ACTA ARITHMETICA
180.1 (2017)

On the Hausdorff dimension faithfulness of Oppenheim expansion

by

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1. Introduction. Hausdorff dimension of sets satisfying certain arithmetic properties has been extensively studied in connection with various aspects of expansions of real numbers. However, determining the exact Hausdorff dimension is always a non-trivial problem. In many cases, the difficult part is to determine a lower bound of the Hausdorff dimension. For this, narrowing the family of admissible coverings may shed some light on this complicated problem.

Let Φ be a fine covering family on $[0,1]$, i.e., a family of subsets of $[0,1]$ such that for any $\epsilon > 0$, there exists an at most countable ϵ-covering $\{E_j\}$ of $[0,1]$ with $E_j \in \Phi$. If for any set $E \subset [0,1]$, the Hausdorff dimension of E determined by only considering the family Φ equals the actual Hausdorff dimension of E, we say that Φ is *Hausdorff dimension faithful*. This notion was introduced by S. Albeverio et al. [1, 2].

To be more precise, the *α-dimensional Hausdorff measure* of a set $E \subset [0,1]$ with respect to a fine family Φ of coverings is defined by

$$H^\alpha(E,\Phi) = \lim_{\epsilon\to 0}\inf\Big\{\sum_j |E_j|^\alpha \;\Big|\; \{E_j\} \text{ is an } \epsilon\text{-covering of } E \text{ with } E_j \in \Phi\Big\},$$

where $|A|$ denotes the length of A. The *Hausdorff dimension* of E with respect to Φ is

$$\dim_{\mathrm{H}}(E,\Phi) = \inf\{\alpha \mid H^\alpha(E,\Phi) = 0\}.$$

If we take for Φ the family of all subsets of $[0,1]$, then we denote $\dim_{\mathrm{H}}(E,\Phi)$ by $\dim_{\mathrm{H}} E$, which is the classical Hausdorff dimension of E. For more properties of Hausdorff dimension, one is referred to [6, 10]. Then a fine covering

2010 *Mathematics Subject Classification*: Primary 11K55; Secondary 28A80.
Key words and phrases: Oppenheim expansion, faithfulness, Hausdorff dimension.
Received 29 October 2016; revised 28 February 2017.
Published online 25 July 2017.

DOI: 10.4064/aa8648-2-2017

family Φ is said to be *Hausdorff dimension faithful* if $\dim_{\mathrm{H}}(E, \Phi) = \dim_{\mathrm{H}} E$ for any $E \subset [0,1]$.

Whether the family of cylinders generated by various expansions of real numbers is Hausdorff dimension faithful has been considered by many authors. A. Besicovitch initiated the study of this problem and proved that the family of cylinders of a binary expansion is Hausdorff dimension faithful. P. Billingsley [5] extended this result to the case of s-adic expansion. Recently, S. Albeverio et al. have produced a series of works in this direction. They gave a general sufficient and necessary condition for the family of cylinders of the famous Cantor series expansion to be faithful. In particular, they gave a sufficient condition for the faithfulness of the families of cylinders generated by infinite linear iterated function systems, which covers the classical Lüroth expansions. The readers are referred to [1, 2, 3, 4, 9]. J. Liu and Z. L. Zhang [8] gave a general sufficient and necessary condition for the family of cylinders generated by infinite linear iterated function systems to be faithful.

In this paper, we focus on the family of cylinders generated by Oppenheim expansion and the family of finite unions of cylinders. We show that the family of cylinders generated by Oppenheim expansion is not faithful for Hausdorff dimension calculation on the unit interval. On the other hand, we prove that the family of all finite unions of consecutive cylinders of the same rank is faithful. Some special cases such as Lüroth expansion, Engel expansion and Sylvester expansion are included.

In the process of obtaining a lower bound for the Hausdorff dimension of some fractal sets associated with Oppenheim expansion, various authors usually take (countable) unions of cylinders instead of cylinders themselves: see, for example, [11, 12, 13]. The faithfulness of the family of all finite unions of consecutive cylinders is likely to disclose the reasons.

2. Statement of main results. First, we briefly recall some basic properties and known results on Oppenheim expansion.

Let $\gamma_n(j)$, $n \geq 1$, be a sequence of positive-rational-valued functions of the positive integer $j \geq 1$. The algorithm $0 < x \leq 1$, $x = x_1$, and for any $n \geq 1$, with positive integers $d_n(x)$,

$$\frac{1}{d_n(x)} < x_n \leq \frac{1}{d_n(x)-1}, \qquad x_n = \frac{1}{d_n(x)} + \gamma_n(d_n(x))x_{n+1} \tag{2.1}$$

leads to the series expansion

$$x = \frac{1}{d_1(x)} + \sum_{n=1}^{\infty} \frac{\gamma_1(d_1(x))\gamma_2(d_2(x)) \cdots \gamma_n(d_n(x))}{d_{n+1}(x)}, \tag{2.2}$$

which is called the *Oppenheim expansion* of x. Set

$$h_n(j) = \gamma_n(j)j(j-1), \quad j \geq 2.$$

If $h_n(j)$ is integer-valued ($n \geq 1, j \geq 2$), then (2.2) is termed the *restricted Oppenheim expansion* of x. Here and in what follows, we always assume h_n is integer-valued for all $n \geq 1$.

Definition 2.1. A vector $(k_1, \ldots, k_N)$ with positive integer components is called *realizable* if there exists an $x \in (0,1]$ such that $d_n(x) = k_n$ for any $1 \leq n \leq N$. An infinite sequence $\{k_n\}_{n\geq 1}$ of positive integers is called *realizable* if $(k_1, \ldots, k_N)$ is realizable for each N.

The algorithm (2.1) implies

$$d_1(x) \geq 2, \quad d_{n+1}(x) \geq h_n(d_n(x)) + 1 \quad \text{for any } n \geq 1. \tag{2.3}$$

On the other hand, any integer sequence $\{d_n\}_{n\geq 1}$ satisfying (2.3) is realizable: there exists a unique $x \in (0,1]$ such that $d_n(x) = d_n$ for any $n \geq 1$. The representation (2.2) under (2.1) is unique.

Oppenheim expansion was first studied by A. Oppenheim who established its arithmetical properties, including the question of rationality of the expansion. The foundations of the metric theory were laid down by J. Galambos [7]. Some fractal sets associated with the Oppenheim expansion were discussed in [11, 12, 13].

For any $n \geq 1$ and any realizable vector $(k_1, \ldots, k_n) \in \mathbb{N}^n$, let

$$B_n(k_1, \ldots, k_n) = \mathrm{cl}\{x \in [0,1) \colon d_1(x) = k_1, \ldots, d_n(x) = k_n\},$$

which is called a *cylinder of rank* n and where 'cl' denotes closure. Let $\mathcal{A}_n$ be the family of all such cylinders:

$$\mathcal{A}_n = \{B_n(k_1, \ldots, k_n) \colon k_1 \geq 2,\, k_{j+1} \geq h_j(k_j) + 1,\, j \geq 1\}.$$

Define

$$\mathcal{A} = \bigcup_{k\geq 1} \mathcal{A}_k.$$

Theorem 2.2. *The covering family $\mathcal{A}$ of all cylinders of Oppenheim expansion is not faithful for Hausdorff dimension calculation on the unit interval.*

Though the family of all cylinders is not faithful, all possible finite unions of consecutive cylinders show different properties. Let $\mathcal{B}_n$ be the family of all possible finite unions of consecutive intervals of rank n, i.e.,

$$\mathcal{B}_n = \Big\{ \bigcup_{i=s}^{s+m} B(k_1, \ldots, k_{n-1}, i) : s \geq h_{n-1}(k_{n-1}(x)) + 1 \text{ and } s \in \mathbb{N}^+, \\ m \in \mathbb{N},\, k_1 \geq 2,\, k_{j+1} \geq h_j(k_j(x)) + 1,\, j \geq 1 \Big\}.$$

Define

$$\mathcal{B} = \bigcup_{k\ge 1} \mathcal{B}_k.$$

The we have the following theorem.

THEOREM 2.3. *The family $\mathcal{B}$ is faithful for Hausdorff dimension calculation on the unit interval.*

3. Proofs

3.1. Proof of Theorem 2.2. To prove that the covering family $\mathcal{A}$ of all cylinders of Oppenheim expansion is not faithful, we shall construct a Cantor-like set E such that $\dim_{\mathrm{H}}(E,\mathcal{A}) > \dim_{\mathrm{H}}(E)$. Let

$$E = \{x \in (0,1] \colon d_1(x) = 8,\ [d_1(x)\cdots d_j(x)(h_j(d_j(x))+1)]^5 < d_{j+1}(x) \le 2[d_1(x)\cdots d_j(x)(h_j(d_j(x))+1)]^5 \text{ for } j \ge 1\}.$$

For brevity, we introduce a symbolic space defined as follows: For any $k \ge 3$, let

$$D_k = \{(\sigma_1,\dots,\sigma_k) \in \mathbb{N}^k \colon \sigma_1 = 8,\ [\sigma_1\cdots\sigma_j(h_j(\sigma_j)+1)]^5 < \sigma_{j+1} \le 2[\sigma_1\cdots\sigma_j(h_j(\sigma_j)+1)]^5 \text{ for } 1 \le j < k\}$$

and define

$$D^* = \bigcup_{k=3}^{\infty} D_k.$$

For any $k \ge 3$ and $\sigma = (\sigma_1,\dots,\sigma_k) \in D_k$, let $\Theta(\sigma) = [\sigma_1\cdots\sigma_k(h_k(\sigma_k)+1)]^5$, and let I_σ and J_σ denote the following closed subintervals of $(0,1]$:

$$I_\sigma = \mathrm{cl}\{x \in [0,1) \colon d_1(x) = \sigma_1,\dots,d_k(x) = \sigma_k\},$$

$$J_\sigma = \bigcup_{\Theta(\sigma)<d\le 2\Theta(\sigma)} \mathrm{cl}\{x \in [0,1) \colon d_1(x) = \sigma_1,\dots,d_k(x) = \sigma_k, d_{k+1}(x) = d\}.$$

Each J_σ is called an *interval of kth order*. It is obvious that

$$E = \bigcap_{k=3}^{\infty} \bigcup_{\sigma\in D_k} J_\sigma.$$

From [7, proof of Theorem 6.1] we have, for any $k \ge 3$ and $\sigma \in D_k$,

$$|I_\sigma| = \gamma_1(\sigma_1)\cdots\gamma_{k-1}(\sigma_{k-1})\frac{1}{\sigma_k(\sigma_k-1)}, \tag{3.1}$$

$$|J_\sigma| = \sum_{\Theta(\sigma)<d\le 2\Theta(\sigma)} \gamma_1(\sigma_1)\cdots\gamma_k(\sigma_k)\frac{1}{d(d-1)} = \gamma_1(\sigma_1)\cdots\gamma_k(\sigma_k)\frac{1}{2\Theta(\sigma)}. \tag{3.2}$$

Then for any $s > 1/5$,

$$\begin{aligned} H^s(E) &\le \liminf_{k\to\infty} \sum_{\sigma\in D_k} |J_\sigma|^s = \liminf_{k\to\infty} \sum_{\sigma\in D_k} \left(\gamma_1(\sigma_1)\cdots\gamma_k(\sigma_k)\frac{1}{2\Theta(\sigma)}\right)^s \\ &\le \liminf_{k\to\infty} \sum_{\sigma\in D_{k-1}} |J_\sigma|^s \sum_{\Theta(\sigma)<\sigma_k\le 2\Theta(\sigma)} \left(\frac{\gamma_k(\sigma_k)(h_{k-1}(\sigma_{k-1})+1)^5}{\sigma_k^5(h_k(\sigma_k)+1)^5}\right)^s. \end{aligned}$$

Since $h_k(\sigma_k) = \gamma_k(\sigma_k)\sigma_k(\sigma_k - 1)$ and

$$[\sigma_1\cdots\sigma_{k-1}(h_{k-1}(\sigma_{k-1})+1)]^5 < \sigma_k \le 2[\sigma_1\cdots\sigma_{k-1}(h_{k-1}(\sigma_{k-1})+1)]^5,$$

it follows that

$$\begin{aligned} H^s(E) &\le \liminf_{k\to\infty} \sum_{\sigma\in D_{k-1}} |J_\sigma|^s \sum_{\Theta(\sigma)<\sigma_k\le 2\Theta(\sigma)} \frac{1}{\sigma_k^{6s}} \\ &\le \liminf_{k\to\infty} \sum_{\sigma\in D_{k-1}} |J_\sigma|^s 5\Theta(\sigma)^{1-6s} \\ &\le \liminf_{k\to\infty} \sum_{\sigma\in D_{k-1}} |J_\sigma|^s \le \cdots \le \liminf_{k\to\infty} \sum_{\sigma\in D_3} |J_\sigma|^s < \infty. \end{aligned}$$

Thus

$$\dim_{\mathrm{H}} E \le 1/5.$$

On the other hand, we will prove that $\dim_{\mathrm{H}}(E, \mathcal{A}) > 1/5$. To this end, we define a probability measure supported on E. First of all, we define a set function $\mu\colon \{I_\sigma \mid \sigma \in D \setminus D_0\} \to \mathbb{R}^+$ as follows. For $\sigma_1 = 8$, let

$$\mu(I_{\sigma_1}) = 1.$$

For any $\sigma = (\sigma_1, \ldots, \sigma_k) \in D_k$ and $k \ge 2$, let

$$\mu(B(\sigma_1,\ldots,\sigma_k)) = \mu(B(\sigma_1,\ldots,\sigma_{k-1}))\frac{|B(\sigma_1,\ldots,\sigma_k)|}{|J(\sigma_1,\ldots,\sigma_{k-1})|}.$$

By Kolmogorov's extension theorem, the set function μ can be extended to a probability measure supported on E, still denoted by μ. From the definition of μ we have, for any $k \ge 2$ and $\sigma = (\sigma_1, \ldots, \sigma_k) \in D_k$,

$$\begin{aligned} \mu(I_\sigma) &\le \frac{|B(\sigma_1,\ldots,\sigma_k)|}{|J(\sigma_1,\ldots,\sigma_{k-1})|} \\ &\le \frac{\gamma_1(\sigma_1)\cdots\gamma_{k-1}(\sigma_{k-1})}{\sigma_k(\sigma_k-1)} \frac{2[\sigma_1\cdots\sigma_{k-1}(h_{k-1}(\sigma_{k-1})+1)]^5}{\gamma_1(\sigma_1)\cdots\gamma_{k-1}(\sigma_{k-1})} \\ &\le \frac{4}{[\sigma_1\cdots\sigma_{k-1}(h_{k-1}(\sigma_{k-1})+1)]^5}. \end{aligned}$$

Then for any $x \in E$,

$$\frac{\mu(B(d_1(x),\dots,d_k(x)))}{|B(d_1(x),\dots,d_k(x))|^\alpha} \le \frac{4}{[d_1(x)\cdots d_{k-1}(x)(h_{k-1}(d_{k-1}(x))+1)]^5}\cdot\frac{(d_k(x)(d_k(x)-1))^\alpha}{(\gamma_1(d_1(x))\cdots\gamma_{k-1}(d_{k-1}(x)))^\alpha}.$$

Since

$$\frac{1}{\gamma_k(d_k(x))} \le d_k(x)(d_k(x)-1) < d_k(x)^2,$$

we have

$$\begin{aligned}\frac{\mu(B(d_1(x),\dots,d_k(x)))}{|B(d_1(x)\cdots,d_k(x))|^\alpha} &\le \frac{4^{1+\alpha}[d_1(x)\cdots d_{k-1}(x)(h_{k-1}(d_{k-1}(x))+1)]^{12\alpha}}{[d_1(x)\cdots d_{k-1}(x)(h_{k-1}(d_{k-1}(x))+1)]^5}\\ &\le \frac{4^{1+\alpha}}{[d_1(x)\cdots d_{k-1}(x)(h_{k-1}(d_{k-1}(x))+1)]^{5-12\alpha}}.\end{aligned}$$

So,

$$\lim_{k\to\infty}\frac{\mu(B(d_1(x),\dots,d_k(x)))}{|B(d_1(x),\dots,d_k(x))|^\alpha}=0$$

for any $0<\alpha<5/12$ and $x\in E$. Therefore,

$$\dim_{\mathrm{H}}(E,\mathcal{A}) \ge 5/12 > \dim_{\mathrm{H}} E,$$

which completes the proof of the theorem.

3.2. Proof of Theorem 2.3. Before the proof of Theorem 2.3, we give the following useful lemma, which can be easily proved by using the equality (3.1).

LEMMA 3.1. *Let $(d_1,\dots,d_n,s)$ be a realizable vector and $B_n(s) := B(d_1,\dots,d_n,s)$. Then*

$$|B_n(s+1)| \le |B_n(s)| \le 6|B_n(s+1)|.$$

To prove Theorem 2.3, we shall find a proper family consisting of finite unions of consecutive cylinders such that the Hausdorff measure of any set with respect to this family is comparable to its classical Hausdorff measure.

For a given set $E\subset(0,1)$, and $\alpha>0$, suppose that $E_j=(a_j,b_j)$, $j\ge1$, is an open ϵ-covering of E. Then for any $j\in\mathbb{N}$, there exists a cylinder $B_{n_j}(d_1,\dots,d_{n_j})$ of rank n_j such that $E_j\subset B_{n_j}(d_1,\dots,d_{n_j})$, and no cylinder of rank n_j+1 contains E_j.

According to the number of cylinders contained in E_j, we divide the proof into two cases.

CASE 1: *E_j contains at least one cylinder of rank n_j+1.*

(1) If only finitely many cylinders of rank n_j+1 intersect E_j, say

$$B_{n_j+1}(d_1,\dots,d_{n_j},s),\dots,B_{n_j+1}(d_1,\dots,d_{n_j},s+t)$$

(where s, t are positive integers), then

$$B_{n_j+1}(d_1, \ldots, d_{n_j}, i) \subset E_j, \quad s+1 \le i \le s+t-1,$$

and

$$E_j \subset \bigcup_{s \le i \le s+t} B_{n_j+1}(d_1, \ldots, d_{n_j}, i).$$

Set

$$J_0 = B_{n_j+1}(d_1, \ldots, d_{n_j}, s),$$
$$J_1 = \bigcup_{i=s+1}^{s+k-1} B_{n_j+1}(d_1, \ldots, d_{n_j}, i),$$
$$J_2 = B_{n_j+1}(d_1, \ldots, d_{n_j}, s+t).$$

In this way, $\mathcal{F}_{E_j} = \{J_0, J_1, J_2\} \subset \mathcal{B}$ is a covering of E_j. By Lemma 3.1, we thus get

$$|E_j|^\alpha \ge \frac{1}{2+6^\alpha} \sum_{J \in \mathcal{F}_{E_j}} |J|^\alpha \quad \text{and} \quad |J_k| \le 6|E_j|, \quad k = 0, 1, 2. \tag{3.3}$$

(2) If there are infinitely many cylinders of rank n_j+1 which intersect E_j, it is easily seen that

$$B_{n_j+1}(d_1, \ldots, d_{n_j}, i) \subset E_j, \quad i \ge d+1,$$

and

$$E_j \subset \bigcup_{i \ge d} B_{n_j+1}(d_1, \ldots, d_{n_j}, i)$$

for some positive integer d. For the given α, there exists a sequence $\{l_0 = 0, l_k\}_{k \ge 1}$ of integers such that

$$\sum_{k \ge 0} \Big| \bigcup_{i=1}^{l_{k+1}} B_{n_j+1}\Big(d + \sum_{m=0}^{k} l_m + i\Big) \Big|^\alpha < \infty$$

where $B_{n_j+1}(t) = B_{n_j+1}(d_1, \ldots, d_{n_j}, t)$. In addition, if l_1 is large enough,

$$\Big| \bigcup_{i=1}^{l_1} B_{n_j+1}(d+i) \Big|^\alpha \ge \sum_{k \ge 1} \Big| \bigcup_{i=1}^{l_{k+1}} B_{n_j+1}\Big(d + \sum_{m=0}^{k} l_m + i\Big) \Big|^\alpha.$$

Now, set $J_0 = B_{n_j+1}(d)$ and $J_k = \bigcup_{i=1}^{l_k} B_{n_j+1}(d + \sum_{m=0}^{k-1} l_m + i))$ for $k \ge 1$. It follows that $J_i \in \mathcal{B}$ and

$$E_j \subset \bigcup_{i \ge 0} J_i \quad \text{and} \quad \bigcup_{i \ge 1} J_i \subset E_j.$$

By Lemma 3.1 above, we can obtain $|J_0| = |B_{n_j+1}(d)| \le 6|B_{n_j+1}(d+1)| \le 6|J_1|$. Then if we let $\mathcal{F}_{E_j} = \{J_i\}_{i\ge 0} \subset \mathcal{B}$, we deduce that

$$\sum_{J\in\mathcal{F}_{E_j}} |J|^\alpha = |J_0|^\alpha + |J_1|^\alpha + \sum_{i\ge 2} |J_i|^\alpha \le (6^\alpha+2)|E_j|^\alpha, \tag{3.4}$$
$$|J_k| \le 6|E_j|, \qquad k\in\mathbb{N}.$$

Case 2: *E_j contains no cylinder of rank n_j+1.* This clearly forces that there are only two consecutive cylinders of rank n_j+1 intersecting E_j, say

$$B_{n_j+1}(d_1,\dots,d_{n_j},s), \qquad B_{n_j+1}(d_1,\dots,d_{n_j},s+1).$$

Recalling that $E_j = (a_j, b_j)$, it follows that

$$a_j \in B_{n_j+1}(d_1,\dots,d_{n_j},s+1) \quad\text{and}\quad b_j \in B_{n_j+1}(d_1,\dots,d_{n_j},s).$$

Suppose that c_j is the right endpoint of $B_{n_j+1}(d_1,\dots,d_{n_j},s+1)$. To complete the proof, we next consider the coverings of (a_j,c_j) and $[c_j,b_j)$ separately.

We first consider the coverings of (a_j,c_j). Without loss of generality, we assume that a_j belongs to some cylinder $B_{n_j+2}(d_1,\dots,d_{n_j},s+1,t)$ of rank n_j+2.

(i) If $t \ge h_{n_j+1}(s+1)+2$, we have

$$(a_j,c_j) \subset \bigcup_{h_{n_j+1}(s+1)+1\le i\le t} B_{n_j+2}(d_1,\dots,d_{n_j},s+1,i)$$

and

$$B_{n_j+2}(d_1,\dots,d_{n_j},s+1,i) \subset (a_j,c_j), \qquad h_{n_j+1}(s+1)+1 \le i \le t-1.$$

Set $J = \bigcup_{h_{n_j+1}(s+1)+1\le i\le t} B_{n_j+2}(d_1,\dots,d_{n_j},s+1,i)$. Then $J\in\mathcal{B}$ and $|J| \le 2|E_j|$, hence

$$|J|^\alpha \le 2^\alpha |E_j|^\alpha. \tag{3.5}$$

(ii) If $t = h_{n_j+1}(s+1)+1$, then since $a_j \ne c_j$, there exist integers l and t_0 such that

$$a_j \in B_{n_j+l+1}(d_1,\dots,d_{n_j},s+1,d_{n_j+2},\dots,d_{n_j+l},t_0),$$

where $t_0 > h_{n_j+l}(d_{n_j+l})+1$ and $d_k = h_{k-1}(d_{k-1})+1$ for $n_j+2 \le k \le n_j+l$. Much as above, set

$$J = \bigcup_{h_{n_j+l}(d_{n_j+l})+1\le i\le t_0} B_{n_j+l+1}(d_1,\dots,d_{n_j},s+1,d_{n_j+2},\dots,d_{n_j+l},i).$$

Then $J\in\mathcal{B}$ and $|J| \le 2|E_j|$, and we get

$$|J|^\alpha \le 2^\alpha |E_j|^\alpha. \tag{3.6}$$

Secondly, we consider the coverings of $[c_j, b_j)$. Since

$$b_j \in B_{n_j+1}(a_1, \ldots, a_{n_j}, s),$$

we assume that b_j is in the cylinder $B_{n_j+2}(d_1, \ldots, d_{n_j}, s, m_0)$ of rank $n_j + 2$ for some integer $m_0 \geq h_{n_j+1}(s) + 1$. Thus $B_{n_j+2}(d_1, \ldots, d_{n_j}, s, m_0 + i) \subset [c_j, b_j)$ for all $i \geq 1$ and $[c_j, b_j) \subset \bigcup_{i\geq 0} B_{n_j+2}(d_1, \ldots, d_{n_j}, s, m_0+i)$. Analysis similar to that in case (I) shows that there is a sequence $\{l_0 = 0, l_k\}_{k\geq 0}$ of integers such that

$$\Big|\bigcup_{i=1}^{l_1} B_{n_j+2}(m_0 + i)\Big|^\alpha \geq \sum_{k\geq 1}\Big|\bigcup_{i=1}^{l_{k+1}} B_{n_j+2}\Big(m_0 + \sum_{n=0}^{k} l_n + i\Big)\Big|^\alpha,$$

where $B_{n_j+2}(m) = B_{n_j+2}(d_1, \ldots, d_{n_j}, s, m)$. Set

$$J_0 = B_{n_j+2}(m_0), \qquad J_k = \bigcup_{i=1}^{l_k} B_{n_j+2}\Big(m_0 + \sum_{m=0}^{k-1} l_m + i\Big), \quad k \geq 1,$$

and $\mathcal{F}_{E_j} = \{J_i\}_{i\geq 0}$. We thus get

$$\sum_{J\in\mathcal{F}_{E_j}} |J|^\alpha \leq (6^\alpha + 2)|E_j|^\alpha \quad \text{and} \quad |J| \leq 6|E_j|, \quad J \in \mathcal{F}_{E_j}. \tag{3.7}$$

Summarizing, for a given open interval E_j, there exists a countable subfamily $\mathcal{F}_{E_j}$ of $\mathcal{B}$ which covers E_j such that

$$\sum_{J\in\mathcal{F}_{E_j}} |J|^\alpha \leq K(a)|E_j|^\alpha,$$

where $K(\alpha)$ is a constant independent of j. Consequently, for any $\alpha > 0$ and $E \subset [0, 1]$, it follows that

$$H^\alpha(E) \leq H^\alpha(E, \mathcal{A}) \leq K(\alpha)H^\alpha(E).$$

This gives $\dim_{\mathrm{H}}(E, \mathcal{B}) = \dim_{\mathrm{H}}(E)$, completing the proof of Theorem 2.3.

3.3. Some special cases. We now list some special cases of the main theorems.

Example 3.2 (Engel expansion). Let $\gamma_j(d_j) = 1/d_j$ $(j = 1, 2, \ldots)$. Then (2.2) and algorithm (2.1) lead to the *Engel expansion* of x,

$$x = \frac{1}{d_1(x)} + \frac{1}{d_1(x)d_2(x)} + \cdots + \frac{1}{d_1(x)\cdots d_n(x)} + \cdots.$$

By Theorems 2.2 and 2.3, we have

Corollary 3.3. *The covering family of all cylinders of Engel expansion is not Hausdorff dimension faithful, but the family of all finite unions of consecutive cylinders of the same rank is faithful.*

Example 3.4 (Sylvester expansion). Let $\gamma_j(d_j) = 1$ $(j = 1, 2, \ldots)$. Then (2.2) and algorithm (2.1) lead to the *Sylvester expansion* of x,

$$x = \frac{1}{d_1(x)} + \frac{1}{d_2(x)} + \cdots + \frac{1}{d_n(x)} + \cdots .$$

By Theorems 2.2 and 2.3, we have

Corollary 3.5. *The covering family of all cylinders of Sylvester expansion is not Hausdorff dimension faithful, but the family of all finite unions of consecutive cylinders of the same rank is faithful.*

Example 3.6 (Lüroth expansion). Let $\gamma_j(d_j) = \frac{1}{d_j(d_j-1)}$ $(j = 1, 2, \ldots)$. Then (2.2) and algorithm (2.1) lead to the *Lüroth expansion* of x,

$$x = \frac{1}{d_1(x)} + \frac{1}{d_1(d_1-1)d_2(x)} + \cdots + \frac{1}{d_1(d_1-1)\cdots d_{n-1}(d_{n-1}-1)d_n(x)} + \cdots .$$

By Theorems 2.2 and 2.3, we have the following result, which has been obtained in [2, 8].

Corollary 3.7. *The covering family of all cylinders of Lüroth expansion is not Hausdorff dimension faithful, but the family of all finite unions of consecutive cylinders of the same rank is faithful.*

Acknowledgements. This work is supported by NSFC(Grant Nos. 11501168, 11501255,11626030).

References

[1] S. Albeverio, G. Ivanenko, M. Lebid and G. Torbin, *On the Hausdorff dimension faithfulness and the Cantor series expansion*, arXiv:1305.6036 (2013).

[2] S. Albeverio,Y. Kondratiev, R. Nikiforov and G. Torbin, *On new fractal phenomena connected with infinite linear IFs*, Math. Nachr. 290 (2017), 1163–1176.

[3] S. Albeverio, M. Pratsiovytyi and G. Torbin, *Fractal probability distributions and transformations preserving the Hausdorff–Besicovitch dimension*, Ergodic Theory Dynam. Systems 24 (2004), 1–16.

[4] S. Albeverio and G. Torbin, *Fractal properties of singular probability distributions with independent Q^*-digits*, Bull. Sci. Math. 129 (2005), 356–367.

[5] P. Billingsley, *Ergodic Theory and Information*, Willey, New York, 1965.

[6] K. J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, 1990.

[7] J. Galambos, *Representations of Real Numbers by Infinite Series*, Lecture Notes in Math. 502, Springer, 1976.

[8] J. Liu and Z. L. Zhang, *On the Hausdorff dimension faithfulness connected with Q_∞-expansion*, Nonlinearity 30 (2017), 2268–2278.

[9] M. Pratsiovytyi and G. Torbin, *On analytic (symbolic) representation of one-dimensional continuous transformations preserving the Hausdorff–Besicovitch dimension*, Trans. National Pedagogical Univ. Phys.-Math. 4 (2003), 207–215.

[10] C. A. Rogers, *Hausdorff Measures*, Cambridge Univ. Press, London, 1970.

[11] B. W. Wang and J. Wu, *A problem of Galambos on Oppenheim series expansions*, Publ. Math. Debrecen 70 (2007), 45–58.

[12] J. Wu, *The Oppenheim series expansions and Hausdorff dimensions*, Acta Arith. 107 (2003), 345–355.

[13] J. Wu, *How many points have the same Engel and Sylvester expansions?*, J. Number Theory 103 (2003), 16–26.

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