,1,0,0)^\frown q(v), \\ q(u)^\frown(0,1,0,1)^\frown q(v), & q(u)^\frown(0,1,0,0)^\frown q(v), & q(u)^\frown(0,0,0,0)^\frown q(v), \end{array}$$

where each vertex is adjacent to the vertices immediately above or below it. We then let

$$L_n = L'_n \cup \bigcup\{L_n^{s,t} : s,t \in L_{n-1},\ s,t \text{ differ at one coordinate}\}.$$

3.3. Uniformly convex operators and non-embeddability. Given an operator $A : X \to Y$, we define $\delta : (0, \infty) \to [0, \infty)$ by

$$\delta(\varepsilon) = 1 - \sup\left\{\left\|\frac{x_1 + x_2}{2}\right\| : x_1, x_2 \in B_X,\ \left\|\frac{Ax_1 - Ax_2}{2}\right\| \geq \varepsilon\right\}.$$

Then A is *uniformly convex* if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$.

We recall the following theorem.

Theorem 6 ([3]). *If $A : X \to Y$ is super weakly compact, then there exists an equivalent norm $|\cdot|$ on X making $A : (X, |\cdot|) \to Y$ uniformly convex.*

Theorem 7. *If $A : X \to Y$ is a super weakly compact operator, none of the families $\{B_n\}$, $\{D_n\}$, $\{L_n\}$ factors through A.*

Remark 8. The proofs closely resemble corresponding proofs for spaces. The proof for binary trees resembles the proof for spaces from [9], while the proofs for diamonds and Laakso graphs resemble the proofs for spaces from [8]. We note that the proofs of the results for super weakly compact operators do not follow from the proofs for spaces, since there exist super weakly compact operators not factoring through any superreflexive Banach space. To see this, let us recall that for $2 \leq q < \infty$, an operator $A : X \to Y$ is said to have *Rademacher cotype* q provided there exists a constant $C \geq 0$ such that for any $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in X$,

$$C\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i x_i\right\| \geq \left(\sum_{i=1}^{n} \|Ax_i\|^q\right)^{1/q}.$$

By the Maurey–Pisier theorem [10], if Z is a Banach space such that for every $2 \leq q < \infty$, Z does not have Rademacher cotype q (that is, I_Z does not have Rademacher cotype q), then c_0 is finitely representable in Z. Of course, such a Z is not superreflexive. Furthermore, it is easy to see that if an operator A factors through a Banach space with Rademacher cotype q, then A must have Rademacher cotype q. Let $a_n = 1/\log(n+1)$ for each $n \in \mathbb{N}$. Then the diagonal operator $A : c_0 \to c_0$ given by $A \sum b_n e_n = \sum a_n b_n e_n$ is compact, but fails to have non-trivial Rademacher cotype. Indeed, for any $2 \leq q < \infty$ and $n \in \mathbb{N}$,

$$\left(\sum_{i=1}^{n} \|Ae_i\|^q\right)^{1/q} \geq \frac{n^{1/q}}{\log(n+1)} = \frac{n^{1/q}}{\log(n+1)}\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i e_i\right\|.$$

Therefore A cannot be factored through any Banach space which has non-trivial Rademacher cotype. It follows that every Banach space is finitely representable in any Banach space through which A factors, and therefore A cannot be factored through any superreflexive Banach space.

Proof of Theorem 7. We may of course assume that $\|A\| \leq 1$ and that A is uniformly convex.

We first show that $\{B_n\}$ does not factor through A. First note that if $x, y, z \in B_X$ are such that $\|\frac{Ay-Az}{2}\| \geq 1/D$, then $\min\{\|x+y\|, \|x+z\|\} \leq 2(1-\delta)$, where $\delta = \delta(1/(2D))$. This is because in this case, either $\|\frac{Ax-Ay}{2}\|$ or $\|\frac{Ax-Az}{2}\|$ is at least $1/(2D)$. We next use this observation in the proof of the following claim: For any $D \geq 1$, any $n \in \mathbb{N}$, and $f : B_{2^n} \to X$ such that for all $s, t \in B_{2^n}$,

$$D^{-1}d(s,t) \leq \|Af(s) - Af(t)\| \leq \|f(s) - f(t)\| \leq d(s,t),$$

there exist $t_0, t_1 \in \{0,1\}^{2^n}$ whose first terms are 0 and 1, respectively, such that

$$\max\{\|f(t_0) - f(\emptyset)\|, \|f(t_1) - f(\emptyset)\|\} \leq 2^n(1-\delta)^n.$$

Here, $\delta = \delta(1/(2D))$ as in the observation above. We work by induction. Suppose that $f : B_2 \to X$ satisfies the hypotheses above. By translating, we may assume $f(\emptyset) = 0$. Then to find t_0, we apply the observation above to the vectors $x = f((0))$, $y = f((0,0)) - x$, and $z = f((0,1)) - x$. To find t_1, we apply the observation above to the vectors $x = f((1))$, $y = f((1,0)) - x$, and $z = f((1,1)) - x$. Next, assume the result holds for some $n \in \mathbb{N}$ and suppose $f : B_{2^{n+1}} \to X$ satisfies the hypotheses above. We may again assume that $f(\emptyset) = 0$. Applying the inductive hypothesis to $f|_{B_{2^n}}$, we may find $s_0, s_1 \in \{0,1\}^{2^n}$ whose first terms are 0 and 1, respectively, such that $\max\{\|f(s_0)\|, \|f(s_1)\|\} \leq 2^n(1-\delta)^n$. Then define $f_0, f_1 : B_{2^n} \to X$ by $f_0(t) = f(s_0^\frown t)$ and $f_1(t) = f(s_1^\frown t)$. Then these maps satisfy the hypotheses, and for $\varepsilon \in \{0,1\}$, we may find $s'_{\varepsilon,0}, s'_{\varepsilon,1} \in \{0,1\}^{2^n}$ whose first terms are 0 and 1, respectively, such that

$$\max\{\|f_\varepsilon(s'_{\varepsilon,0}) - f(s_\varepsilon)\|, \|f_\varepsilon(s'_{\varepsilon,1}) - f(s_\varepsilon)\|\} \leq 2^n(1-\delta)^n.$$

Let $s_\emptyset = \emptyset$ and $s_{(\varepsilon,\varepsilon')} = s_\varepsilon^\frown s'_{\varepsilon,\varepsilon'}$. Then $g : B_2 \to X$ defined by $g(t) = f(s_t)/(2^n(1-\delta)^n)$ satisfies the hypotheses of the base case. Therefore there exist $t_0, t_1 \in \{0,1\}^2$ with first terms 0 and 1, respectively, such that $\|g(t_0)\|, \|g(t_1)\| \leq 2(1-\delta)$. Then if $t_0 = (0,\varepsilon)$, we have

$$\|f(s_{0,\varepsilon})\| = 2^n(1-\delta)^n\|g(t_0)\| \leq 2^{n+1}(1-\delta)^{n+1}.$$

We similarly deduce that if $t_1 = (1,\varepsilon)$, then $\|f(s_{1,\varepsilon})\| \leq 2^{n+1}(1-\delta)^{n+1}$.

For the diamond D_n, we let t, b denote $t = (1, \dots, 1)$ and $b = (0, \dots, 0)$. Note that the vertices t, b depend implicitly on n, but it will be clear from the context in which diamond t, b lie. We claim that if $f : D_n \to X$ is any function such that for every $s, s' \in D_n$,

$$D^{-1}d(s,s') \leq \|Af(s) - Af(s')\| \leq \|f(s) - f(s')\| \leq d(s,s'),$$

then there exist adjacent vertices s, s' in D_n such that

$$\|f(t) - f(b)\| \le 2^n(1-\delta)^n\|f(s) - f(s')\| \le 2^n(1-\delta)^n,$$

where $\delta = \delta(1/D)$. From this it follows that $D \ge (1-\delta)^{-n}$, since $d(t, b) = 2^n$. We prove the claim by induction on $n \in \mathbb{N}$.

Base case: Assume $f : D_1 \to X$ is as above. By translating, we may assume $f(b) = 0$. Let $l = (0, 1)$ and $r = (1, 0)$. Let $u = f(l)$, $v = f(r)$, and write $f(t) = u + w = v + x$. Note that $\|u\|, \|v\|, \|w\|, \|x\| \le 1$. Moreover, if $u' = u/(\|u\| \vee \|v\|)$ and $v' = v/(\|u\| \vee \|v\|)$, then

$$\left\|\frac{Au' - Av'}{2}\right\| \ge d(l, r)/(2D(\|u\| \vee \|v\|)) \ge 1/D.$$

Therefore, $\|u+v\| \le 2(1-\delta)(\|u\| \vee \|v\|)$. Similarly, with $w' = w/(\|w\| \vee \|x\|)$ and $x' = x/(\|w\| \vee \|x\|)$, we have

$$\begin{aligned}\left\|\frac{Aw' - Ax'}{2}\right\| &= \left\|\frac{A(f(t) - u) - A(f(t) - v)}{2}\right\|(\|w\| \vee \|x\|)^{-1} \\ &= \left\|\frac{Au - Av}{2}\right\|(\|w\| \vee \|x\|)^{-1} \ge 1/D,\end{aligned}$$

so $\|w + x\| \le 2(1-\delta)(\|w\| \vee \|x\|)$. From this we deduce that

$$\begin{aligned}\|f(t)\| &= \left\|\frac{u + w}{2} + \frac{v + x}{2}\right\| \le \left\|\frac{u + v}{2}\right\| + \left\|\frac{w + x}{2}\right\| \\ &\le 2(1-\delta)(\|u\| \vee \|v\| \vee \|w\| \vee \|x\|) = 2(1-\delta)\|f(s) - f(s')\|\end{aligned}$$

for some pair s, s' of adjacent vertices.

Inductive case: Suppose $f : D_{n+1} \to X$ is as above. Recall that $d : \{0,1\}^{2^n} \to \{0,1\}^{2^{n+1}}$ is the doubling function. Note that the doubling function $d : D_n \to D_{n+1}$ doubles distances as well. Define $g : D_n \to X$ by $g(s) = \frac{1}{2}f(d(s))$. Since g satisfies the inequalities in the inductive hypothesis, we see that there exist adjacent $u, u' \in D_n$ such that $\|g(t') - g(b')\| \le 2^n(1-\delta)^n\|g(u) - g(u')\|$, where t', b' denote the top and bottom of D_n. From this it follows that $\|f(t) - f(b)\| \le 2^n(1-\delta)^n\|f(d(u)) - f(d(u'))\|$. Note that the portion of D_{n+1} which lies pointwise between $d(u)$ and $d(u')$ is isometrically identifiable with D_1. Under this identification, treating the restriction of f to this portion of D_{n+1} as a map of D_1 into X, and using the base case, we deduce that there exist adjacent s, s' in this portion of D_{n+1} such that $\|f(u) - f(u')\| \le 2(1-\delta)\|f(s) - f(s')\|$. From this, we deduce that

$$\|f(t) - f(b)\| \le 2^n(1-\delta)^n\|f(u) - f(u')\| \le 2^{n+1}(1-\delta)^{n+1}\|f(s) - f(s')\|.$$

We perform a similar argument for the Laakso graphs, using a similar self-similarity. We claim that if $f : L_n \to X$ is such that for all $s, t \in L_n$,

$$\frac{1}{D}d(s, t) \le \|Af(s) - Af(t)\| \le \|f(s) - f(t)\| \le d(s, t),$$

then there exist adjacent $s, s' \in L_n$ such that

$$\|f(1) - f(0)\| \le 4^n(1-\delta)^n \|f(s) - f(s')\| \le 4^n(1-\delta)^n,$$

where $\delta = \delta(1/D)$, 1 denotes the constantly 1 sequence, and 0 denotes the constantly 0 sequence. This implies $D \ge (1-\delta)^n$, since $d(0,1) = 4^n$. The $n = 1$ case follows, after translating so that $f(0,0,0,0) = 0$, by considering the four vectors $0 = f(0,0,0,0)$, $u = f(1,1,0,0)$, $v = f(0,1,0,1)$, and $x = f(1,1,1,1) = u + w = v + x$. Each vector has norm at most 2, so arguing as with the diamonds we obtain a pair t, t' from among the four vertices above such that $d(t,t') = 2$ and

$$\|f(1,1,1,1) - f(0,0,0,0)\| \le 2(1-\delta)\|f(t) - f(t')\|.$$

Fix a vertex t'' between t and t' in L_n, and fix adjacent vertices s, s' from among t, t', t'' so that $\|f(s) - f(s')\| = \max\{\|f(t) - f(t'')\|, \|f(t'') - f(t')\|\}$. Then

$$\|f(1,1,1,1) - f(0,0,0,0)\| \le 2(1-\delta)\|f(t) - f(t')\| \le 4(1-\delta)\|f(s) - f(s')\|.$$

For the inductive case, we fix an $f : L_{n+1} \to X$ satisfying the inequalities of the inductive hypothesis. Then $g : L_n \to X$ given by $g(t) = 4^{-1} f(q(t))$, where q is the quadrupling function defined above, also satisfies those inequalities. This is because q also quadruples distances. By the inductive hypothesis applied to g, there exist adjacent $s, s' \in L_n$ such that

$$\|g(1') - g(0')\| \le 4^n(1-\delta)^n \|g(s) - g(s')\|,$$

where $1'$ and $0'$ are the constantly 1 and 0 sequences, respectively, in L_n. This implies that

$$\|f(1) - f(0)\| = \|f(q(1')) - f(q(0'))\| \le 4^n(1-\delta)^n \|f(q(s)) - f(q(s'))\|.$$

Note that the portion of L_{n+1} between $q(s)$ and $q(s')$ is isometrically identifiable with L_1 in a way which associates s with $(0,0,0,0)$ and s' with $(1,1,1,1)$, and f restricted to this portion of L_{n+1} can be thought of as a map of L_1 into X satisfying the inequalities of the base case. Therefore we may find adjacent t, t' in this portion of L_{n+1} such that $\|f(q(s)) - f(q(s'))\| \le 4(1-\delta)\|f(t) - f(t')\|$. Then

$$\begin{aligned}\|f(1) - f(0)\| &\le 4^n(1-\delta)^n \|f(q(s)) - f(q(s'))\| \\ &\le 4^{n+1}(1-\delta)^{n+1}\|f(t) - f(t')\|. \quad \blacksquare\end{aligned}$$

3.4. Positive results. The remainder of this work is devoted to proving the next two theorems regarding positive results on the factorization of certain families of graphs through non-super weakly compact operators.

Theorem 9. *Suppose $A : X \to Y$ is not super weakly compact. Then $\{D_n\}$ and $\{L_n\}$ factor through A.*

Proof. Johnson and Schechtman [8] showed that for each $n \in \mathbb{N}$, for any $\psi > 0$ and $c \geq 1$, there exists a constant $C_0 = C_0(\psi, c) > 0$ such that if Y is a Banach space and if $(y_i)_{i=1}^{2^n} \subset Y$ is c-basic such that every convex combination of $(y_i)_{i=1}^{2^n}$ has norm at least ψ, then $f : D_n \to Y$ given by $f(k_1, \ldots, k_{2^n}) = \sum_{i=1}^{2^n} k_i y_i$ satisfies $C_0 d(s, s') \leq \|f(s) - f(s')\|$. This, combined with Corollary 5, implies that if $A : X \to Y$ fails to be super weakly compact, then there exists $D > 0$ such that for any $n \in \mathbb{N}$, there exists $f : D_n \to X$ such that

$$D^{-1} d(s, s') \leq \|Af(s) - Af(s')\| \quad \text{and} \quad \|f(s) - f(s')\| \leq d(s, s')$$

for all $s, s' \in D_n$. Indeed, by Corollary 5, we may find constants ψ, c depending only on A such that for any $n \in \mathbb{N}$, there exists $(x_i)_{i=1}^{2^n}$ such that $(y_i)_{i=1}^{2^n} = (Ax_i)_{i=1}^{2^n}$ is c-basic and every convex combination of $(y_i)_{i=1}^{2^n}$ has norm at least ψ. We then take $f((k_i)_{i=1}^{2^n}) = \sum_{i=1}^{2^n} k_i x_i$. The upper estimate follows from the triangle inequality together with the fact that the Lipschitz constant is attained on a pair of adjacent vertices. The lower estimate follows with $D = C_0$.

Johnson and Schechtman [8] also outline a proof very similar to the previous argument of the fact that there exists a constant $C_1 = C_1(\psi, c) > 0$ such that if Y is a Banach space and $(y_i)_{i=1}^{4^n} \subset Y$ is c-basic such that every convex combination of $(y_i)_{i=1}^{4^n}$ has norm at least ψ, then $f : L_n \to Y$ given by $f(k_1, \ldots, k_{4^n}) = \sum_{i=1}^{4^n} k_i y_i$ satisfies $C_1 d(s, s') \leq \|f(s) - f(s')\|$. As in the previous paragraph, we obtain a positive factorization result for the Laakso graphs through non-super weakly compact operators. ■

THEOREM 10. *Suppose $A : X \to Y$ is not super weakly compact. Then $\{B_n\}$ and B factor through A.*

Proof. We can and do assume $\|A\| \leq 1$.

Bourgain [4] showed that for each $n \in \mathbb{N}$, there exists an enumeration $(s_i)_{i=0}^{k}$ of B_n and a constant $C = C(\psi, c) > 0$ such that for each $\psi > 0$ and $c \geq 1$, if Y is a Banach space and $(y_i)_{i=1}^{k} \subset Y$ is c-basic such that every convex combination of $(y_i)_{i=1}^{k}$ has norm at least ψ, then if $y_{s_i} = y_i$ and $f(s) = \sum_{\emptyset \prec u \preceq s} y_u$, we have

$$C d(s, s') \leq \|f(s) - f(s')\|$$

for all $s, s' \in B_n$. This, combined with Corollary 5, implies that if $A : X \to Y$ fails to be super weakly compact, then there exists $D > 0$ (depending only on $R(A)$) such that for any $n \in \mathbb{N}$ and any $Z \leq X$ with finite codimension in X, there exists $f : B_n \to Z$ such that

$$D^{-1} d(s, s') \leq \|Af(s) - Af(s')\| \leq \|f(s) - f(s')\| \leq d(s, t) \tag{1}$$

for all $s, s' \in B_n$. We obtain a positive factorization result for $\{B_n\}$ by choosing for each $n \in \mathbb{N}$ some $(x_i)_{i=1}^{k} \subset B_X$ such that $(y_i)_{i=1}^{k} = (Ax_i)_{i=1}^{k}$

satisfies the conditions above with c, ψ depending on $R(A)$ and not on n. We emphasize that the function f can be taken to map into any closed subspace Z of X such that $\dim(X/Z) < \infty$.

We next use the previous paragraph to prove that B factors through A. This proof modifies Baudier's [1] proof of the corresponding result for non-superreflexive Banach spaces. A similar reasoning was given by Baudier, Kalton, and Lancien [2] to prove a positive result regarding the embedding of countably infinitely branching trees rather than binary trees. The arguments from [1] and [2] can be thought of as the local and asymptotic versions of the same phenomenon. The asymptotic argument from [2] also has an operator analogue which appears in [5].

Recall that $r_n = 2^n - 1$ and $\Lambda_n = \{t \in B : r_n \leq |t| < r_{n+1}\}$ for $n = 0, 1, \dots$. Recall also that $\ell(s) = \max\{n : r_n \leq |s|\}$. We define $0 = q_0 < q_1 < \cdots$ and a partition $(S_i)_{i=1}^\infty$ of B such that for each $n \in \mathbb{N}$, $(S_i)_{i=q_n+1}^{q_{n+1}}$ partitions Λ_n. Let $q_0 = 0$, $q_1 = 1$, and $S_1 = \{\emptyset\} = \Lambda_0$. Suppose now that $q_0 < \cdots < q_n$ have been defined and $\bigcup_{i=0}^{n-1} \Lambda_i = \bigcup_{i=0}^{q_n} S_i$. Let $t_1, \dots, t_m$ be an enumeration of the maximal members of Λ_{n-1} and let $q_{n+1} = q_n + m$. For $1 \leq i \leq m$, let S_{q_n+i} consist of those $t \in \Lambda_n$ such that t_i is an initial segment of t. This completes the recursive definition of $(q_i)_{i=0}^\infty$ and $(S_i)_{i=1}^\infty$.

Next, for each $i \in \mathbb{N}$, if n is such that $S_i \subset \Lambda_n$, then we define a map $f_i : B_{2^n} \to X$ and finite-dimensional subspaces $E_1, E_2, \dots$ of Y by induction on i. Let $E_1 = \{0\}$, $f_1(\emptyset) = 0$. Assume that $f_1, \dots, f_{i-1}$ and $E_1, \dots, E_{i-1}$ have been defined. Suppose also that for each $1 \leq j < i-1$, a finite $J_j \subset B_{Y^*}$ has been chosen such that for any $y \in \operatorname{span}(E_1 \cup \dots \cup E_j)$, $2\max_{y^* \in J_j} |y^*(y)| \geq \|y\|$. Note that there exists $n \in \mathbb{N}$ such that $S_i \subset \Lambda_n$, and a maximal member s of Λ_{n-1} such that S_i consists of all members of Λ_n which extend s. Note that B_{2^n} is canonically isometric to $\{s\} \cup S_i$ via the map $t \mapsto s^\frown t$. By compactness of the unit ball of the finite-dimensional space F spanned by $E_1 \cup \dots \cup E_{i-1}$ and homogeneity, there exists a finite subset J_{i-1} of B_{Y^*} such that for any $y \in F$, $\|y\| \leq 2\max_{y^* \in J_{i-1}} |y^*(y)|$, and such that $\bigcup_{j=1}^{i-2} J_j \subset J_{i-1}$. Let $W_i = \bigcap_{y^* \in J_{i-1}} \ker(y^*) \leq Y$ and let $Z_i = \bigcap_{y^* \in J_{i-1}} \ker(A^* y^*) \leq X$. By equation (1) above, there exists a map $f_i : B_{2^n} \to Z_i$ such that $f_i(\emptyset) = 0$ and for each $t, t' \in B_{2^n}$,

$$d(t,t')/D \leq \|Af(t) - Af(t')\| \leq \|f(t) - f(t')\| \leq d(t,t').$$

Let E_i denote the span of $\{Af_i(t) : t \in B_{2^n}\}$. This completes the definitions of f_i and E_i. The purpose of having f_i map into Z_i is that, for W denoting the closed span of $\bigcup_{i=1}^\infty E_i$, E_i is the range of a projection $P_i : W \to E_i$ having norm not more than 4. Indeed, for any $i \leq n \in \mathbb{N}$, if $z_j \in E_j$ for each $1 \leq j \leq n$, then

$$\Big\|\sum_{j=1}^{i} z_j\Big\| \le 2\max_{y^*\in J_i}\Big|y^*\Big(\sum_{j=1}^{i} z_j\Big)\Big| = 2\max_{y^*\in J_i}\Big|y^*\Big(\sum_{j=1}^{n} z_j\Big)\Big| \le 2\Big\|\sum_{j=1}^{n} z_j\Big\|.$$

Here we are using the fact that $J_1 \subset J_2 \subset \cdots$ and $y^*(z_j) = 0$ whenever $i < j$, $y^* \in J_i$, and $z_j \in E_j$. This shows that there exists a projection $Q_i : W \to \operatorname{span}\{E_1, \dots, E_i\}$ with norm not more than 2. We then let $P_i = Q_i - Q_{i-1}$, where $Q_0 = 0$.

Given $\emptyset \neq t$, we may uniquely write $t = t_1^\frown \dots ^\frown t_n$ for some $n \in \mathbb{N}$ and $t_1, \dots, t_n \neq \emptyset$ such that for each $1 \le i < n$, $|t_i| = 2^i$. That is, for each $1 \le i < n$, $t_1^\frown \dots ^\frown t_i$ is maximal in Λ_i. Let us say $t_1^\frown \dots ^\frown t_n$ is the *standard representation* of t. Moreover, there exist $1 \le j_1 < \cdots < j_n$ such that for each $1 \le i \le n$,

$$\{t_1^\frown \dots ^\frown t_{i-1}^\frown u : \emptyset \prec u \preceq t_i\} \subset S_{j_i}.$$

We let $f(t) = \sum_{i=1}^{n} f_{j_i}(t_i)$. We let $f(\emptyset) = 0$. We note that f is 1-Lipschitz. Indeed, it is sufficient to check that $\|f(t) - f(t^-)\| \le 1$ for all $t \neq \emptyset$. First choose i such that $t \in S_i$ and let $t_1^\frown \dots ^\frown t_n$ be the standard representation of t. Then $f(t) - f(t^-) = f_i(t_n) - f_i(t_n^-)$ and $\|f(t) - f(t^-)\| = \|f_i(t_n) - f_i(t_n^-)\| \le 1$.

We now prove that for any $s \neq t \in B$,

$$\frac{d(s,t)}{32D} \le \|Af(s) - Af(t)\|,$$

which will finish the proof. We prove the case $s, t \neq \emptyset$, with the cases $s = \emptyset$ or $t = \emptyset$ requiring only notational changes. Let $s = s_1^\frown \dots ^\frown s_m$, $t = t_1^\frown \dots ^\frown t_n$ be the standard representations and assume without loss of generality that $m \le n$. Let u be the largest common initial segment of s and t. We fix $k_1, \dots, k_m$, $j_1, \dots, j_n$ such that $s_1^\frown \dots ^\frown s_i \in S_{k_i}$ for each $1 \le i \le m$ and $t_1^\frown \dots ^\frown t_i \in S_{j_i}$ for each $1 \le i \le n$. We consider three cases.

Case 1: $\ell(u) < n-1$. Then $P_{j_{n-1}} Af(s) = 0$, $P_{j_{n-1}} Af(t) = Af_{j_{n-1}}(t_{n-1})$,

$$4\|Af(s) - Af(t)\| \ge \|Af_{n_{j-1}}(t_{n-1})\| \ge |t_{n-1}|/D = 2^{n-1}/D,$$

and

$$d(s,t) \le |s| + |t| \le 2^{m+1} + 2^{n+1} \le 8 \cdot 2^{n-1}.$$

Therefore in this case, $d(s,t)/(32D) \le \|Af(s) - Af(t)\|$.

Case 2: $\ell(u) = n-1$. Then either $m = n$ or $m = n-1$. If $m = n-1$, let $s_n = \emptyset$, so that

$$d(s,t) = d(s_{n-1}, t_{n-1}) + |s_n| + |t_n|,$$

and $3\max\{d(s_{n-1}, t_{n-1}), |s_n|, |t_n|\} \ge d(s,t)$. Furthermore, $k_{n-1} = j_{n-1}$ and $k_n \neq j_n$. We observe that

$$d(s_{n-1}, t_{n-1})/D \le \|P_{j_{n-1}} Af(s) - P_{j_{n-1}} Af(t)\| \le 4\|Af(s) - Af(t)\|,$$

$|s_n| = 0$ if $m = n-1$ and otherwise

$$|s_n|/D \le \|P_{k_n} Af(s)\| = \|P_{k_n}(Af(s) - Af(t))\| \le 4\|Af(s) - Af(t)\|,$$
$$|t_n|/D \le \|P_{j_n} Af(t)\| = \|P_{j_n}(Af(s) - Af(t))\| \le 4\|Af(s) - Af(t)\|.$$

These inequalities together imply that

$$d(s,t)/(12D) \le \|Af(s) - Af(t)\|.$$

CASE 3: $\ell(u) = n$. Then $m = n$, $s_1^\frown \ldots ^\frown s_{n-1} = t_1^\frown \ldots ^\frown t_{n-1}$, $k_n = j_n$, and $s, t \in S_{j_n}$. Then

$$d(s,t)/D \le \|P_{j_n}(Af(s) - Af(t))\| \le 4\|Af(s) - Af(t)\|,$$
$$d(s,t)/(4D) \le \|Af(s) - Af(t)\|. \ \blacksquare$$

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