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Truncated second main theorems and uniqueness theorems for non-Archimedean meromorphic maps

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Abstract. In this paper, several second main theorems are given for the non-Archimedean meromorphic map $f : \mathbb{F}^m \to \mathbb{P}^n$ intersecting hyperplanes in $\mathbb{P}^n$ in terms of the truncated counting functions defined by W. Cherry and C. Toropu, where $\mathbb{F}$ is an algebraically closed field of characteristic $p \geq 0$ complete with respect to a non-Archimedean absolute value. As an application, the uniqueness problem for non-Archimedean meromorphic maps sharing hyperplanes is also discussed.

1. Introduction. In 1920's, R. Nevanlinna [24] established the so-called value distribution theory for meromorphic functions over the complex number field. For each meromorphic function f, he introduced three functions of r, the distance from the origin. The characteristic function $T_f(r)$ measures the growth of f, the counting function $N_f(a,r)$ counts the number of times f takes the value a in the disc of radius r, and the proximity function $m_f(a,r)$ measures how close f is to a on the circle of radius r. As the generalization of the fundamental theorem of algebra, Nevanlinna showed two main theorems in terms of these functions. The first main theorem is just a reformulation of the Poisson–Jensen formula. However, the second main theorem is an elegant, deeper theorem which makes Nevanlinna theory a rich, non-trivial theory. For example, as an application of his second main theorem, Nevanlinna [25] showed a uniqueness theorem for meromorphic functions which is known as Nevanlinna's five-value theorem.

The classical Nevanlinna theory for meromorphic functions has been generalized to higher dimensional cases. For holomorphic curves in $\mathbb{P}^n$, H. Cartan [8] and Ahlfors [2] proved the second main theorem with hyperplanes in $\mathbb{P}^n$ located in general position, in which those holomorphic curves

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are assumed to be linearly non-degenerate. Nochka [26–28] considered the linearly degenerate case and solved Cartan's conjecture by establishing the so-called Nochka weights, and Ru [34, 35] gave the second main theorem for algebraically non-degenerate holomorphic curves intersecting hypersurface divisors. For meromorphic maps of several variables, Nevanlinna theory was generalized by Carlson, Griffiths, King, Stoll, Fujimoto and others (see [7, 17, 19, 30, 37] for instance). The higher dimensional uniqueness problem has also been much discussed (see e.g. [9, 15, 16, 18, 36, 38, 41]).

Motivated by the analogy between Nevanlinna theory and Diophantine approximation, the study of Nevanlinna theory over non-Archimedean fields has been developed over the past few decades (see e.g. [6, 11, 14, 22, 23]).

For the case of several variables, Cherry and Ye [13] developed non-Archimedean Nevanlinna theory and gave the following second main theorem for non-Archimedean meromorphic maps intersecting hyperplanes in $\mathbb{P}^n$.

Theorem A ([13]). *Let $\mathbb{F}$ be an algebraically closed field of characteristic 0 complete with respect to a non-Archimedean absolute value. Let $f : \mathbb{F}^m \to \mathbb{P}^n$ be a linearly non-degenerate non-Archimedean meromorphic map, represented by analytic coordinate functions $(f_0, \ldots, f_n)$ without common factors. Let $\gamma^1, \ldots, \gamma^n$ be multi-indices such that*

$$W = \det \begin{pmatrix} f_0 & \cdots & f_n \\ \partial^{\gamma^1} f_0 & \cdots & \partial^{\gamma^1} f_n \\ \vdots & \cdots & \vdots \\ \partial^{\gamma^n} f_0 & \cdots & \partial^{\gamma^n} f_n \end{pmatrix} \not\equiv 0.$$

Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n$ in general position. Then, for all $r > 1$,

$$(q-(n+1))T_f(r) \le \sum_{j=1}^{q} N_f(H_j, r) - N_W(0,r) - B\log r + O(1),$$

where $B = \sum_{i=1}^n |\gamma^i|$ and $O(1)$ depends only on $H_1, \ldots, H_q$ and f.

Cherry and Toropu [12] defined truncated counting functions for arbitrary characteristic and obtained the ABC theorems. They also pointed out that one could obtain the positive characteristic Cartan-type second main theorem for non-Archimedean meromorphic maps by using the proof of Theorem A.

The purpose of this paper is twofold. We first prove a general second main theorem for non-Archimedean meromorphic maps in several variables for arbitrary characteristic in terms of the truncated counting functions. Our main result is as follows.

Main Theorem. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $p \ge 0$, complete with respect to a non-Archimedean absolute value. Let*

$f = (f_0, \dots, f_n) : \mathbb{F}^m \to \mathbb{P}^n$ be a linearly non-degenerate non-Archimedean meromorphic map with index of independence s and with $\operatorname{rank} f = k$. *Let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^n$ in N-subgeneral position ($N \geq n$). Then, for all $r > 1$,*

$$(q - 2N + n - 1)T_f(r) \leq \sum_{j=1}^{q} N_f^{(\mathrm{a})}(H_j, r) - \frac{N+1}{n+1} \log r + O(1),$$

where

$$\mathrm{a} = \begin{cases} n - k + 1 & \text{if } p = 0, \\ p^{s-1}(n - k + 1) & \text{if } p > 0. \end{cases}$$

On the other hand, as an application of our main theorem, we will discuss the uniqueness problem for non-Archimedean meromorphic maps.

We organize our paper as follows. In Sections 2 and 3, we recall the basic concepts and notation for non-Archimedean analysis and value distribution theory of several variables introduced in [12] and [13]. In Sections 4 and 5 we state and prove several truncated second main theorems for non-Archimedean meromorphic maps in arbitrary characteristic. Finally, in Section 6 we discuss the uniqueness problem for non-Archimedean meromorphic maps.

2. Preliminaries on non-Archimedean analysis. We first give a brief introduction on some definitions and notation of non-Archimedean analysis. For more details, one can refer to [5, 12, 13].

2.1. Non-Archimedean number fields. Let $\mathbb{F}$ be an algebraically closed field of characteristic p, complete with respect to a non-Archimedean absolute value $|\ |$. The *non-Archimedean absolute value* $|\ |$ is a non-negative real-valued function satisfying the following three properties:

(i) $|a| = 0$ if and only if $a = 0$;
(ii) $|ab| = |a|\,|b|$ for all $a, b \in \mathbb{F}$;
(iii) $|a + b| \leq \max\{|a|, |b|\}$ for all $a, b \in \mathbb{F}$.

Let $\mathbb{F}^\times$ denote $\mathbb{F} \setminus \{0\}$, and let $|\mathbb{F}^\times| = \{|a| : a \in \mathbb{F}^\times\}$ be a subset of $\mathbb{R}$.

2.2. Analytic functions and meromorphic functions

2.2.1. *Notation.* Let $\mathbb{F}^m$ be the mth Cartesian product of $\mathbb{F}$. Let

$$B^m(r) = \Big\{(z_1, \dots, z_m) \in \mathbb{F}^m : \max_j |z_j| \leq r\Big\}$$

be the "closed" ball of radius r, which is actually both open and closed. For $z = (z_1, \dots, z_m) \in \mathbb{F}^m$ and a multi-index $\gamma = (\gamma_1, \dots, \gamma_m) \in \mathbb{Z}^m_{\geq 0}$, define

$$z^\gamma = z_1^{\gamma_1} \cdots z_m^{\gamma_m}, \qquad |\gamma| = \gamma_1 + \cdots + \gamma_m, \qquad \gamma! = \gamma_1! \cdots \gamma_m!.$$

2.2.2. *Analytic functions on $B^m(r)$.* By an *analytic function f on $B^m(r)$*, we mean a formal power series

$$\sum_\gamma a_\gamma z^\gamma$$

with $a_\gamma \in \mathbb{F}$ such that

$$\lim_{|\gamma|\to\infty} |a_\gamma| r^\gamma = 0.$$

If $f = \sum_\gamma a_\gamma z^\gamma$ is an analytic function on $B^m(r)$, define

$$|f|_r = \sup_\gamma |a_\gamma| r^{|\gamma|}.$$

If $r \in |\mathbb{F}^\times|$, then the ring of analytic functions on $B^m(r)$ is a Noetherian and unique factorization domain (see [5, §5.2.6, Theorem 1]).

2.2.3. *Analytic functions on $\mathbb{F}^m$.* We call the power series expansion $f = \sum_\gamma a_\gamma z^\gamma$ an *analytic function* on $\mathbb{F}^m$ if

$$\lim_{|\gamma|\to\infty} |a_\gamma| r^\gamma = 0$$

for all non-negative r. Denote by $\mathcal{E}_m$ the ring of analytic functions on $\mathbb{F}^m$. Since $f \in \mathcal{E}_m$ implies f is analytic on $B^m(r)$ for all r, we can define

$$|f|_r = \sup_\gamma |a_\gamma| r^{|\gamma|},$$

which is a non-decreasing function of r (see [12, Proposition 2.1]). Now we list some ring-theoretic properties of $\mathcal{E}_m$.

(i) The only *units* in $\mathcal{E}_m$ are the non-zero constant functions.
(ii) An element $P \in \mathcal{E}_m$ is called *irreducible* if whenever we write $P = fg$ with $f, g \in \mathcal{E}_m$, at least one of f and g is a unit in $\mathcal{E}_m$.
(iii) We say an analytic function g *divides* an analytic function f if $f = gh$ for some analytic function h.
(iv) Suppose f, g are non-unit in $\mathcal{E}_m$ such that g divides f in $\mathcal{E}_m$. If there exists a positive integer e such that

$$g^e \mid f \quad \text{and} \quad g^{e+1} \nmid f,$$

then we say g divides f with *multiplicity* e. (The existence of e is shown in [12].)
(v) Given two entire functions f_1 and f_2, denote by

$$g = \gcd(f_1, f_2)$$

the *greatest common divisor* of f_1 and f_2. (Although $\mathcal{E}_m$ is not factorial, the notion of greatest common divisor does make sense in $\mathcal{E}_m$—see [13, appendix].) And g is only defined up to a unit, hence a multiplicative constant.

2.2.4. *Meromorphic functions.* We define a *meromorphic function* f on $\mathbb{F}^m$ (or $B^m(r)$) to be the quotient of two analytic functions g, h such that g and h have no common factors in the ring of analytic functions on $\mathbb{F}^m$ (or $B^m(r)$), i.e.,

$$f = \frac{g}{h}.$$

Define

$$|f|_r = \frac{|g|_r}{|h|_r}.$$

For meromorphic functions f_1, f_2 on $\mathbb{F}^m$, we have

$$|f_1 + f_2|_r \leq \max\{|f_1|_r, |f_2|_r\} \quad \text{and} \quad |f_1 f_2|_r = |f_1|_r |f_2|_r.$$

We denote by $\mathcal{M}_m$ the field of meromorphic functions on $\mathbb{F}^m$, which is defined to be the fraction field of $\mathcal{E}_m$.

2.3. Derivatives and Hasse derivatives. For an analytic (or meromorphic) function f of m variables and a multi-index γ, denote by $\partial^\gamma f$ the ordinary partial derivative $\frac{\partial^{|\gamma|} f}{\partial z_1^{\gamma_1} \cdots \partial z_m^{\gamma_m}}$.

In positive characteristic, the so-called Hasse derivative is more useful.

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_m)$ be multi-indices. Set

$$\alpha + \beta := (\alpha_1 + \beta_1, \ldots, \alpha_m + \beta_m).$$

We say that $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for all $i \in \{1, \ldots, m\}$. For $\alpha \geq \beta$, set

$$\alpha - \beta := (\alpha_1 - \beta_1, \ldots, \alpha_m - \beta_m), \qquad \binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_m}{\beta_m}.$$

Given an analytic function $f = \sum_\alpha a_\alpha z^\alpha \in \mathcal{E}_m$ and a multi-index γ, we define the *Hasse derivative* of multi-index γ of f by

$$D^\gamma f = \sum_{\alpha \geq \gamma} \binom{\alpha}{\gamma} a_\alpha z^{\alpha - \gamma}.$$

We see that $D^\gamma f$ is also in $\mathcal{E}_m$.

In particular, if $\gamma = (0, \ldots, 0)$, then $D^\gamma f = f$; if $|\gamma| = 1$, then $D^\gamma f = \partial^\gamma f$; if $\operatorname{Char} \mathbb{F} = 0$, then $D^\gamma f = \partial^\gamma f / \gamma!$.

For $\gamma = (\gamma_1, \ldots, \gamma_m)$ with $\gamma_j = k$ and $\gamma_i = 0$ for $i \neq j$, set $D_j^k f := D^\gamma f$.

The Hasse derivative D^γ can be extended to meromorphic functions. For example, for $f = g/h \in \mathcal{M}_m$,

$$D_i^1 f = \frac{h D_i^1 g - g D_i^1 h}{h^2}, \qquad i = 1, \ldots, m.$$

The Hasse derivatives satisfy the following properties:

(i) $D^\gamma(f + g) = D^\gamma f + D^\gamma g$, $f, g \in \mathcal{M}_m$.
(ii) $D^\gamma(fg) = \sum_{\alpha + \beta = \gamma} D^\alpha f D^\beta g$, $f, g \in \mathcal{M}_m$.

(iii) $D^{\alpha}D^{\beta}f = \binom{\alpha+\beta}{\beta}D^{\alpha+\beta}f$, $f \in \mathcal{M}_m$.
(iv) (Logarithmic derivative lemma) For $f \in \mathcal{E}_m$,

$$|D^{\gamma}f|_r \leq \frac{|f|_r}{r^{|\gamma|}}, \qquad |\partial^{\gamma}f|_r \leq \frac{|f|_r}{r^{|\gamma|}}.$$

(v) Let f be an entire function in $\mathcal{E}_m$. Let γ be a multi-index. Let P be an irreducible element of $\mathcal{E}_m$ that divides f with exact multiplicity e. If $e > |\gamma|$, then $P^{e-|\gamma|}$ divides $D^{\gamma}f$.

For each integer $k \geq 2$, let

$$\mathcal{M}_m[k] = \{Q \in \mathcal{M}_m : D_j^i Q \equiv 0 \text{ for all } 0 < i < k \text{ and } 1 \leq j \leq m\}.$$

If $\mathbb{F}$ has characteristic 0, then $\mathcal{M}_m[k] = \mathbb{F}$ for all $k \geq 2$. If $\mathbb{F}$ has characteristic $p > 0$ and if $s \geq 1$ is an integer, then $\mathcal{M}_m[p^s]$ is the fraction field of $\mathcal{E}_m[p^s]$ where $\mathcal{E}_m[p^s] = \{g^{p^s} : g \in \mathcal{E}_m\}$ is a subring of $\mathcal{E}_m$. Moreover,

$$\mathcal{M}_m[p^{s-1}+1] = \mathcal{M}_m[p^s].$$

3. Non-Archimedean value distribution theory

3.1. Value distribution theory for non-Archimedean meromorphic functions

3.1.1. *Counting functions.* Let $f = \sum_{\gamma} a_{\gamma} z^{\gamma} \in \mathcal{E}_m$ be an entire function. For $r > 0$, define

$$n_f(0,r) = \sup\{|\gamma| : |a_{\gamma}|r^{|\gamma|} = |f|_r\},$$
$$n_f(0,0) = \lim_{r\to 0} n_f(0,r) = \min\{|\gamma| : a_{\gamma} \neq 0\}.$$

We define the counting function of zeros of f by

$$N_f(0,r) = n_f(0,0)\log r + \int_0^r (n_f(0,t) - n_f(0,0))\,\frac{dt}{t}.$$

If $f \in \mathcal{E}_m$ is non-constant, then

$$N_f(0,r) \geq \log r + O(1) \quad \text{ for } r \geq 1.$$

If $f = g/h$ is a meromorphic function in which g and h have no common factors, then define

$$N_f(\infty,r) = N_h(0,r) \quad \text{and} \quad N_f(0,r) = N_g(0,r).$$

If f is not identically equal to $a \in \mathbb{F}$, then define

$$N_f(a,r) = N_{f-a}(0,r).$$

We now list some properties which can be found in [12] and [13].

(i) Let $f_1, f_2 \in \mathcal{E}_m$ be entire functions such that $f_1 \mid f_2$. Then

$$N_{f_1}(0,r) \leq N_{f_2}(0,r) \quad \text{ for all } r \geq 1.$$

(ii) Let f, g be meromorphic functions on $\mathbb{F}^m$. Then for all $r > 0$,

$$N_{fg}(0,r) = N_f(0,r) + N_g(0,r),$$
$$N_{f+g}(\infty,r) \le N_f(\infty,r) + N_g(\infty,r),$$
$$N_{fg}(\infty,r) = N_f(\infty,r) + N_g(\infty,r).$$

(iii) (Poisson–Jensen–Green formula) Let f be a meromorphic function. Then

$$N_f(0,r) = \log|f|_r + C_f \quad \text{for all } r,$$

where C_f is a constant depending on f but not r.

3.1.2. *Proximity functions.* Let f be a meromorphic function and a be an element of $\mathbb{F}$. We define the *proximity functions* of ∞ and a by

$$m_f(\infty,r) = \max\{0, \log|f|_r\} = \log^+|f|_r, \qquad m_f(a,r) = m_{1/(f-a)}(\infty,r).$$

3.1.3. *Characteristic functions.* For $f \in \mathcal{M}_m$, we define the *characteristic function* of f by

$$T_f(r) = m_f(\infty,r) + N_f(\infty,r).$$

Note that $T_f(r) = \max\{\log|g|_r, \log|h|_r\} + O(1)$ if $f = g/h$. We have the following first main theorem.

The first main theorem for meromorphic functions. *Let f be a meromorphic function and let $a \in \mathbb{F}$. Then*

$$T_f(r) = m_f(a,r) + N_f(a,r) + O(1).$$

3.2. Truncated counting functions. We note that $N_f(0,r)$ counts the zeros of f with multiplicities. Now we will introduce the truncated counting functions defined in [12].

3.2.1. *Radical of f.* Let $f \in \mathcal{E}_m$. For $j = 1, \dots, m$, define

$$g_j = \gcd(f, D_j^1 f) \quad \text{and} \quad h_j = \frac{f}{g_j}.$$

Define $R(f)$ to be the least common multiple of the h_j, which is called the *radical* of f. We note that $R(f)$ has the following properties:

(i) $R(f)$ is square free, i.e., $g^2 \nmid R(f)$ for any non-constant $g \in \mathcal{E}_m$.
(ii) $R(f) \mid f$.
(iii) Let $P \in \mathcal{E}_m$ be an irreducible element such that $P \mid f$ with multiplicity e. Then $P \mid R(f)$ if and only if $p \nmid e$, where p is the characteristic of $\mathbb{F}$. For $p = 0$, $R(f)$ is the square free part of f; for $p > 0$, $R(f)$ does not contain those irreducible factors of f which divide f with multiplicity divisible by p.
(iv) For $p = 0$ and an integer $\ell \ge 1$, if $P \in \mathcal{E}_m$ is irreducible such that $P \mid f$ with multiplicity e, then $P \mid \gcd(f, R(f)^\ell)$ with multiplicity $\min\{\ell, e\}$.

Definition 3.1. For $p=0$, set

$$N_f^{(\ell)}(0,r)=N_{\gcd(f,R(f)^\ell)}(0,r).$$

In particular,

$$N_f^{(1)}(0,r)=N_{R(f)}(0,r).$$

Remark 3.2. If $p>0$, then $R(f)$ does not contain those irreducible factors of f which divide f with multiplicity divisible by p, i.e., if P is an irreducible element in $\mathcal{E}_m$ such that $P\,|\,f$ with multiplicity e, then P divides $\gcd(f,R(f)^\ell)$ with multiplicity

$$\begin{cases}\min\{\ell,e\}, & p\nmid e,\\ 0, & p\,|\,e.\end{cases}$$

Hence, in positive characteristic, $N_{\gcd(f,R(f)^\ell)}(0,r)$ might be called the overly truncated counting function.

3.2.2. *Higher p^s-radicals of f.* Let $f\in\mathcal{E}_m$. For $s=0$, set $R_{p^0}(f)=R(f)$. We define $R_{p^s}(f)$ by induction. For $s\geq 1$, assume that $R_{p^{s-1}}(f)$ has been defined. Set

$$\bar{f}=\frac{f}{\gcd(f,R_{p^{s-1}}(f)^{p^s})},\qquad g_i=\gcd(\bar{f},D_i^{p^s}\bar{f}),\quad h_i=\frac{\bar{f}}{g_i}$$

for $i=1,\dots,m$. Let H be the least common multiple of the h_i, and set

$$G=\frac{H}{\gcd(H,R_{p^{s-1}}(H)^{p^{s-1}})},$$

which is a p^sth power. Let R be a p^sth root of G. Define $R_{p^s}(f)$ to be the least common multiple of $R_{p^{s-1}}(f)$ and R, which is called the *higher p^s-radical* of f. It has the following properties:

(i) $R_{p^s}(f)$ is square free.
(ii) If $P\in\mathcal{E}_m$ is irreducible such that $P\,|\,f$ with multiplicity e, then $P\,|\,R_{p^s}(f)$ if and only if $p^{s+1}\,|\,e$.

3.2.3. *Square free part of f.* Take a sequence $\{r_\iota\}_{\iota\in\mathbb{N}}\subset|\mathbb{F}^\times|$ such that $r_\iota\to\infty$. For $B^m(r_\iota)$, take s_ι such that if $P\in\mathcal{E}_m$ is irreducible such that $P\,|\,f$ and P is not a unit on $B^m(r_\iota)$, then $P\,|\,R_{p^s}(f)$ for $s\geq s_\iota$. Let $u_{\iota,\iota+1}$ be a unit on $B^m(r_\iota)$ such that

$$R_{p^{s_\iota}}(f)=u_{\iota,\iota+1}R_{p^{s_{\iota+1}}}(f).$$

Define $v_\iota=\prod_{\eta=\iota}^{\infty}u_{\eta,\eta+1}$, which is a unit on $B^m(r_\iota)$. Define

$$S(f)=\lim_{\iota\to\infty}\frac{R_{p^{s_\iota}}(f)}{v_\iota}\in\mathcal{E}_m,$$

which is called the *square free part* of f. Thus, $S(f)$ has the following properties:

(i) $S(f)$ is square free.
(ii) Let $P \in \mathcal{E}_m$ be an irreducible element. Then $P \mid f$ if and only if $P \mid S(f)$. Hence, $S(f)$ contains all factors of f. Moreover, for $\ell \geq 1$, if $P \mid f$ with multiplicity e, then $P \mid \gcd(f, S(f)^\ell)$ with multiplicity $\min\{\ell, e\}$.

DEFINITION 3.3. For any characteristic and $\ell \geq 1$, define the *ℓth truncated counting function* of the zeros of $f \in \mathcal{E}_m$ by

$$N_f^{(\ell)}(0, r) = N_{\gcd(f, S(f)^\ell)}(0, r).$$

3.3. Value distribution theory for non-Archimedean meromorphic maps. Let $\mathbb{P}^n = \mathbb{P}^n(\mathbb{F})$ denote the projective n-space over $\mathbb{F}$. Let $f : \mathbb{F}^m \to \mathbb{P}^n$ be a non-Archimedean meromorphic map defined by $(f_0, \ldots, f_n)$ such that $f_0, \ldots, f_n \in \mathcal{E}_m$ have no common factors and not all of the f_i are identically zero. We note that $(f_0, \ldots, f_n)$ and $(g_0, \ldots, g_n)$ define the same meromorphic map if there exists a constant C such that $f_i = Cg_i$, $i = 0, \ldots, n$.

The *characteristic function* of $f = (f_0, \ldots, f_n) : \mathbb{F}^m \to \mathbb{P}^n$ is defined by

$$T_f(r) = \log \|f\|_r,$$

where $\|f\|_r := \max_i |f_i|_r$.

Let $H = \{(x_0, \ldots, x_n) \in \mathbb{P}^n : a_0 x_0 + \cdots + a_n x_n = 0,\ a_i \in \mathbb{F} \text{ for } i = 0, \ldots, n\}$ be a hyperplane in $\mathbb{P}^n$. Assume that the image of f is not contained in H, i.e., $(f, H) = \sum_i a_i f_i \not\equiv 0$.

Define the *proximity function* of f and H by

$$m_f(H, r) = \log \frac{\|f\|_r \|H\|}{|(f, H)|_r},$$

where $\|H\| := \max_i |a_i|$.

Define the *counting function* (counted with multiplicity) of f and H by

$$N_f(H, r) := N_{(f,H)}(0, r) \ (= \log |(f, H)|_r + O(1)).$$

Similarly, for $\ell \geq 1$, we can define the ℓth truncated counting function $N_f^{(\ell)}(H, r)$.

THE FIRST MAIN THEOREM FOR MEROMORPHIC MAPS. *Let f be a non-Archimedean meromorphic map to $\mathbb{P}^n$, and H be a hyperplane in $\mathbb{P}^n$ such that $(f, H) \not\equiv 0$. Then*

$$T_f(r) = m_f(H, r) + N_f(H, r) + O(1).$$

4. Second main theorems for non-Archimedean meromorphic maps. In this section, we will state some of our main results on the second main theorem for non-Archimedean meromorphic maps.

4.1. Linear degeneracy and Wronskians. A non-Archimedean meromorphic map $f=(f_0,\dots,f_n):\mathbb{F}^m\to\mathbb{P}^n$ is called *linearly non-degenerate* if the image of f is not contained in any proper subspace of $\mathbb{P}^n$ or $f_0,\dots,f_n$ are linearly independent over $\mathbb{F}$. If $\operatorname{Char}\mathbb{F}=p>0$, then there exists an integer $s\geq 1$ such that $f_0,\dots,f_n$ are linearly independent over $\mathcal{M}_m[p^s]$ (see [12, Lemma 5.2]). Hence, $f_0,\dots,f_n$ are also linearly independent over $\mathcal{M}_m[p^{s+1}]$.

Definition 4.1.

(i) Let $f=(f_0,\dots,f_n):\mathbb{F}^m\to\mathbb{P}^n$ be a linearly non-degenerate non-Archimedean meromorphic map. Denote by s the smallest integer such that $f_0,\dots,f_n$ are linearly independent over $\mathcal{M}_m[p^s]$.
(ii) Let $f=(f_0,\dots,f_n):\mathbb{F}^m\to\mathbb{P}^n$ be a non-constant non-Archimedean meromorphic map. Denote by s the smallest integer such that any subset of functions in $\{f_0,\dots,f_n\}$ linearly independent over $\mathbb{F}$ remains linearly independent over $\mathcal{M}_m[p^s]$.

We call the s in (i) and (ii) the *index of independence* of f.

Let f be a non-Archimedean meromorphic map from $\mathbb{F}^m$ to $\mathbb{P}^n$ defined by $f=(f_0,\dots,f_n)$ with $f_j\in\mathcal{E}_m$ for all j. For a multi-index γ, set

$$D^\gamma f=(D^\gamma f_0,\dots,D^\gamma f_n)\in\mathcal{M}_m^{n+1}.$$

For $\kappa=0,1,\dots,$ denote by $\mathcal{F}^\kappa$ the $\mathcal{M}_m$-subspace of $\mathcal{M}_m^{n+1}$ which is generated by $\{D^\gamma f:|\gamma|\leq\kappa\}$. Set $l(\kappa):=\dim_{\mathcal{M}_m}\mathcal{F}^\kappa$. Obviously, $\mathcal{F}^{\kappa_1}\supseteq\mathcal{F}^{\kappa_2}$ and $l(\kappa_1)\geq l(\kappa_2)$ if $\kappa_1\geq\kappa_2$.

For $\kappa=0$, the subspace generated by $\{f\}$ has dimension 1, i.e., $l(0)=1$.

For $\kappa=1$, the dimension of the subspace generated by $\{f,D_i^1 f:1\leq i\leq m\}$ is equal to the rank of

$$\begin{pmatrix} f_0 & \cdots & f_n\\ D_1^1 f_0 & \cdots & D_1^1 f_n\\ \vdots & \cdots & \vdots\\ D_m^1 f_0 & \cdots & D_m^1 f_n\end{pmatrix}=\begin{pmatrix} f_0 & \cdots & f_n\\ \frac{\partial f_0}{\partial z_1} & \cdots & \frac{\partial f_n}{\partial z_1}\\ \vdots & \cdots & \vdots\\ \frac{\partial f_0}{\partial z_m} & \cdots & \frac{\partial f_n}{\partial z_m}\end{pmatrix}.$$

Remark 4.2. We note that for a meromorphic map $f=(f_0,\dots,f_n):\mathbb{C}^m\to\mathbb{P}^n(\mathbb{C})$, the rank of df is equal to

$$\operatorname{rank}\begin{pmatrix} f_0 & \cdots & f_n\\ \frac{\partial f_0}{\partial z_1} & \cdots & \frac{\partial f_n}{\partial z_1}\\ \vdots & \cdots & \vdots\\ \frac{\partial f_0}{\partial z_m} & \cdots & \frac{\partial f_n}{\partial z_m}\end{pmatrix}-1,$$

where $\mathbb{C}$ is the complex number field. Hence, for a non-Archimedean meromorphic map $f : \mathbb{F}^m \to \mathbb{P}^n$, we can similarly define the *rank* of f, written as $\operatorname{rank} f$, such that

$$l(1) = \operatorname{rank} f + 1 \leq \max\{m+1, n+1\}.$$

For $j = 0, \dots, n$, denote by $(D^\gamma f_j)_\kappa$ the vector including all Hasse derivatives of f_j with $|\gamma| \leq \kappa$ arranged in some fixed order. We note that if $l(\kappa) < n+1$, then

$$(D^\gamma f_0)_\kappa, \dots, (D^\gamma f_n)_\kappa$$

are linearly dependent over $\mathcal{M}_m$.

PROPOSITION 4.3. *A non-Archimedean meromorphic map f is linearly non-degenerate if and only if there exists κ_0 such that $l(\kappa_0) = n+1$.*

Proof. If f is linearly degenerate, then there exists $(a_0, \dots, a_n) \neq (0, \dots, 0)$ such that

$$a_0 f_0 + \dots + a_n f_n \equiv 0.$$

Hence, for any γ,

$$a_0 D^\gamma f_0 + \dots + a_n D^\gamma f_n \equiv 0.$$

This implies that $l(\kappa) < n+1$ for all κ.

On the other hand, assume that f is linearly non-degenerate. We claim that there exists κ_0 such that $l(\kappa_0) = n+1$.

Indeed, otherwise $\max_\kappa l(\kappa) < n+1$. Set $\max_\kappa l(\kappa) := L+1$ and $l(\kappa) = L+1$ for $\kappa \geq \kappa_0$. We may assume that

$$(D^\gamma f_0)_{\kappa_0}, \dots, (D^\gamma f_L)_{\kappa_0}$$

are linearly independent over $\mathcal{M}_m$. Moreover, we may assume that, for any fixed κ $(> \kappa_0)$, there exist $Q_0, \dots, Q_L \in \mathcal{M}_m$ not all zero such that

$$Q_0 D^\gamma f_0 + \dots + Q_L D^\gamma f_L + D^\gamma f_{L+1} \equiv 0 \tag{4.1}$$

for every γ with $|\gamma| \leq \kappa$.

Now we show $Q_j \in \mathcal{M}_m[K]$ for all $K \leq \kappa - \kappa_0 + 1$, which contradicts f being linearly non-degenerate.

Taking the Hasse derivative D_i^1 of both sides of (4.1), we obtain

$$\begin{aligned} 0 &= D_i^1(Q_0 D^\gamma f_0 + \dots + Q_L D^\gamma f_L + D^\gamma f_{L+1}) \\ &= D_i^1 Q_0 \cdot D^\gamma f_0 + \dots + D_i^1 Q_L \cdot D^\gamma f_L + Q_0 D_i^1 D^\gamma f_0 + \dots + Q_L D_i^1 D^\gamma f_L \\ &\quad + D_i^1 D^\gamma f_{L+1} \\ &= D_i^1 Q_0 \cdot D^\gamma f_0 + \dots + D_i^1 Q_L \cdot D^\gamma f_L \end{aligned}$$

for $|\gamma| \leq \kappa_0$ and $i = 1, \dots, m$. By the linear independence of

$$(D^\gamma f_0)_{\kappa_0}, \dots, (D^\gamma f_L)_{\kappa_0},$$

we have

$$D_i^1 Q_0 = \cdots = D_i^1 Q_L \equiv 0 \quad \text{for } i = 1, \ldots, m.$$

Hence $Q_0, \ldots, Q_L \in \mathcal{M}_m[2]$.

Assume that we have proved $Q_0, \ldots, Q_L \in \mathcal{M}_m[K-1]$; we now show $Q_0, \ldots, Q_L \in \mathcal{M}_m[K]$.

Taking the Hasse derivative D_i^{K-1} of both sides of (4.1), we obtain, for $|\gamma| \leq \kappa_0$ and all i,

$$\begin{aligned} 0 &= D_i^{K-1}(Q_0 D^\gamma f_0 + \cdots + Q_L D^\gamma f_L + D^\gamma f_{L+1}) \\ &= D_i^{K-1} Q_0 \cdot D^\gamma f_0 + \cdots + D_i^{K-1} Q_L \cdot D^\gamma f_L + Q_0 D_i^{K-1} D^\gamma f_0 + \cdots \\ &\quad + Q_L D_i^{K-1} D^\gamma f_L + D_i^{K-1} D^\gamma f_{L+1} \\ &= D_i^{K-1} Q_0 \cdot D^\gamma f_0 + \cdots + D_i^{K-1} Q_L \cdot D^\gamma f_L. \end{aligned}$$

Hence $D_i^{K-1} Q_0 = \cdots = D_i^{K-1} Q_L \equiv 0$ for $i = 1, \ldots, m$, i.e., $Q_0, \ldots, Q_L \in \mathcal{M}_m[K]$. ∎

Assume that f is linearly non-degenerate with index of independence s. Set

$$\kappa_0 = \min\{\kappa : l(\kappa) = n+1\}.$$

Pick $\kappa^0 := 0 < \kappa^1 < \cdots < \kappa^l = \kappa_0$ such that

$$1 = l(0) < l(\kappa^1) < l(\kappa^2) < \cdots < l(\kappa_0) = n+1,$$

so

$$\mathcal{F}^0 \subsetneq \mathcal{F}^{\kappa^1} \subsetneq \mathcal{F}^{\kappa^2} \subsetneq \cdots \subsetneq \mathcal{F}^{\kappa_0} = \mathcal{M}_m^{n+1}.$$

We now show $\kappa^u \leq \kappa^{u-1} + p^{s-1}$.

Set $\dim_{\mathcal{M}_m} \mathcal{F}^{\kappa^{u-1}} := M$. The rank of the set of vectors $(D^\gamma f_0)_{\kappa^{u-1}}, \ldots, (D^\gamma f_n)_{\kappa^{u-1}}$ is M. Without loss of generality, $(D^\gamma f_0)_{\kappa^{u-1}}, \ldots, (D^\gamma f_{M-1})_{\kappa^{u-1}}$ are linearly independent over $\mathcal{M}_m$.

Claim. *There exists an integer κ^u with $\kappa^u \leq \kappa^{u-1} + p^{s-1}$ such that*

$$\dim_{\mathcal{M}_m} \mathcal{F}^{\kappa^u} = M + 1.$$

Proof. Otherwise, for $l' := \kappa^{u-1} + p^{s-1}$, we may assume there exist

$$Q_0, \ldots, Q_{M-1}$$

in $\mathcal{M}_m$ not all zero such that

$$Q_0 D^\gamma f_0 + \cdots + Q_{M-1} D^\gamma f_{M-1} + D^\gamma f_M \equiv 0 \quad \text{for all } |\gamma| \leq l'. \tag{4.2}$$

Now we show $Q_0, \ldots, Q_{M-1} \in \mathcal{M}_m[p^s]$.

Taking the Hasse derivative D_i^1 of both sides of (4.2), we obtain

$$\begin{aligned}0 &= D_i^1(Q_0 D^\gamma f_0 + \cdots + Q_{M-1} D^\gamma f_{M-1} + D^\gamma f_M)\\ &= D_i^1 Q_0 \cdot D^\gamma f_0 + \cdots + D_i^1 Q_{M-1} \cdot D^\gamma f_{M-1} + Q_0 D_i^1 D^\gamma f_0 + \cdots\\ &\quad + Q_{M-1} D_i^1 D^\gamma f_{M-1} + D_i^1 D^\gamma f_M\\ &= D_i^1 Q_0 \cdot D^\gamma f_0 + \cdots + D_i^1 Q_{M-1} \cdot D^\gamma f_{M-1}\end{aligned}$$

for $|\gamma| \le \kappa^{u-1}$ and $i = 1, \ldots, m$. We have

$$D_i^1 Q_0 = \cdots = D_i^1 Q_{M-1} \equiv 0 \quad \text{for } i = 1, \ldots, m.$$

Hence $Q_0, \ldots, Q_{M-1} \in \mathcal{M}_m[2] = \mathcal{M}_m[p^0 + 1] = \mathcal{M}_m[p]$. Similarly, we can show that if $Q_0, \ldots, Q_{M-1} \in \mathcal{M}_m[p^{s-1}]$, then $Q_0, \ldots, Q_{M-1} \in \mathcal{M}_m[p^s]$. Taking the Hasse derivative $D_i^{p^{s-1}}$ of both sides of (4.2), we get

$$\begin{aligned}0 &= D_i^{p^{s-1}}(Q_0 D^\gamma f_0 + \cdots + Q_{M-1} D^\gamma f_{M-1} + D^\gamma f_M)\\ &= D_i^{p^{s-1}} Q_0 \cdot D^\gamma f_0 + \cdots + D_i^{p^{s-1}} Q_{M-1} \cdot D^\gamma f_{M-1} + Q_0 D_i^{p^{s-1}} D^\gamma f_0 + \cdots\\ &\quad + Q_{M-1} D_i^{p^{s-1}} D^\gamma f_{M-1} + D_i^{p^{s-1}} D^\gamma f_M\\ &= D_i^{p^{s-1}} Q_0 \cdot D^\gamma f_0 + \cdots + D_i^{p^{s-1}} Q_{M-1} \cdot D^\gamma f_{M-1}\end{aligned}$$

for $|\gamma| \le \kappa^{u-1}$ and $i = 1, \ldots, m$. We have $D_i^{p^{s-1}} Q_0 = \cdots = D_i^{p^{s-1}} Q_{M-1} \equiv 0$. Hence $Q_0, \ldots, Q_{M-1} \in \mathcal{M}_m[p^{s-1} + 1] = \mathcal{M}_m[p^s]$, which contradicts f being linearly non-degenerate over $\mathcal{M}_m[p^s]$. ∎

Now, we estimate l and κ_0. For any non-Archimedean meromorphic map f with index of independence s, it is easy to see that

$$n + 1 = l(\kappa_0) = \sum_{u=1}^{l} (l(\kappa^u) - l(\kappa^{u-1})) + l(0) \ge l + 1,$$

i.e., $l \le n$ and

$$\kappa_0 \le \begin{cases} p^{s-1} n & \text{if } p > 0,\\ n & \text{if } p = 0.\end{cases}$$

In particular, for a non-Archimedean meromorphic map with positive rank, we have the following proposition.

Proposition 4.4. *If* $\operatorname{rank} f = k$, *then* $l \le n - k + 1$.

Proof. If $k > 0$, then $\kappa^1 = 1$ and

$$n + 1 = l(\kappa_0) = \sum_{u=2}^{l} (l(\kappa^u) - l(\kappa^{u-1})) + l(1) \ge (l - 1) + k + 1,$$

i.e., $l \le n - k + 1$. ∎

We have

$$\kappa_0 \leq \begin{cases} p^{s-1}(n-k+1) & \text{if } p>0, \\ n-k+1 & \text{if } p=0. \end{cases} \tag{4.3}$$

Assume that $f=(f_0,\dots,f_n):\mathbb{F}^m\to\mathbb{P}^n$ is linearly non-degenerate over $\mathbb{F}$ (or $\mathcal{M}_m[p^s]$). Set $\kappa_0=\min\{\kappa : l(\kappa)=n+1\}$. We can take multi-indices $\gamma^0=(0,\dots,0),\gamma^1,\dots,\gamma^n$ with

$$|\gamma^0|\leq\cdots\leq|\gamma^n|$$

such that $\{D^{\gamma^0}f,\dots,D^{\gamma^n}f\}$ is a basis of $\mathcal{M}_m^{n+1}$. We have $|\gamma^i|\leq\kappa_0$, $i=0,1,\dots,n$. Then the *generalized Wronskian* satisfies

$$W_{\gamma^0\dots\gamma^n}(f_0,\dots,f_n)=\det\begin{pmatrix} D^{\gamma^0}f_0 & \cdots & D^{\gamma^0}f_n \\ \vdots & \cdots & \vdots \\ D^{\gamma^n}f_0 & \cdots & D^{\gamma^n}f_n \end{pmatrix}\not\equiv 0.$$

For $n+1$ hyperplanes $H_j=\{(x_0,\dots,x_n)\in\mathbb{P}^n : a_{j0}x_0+\cdots+a_{jn}x_n=0$ with $a_{ji}\in\mathbb{F}\}$, $j=1,\dots,n+1$, we have

$$\det\begin{pmatrix} D^{\gamma^0}(f,H_1) & \cdots & D^{\gamma^0}(f,H_{n+1}) \\ \vdots & \cdots & \vdots \\ D^{\gamma^n}(f,H_1) & \cdots & D^{\gamma^n}(f,H_{n+1}) \end{pmatrix}$$
$$=\det\begin{pmatrix} D^{\gamma^0}f_0 & \cdots & D^{\gamma^0}f_n \\ \vdots & \cdots & \vdots \\ D^{\gamma^n}f_0 & \cdots & D^{\gamma^n}f_n \end{pmatrix}\cdot\det\begin{pmatrix} a_{10} & \cdots & a_{n+1,0} \\ \vdots & \cdots & \vdots \\ a_{1n} & \cdots & a_{n+1,n} \end{pmatrix},$$

where $D^{\gamma^i}(f,H_j)=a_{j0}D^{\gamma^i}f_0+\cdots+a_{jn}D^{\gamma^i}f_n$, $j=1,\dots,n+1$, $i=1,\dots,n$. If $(a_{j0},\dots,a_{jn})$, $j=1,\dots,n+1$, are linearly independent, then

$$\det\begin{pmatrix} D^{\gamma^0}(f,H_1) & \cdots & D^{\gamma^0}(f,H_{n+1}) \\ \vdots & \cdots & \vdots \\ D^{\gamma^n}(f,H_1) & \cdots & D^{\gamma^n}(f,H_{n+1}) \end{pmatrix}=C\cdot\det\begin{pmatrix} D^{\gamma^0}f_0 & \cdots & D^{\gamma^0}f_n \\ \vdots & \cdots & \vdots \\ D^{\gamma^n}f_0 & \cdots & D^{\gamma^n}f_n \end{pmatrix}$$

with $C\neq 0$. (See [20] for more properties of the generalized Wronskian.)

4.2. Second main theorems for a non-Archimedean meromorphic map intersecting hyperplanes. We first introduce the definition of general position.

Definition 4.5. Let $N\geq n$ be an integer and let H_j, $j=1,\dots,q$, with $q\geq N+1$, be hyperplanes in $\mathbb{P}^n$ defined by $H_j=\{(x_0,\dots,x_n)\in\mathbb{P}^n : a_{j0}x_0+\cdots+a_{jn}x_n=0$ with $a_{ji}\in\mathbb{F}\}$. We say that $H_1,\dots,H_q$ are *in N-subgeneral position* if for any $1\leq j_0<\cdots<j_N\leq q$, we have $\bigcap_{u=0}^{N}H_{j_u}=\emptyset$.

If the hyperplanes are in n-subgeneral position, we simply say that they are *in general position.*

Let $L_j = (a_{j0}, \dots, a_{jn})$ be the non-zero vector in $\mathbb{F}^{n+1}$ defined by the coefficients of H_j for $1 \le j \le q$. We note that if $H_1, \dots, H_q$ are in N-subgeneral position, then for any $1 \le j_0 < \cdots < j_N \le q$, there exists a subset I of $\{j_0, \dots, j_N\}$ such that $\sharp I = n+1$ and L_j, $j \in I$, are linearly independent.

We obtain the following second main theorem.

THEOREM 4.6. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $p \ge 0$ complete with respect to a non-Archimedean absolute value. Let $f : \mathbb{F}^m \to \mathbb{P}^n$ be a linearly non-degenerate non-Archimedean meromorphic map with index of independence s and* $\operatorname{rank} f = k$. *Let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^n$ in N-subgeneral position $(N \ge n)$. Then, for all $r > 1$,*

$$(q - 2N + n - 1)T_f(r) \le \sum_{j=1}^{q} N_f^{(\mathrm{a})}(H_j, r) - \frac{N+1}{n+1}\log r + O(1),$$

where

$$\mathrm{a} = \begin{cases} n-k+1 & \text{if } p = 0, \\ p^{s-1}(n-k+1) & \text{if } p > 0. \end{cases}$$

The proof of Theorem 4.6 will be given in the next section.

REMARK 4.7. This result is a non-Archimedean generalization of [29, Theorem 10.1]. For $m = 1$ and $\operatorname{Char}\mathbb{F} = 0$, Theorem 4.6 is just [21, Theorem 6.26].

Moreover, we have the following result.

THEOREM 4.8. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $p \ge 0$ complete with respect to a non-Archimedean absolute value. Let $f : \mathbb{F}^m \to \mathbb{P}^n$ be a non-constant non-Archimedean meromorphic map with index of independence s and* $\operatorname{rank} f = k$. *Let $H_1, \dots, H_q$ with $q \ge 2n+1$ be hyperplanes in $\mathbb{P}^n$ in general position. Assume that $(f, H_j) \not\equiv 0$ for $j = 1, \dots, q$. Then, for all $r > 1$,*

$$(q - 2n)T_f(r) \le \sum_{j=1}^{q} N_f^{(\mathrm{a})}(H_j, r) - \log r + O(1),$$

where a *is as in Theorem* 4.6.

Proof. Assume that the image of f is contained in a subspace of dimension l, but in no subspace of smaller dimension. Without loss of generality, we assume that $f(\mathbb{F}^m) \subset \mathbb{P}^l$. Then $f : \mathbb{F}^m \to \mathbb{P}^l$ is linearly non-degenerate. Let $\hat{H}_j = H_j \cap \mathbb{P}^l$, a hyperplane in $\mathbb{P}^l$. Since $H_1, \dots, H_q$ are in general position in $\mathbb{P}^n$, it is easy to see that $\hat{H}_1, \ldots, \hat{H}_q$ are in n-subgeneral position in $\mathbb{P}^l$.

Applying Theorem 4.6 to f and $\hat{H}_1, \ldots, \hat{H}_q$, we obtain

$$(q - 2n + l - 1)T_f(r) \leq \sum_{j=1}^{q} N_f^{(\hat{a})}(\hat{H}_j, r) - \frac{n+1}{l+1} \log r + O(1),$$

where

$$\hat{a} = \begin{cases} l - k + 1 & \text{if } p = 0, \\ p^{s-1}(l - k + 1) & \text{if } p > 0. \end{cases}$$

Hence the assertion follows. ■

4.3. Defect relation

Definition 4.9. Let $f : \mathbb{F}^m \to \mathbb{P}^n$ be a non-Archimedean meromorphic map and let H be a hyperplane in $\mathbb{P}^n$. We define the *defect* of H for f to be

$$\delta_f(H) = \liminf_{r \to \infty} \left\{ 1 - \frac{N_f(H, r)}{T_f(r)} \right\}.$$

By Theorems 4.6 and 4.8, we have the following defect relation.

Corollary 4.10.

(i) *Let $f : \mathbb{F}^m \to \mathbb{P}^n$ be a linearly non-degenerate non-Archimedean meromorphic map, and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n$ in N-subgeneral position $(N \geq n)$. Then*

$$\sum_{j=1}^{q} \delta_f(H_j) \leq 2N - n + 1.$$

(ii) *Let $f : \mathbb{F}^m \to \mathbb{P}^n$ be a non-constant non-Archimedean meromorphic map, and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n$ in general position. Then*

$$\sum_{j=1}^{q} \delta_f(H_j) \leq 2n.$$

Remark 4.11. (i) Corollary 4.10 is actually not sharp. If $H_1, \ldots, H_q$ are hyperplanes in general position, then

$$\sum_{j=1}^{q} \delta_f(H_j) \leq n$$

(see [13], [32] and [4]). Hence, if the image of f omits $n + 1$ hyperplanes in general position, then f must be constant.

(ii) A non-Archimedean meromorphic map $f : \mathbb{F}^m \to \mathbb{P}^n$ is said to be *d-degenerate* if the image of f is contained in a linear subspace of dimension $< d$ with $1 \leq d \leq n$. One can show that if $H_1, \ldots, H_q$ are hyperplanes in $\mathbb{P}^n$, then any non-Archimedean meromorphic map $f : \mathbb{F}^m \to \mathbb{P}^n \setminus \{H_j\}_{j=1}^q$ is d-degenerate if and only if $\dim \bigcap_{j=1}^q H_j < d - 1$.

(iii) The fact that Corollary 4.10 is not sharp does not mean Theorems 4.6 and 4.8 are not sharp, because the counting functions in Theorems 4.6 and 4.8 are truncated.

Definition 4.12. Let f be a non-Archimedean meromorphic map and let H be a hyperplane. Define

$$\delta_{f,\mathrm{a}}(H) = \liminf_{r\to\infty}\left\{1 - \frac{N_f^{(\mathrm{a})}(H,r)}{T_f(r)}\right\},$$

where a is an integer.

Obviously, $0 \le \delta_f(H) \le \delta_{f,\mathrm{a}}(H) \le 1$.

If we consider this defect, we have the following defect relation.

Corollary 4.13.

(i) *Let $f : \mathbb{F}^m \to \mathbb{P}^n$ be a linearly non-degenerate non-Archimedean meromorphic map with index of independence s and rank k, and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n$ in N-subgeneral position ($N \ge n$). Then*

$$\sum_{j=1}^q \delta_{f,\mathrm{a}}(H_j) \le 2N - n + 1 \quad \textit{where} \quad \mathrm{a} = \begin{cases} n-k+1 & \textit{if } p = 0,\\ p^{s-1}(n-k+1) & \textit{if } p > 0.\end{cases}$$

(ii) *Let $f : \mathbb{F}^m \to \mathbb{P}^n$ be a non-constant non-Archimedean meromorphic map with index of independence s and rank k, and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n$ in general position. Then*

$$\sum_{j=1}^q \delta_{f,\mathrm{a}}(H_j) \le 2n,$$

where a *is as in* (i).

5. Proof of Theorem 4.6. We first give a general form of the second main theorem without the assumption that $H_1, \ldots, H_q$ are in general position, which is a generalization of the result given by Ru [31] and Vojta [39] for the non-Archimedean case.

Theorem 5.1. *Let $f = (f_0, \ldots, f_n) : \mathbb{F}^m \to \mathbb{P}^n$ be a linearly non-degenerate non-Archimedean meromorphic map. Let $\gamma^0 = (0, \ldots, 0), \gamma^1, \ldots, \gamma^n$ be multi-indices such that*

$$W = W_{\gamma^0\cdots\gamma^n}(f_0, \ldots, f_n) = \det\begin{pmatrix} D^{\gamma^0} f_0 & \cdots & D^{\gamma^0} f_n \\ \vdots & \cdots & \vdots \\ D^{\gamma^n} f_0 & \cdots & D^{\gamma^n} f_n \end{pmatrix} \not\equiv 0.$$

Let $B = \sum_{i=1}^{n} |\gamma^i|$. *Let* $H_1, \dots, H_q$ *be hyperplanes in* $\mathbb{P}^n$, *and let* $L_1, \dots, L_q$ *be the vectors in* $\mathbb{F}^{n+1}$ *defined by* $H_1, \dots, H_q$. *Then, for all* $r > 1$,

$$\max_{\mathcal{K}} \sum_{k\in\mathcal{K}} m_f(H_k, r) + N_W(0, r) \le (n+1)T_f(r) - B\log r + O(1),$$

where the maximum is taken over all subsets $\mathcal{K}$ *of* $\{1, \dots, q\}$ *such that* L_k, $k \in \mathcal{K}$, *are linearly independent.*

Proof. Without loss of generality, we may assume that $q \ge n+1$ and $\sharp\mathcal{K} = n+1$. Let T be the set of all injective maps $\mu : \{0, \dots, n\} \to \{1, \dots, q\}$ such that $L_{\mu(0)}, \dots, L_{\mu(n)}$ are linearly independent. We have

$$\begin{aligned}\max_{\mu\in T} \sum_{j=0}^{n} m_f(H_{\mu(j)}, r) &= \max_{\mu\in T} \sum_{j=0}^{n} \log \frac{\|f\|_r \|H_{\mu(j)}\|}{|(f, H_{\mu(j)})|_r} \\ &= \max_{\mu\in T} \log \frac{\|f\|_r^{n+1}}{\prod_{j=0}^{n} |(f, H_{\mu(j)})|_r} + O(1) \\ &= \max_{\mu\in T} \log \frac{|W|_r}{\prod_{j=0}^{n} |(f, H_{\mu(j)})|_r} + \log \frac{\|f\|_r^{n+1}}{|W|_r} + O(1) \\ &= \max_{\mu\in T} \log \frac{|W_{\gamma^0\dots\gamma^n}((f, H_{\mu(0)}), \dots, (f, H_{\mu(n)}))|_r}{\prod_{j=0}^{n} |(f, H_{\mu(j)})|_r} + \log \frac{\|f\|_r^{n+1}}{|W|_r} + O(1) \\ &\le (n+1)T_f(r) - N_W(0, r) - B\log r + O(1).\end{aligned}$$

Here, by the logarithmic derivative lemma,

$$\max_{\mu\in T} \log \frac{|W_{\gamma^0\dots\gamma^n}((f, H_{\mu(0)}), \dots, (f, H_{\mu(n)}))|_r}{\prod_{j=0}^{n} |(f, H_{\mu(j)})|_r} \le -\sum_{i=0}^{n} |\gamma^i| \log r. \ \blacksquare$$

REMARK 5.2. (i) For $\operatorname{Char}\mathbb{F} = 0$, Theorem 5.1 is a generalization of Theorem A.

(ii) The existence of W can be found in Section 4.1.

To prove Theorem 4.6, we recall a lemma due to Nochka [28].

LEMMA 5.3. *Let* $H_1, \dots, H_q$ *be hyperplanes in* $\mathbb{P}^n$ *in* N*-subgeneral position with* $q \ge 2N - n + 1$. *There exist positive rational numbers* $\omega(j)$, $j = 1, \dots, q$, *called the Nochka weights, and* $\theta \ge 1$, *called the Nochka constant, satisfying the following conditions:*

- (i) $0 \le \omega(j)\theta \le 1$ *for all* $j \in \{1, \dots, q\}$.
- (ii) $q - 2N + n - 1 = \theta(\sum_{j=1}^{q} \omega(j) - n - 1)$.
- (iii) $1 \le (N+1)/(n+1) \le \theta \le (2N - n + 1)/(n+1)$.
- (iv) *If* $\emptyset \ne S \subset \{1, \dots, q\}$ *with* $\sharp S \le N+1$, *then* $\sum_{j\in S} \omega(j) \le \dim L(S)$, *where* $L(S)$ *is the linear space generated by* $\{L_j : j \in S\}$ *and* L_j *are the non-zero vectors defined by the coefficients of the hyperplanes* H_j.

(v) *Given real numbers $E_1, \dots, E_q$ with $E_j \geq 1$ for $1 \leq j \leq q$, and given any $J \subset \{1, \dots, q\}$ with $0 < \sharp J \leq N+1$, there exists a subset $I \subset J$ with $\sharp I = \dim L(J)$ such that $\{L_j\}_{j \in I}$ is a basis for $L(J)$, and*

$$\prod_{j \in J} E_j^{\omega(j)} \leq \prod_{j \in I} E_j.$$

Proof of Theorem 4.6. For any fixed $r > 1$, we take $\mu(r,0), \dots, \mu(r,N) \in \{1, \dots, q\}$ such that

$$0 < |(f, H_{\mu(r,0)})|_r \leq \dots \leq |(f, H_{\mu(r,N)})|_r \leq |(f, H_j)|_r \tag{5.1}$$

for $j \in \{1, \dots, q\} \setminus \{\mu(r,0), \dots, \mu(r,N)\}$, where

$$(f, H_{\mu(r,v)}) = a_{\mu(r,v)0} f_0 + \dots + a_{\mu(r,v)n} f_n, \quad 0 \leq v \leq N.$$

As $H_1, \dots, H_q$ are in N-subgeneral position, there exists a subset I of $\{\mu(r,0), \dots, \mu(r,N)\}$ such that $\sharp I = n+1$ and $\{L_j\}_{j \in I}$ are linearly independent. By solving the system of linear equations

$$(f, H_j) = 0, \quad j \in I,$$

we have

$$f_v = \sum_{j \in I} \tilde{a}_{vj} (f, H_j), \quad 0 \leq v \leq n,$$

where $\tilde{a}_{vj} \in \mathbb{F}$. Thus, for any r, there exists a $C > 0$ such that

$$\|f\|_r \leq C \max_{j \in I} |(f, H_j)|_r \leq C \max_{0 \leq v \leq N} |(f, H_{\mu(r,v)})|_r.$$

By (5.1),

$$\frac{\|f\|_r}{|(f, H_j)|_r} \leq C \quad \text{for } j \neq \mu(r,0), \dots, \mu(r,N).$$

Since there are only finitely many choices of $N+1$ hyperplanes in $\{H_j\}_{j=1}^q$, we can find a positive constant C such that, for each r,

$$\prod_{j=1}^{q} \left(\frac{\|f\|_r \|H_j\|}{|(f, H_j)|_r} \right)^{\omega(j)} \leq C \prod_{v=0}^{N} \left(\frac{\|f\|_r \|H_{\mu(r,v)}\|}{|(f, H_{\mu(r,v)})|_r} \right)^{\omega(\mu(r,v))}. \tag{5.2}$$

By Lemma 5.3(v), there is a subset I' of $\{\mu(r,0), \dots, \mu(r,N)\}$ with $\sharp I' = n+1$ and with $\{L_j\}_{j \in I'}$ linearly independent such that

$$\prod_{v=0}^{N} \left(\frac{\|f\|_r \|H_{\mu(r,v)}\|}{|(f, H_{\mu(r,v)})|_r} \right)^{\omega(\mu(r,v))} \leq \prod_{j \in I'} \frac{\|f\|_r \|H_j\|}{|(f, H_j)|_r}.$$

Hence,

$$\sum_{v=0}^{N}\omega(\mu(r,v))\log\frac{\|f\|_r\|H_{\mu(r,v)}\|}{|(f,H_{\mu(r,v)})|_r}\le\sum_{j\in I'}\log\frac{\|f\|_r\|H_j\|}{|(f,H_j)|_r}\le\max_{\mu\in T}\sum_{j=0}^{n}\log\frac{\|f\|_r\|H_{\mu(j)}\|}{|(f,H_{\mu(j)})|_r},\tag{5.3}$$

where T is the set of all injective maps $\mu:\{0,\dots,n\}\to\{1,\dots,q\}$ such that $L_{\mu(0)},\dots,L_{\mu(n)}$ are linearly independent.

By (5.2), (5.3) and Theorem 5.1, we have

$$\begin{aligned}\sum_{j=1}^{q}\omega(j)m_f(H_j,r)&\le\max_{\mu\in T}\sum_{j=0}^{n}m_f(H_{\mu(j)},r)+O(1)\\&\le(n+1)T_f(r)-N_W(0,r)-B\log r+O(1),\end{aligned}$$

where $W=W_{\gamma^0\dots\gamma^n}(f_0,\dots,f_n)$ and $B=\sum_{i=1}^{n}|\gamma^i|$.

Using the first main theorem, we obtain

$$\begin{aligned}\sum_{j=1}^{q}\omega(j)T_f(r)\le(n+1)T_f(r)&+\sum_{j=1}^{q}\omega(j)N_{(f,H_j)}(0,r)\\&-N_W(0,r)-B\log r+O(1).\end{aligned}\tag{5.4}$$

Now, we give the following estimate for $N_W(0,r)$.

Lemma 5.4. *We have*

$$\sum_{j=1}^{q}\omega(j)N_{(f,H_j)}(0,r)-N_{W_{\gamma^0\dots\gamma^n}(f_0,\dots,f_n)}(0,r)\le\sum_{j=1}^{q}\omega(j)N^{(|\gamma^n|)}_{(f,H_j)}(0,r)+O(1).\tag{5.5}$$

Proof. Set $\tilde G_j=\gcd((f,H_j),S((f,H_j))^{|\gamma^n|})$, $j=1,\dots,q$. It suffices to show that

$$\sum_{j=1}^{q}\omega(j)\log|(f,H_j)|_r-\log|W_{\gamma^0\dots\gamma^n}(f_0,\dots,f_n)|_r\le\sum_{j=1}^{q}\omega(j)\log|\tilde G_j|_r+O(1).$$

Since the $\omega(j)$ are rational numbers, there exists an integer A such that $\tilde\omega(j):=A\omega(j)$ are integers. We now show that $\prod_{j=1}^{q}(f,H_j)^{\tilde\omega(j)}$ divides

$$(W_{\gamma^0\dots\gamma^n}(f_0,\dots,f_n))^A\prod_{j=1}^{q}\tilde G_j^{\tilde\omega(j)}.$$

Suppose that $P \in \mathcal{E}_m$ is an irreducible element with $P \mid \prod_{j=1}^{q}(f, H_j)^{\tilde{\omega}(j)}$. There exists a subset R of $\{1, \dots, q\}$ such that $P \mid (f, H_j)$ with exact multiplicity e_j, $j \in R$. By the assumption, $0 < \sharp R \le N$. We have

$$P^{\min\{e_j, |\gamma^n|\}} \mid \tilde{G}_j$$

by property (ii) of the square free part. Let $S \subset R$ be such that $e_j > |\gamma^n|$ for $j \in S$.

Case 1. If $j \in R \setminus S$, i.e., $e_j \le |\gamma^n|$, then $P^{e_j} \mid \tilde{G}_j$. We have

$$P^{\sum_{j \in R\setminus S} \tilde{\omega}(j) e_j} \mid \prod_{j \in R \setminus S} \tilde{G}_j^{\tilde{\omega}(j)}.$$

Case 2. If $j \in S$, then $e_j > |\gamma^n|$. We consider S_τ, $0 \le \tau \le t$, such that

$$S_0 := \emptyset \ne S_1 \subset S_2 \subset \cdots \subset S_t := S$$

and $e_j = m_\tau$ for $j \in S_\tau - S_{\tau-1}$, where $m_1 > m_2 > \cdots > m_t$ $(> |\gamma^n|)$. Set $m_\tau^* = m_\tau - |\gamma^n|$. By Lemma 5.3(iv), we have

$$\begin{aligned}
\sum_{j \in S} \omega(j)(e_j - |\gamma^n|) &= \sum_{\tau=1}^{t} \sum_{j \in S_\tau - S_{\tau-1}} \omega(j) m_\tau^* \\
&= \sum_{\tau=1}^{t} \sum_{j \in S_\tau - S_{\tau-1}} \omega(j) \Big(\sum_{\sigma=\tau}^{t-1} (m_\sigma^* - m_{\sigma+1}^*) + m_t^* \Big) \\
&= (m_1^* - m_2^*) \sum_{j \in S_1} \omega(j) + (m_2^* - m_3^*) \sum_{j \in S_2} \omega(j) + \cdots + m_t^* \sum_{j \in S_t} \omega(j) \\
&\le \dim L(S_1)(m_1^* - m_2^*) + \dim L(S_2)(m_2^* - m_3^*) + \cdots + \dim L(S_t) m_t^* \\
&= \dim L(S_1) m_1^* + (\dim L(S_2) - \dim L(S_1)) m_2^* + \cdots \\
&\quad + (\dim L(S_t) - \dim L(S_{t-1})) m_t^*.
\end{aligned}$$

Take $j_1, \dots, j_{\dim L(S)} \in S$ such that $L_{j_1}, \dots, L_{j_{\dim L(S)}}$ are linearly independent and $\{j_1, \dots, j_{\dim L(S_i)}\} \subset S_i$. Since $H_1, \dots, H_q$ are in N-subgeneral position, we can find $L_{j_{\dim L(S)+1}}, \dots, L_{j_{n+1}}$ such that

$$L_{j_1}, \dots, L_{j_{\dim L(S)}}, L_{j_{\dim L(S)+1}}, \dots, L_{j_{n+1}}$$

are linearly independent. We have

$$\begin{aligned}
&W_{\gamma^0 \dots \gamma^n}(f_0, \dots, f_n) \\
&\quad = C \cdot W_{\gamma^0 \dots \gamma^n}\big((f, H_{j_1}), \dots, (f, H_{j_{\dim L(S)}}), (f, H_{j_{\dim L(S)+1}}), \dots, (f, H_{j_{n+1}})\big)
\end{aligned}$$

for a non-zero constant C.

By property (v) of Hasse derivatives, it is easy to see that P divides $W_{\gamma^0 \dots \gamma^n}(f_0, \dots, f_n)$ with multiplicity at least

$$
\begin{aligned}
e_{j_1} + \cdots + e_{j_{\dim L(S)}} &- \dim L(S) \cdot |\gamma^n| \\
&= \dim L(S_1) m_1^* + (\dim L(S_2) - \dim L(S_1)) m_2^* + \cdots \\
&\quad + (\dim L(S_t) - \dim L(S_{t-1})) m_t^* \\
&\geq \sum_{j \in S} \omega(j)(e_j - |\gamma^n|),
\end{aligned}
$$

which means $P^{\sum_{j\in S} \tilde{\omega}(j)(e_j - |\gamma^n|)} \mid W^A$ and $P^{\sum_{j\in S}\tilde{\omega}(j)e_j} \mid W^A \prod_{j\in S} \tilde{G}_j^{\tilde{\omega}(j)}$.

Combining Cases 1 and 2, we see that

$$P^{\sum_{j\in R}\tilde{\omega}(j)e_j} \mid W^A \prod_{j\in R} \tilde{G}_j^{\tilde{\omega}(j)}.$$

Hence,

$$\prod_{j=1}^{q} (f, H_j)^{\tilde{\omega}(j)} \mid (W_{\gamma^0 \ldots \gamma^n}(f_0, \ldots, f_n))^A \prod_{j=1}^{q} \tilde{G}_j^{\tilde{\omega}(j)}. \ \blacksquare$$

Now, by (5.4) and (5.5), we have

$$\Big(\sum_{j=1}^{q} \omega(j) - (n+1)\Big) T_f(r) \leq \sum_{j=1}^{q} \omega(j) N_{(f,H_j)}^{(|\gamma^n|)}(0,r) - B \log r + O(1).$$

By Lemma 5.3(i), (ii),

$$\frac{q - 2N + n - 1}{\theta} T_f(r) \leq \frac{1}{\theta} \sum_{j=1}^{q} N_{(f,H_j)}^{(|\gamma^n|)}(0,r) - B \log r + O(1).$$

Hence

$$(q - 2N + n - 1) T_f(r) \leq \sum_{j=1}^{q} N_{(f,H_j)}^{(|\gamma^n|)}(0,r) - B\theta \log r + O(1)$$

with $1 \leq \frac{N+1}{n+1} \leq \theta \leq \frac{2N-n+1}{n+1}$.

We note that $|\gamma^n| \leq$ a by (4.3), and $B\theta \geq \theta \geq \frac{N+1}{n+1}$. This finishes the proof of Theorem 4.6.

6. Uniqueness theorems for non-Archimedean meromorphic maps. In this section, as an application of Theorem 4.6, we discuss the uniqueness problems for non-Archimedean meromorphic maps.

Adams and Straus [1] proved the following uniqueness theorem for non-Archimedean meromorphic functions, which is sharp.

THEOREM B ([1]). *Let f and g be non-constant non-Archimedean meromorphic functions on $\mathbb{F}$, where $\mathbb{F}$ has characteristic* 0. *Let a_1, a_2, a_3, a_4 be four distinct values. Assume that* $\mathrm{Zero}(f, a_j) = \mathrm{Zero}(g, a_j)$ *for $j = 1, 2, 3, 4$, where* $\mathrm{Zero}(f, a)$ *denotes the set of zeros of $f - a$. Then $f \equiv g$.*

In 2001, Ru [33] extended Theorem B to non-Archimedean holomorphic curves in projective space.

THEOREM C ([33]). *Let $f, g : \mathbb{F} \to \mathbb{P}^n$ be non-Archimedean holomorphic curves, where $\mathbb{F}$ has characteristic* 0. *Assume that f and g are linearly non-degenerate. Let $H_1, \ldots, H_{3n+1}$ be hyperplanes in $\mathbb{P}^n$ in general position. Assume that* $\mathrm{Zero}(f, H_j) = \mathrm{Zero}(g, H_j)$ *for* $j = 1, \ldots, 3n+1$ *and* $\mathrm{Zero}(f, H_i) \cap \mathrm{Zero}(f, H_j) = \emptyset$ *for $i \neq j$, where* $\mathrm{Zero}(f, H)$ *denotes the set of zeros of (f, H). If $f(z) = g(z)$ for every $z \in \bigcup_{j=1}^{3n+1} \mathrm{Zero}(f, H_j)$, then $f \equiv g$.*

We improved Theorem C in [40] as follows.

THEOREM D ([40]). *Let $f, g : \mathbb{F} \to \mathbb{P}^n$ be non-Archimedean holomorphic curves, where $\mathbb{F}$ has characteristic* 0. *Assume that f and g are linearly non-degenerate. Let $H_1, \ldots, H_{2n+2}$ be hyperplanes in $\mathbb{P}^n$ in general position. Assume that* $\mathrm{Zero}(f, H_j) = \mathrm{Zero}(g, H_j)$ *for* $j = 1, \ldots, 2n+2$ *and* $\mathrm{Zero}(f, H_i) \cap \mathrm{Zero}(f, H_j) = \emptyset$ *for $i \neq j$. If $f(z) = g(z)$ for every $z \in \bigcup_{j=1}^{2n+2} \mathrm{Zero}(f, H_j)$, then $f \equiv g$.*

We also considered the positive characteristic case.

THEOREM E ([40]). *Let $\mathbb{F}$ have positive characteristic p, and $f, g : \mathbb{F} \to \mathbb{P}^n$ be linearly non-degenerate non-Archimedean holomorphic curves with index of independence $\leq s$. Let $H_1, \ldots, H_{2p^{s-1}n+2}$ be hyperplanes in $\mathbb{P}^n$ in general position. Assume* $\mathrm{Zero}(f, H_j) = \mathrm{Zero}(g, H_j)$ *for* $j = 1, \ldots, 2p^{s-1}n + 2$ *and* $\mathrm{Zero}(f, H_i) \cap \mathrm{Zero}(f, H_j) = \emptyset$ *for $i \neq j$. If $f(z) = g(z)$ for every $z \in \bigcup_{j=1}^{2p^{s-1}n+2} \mathrm{Zero}(f, H_j)$, then $f \equiv g$.*

Actually, by using Theorem 4.6, one can obtain more general uniqueness theorems for non-Archimedean meromorphic maps similar to the results given in [3] and [10] for the complex case. However, in this section, we only consider some interesting particular cases.

For $\mathrm{Char}\,\mathbb{F} = 0$, we have the following result.

THEOREM 6.1. *Let $f, g : \mathbb{F}^m \to \mathbb{P}^n$ $(m \geq n)$ be linearly non-degenerate non-Archimedean meromorphic maps with rank n. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n$ in general position. Assume that*

(i) $\mathrm{Zero}(f, H_j) = \mathrm{Zero}(g, H_j)$, *$j = 1, \ldots, q$;*
(ii) *(f, H_i) and (f, H_j) have no common divisors for $i \neq j$;*
(iii) *$f = g$ on* $\bigcup_{j=1}^{q} \mathrm{Zero}(f, H_j)$.

If $q \geq n + 3$, then $f \equiv g$.

For $\mathrm{Char}\,\mathbb{F} > 0$, we show another one.

THEOREM 6.2. *Let $\mathbb{F}$ have positive characteristic p, and $f, g : \mathbb{F}^m \to \mathbb{P}^n$ be linearly non-degenerate non-Archimedean meromorphic maps with index* *of independence* $\leq s$ *and rank* k. *Let* $H_1, \ldots, H_q$ *be hyperplanes in* $\mathbb{P}^n$ *in general position. Assume that*

(i) $\mathrm{Zero}(f, H_j) = \mathrm{Zero}(g, H_j)$, $j = 1, \ldots, q$;
(ii) (f, H_i) *and* (f, H_j) *have no common divisors for* $i \neq j$;
(iii) $f = g$ *on* $\bigcup_{j=1}^{q} \mathrm{Zero}(f, H_j)$.

Let $\mathrm{a} = p^{s-1}(n - k + 1)$. *If* $\mathrm{a} \geq n$ *and* $q \geq 2\mathrm{a} + 2$, *then* $f \equiv g$.

Remark 6.3. The method of proof is similar to that used in the complex case, which is based on Theorem 4.6 and different from the method of [33] and [40].

Lemma 6.4. *Let* f *and* g *be linearly non-degenerate non-Archimedean meromorphic maps. Let* $H_1, \ldots, H_q$ *be hyperplanes in* $\mathbb{P}^n$ *in general position such that*

(i) $\mathrm{Zero}(f, H_j) = \mathrm{Zero}(g, H_j)$, $j = 1, \ldots, q$;
(ii) (f, H_i) *and* (f, H_j) *have no common divisors for* $i \neq j$;
(iii) $f = g$ *on* $\bigcup_{j=1}^{q} \mathrm{Zero}(f, H_j)$.

Assume that $\Phi = (f, H_i)(g, H_j) - (f, H_j)(g, H_i) \not\equiv 0$ *for some* $i \neq j$. *Take a sequence* $\{r_\iota\}_{\iota \in \mathbb{N}}$ *with* $1 < r_\iota \in |\mathbb{F}^\times|$ *and* $r_\iota \to \infty$ $(\iota \to \infty)$. *Then, for any positive integer* a,

$$
\begin{aligned}
(6.1) \qquad \sum_{\substack{l=1 \\ l \neq i,j}}^{q} N_f^{(1)}(H_l, r_\iota) &+ N_f^{(\mathrm{a})}(H_i, r_\iota) + N_g^{(\mathrm{a})}(H_i, r_\iota) - \mathrm{a}N_f^{(1)}(H_i, r_\iota) \\
&+ N_f^{(\mathrm{a})}(H_j, r_\iota) + N_g^{(\mathrm{a})}(H_j, r_\iota) - \mathrm{a}N_f^{(1)}(H_j, r_\iota) \\
&\leq N_\Phi(0, r_\iota) \leq T(r_\iota) + O(1),
\end{aligned}
$$

where $T(r_\iota) := T_f(r_\iota) + T_g(r_\iota)$.

Proof. Firstly, we show the first inequality in (6.1). Suppose P is an irreducible element in $\mathcal{E}_m$ which divides (f, H_l) for some l, $1 \leq l \leq q$. (We note that any two elements in $\{(f, H_j)\}_{j=1}^{q}$ have no common divisors.)

For $l \neq i, j$, since $f = g$ on $\mathrm{Zero}(f, H_l)$, we have $\mathrm{Zero}(f, H_l) \subset \mathrm{Zero}(\Phi, 0)$. Due to Hilbert's Nullstellensatz (see [5, §7.1.2, Theorem 3]), we have

$$P \mid \Phi \quad \text{on } B^m(r_\iota).$$

For $l = i, j$, assume that P is an irreducible element of $\mathcal{E}_m$ which divides (f, H_l) with exact multiplicity e_l^f and divides (g, H_l) with exact multiplicity e_l^g. Hence P divides Φ with multiplicity at least $\min\{e_l^f, e_l^g\}$. We note that

$$\min\{e_l^f, e_l^g\} \geq \min\{e_l^f, \mathrm{a}\} + \min\{e_l^g, \mathrm{a}\} - \mathrm{a}.$$

We see that P divides $\gcd((f,H_l),S((f,H_l))^{\mathrm{a}})\cdot\gcd((g,H_l),S((g,H_l))^{\mathrm{a}})$ with multiplicity

$$\min\{e_l^f,\mathrm{a}\}+\min\{e_l^g,\mathrm{a}\},$$

and P divides $\Phi\cdot(\gcd((f,H_l),S((f,H_l))))^{\mathrm{a}}$ with multiplicity at least

$$\min\{e_l^f,e_l^g\}+\mathrm{a}.$$

Hence,

$$\sum_{\substack{l=1\\ l\neq i,j}}^{q} N^{(1)}_{(f,H_l)}(0,r_\iota)+N^{(\mathrm{a})}_{(f,H_i)}(0,r_\iota)+N^{(\mathrm{a})}_{(g,H_i)}(0,r_\iota)+N^{(\mathrm{a})}_{(f,H_j)}(0,r_\iota)+N^{(\mathrm{a})}_{(g,H_j)}(0,r_\iota)$$
$$\leq N_\Phi(0,r_\iota)+\mathrm{a}N^{(1)}_{(f,H_i)}(0,r_\iota)+\mathrm{a}N^{(1)}_{(f,H_j)}(0,r_\iota).$$

On the other hand, for all $r>1$,

$$\begin{aligned}
N_\Phi(0,r)&=\log|\Phi|_r+O(1)\\
&=\log|(f,H_i)(g,H_j)-(f,H_j)(g,H_i)|_r+O(1)\\
&\leq \log\max\{|(f,H_i)|_r|(g,H_j)|_r,|(f,H_j)|_r|(g,H_i)|_r\}+O(1)\\
&\leq \log\max_i|f_i|_r+\log\max_i|g_i|_r+O(1),
\end{aligned}$$

i.e., $N_\Phi(0,r)\leq T(r)+O(1)$. ∎

REMARK 6.5. The following examples show that assumptions (ii) and (iii) in Lemma 6.4 cannot be removed.

(i) Let $f=(z,z^2,z^3,1)$ and $g=(z^2,z,z^3,1)$. Let H_1, H_2, H_3 and H_4 be the coordinate hyperplanes in $\mathbb{P}^3$. They satisfy assumptions (i) and (iii) in Lemma 6.4, but (f,H_1), (f,H_2) and (f,H_3) have common divisors. If we take $i=1$, $j=4$ and $\Phi=z(1-z)$, it is easy to check (6.1) is not true.

(ii) Let $f=(z^2,1-z,h)$ and $g=(z,(1-z)^2,h)$, where h is a transcendental non-Archimedean entire function not vanishing at 0 and 1. Let H_1, H_2 and H_3 be the coordinate hyperplanes in $\mathbb{P}^2$. They satisfy assumptions (i) and (ii) in Lemma 6.4, but f may not be equal to g on the zeros of h. If we take $i=1$, $j=2$ and $\Phi=z^2(1-z)^2-(1-z)z$, then (6.1) is also false.

Proof of Theorem 6.1. Suppose that $f\not\equiv g$. There exist $i,j\in\{1,\ldots,q\}$ with $i\neq j$ such that

$$\Phi=(f,H_i)(g,H_j)-(f,H_j)(g,H_i)\not\equiv 0.$$

By Lemma 6.4, we have

$$\sum_{l=1}^{q} N^{(1)}_{(f,H_l)}(0,r_\iota)\leq N_\Phi(0,r_\iota)\leq T(r_\iota)+O(1), \tag{6.2}$$

$$\sum_{l=1}^{q} N^{(1)}_{(g,H_l)}(0,r_\iota)\leq N_\Phi(0,r_\iota)\leq T(r_\iota)+O(1). \tag{6.3}$$

Summing (6.2) and (6.3), we obtain

$$\sum_{l=1}^{q}(N^{(1)}_{(f,H_l)}(0,r_\iota)+N^{(1)}_{(g,H_l)}(0,r_\iota))\le 2T(r_\iota)+O(1).$$

By Theorem 4.6,

$$(q-n-1)T(r_\iota)\le 2T(r_\iota)-\log r_\iota+O(1).$$

This is a contradiction for $q\ge n+3$ and $\iota\to\infty$.

Proof of Theorem 6.2. Suppose that $f\not\equiv g$. By changing indices if necessary, we may assume that

$$\underbrace{\frac{(f,H_1)}{(g,H_1)}\equiv\cdots\equiv\frac{(f,H_{k_1})}{(g,H_{k_1})}}_{\text{group 1}}\not\equiv\underbrace{\frac{(f,H_{k_1+1})}{(g,H_{k_1+1})}\equiv\cdots\equiv\frac{(f,H_{k_2})}{(g,H_{k_2})}}_{\text{group 2}}$$
$$\not\equiv\cdots\not\equiv\underbrace{\frac{(f,H_{k_{u-1}+1})}{(g,H_{k_{u-1}+1})}\equiv\cdots\equiv\frac{(f,H_{k_u})}{(g,H_{k_u})}}_{\text{group } u},$$

where $k_u=q$.

Since $f\not\equiv g$, the number of elements of every group is at most n.

We define $\sigma:\{1,\dots,q\}\to\{1,\dots,q\}$ by

$$\sigma(i)=\begin{cases} i+n & \text{if } i\le q-n,\\ i+n-q & \text{if } i>q-n.\end{cases}$$

It is easy to see that σ is bijective and $|\sigma(i)-i|\ge n$ (note that $q>2n$). Hence $(f,H_i)/(g,H_i)$ and $(f,H_{\sigma(i)})/(g,H_{\sigma(i)})$ belong to distinct groups.

We consider $\Phi=(f,H_i)(g,H_{\sigma(i)})-(g,H_i)(f,H_{\sigma(i)})\not\equiv 0$, $i=1,\dots,q$.

Using Lemma 6.4 for Φ, we get

$$\begin{aligned}(6.4)\qquad \sum_{\substack{j=1\\ j\ne i,\sigma(i)}}^{q} N^{(1)}_{(f,H_j)}(0,r_\iota)&+N^{(\mathrm{a})}_{(f,H_i)}(0,r_\iota)+N^{(\mathrm{a})}_{(g,H_i)}(0,r_\iota)\\ &+N^{(\mathrm{a})}_{(f,H_{\sigma(i)})}(0,r_\iota)+N^{(\mathrm{a})}_{(g,H_{\sigma(i)})}(0,r_\iota)\\ &\le T(r_\iota)+\mathrm{a}N^{(1)}_{(f,H_i)}(0,r_\iota)+\mathrm{a}N^{(1)}_{(f,H_{\sigma(i)})}(0,r_\iota)+O(1).\end{aligned}$$

Taking the sum of (6.4) over $i=1,\dots,q$, we obtain

$$(q-2)\sum_{j=1}^{q} N_f^{(1)}(H_j,r_\iota) - \mathrm{a}\sum_{i=1}^{q}\big(N_f^{(1)}(H_i,r_\iota)+N_f^{(1)}(H_{\sigma(i)},r_\iota)\big)$$
$$+\sum_{i=1}^{q}\big(N_f^{(\mathrm{a})}(H_i,r_\iota)+N_f^{(\mathrm{a})}(H_{\sigma(i)},r_\iota)\big)+\sum_{i=1}^{q}\big(N_g^{(\mathrm{a})}(H_i,r_\iota)+N_g^{(\mathrm{a})}(H_{\sigma(i)},r_\iota)\big)$$
$$\le qT(r_\iota)+O(1).$$

Since σ is bijective, this gives

$$(q-2\mathrm{a}-2)\sum_{j=1}^{q} N_f^{(1)}(H_j,r_\iota) + 2\sum_{i=1}^{q}\big(N_f^{(\mathrm{a})}(H_i,r_\iota)+N_g^{(\mathrm{a})}(H_i,r_\iota)\big)$$
$$\le qT(r_\iota)+O(1).$$

By Theorem 4.6 and $q \ge 2\mathrm{a}+2$, we have

$$2(q-n-1)T(r_\iota) \le qT(r_\iota) - \log r_\iota + O(1),$$

i.e.,

$$(q-2n-2)T(r_\iota) \le -\log r_\iota + O(1).$$

This is a contradiction for $\mathrm{a} \ge n$.

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