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Triple correlations of multiplicative functions

by

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1. Introduction. Let $g_j : \mathbb{N} \to \mathbb{C}$ be multiplicative functions such that $|g_j(n)| \le 1$ for all n and $j = 1, 2, 3$. Let $F_1(x), F_2(x), F_3(x)$ be polynomials with integer coefficients.

Consider the following triple correlation function:

$$M_x(g_1, g_2, g_3) = \frac{1}{x} \sum_{n \le x} g_1(F_1(n)) g_2(F_2(n)) g_3(F_3(n)). \tag{1}$$

Kátai [KAT] studied the asymptotic behaviour of the above sum (1) when $F_j(x)$, $j = 1, 2, 3$, are special polynomials and under some assumptions on g_j, $j = 1, 2, 3$ but did not provide the error term. Stepanauskas [ST4] studied the asymptotic formula for the sum (1) with an explicit error term when $F_j(x)$, $j = 1, 2, 3$, are linear polynomials and g_1, g_2, g_3 are close to 1 (see Definition 1.1). Recently, Klurman [KL] studied the double correlation function (i.e. the sum (1) with $g_3 = 1$).

Estimation (1) can be used to get information on the behaviour of the distribution of the sum

$$f_1(F_1(n)) + f_2(F_2(n)) + f_3(F_3(n)), \tag{2}$$

where f_1, f_2 and f_3 are real-valued additive functions.

From now on, let $F(n)$ and $F_1(n), F_2(n), F_3(n)$ be positive integer-valued polynomials with integer coefficients, which are not divisible by the square of any irreducible polynomial. Also suppose that $F_j(n)$ and $F_k(n)$ are relatively prime for $j \ne k$ and for all n. Let v and v_j denote the degrees of the polynomials $F(n)$ and $F_j(n)$ respectively.

In this paper, we will investigate the following sums with various assumptions on g_j, $j = 1, 2, 3$:

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$$M'_x(g_1,g_2,g_3)=\frac{1}{x}\sum_{n\le x} g_1(n+3)g_2(n+2)g_3(n+1) \tag{3}$$

and find an asymptotic formula for the following triple correlation function with an explicit error term which is an improvement of a theorem of Kátai [KAT, Theorem 5]:

$$M''_x(g_1,g_2,g_3)=\frac{1}{x}\sum_{n\le x} g_1(F_1(n))g_2(F_2(n))g_3(F_3(n)), \tag{4}$$

where $F_1(x), F_2(x)$ and $F_3(x)$ are polynomials as above of degree ≥ 2.

DEFINITION 1.1. A multiplicative function g is said to be *close to* 1 if

$$\sum_p \frac{g(p)-1}{p}<\infty. \tag{5}$$

DEFINITION 1.2. A multiplicative function g is called *good* if there exists a $\kappa\in\mathbb{C}$ such that for each $u>0$,

$$\sum_{p\le x}|g(p)-\kappa|\ll\frac{x}{(\log x)^u}. \tag{6}$$

Outline. In Section 5, we prove an asymptotic formula for (4) with an explicit error term.

The asymptotic behaviour of the sum (3) is very difficult to obtain if any two of g_1, g_2, and g_3 are the Möbius function and the other one is close to some fixed complex number.

In Section 6, first we investigate the asymptotic behaviour of (3) when g_1, g_2 are close to 1 and g_3 is the Möbius function. Secondly, we obtain an asymptotic formula for (3) when g_3 is a good function.

In Section 7, under some assumption we investigate the asymptotic behaviour of (3) when g_1 is close to 1 and g_2, g_3 are the Möbius function.

In Section 8, we formulate some applications of the sums (3) and (4).

2. Notation. Throughout the paper, p and q denote primes; j,k,l,m and n are natural numbers; $c,c',c_1,c_2,\ldots$ are absolute constants; $c_F, c_{F_1}, c_{F_2}, c_{F_3}$ are constants depending on F, F_1, F_2, F_3; $\wp$ is the set of all primes; $\exists$ and $\ni$ stand for 'there exists' and 'such that' respectively.

In Section 3, we use the following notation for Theorem 3.1:

$$S(r,x,g_j)=\sum_{r<p\le x+4-j}\frac{|g_j(p)-1|^2}{p},\qquad S(r,x)=\sum_{j=1}^{3}S(r,x,g_j),\quad r>0,$$

$$h_j=\mu*g_j,\quad j=1,2,3,\quad P(x)=\prod_{p\le x}w_p,\qquad P(r,x)=\prod_{r<p\le x}w_p,$$

where

$$w_p = \sum_{\substack{m_1=0\\(p^{m_1},3)=1}}^{\infty} \sum_{\substack{m_2=0\\(p^{m_2},2)=1\\(p^{m_1},p^{m_2})=1\\(p^{m_2},p^{m_3})=1\\(p^{m_1},p^{m_3})|2}}^{\infty} \sum_{m_3=0}^{\infty} \frac{h_1(p^{m_1})h_2(p^{m_2})h_3(p^{m_3})}{[p^{m_1},p^{m_2},p^{m_3}]}. \tag{7}$$

We use the following notation for Theorem 3.2:

Let $\varrho(d)$ and $\varrho_j(d)$ denote the number of solutions of the congruences $F(n) \equiv 0 \pmod d$ and $F_j(n) \equiv 0 \pmod d$ respectively.

Let $\varrho(d_1, d_2, d_3)$ be the number of solutions of the congruence system

$$F_j(n) \equiv 0 \pmod{d_j}, \quad j = 1, 2, 3.$$

Let D_γ denote the set of those tuples $\{d_1, d_2, d_3\}$ of natural numbers for which no prime factor of d_i exceeds γ. Set

$$S_1(r,x) = \sum_{r<p\le x} \sum_{j=1}^{3} \frac{|g_j(p)-1|^2 \varrho_j(p)}{p}, \quad T(x) = \sum_{\substack{j=1\\x<p\le F_j(x)}}^{3} \frac{|g_j(p)-1|^2 \varrho_j(p)}{p},$$

$$C(r,x) = \sum_{j=1}^{3} \sum_{m=1}^{v_j-1} \sum_{\substack{p^m \le F_j(x)\\p>r}} |g_j(p^m)-1| \varrho_j(p^m),$$

$$P'(x) = P_1(\gamma)P_2(\gamma,x), \qquad P''(r,x) = \prod_{r<p\le x} w'_p,$$

where

$$P_1(\gamma) = \sum_{\{d_1,d_2,d_3\}\in D_\gamma} \frac{h_1(d_1)h_2(d_2)h_3(d_3)}{[d_1,d_2,d_3]} \varrho(d_1,d_2,d_3), \tag{8}$$

$$P_2(\gamma,x) = \prod_{\gamma<p\le x} \biggl(1 + \sum_{m=1}^{\infty} \sum_{j=1}^{3} \frac{h_j(p^m)\varrho_j(p^m)}{p^m}\biggr) =: \prod_{\gamma<p\le x} w'_p. \tag{9}$$

For Theorems 3.3 and 3.4 we need the following notation:

$$\theta_\tau(n) = \prod_{p|n} \biggl(1 + \sum_{m=1}^{\infty} \frac{g_3(p^m)}{p^{m(1+i\tau)}}\biggr)^{-1}, \quad \tau \in \mathbb{R},$$

$$Q_\tau(r) = \prod_{p\le r} \biggl(1 - \frac{2\theta_\tau(p)}{p-1} + \theta_\tau(p) \sum_{m=1}^{\infty} \frac{g_1(p^m)+g_2(p^m)}{p^m}\biggr),$$

$$P_3(r,x) = \prod_{r<p\le x} \biggl(1 - \frac{2}{p} + \Bigl(1-\frac{1}{p}\Bigr) \sum_{m=1}^{\infty} \frac{g_1(p^m)+g_2(p^m)}{p^m}\biggr),$$

$$S_2(r,x) = \sum_{j=1}^{2} \sum_{r<p\le x+4-j} \frac{|g_j(p)-1|^2}{p}, \qquad M_x(g_3) = \frac{1}{x} \sum_{n\le x} g_3(n).$$

3. Statements of theorems. We begin with an asymptotic formula for the sum (3) with an explicit error term, which is a special case of a theorem of Stepanauskas [ST4].

Theorem 3.1. *Let g_1, g_2 and g_3 be multiplicative functions with modulus less than or equal to* 1. *Then there exists a positive absolute constant c such that for all $x \geq r \geq 2$ and all $2/3 < \alpha < 1$, we have*

$$M'_x(g_1, g_2, g_3) = P(x) + O\biggl(x^{2-3\alpha} \exp\biggl(c\frac{r^\alpha}{\log r}\biggr) + (S(r,x))^{1/2} + (r\log r)^{-1/2}\biggr).$$

The aim of this paper is to prove the following statements:

Theorem 3.2. *Let $F_j(x)$, $j = 1, 2, 3$, be polynomials as above of degree greater than or equal to* 2. *Let g_1, g_2 and g_3 be multiplicative functions close to* 1 *and whose modulus does not exceed* 1. *Then there exists a positive absolute constant c and a natural number γ such that for all $x \geq r \geq \gamma$ and all $1 - \frac{1}{v_1+v_2+v_3} < \alpha < 1$, we have*

$$\begin{aligned} M''_x(g_1, g_2, g_3) - P'(x) \ll{}& \frac{1}{x}(F_1(x)F_2(x)F_3(x))^{1-\alpha} \exp\biggl(\frac{cr^\alpha}{\log r}\biggr) + (T(x))^{1/2} \\ &+ (S_1(r,x))^{1/2} + (r\log r)^{-1/2} + \frac{1}{x}C(r,x) + \frac{1}{\log x}. \end{aligned}$$

Remark. Theorem 3.2 can be extended to higher correlations.

Theorem 3.3. *Let g_1 and g_2 be multiplicative functions which do not exceed* 1 *and*

$$\sum_p \sum_{j=1}^{2} \frac{|g_j(p) - 1|^2}{p} < \infty. \tag{10}$$

Then as $x \to \infty$,

$$M'_x(g_1, g_2, \mu) = \frac{1}{x}\sum_{n\leq x} g_1(n+3)g_2(n+2)\mu(n+1) = o(1).$$

Theorem 3.4. *Let g_1, g_2 and g_3 be multiplicative functions whose modulus does not exceed* 1 *and suppose g_3 is a good function. Assume further that there exists a positive constant c_1 such that*

$$\biggl|1 + \sum_{k=1}^{\infty} \frac{g_3(2^k)}{2^{k(1+i\xi)}}\biggr| \geq c_1 \tag{11}$$

for $\xi = 0$ if g_3 is real-valued, and for all $\xi \in \mathbb{R}$ otherwise. Then there exist positive absolute constants c, c' and a real τ with $|\tau| \leq (\log x)^{1/19}$ such that *for all* $x \geq r \geq 2$ *and all* $1/2 < \alpha < 5/9$, *we have*

$$M_x'(g_1, g_2, g_3) - M_x(g_3)P_3(r,x)Q_\tau(r) \ll x^{1-2\alpha} \exp\left(c\frac{r^\alpha}{\log r}\right) + \frac{(\log r)^c}{(\log x)^{c'}} + \frac{\exp(c(\log\log r)^2)}{(\log x)^{1/19}} + (S_2(r,x))^{1/2} + (r \log r)^{-1/2}.$$

For real-valued g_3 *we may set* $\tau = 0$ *in the expression of* $Q_\tau(r)$.

Assumption 3.5 (2-point Chowla type conjecture). For every given $A > 0$,

$$\sum_{n \leq x} \mu(n+2)\mu(n+1)\exp(2\pi i n\alpha) = O\left(\frac{x}{(\log x)^A}\right)$$

uniformly for all real α, and the implied constant depends on A.

Theorem 3.6. *Let* g_1 *be a multiplicative function such that* $|g_1(n)| \leq 1$ *for all* n *and*

$$\sum_p \frac{|g_1(p) - 1|^2}{p} < \infty. \tag{12}$$

Suppose that Assumption 3.5 *holds. Then as* $x \to \infty$,

$$M_x'(g_1, \mu, \mu) = \frac{1}{x}\sum_{n \leq x} g_1(n+3)\mu(n+2)\mu(n+1) = o(1).$$

4. Corollaries. In this section we state several corollaries of our results. Corollary 4.1 below is a linear version of a theorem of Kátai [KAT, Theorem 1].

Corollary 4.1. *Let* g_1, g_2, g_3 *be multiplicative functions such that for all* $j = 1, 2, 3$, $|g_j| \leq 1$ *and* g_j *is close to* 1. *Then as* $x \to \infty$,

$$M_x'(g_1, g_2, g_3) = \prod_p w_p + o(1),$$

where w_p *is defined in* (7).

Corollary 4.2 is a polynomial version, with the degree of the polynomial greater than or equal to 2, of a theorem of Kátai [KAT, Theorem 5].

Corollary 4.2. *Let* $F_j(n)$ $(j = 1, 2, 3)$ *be as above of degree* $v_j \geq 2$. *Let* g_j $(j = 1, 2, 3)$ *be as above and*

$$\sum_p \frac{(g_j(p) - 1)\varrho_j(p)}{p} < \infty. \tag{13}$$

Suppose that as $p \to \infty$,

$$(g_j(p^\alpha) - 1)\varrho_j(p^\alpha) \to 0, \tag{14}$$

for $\alpha = 1$ when $v_j \geq 2$, and for $\alpha = 1, \dots, v_j - 2$ when $v_j \geq 3$. Then as $x \to \infty$,

$$M''_x(g_1, g_2, g_3) \to P_1(\gamma) P_2(\gamma),$$

where $P_1(\gamma)$ is defined by (8) *and*

$$P_2(\gamma) := \prod_{p>\gamma} \biggl(1 + \sum_{m=1}^{\infty} \sum_{j=1}^{3} \frac{h_j(p^m) \varrho_j(p^m)}{p^m}\biggr). \tag{15}$$

Corollaries 4.3, 4.4, 4.5 and 4.6 are direct applications of Theorems 3.1, 3.3, 3.6 and 3.2 respectively.

Corollary 4.3. *Let $\phi(n) = n \prod_{p|n}(1 - 1/p)$ be Euler's totient function and $\sigma(n) = \sum_{d|n} d$. Then for $x \geq 2$ and $0 < A < 1$ we have*

$$\frac{1}{x} \sum_{n \leq x} \frac{\phi(n+3)\phi(n+2)\phi(n+1)}{\sigma(n+3)\sigma(n+2)\sigma(n+1)} = w_2 \prod_{p>2} w_p + O\biggl(\frac{1}{(\log x)^A}\biggr),$$

$$\frac{1}{x} \sum_{n \leq x} \frac{\phi(n+3)\phi(n+2)\phi(n+1)}{(n+3)(n+2)(n+1)} = w_2 \prod_{p>2} \biggl\{1 - \frac{3}{p^2}\biggr\} + O\biggl(\frac{1}{(\log x)^A}\biggr),$$

where

$$w_p = \biggl(1 - \frac{3}{p} + 3\biggl(1 - \frac{1}{p}\biggr)^2 \sum_{m=1}^{\infty} \frac{1}{1 + p + \dots + p^m}\biggr) \quad \textit{for } p > 2,$$

and w_2 is defined by (7) *with $p = 2$, where $g_j(n)$, $j = 1, 2, 3$, are replaced by $\phi(n)/\sigma(n)$ and $\phi(n)/n$ respectively.*

Corollary 4.4. *If ϕ, μ and σ are as above then as $x \to \infty$,*

$$\frac{1}{x} \sum_{n \leq x} \frac{\phi(n+3)}{(n+3)} \frac{\phi(n+2)}{(n+2)} \mu(n+1) = o(1),$$

$$\frac{1}{x} \sum_{n \leq x} \frac{\phi(n+3)}{\sigma(n+3)} \frac{\phi(n+2)}{\sigma(n+2)} \mu(n+1) = o(1).$$

Corollary 4.5. *If ϕ, μ and σ are as above, and under Assumption* 3.5, *we have as $x \to \infty$,*

$$\frac{1}{x} \sum_{n \leq x} \frac{\phi(n+3)}{(n+3)} \mu(n+2)\mu(n+1) = o(1),$$

$$\frac{1}{x} \sum_{n \leq x} \frac{\phi(n+3)}{\sigma(n+3)} \mu(n+2)\mu(n+1) = o(1).$$

Corollary 4.6. *Let $F_1(x) = x^2 + b$, $F_2(x) = x^2 + c$, $F_3(x) = x^2 + d$, $0 < t < 1$, where b, c, d are taken such that $F_j(x)$, $j = 1, 2, 3$, satisfies the* *assumption of Theorem* 3.2 *and is a quadratic residue for all odd prime* p. *Then there exists a natural number* γ *such that for all* $x \geq \gamma$,

$$\frac{1}{x}\sum_{n\leq x}\frac{\phi(n^2+b)\phi(n^2+c)\phi(n^2+d)}{\sigma(n^2+b)\sigma(n^2+c)\sigma(n^2+d)} = P_1'(\gamma)\prod_{p>\gamma} w_p' + O\Big(\frac{1}{(\log x)^t}\Big),$$

$$\frac{1}{x}\sum_{n\leq x}\frac{\phi(n^2+b)\phi(n^2+c)\phi(n^2+d)}{(n^2+b)(n^2+c)(n^2+d)} = P_1''(\gamma)\prod_{p>\gamma}\Big(1-\frac{6}{p^2}\Big) + O\Big(\frac{1}{(\log x)^t}\Big),$$

where

$$w_p' = \bigg(1-\frac{6}{p}+6\Big(1-\frac{1}{p}\Big)^2\sum_{m=1}^{\infty}\frac{1}{1+p+\dots+p^m}\bigg),$$

and $P_1'(\gamma)$ *and* $P_2''(\gamma)$ *are defined by* (8) *in which* $g_j(n), j = 1,2,3$, *are replaced by* $\phi(n)/\sigma(n)$ *and* $\phi(n)/n$ *respectively.*

Corollaries 4.7 and 4.8 below concern the behaviour of the distribution of the sum (2).

Corollary 4.7. *Let* f_1, f_2 *and* f_3 *be real-valued additive functions and*

$$\sum_{|f_j(p)|\leq 1}\frac{f_j^2(p)}{p} < \infty, \qquad j = 1,2,3, \tag{16}$$

$$\sum_{|f_j(p)|>1}\frac{1}{p} < \infty, \qquad j = 1,2,3, \tag{17}$$

$$\sum_{j=1}^{3}\sum_{|f_j(p)|\leq 1}\frac{f_j(p)}{p} < \infty. \tag{18}$$

Then the distribution function

$$\frac{1}{[x]}\#\{n \mid n \leq x,\ f_1(n+3)+f_2(n+2)+f_3(n+1) \leq z\} \tag{19}$$

converges weakly towards a limit distribution [TEN, Chapter III.2] *as* $x \to \infty$, *and the characteristic function of this limit distribution is equal to*

$$w_2\prod_{p>2}\bigg(1-\frac{3}{p}+\Big(1-\frac{1}{p}\Big)\sum_{m=1}^{\infty}\sum_{k=1}^{3}\frac{\exp(itf_k(p^m))}{p^m}\bigg), \tag{20}$$

where w_2 *is defined by* (7) *with* $p = 2$ *with* g_k *is replaced by* $\exp(itf_k)$ *for* $k = 1,2,3$.

Corollary 4.8. *Let* f_1, f_2 *and* f_3 *be real-valued additive functions and* $F_j(n)$, $j = 1,2,3$, *are as above of degree* $v_j \geq 2$. *Assume that*

$$\sum_{|f_j(p)|\le 1} \frac{f_j^2(p)}{p}\varrho_j(p) < \infty, \qquad j = 1,2,3, \tag{21}$$

$$\sum_{|f_j(p)|>1} \frac{\varrho_j(p)}{p} < \infty, \qquad j = 1,2,3, \tag{22}$$

$$\sum_{j=1}^{3} \sum_{|f_j(p)|\le 1} \frac{f_j(p)\varrho_j(p)}{p} < \infty, \tag{23}$$

$$f_j(p^m)\varrho_j(p^m) \to 0, \tag{24}$$

for $m = 1$ *when* $v_j \ge 2$, *and for* $m = 1, \ldots, v_j - 2$ *when* $v_j \ge 3$. *Then the distribution functions*

$$\frac{1}{[x]}\#\{n \mid n \le x,\ f_1(F_1(n)) + f_2(F_2(n)) + f_3(F_3(n)) \le z\} \tag{25}$$

converges weakly towards a limit distribution as $x \to \infty$, *and the characteristic function of this limit distribution is equal to* $P_1(\gamma)P_2(\gamma)$, *where* $P_1(\gamma)$ *and* $P_2(\gamma)$ *are defined by* (8) *and* (15) *respectively with* g_j *replaced by* $\exp(itf_j)$, $j = 1, 2, 3$.

5. Proof of Theorem 3.2.

We begin with some lemmas.

Lemma 5.1 ([ERD, Lemma 3]). *Let* $F(m)$ *be primitive polynomial of degree* v *with integer coefficients and with discriminant* D. *Suppose* $D \ne 0$. *Then the number of solutions of the congruence* $F(m) \equiv 0 \pmod{p^\alpha}$ *is* $\varrho(p)$ *when* $p \nmid D$, *and smaller than* vD^2 *when* $p \mid D$. *Further,* ϱ *is a multiplicative function and* $\varrho(p^\alpha) \le c$, *where* c *depends only on* F.

Now we prove a polynomial version of the classical Turán–Kubilius inequality.

Lemma 5.2. *Let* $f(p^m)$ *be the sequence of complex numbers for all* $p \in \wp$, $m \ge 1$, *and let* $F(n)$ *be a polynomial as above of degree* v. *Then*

$$\sum_{n\le x} |K(F(n)) - A(x)| \ll xB(F(x)) + \sum_{m=1}^{v-1} \sum_{p^m \le F(x)} |f(p^m)|\varrho(p^m) + \frac{x}{\log x},$$

where

$$K(n) := \sum_{p^m \| n} f(p^m), \quad A(x) := \sum_{p^m \le x} \frac{f(p^m)\varrho(p^m)}{p^m},$$

$$B^2(x) := \sum_{p^m \le x} \frac{|f(p^m)|^2\varrho(p^m)}{p^m}.$$

Proof. We write $K(F(n)) = \sum_{p^m \| F(n)} f(p^m) = g(F(n)) + h(F(n))$, where

$$g(y) = \sum_{\substack{p^m \| y \\ p^m \le y}} f(p^m) \quad \text{and} \quad h(y) = \sum_{\substack{p^m \| y \\ p^m > y}} f(p^m).$$

Now,

$$\begin{aligned}\sum_{n\le x} |K(F(n)) - A(x)| \le \sum_{n\le x} |g(F(n)) - A(x^{1/2})| &+ \sum_{n\le x} |h(F(n))| \\ &+ \sum_{n\le x} |A(x^{1/2}) - A(x)|.\end{aligned}$$

From the Turán–Kubilius inequality [ELL1, Lemma 4.11], we have

$$\sum_{n\le x} |g(F(n)) - A(x^{1/2})| \ll xB(x^{1/2}).$$

From Lemma 5.1 and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}|A(x) - A(x^{1/2})| &\le \sum_{x^{1/2} < p^m \le x} \frac{|f(p^m)|\varrho(p^m)}{p^m} \\ &\le \biggl(\sum_{x^{1/2} < p^m \le x} \frac{|f(p^m)|^2 \varrho(p^m)}{p^m}\biggr)^{1/2} \biggl(\sum_{x^{1/2} < p^m \le x} \frac{\varrho(p^m)}{p^m}\biggr)^{1/2} = O(B(x)).\end{aligned}$$

Again by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\sum_{n\le x} |h(F(n))| &= \sum_{n\le x} \Bigl| \sum_{\substack{p^m \| F(n) \\ p^m > x^{1/2}}} f(p^m) \Bigr| \\ &\ll \sum_{x^{1/2} < p^m \le F(x)} \frac{x|f(p^m)|\varrho(p^m)}{p^m} + \sum_{x^{1/2} < p^m \le F(x)} |f(p^m)|\varrho(p^m) \\ &\ll x\biggl(\sum_{x^{1/2} < p^m \le F(x)} \frac{|f(p^m)|^2 \varrho(p^m)}{p^m}\biggr)^{1/2} \biggl(\sum_{x^{1/2} < p^m \le F(x)} \frac{\varrho(p^m)}{p^m}\biggr)^{1/2} + \frac{x}{\log x} \\ &\quad + \sum_{x^{1/2} < p^m \le F(x)} |f(p^m)|\varrho(p^m) \\ &\ll xB(F(x)) + \sum_{m=1}^{v-1} \sum_{p^m \le F(x)} |f(p^m)|\varrho(p^m) + \frac{x}{\log x},\end{aligned}$$

which proves the lemma. ■

The following lemma ensures the existence of γ as in Theorem 3.2.

Lemma 5.3 ([TAN, Lemma 2.1]). *If* $F_1(m)$ *and* $F_2(m)$ *are relatively prime polynomials with integer coefficients, then the congruence* $F_1(m) \equiv 0 \pmod{a}$ *and* $F_2(m) \equiv 0 \pmod{a}$ *have common roots for at most finitely many values of* a.

Define multiplicative functions g_{jr} and g^*_{jr}, $j = 1, 2, 3$, by

$$g_{jr}(p^m) = \begin{cases} g_j(p^m) & \text{if } p \le r, \\ 1 & \text{if } p > r, \end{cases} \qquad g^*_{jr}(p^m) = \begin{cases} 1 & \text{if } p \le r, \\ g_j(p^m) & \text{if } p > r. \end{cases}$$

and multiplicative functions h_{jr}, $j = 1, 2, 3$, by

$$h_{jr}(p^m) = \begin{cases} g_j(p^m) - g_j(p^{m-1}) & \text{if } p \le r, \\ 0 & \text{if } p > r, \end{cases}$$

so that $g_{jr} = 1 * h_{jr}$, $j = 1, 2, 3$.

Proof of Theorem 3.2. We can write

$$\begin{aligned} M''_x(g_1, g_2, g_3) - P'(x) &= P'(r, x)\biggl(\frac{1}{x}\sum_{n\le x}\prod_{j=1}^{3} g_{jr}(F_j(n)) - P'(r)\biggr) \\ &\quad + \frac{1}{x}\sum_{n\le x}\prod_{j=1}^{3} g_{jr}(F_j(n))(g^*_{1r}(F_1(n))g^*_{2r}(F_3(n))g^*_{3r}(F_3(n)) - P'(r, x)). \end{aligned}$$

So,

$$\begin{aligned} |M''_x(g_1, g_2, g_3) - P'(x)| &\ll \biggl|\frac{1}{x}\sum_{n\le x} g_{1r}(F_1(n))g_{2r}(F_2(n))g_{3r}(F_3(n)) - P'(r)\biggr| \\ &\quad + \frac{1}{x}\sum_{n\le x} |g^*_{1r}(F_1(n))g^*_{2r}(F_2(n))g^*_{3r}(F_3(n)) - P'(r, x)| =: E_1 + E_2. \end{aligned}$$

Estimation of E_1. We have

$$\begin{aligned} \frac{1}{x}\sum_{n\le x}\prod_{j=1}^{3} g_{jr}(F_j(n)) &= \frac{1}{x}\sum_{\substack{d_j\le F_j(x)\\ j=1,2,3}}\prod_{j=1}^{3} h_{jr}(d_j)\sum_{\substack{n\le x\\ d_j\mid F_j(n)\\ j=1,2,3}} 1 \\ &= \frac{1}{x}\sum_{d_1\le F_1(x)}\sum_{d_2\le F_2(x)}\sum_{d_3\le F_3(x)} h_{1r}(d_1)h_{2r}(d_2)h_{3r}(d_3)\frac{x}{[d_1, d_2, d_3]}\varrho(d_1, d_2, d_3) \\ &\quad + O\biggl(\frac{1}{x}\sum_{\substack{d_j\le F_j(x)\\ j=1,2,3}}\prod_{j=1}^{3} h_{jr}(d_j)\varrho(d_1, d_2, d_3)\biggr) =: P'_1 + E_3. \end{aligned}$$

Now we observe that

$$\sum_{d_j=1}^{\infty} \frac{|h_{jr}(d_j)|\varrho_j(d_j)}{d_j} \le \exp\biggl(c_{F_j}\sum_{p\le r}\frac{1}{p}\biggr) \ll (\log r)^{c_{F_j}},$$

and for $0<\alpha<1$,

$$\sum_{d_j=1}^{\infty} \frac{|h_{jr}(d_j)|\varrho_j(d_j)}{d_j^{1-\alpha}} \le \prod_{p\le r}\biggl(1+\sum_{m=1}^{\infty}\frac{|h_{jr}(p^m)|\varrho_j(p^m)}{p^{m(1-\alpha)}}\biggr) \le \exp\biggl(c_{F_j}\frac{r^\alpha}{\log r}\biggr).$$

Therefore

$$\begin{aligned} E_3 &\ll \frac{1}{x}\sum_{\substack{d_j\le F_j(x)\\ j=1,2,3}}\prod_{j=1}^{3}|h_{jr}(d_j)|\varrho_j(d_j) \\ &\ll \frac{1}{x}(F_1(x)F_2(x)F_3(x))^{1-\alpha}\sum_{\substack{d_j=1\\ j=1,2,3}}^{\infty}\prod_{j=1}^{3}\frac{|h_{jr}(d_j)|}{d_j^{1-\alpha}}\varrho_j(d_j) \\ &\ll \frac{1}{x}(F_1(x)F_2(x)F_3(x))^{1-\alpha}\exp\biggl(c_F\frac{r^\alpha}{\log r}\biggr). \end{aligned}$$

Now,

$$\begin{aligned} P_1' &= P'(r) + O\biggl(\sum_{k=1}^{3}\sum_{\substack{d_j=1\\ j=1,2,3\\ d_k>F_k(x)}}^{\infty}\frac{|h_{1r}(d_1)h_{2r}(d_2)h_{3r}(d_3)|}{[d_1,d_2,d_3]}\varrho(d_1,d_2,d_3)\biggr) \\ &=: P'(r)+E_4. \end{aligned}$$

Again from the above observations, we have

$$\begin{aligned} E_4 &\ll \sum_{k=1}^{3}\sum_{\substack{d_j=1\\ j=1,2,3\\ d_k>F_k(x)}}^{\infty}\frac{|h_{1r}(d_1)h_{2r}(d_2)h_{3r}(d_3)|}{d_1d_2d_3}\varrho_1(d_1)\varrho_2(d_2)\varrho_3(d_3) \\ &\ll (F_1(x)^{-\alpha}+F_2(x)^{-\alpha}+F_3(x)^{-\alpha})\exp\biggl(c_F\frac{r^\alpha}{\log r}\biggr). \end{aligned}$$

Estimation of E_2. To estimate of E_2 we will use a technique of R. Warlimont [WAR]. Let

$$N_r' = \{n\le x \mid \exists k\in\{1,2,3\}\ \exists p>r \ni p^m\|F_k(n),\ |1-g_k(p^m)|>1/2\}.$$

Decompose E_2 into

$$\begin{aligned}
E_2 &= \frac{1}{x}\sum_{n\in N_r'} |g_{1r}^*(F_1(n))g_{2r}^*(F_2(n))g_{3r}^*(F_3(n)) - P'(r,x)| \\
&\quad + \frac{1}{x}\sum_{n\notin N_r'} |g_{1r}^*(F_1(n))g_{2r}^*(F_2(n))g_{3r}^*(F_3(n)) - P'(r,x)| \\
&=: E_5 + E_6.
\end{aligned}$$

Set

$$\eta_j(p) := \sum_{m=1}^{\infty} \frac{(g_j(p^m)) - g_j(p^{m-1}))\varrho_j(p^m)}{p^m}, \qquad j=1,2,3.$$

From Lemma 5.1, we have

$$|\eta_j(p)| \le 2c_{F_j}\frac{1}{p-1} \le \frac{1}{6} \qquad \text{if } p \ge 1 + 12c_{F_j} =: p_j.$$

Let $p_4 := \max(p_1, p_2, p_3)$. If $r \ge p_4$, then

$$\begin{aligned}
P'(r,x) &= \prod_{r<p\le x} (1 + \eta_1(p) + \eta_2(p) + \eta_3(p)) \\
&= \exp\Big(\sum_{r<p\le x}\sum_{j=1}^{3}(\eta_j(p) + O(|\eta_j(p)|^2))\Big) \\
&= \exp\Big(\sum_{r<p\le x}\sum_{j=1}^{3}\frac{(g_j(p)-1)\varrho_j(p)}{p} + O\Big(\sum_{r<p\le x}\frac{1}{p^2}\Big)\Big) \ll 1.
\end{aligned}$$

Without loss of generality we may assume that $r \ge p_4$. Now

$$\begin{aligned}
E_5 &\ll \frac{1}{x}\sum_{n\in N_r'} 1 \ll \frac{1}{x}\sum_{\substack{p^m\le F_j(x)\\ |1-g_j(p^m)|>1/2\\ p>r}} \left(\frac{x\varrho_j(p^m)}{p^m} + \varrho_j(p^m)\right) \\
&\ll \frac{1}{x}\sum_{\substack{p^m\le F_j(x)\\ |1-g_j(x)|>1/2\\ p>r}} \frac{x\varrho_j(p^m)}{p^m} + \frac{1}{x}\sum_{\substack{p^m\le F_j(x)\\ |1-g_j(p^m)|>1/2\\ p>r}} \varrho_j(p^m) \\
&\ll \sum_{r<p\le F_j(x)} \frac{|1-g_j(p)|\varrho_j(p)}{p} + \sum_{p>r}\frac{1}{p^2} + \frac{1}{x}\sum_{\substack{p^m\le F_j(x)\\ p>r,\, m<v_j}} |1-g_j(p^m)|\varrho_j(p^m) \\
&\quad + \frac{1}{\log x} \\
&\ll S_1(r,x) + T(x) + (r\log r)^{-1} + \frac{1}{x}C(r,x) + \frac{1}{\log x}.
\end{aligned}$$

Since we know that if $\Re(u), \Re(v) \le 0$, then

$$|\exp(u) - \exp(v)| \le |u - v|, \tag{26}$$

$$\log(1 + z) = z + O(|z|^2) \quad \text{for } |z| \le 1,\ |\arg(z)| \le \pi/2, \tag{27}$$

we obtain

$$\begin{aligned} E_6 \ll{}& \frac{1}{x} \sum_{n \le x} \sum_{j=1}^{3} \Bigg| \sum_{\substack{p^m \| F_j(n) \\ p > r}} (g_j(p^m) - 1) - \sum_{\substack{p^m \le x \\ p > r}} \frac{g_j(p^m) - 1}{p^m} \varrho_j(p^m) \Bigg| \\ &+ \frac{1}{x} \sum_{n \le x} \Bigg| \sum_{\substack{p^m \le x \\ p > r}} \sum_{j=1}^{3} \frac{(g_j(p^m) - 1)\varrho_j(p^m)}{p^m} - \log P'(r, x) \Bigg| \\ &+ O\Bigg(\frac{1}{x} \sum_{n \le x} \sum_{j=1}^{3} \sum_{\substack{p^m \| F_j(n) \\ p > r}} |g_j(p^m) - 1|^2 \Bigg) =: E_{61} + E_{62} + E_{63}. \end{aligned}$$

From Lemma 5.2, we have

$$\begin{aligned} E_{61} &\ll \sum_{j=1}^{3} \Bigg(\sum_{\substack{p^m \le F_j(x) \\ p > r}} \frac{|g_j(p^m) - 1|^2 \varrho_j(p^m)}{p^m} \Bigg)^{1/2} + \frac{1}{x} C(r, x) + \frac{1}{\log x} \\ &\ll (S_1(r, x))^{1/2} + (T(x))^{1/2} + (r \log r)^{-1/2} + \frac{1}{x} C(r, x) + \frac{1}{\log x}, \end{aligned}$$

and

$$\begin{aligned} E_{62} &= \Bigg| \sum_{j=1}^{3} \sum_{r < p \le x} \frac{(g_j(p) - 1)\varrho_j(p)}{p} + O\Bigg(\sum_{p > r} \frac{1}{p^2} \Bigg) - \sum_{r < p \le x} \log w'_p \Bigg| \\ &= \Bigg| \sum_{j=1}^{3} \sum_{r < p \le x} \frac{(g_j(p) - 1)\varrho_j(p)}{p} + O\Bigg(\sum_{p > r} \frac{1}{p^2} \Bigg) - \sum_{r < p \le x} \sum_{j=1}^{3} \frac{(g_j(p) - 1)\varrho_j(p)}{p} \Bigg| \\ &\ll \sum_{p > r} \frac{1}{p^2} \ll (r \log r)^{-1}, \end{aligned}$$

and

$$\begin{aligned} E_{63} &\ll \sum_{j=1}^{3} \sum_{\substack{p^m \le F_j(x) \\ p > r}} \frac{|g_j(p^m) - 1|^2 \varrho_j(p^m)}{p^m} + \frac{1}{x} C(r, x) + \frac{1}{\log x} \\ &\ll S_1(r, x) + T(x) + (r \log r)^{-1} + \frac{1}{x} C(r, x) + \frac{1}{\log x}. \end{aligned}$$

Combining all these estimates for all $1 - \frac{1}{v_1 + v_2 + v_3} < \alpha < 1$, we get the conclusion of the theorem.

6. Proofs of Theorems 3.3 and 3.4. We begin with some lemmas. The first lemma will be used to prove Theorem 3.3.

Lemma 6.1 ([DAV, Theorem 1]). *For any given $K > 0$,*

$$\sum_{n \le x} \mu(n) \exp(2\pi i n\theta) = O\Big(\frac{x}{(\log x)^K}\Big)$$

uniformly in θ, where the implied constant depends on K.

The next lemmas will be used to prove Theorem 3.4.

Lemma 6.2 ([ELL2, Theorem 2]). *Let g be a multiplicative function whose modulus does not exceed* 1. *Then there is a real τ with $|\tau| \le (\log x)^{1/19}$ such that*

$$\sum_{\substack{n \le x \\ (n,D)=1}} g(n) = \theta_\tau(D) \sum_{n \le x} g(n) + O\Big(\frac{x(\log\log 3D)^2}{(\log x)^{1/19}}\Big) \tag{28}$$

uniformly for $x \ge 2$ for odd integers D. If, in addition, the condition (11) *is satisfied then* (28) *holds for even integers as well. For real-valued g we may set $\tau = 0$.*

The following lemma is a special case of a theorem of Wolke [WOL].

Lemma 6.3 ([WOL, Theorem 1]). *Let g be as above and suppose g is a good function. Then for given any $A > 0$ there is a corresponding $A_1 > 0$, possibly depending on g, such that for $x \ge 2$ and $Q = x^{1/2}(\log x)^{-A_1}$, we have*

$$\sum_{d \le Q} \max_{(l,d)=1} \max_{u \le x} \Big| \sum_{\substack{n \le u \\ n \equiv l\,(d)}} g(n) - \frac{1}{\phi(d)} \sum_{\substack{n \le u \\ (n,d)=1}} g(n) \Big| \ll \frac{x}{(\log x)^A}.$$

If $-\tau \in \mathbb{N}$ or $\tau = 0$ then

$$\sum_{d \le Q} \max_{l} \max_{u \le x} \Big| \sum_{\substack{n \le u \\ n \equiv l\,(d)}} g(n) \Big| \ll \frac{x}{(\log x)^A}.$$

The following lemma is a two-dimensional version of the standard Cauchy–Schwarz inequality:

Lemma 6.4. *If x_j, x_k and c_{jk} are non-negative real numbers, then*

$$\sum_{j \le y} \sum_{k \le y} x_j x_k c_{jk} \le \Big(\sum_{j \le y} \sum_{k \le y} x_j^2 x_k^2 c_{jk}\Big)^{1/2} \Big(\sum_{j \le y} \sum_{k \le y} c_{jk}\Big)^{1/2}.$$

Proof. By applying the Cauchy–Schwarz inequality, we have

$$\sum_{j \le y} \sum_{k \le y} x_j x_k c_{jk} \le \sum_{j \le y} \Big(\sum_{k \le y} x_j^2 x_k^2 c_{jk}\Big)^{1/2} \Big(\sum_{k \le y} c_{jk}\Big)^{1/2} =: \sum_{j \le y} a_j b_j.$$

Again by applying the Cauchy–Schwarz inequality, we have

$$\sum_{j\le y} a_j b_j \le \Big(\sum_{j\le y} a_j^2\Big)^{1/2}\Big(\sum_{j\le y} b_j^2\Big)^{1/2} = \Big(\sum_{j,k\le y} x_j^2 x_k^2 c_{jk}\Big)^{1/2}\Big(\sum_{j,k\le y} c_{jk}\Big)^{1/2}. \blacksquare$$

Proof of Theorem 3.3. Set

$$R(r,x) = \prod_{r<p\le x}\Big\{1-\frac{2}{p}+\Big(1-\frac{1}{p}\Big)\sum_{m=1}^{\infty}\frac{g_1(p^m)+g_2(p^m)}{p^m}\Big\}.$$

It is easy to see that $|R(r,x)|\le 1$. Therefore,

$$\begin{aligned} M_x'(g_1,g_2,\mu) &= R(r,x)\frac{1}{x}\sum_{n\le x} g_{1r}(n+3)g_{2r}(n+2)\mu(n+1) \\ &\quad + \frac{1}{x}\sum_{n\le x} g_{1r}(n+3)g_{2r}(n+2)\mu(n+1)\big(g_{1r}^*(n+3)g_{2r}^*(n+2)-R(r,x)\big). \end{aligned}$$

So,

$$\begin{aligned} |M_x'(g_1,g_2,\mu)| &\le \frac{1}{x}|M_x'(g_{1r},g_{2r},\mu)| + \frac{1}{x}\sum_{n\le x}\Big|\prod_{j=1}^{2} g_{jr}^*(n+4-j)-R(r,x)\Big| \\ &=: T_1+T_2. \end{aligned}$$

Estimation of T_1. We have

$$\begin{aligned} \sum_{n\le x} g_{1r}(n+3)g_{2r}(n+2)\mu(n+1) &= \sum_{n\le x}\sum_{d_1|n+3}\sum_{d_2|n+2} h_{1r}(d_1)h_{2r}(d_2)\mu(n+1) \\ = \sum_{\substack{d_1\le x+3\\ d_2\le x+2}} h_{1r}(d_1)h_{2r}(d_2)\sum_{\substack{2\le n\le x+1\\ d_1|n+2\\ d_2|n+1}}\mu(n) &= \sum_{\substack{d_1\le x+3\\ d_2\le x+2\\ (d_1,d_2)=1}} h_{1r}(d_2)h_{2r}(d_2)\sum_{\substack{2\le n\le x+1\\ n\equiv v\ (d_1d_2)}}\mu(n) \\ = \sum_{\substack{d_1\le y\\ d_2\le y\\ (d_1,d_2)=1}}\prod_{j=1}^{2} h_{jr}(d_j)\sum_{\substack{2\le n\le x+1\\ n\equiv v\ (d_1d_2)}}\mu(n) &+ \sum_{k=1}^{2}\sum_{\substack{d_j\le x+4-j\\ j=1,2\\ d_k>y\\ (d_1,d_2)=1}}\prod_{j=1}^{2} h_{jr}(d_j)\sum_{\substack{2\le n\le x+1\\ n\equiv v\ (d_1d_2)}}\mu(n), \end{aligned}$$

where v is the unique solution of the system of linear congruence $n\equiv -2 \pmod{d_1}$, $n\equiv -1 \pmod{d_2}, 0\le v\le d_1d_2-1$ and $y:=\log x$. So,

$$\begin{aligned} xT_1 &\ll \sum_{\substack{d_j\le y\\ j=1,2}} |h_{1r}(d_1)h_{2r}(d_2)|\Big|\sum_{\substack{2\le n\le x+1\\ n\equiv v\ (d_1d_2)}}\mu(n)\Big| \\ &\quad + \sum_{k=1}^{2}\sum_{\substack{d_j\le x+4-j\\ j=1,2\\ d_k>y}} |h_{1r}(d_1)h_{2r}(d_2)|\Big(\frac{x}{d_1d_2}+1\Big) =: T_{11}+T_{12}. \end{aligned}$$

From Lemma 6.1, we have

$$\sum_{\substack{n\le x+1\\ n\equiv v\,(d_1d_2)}}\mu(n)=\sum_{n\le x+1}\mu(n)\frac{1}{d_1d_2}\sum_{l=1}^{d_1d_2}\exp\biggl(\frac{(n-v)l}{d_1d_2}\biggr)$$

$$=\frac{1}{d_1d_2}\sum_{l=1}^{d_1d_2}\exp\biggl(\frac{-vl}{d_1d_2}\biggr)\sum_{n\le x+1}\mu(n)\exp\biggl(\frac{nl}{d_1d_2}\biggr)\ll\frac{x}{(\log x)^K}.$$

So,

$$T_{11}\ll\frac{x}{(\log x)^K}\sum_{d_1,d_2\le y}|h_{1r}(d_1)h_{2r}(d_2)|$$

$$\ll\frac{x}{(\log x)^K}y^4\sum_{d_1,d_2=1}^{\infty}\frac{|h_{1r}(d_1)h_{2r}(d_2)|}{d_1^2d_2^2}\ll\frac{x}{\log x}\quad\text{if } K\ge 5.$$

Now from the two estimates

$$\sum_{d_j=1}^{\infty}\frac{|h_{jr}(d_j)|}{d_j^{\alpha}}=\prod_{p\le r}\biggl(1+\sum_{m=1}^{\infty}\frac{|h_{jr}(p^m)|}{p^{m\alpha}}\biggr)\le\prod_{p\le r}\biggl(1+\frac{2}{p^{\alpha}-1}\biggr)\tag{29}$$

$$\le\exp\biggl(c_1\sum_{p\le r}\frac{1}{p^{\alpha}}\biggr)\le\exp\biggl(c_2\frac{r^{1-\alpha}}{\log r}\biggr)$$

and

$$\sum_{d_j=1}^{\infty}\frac{|h_{jr}(d_j)|}{d_j}\le\exp\biggl(c_3\sum_{p\le r}\frac{1}{p}\biggr)\ll(\log r)^{c_4}\tag{30}$$

we obtain

$$T_{12}\ll x\sum_{\substack{d_j\le x+4-j\\ j=1,2\\ d_k>y}}\frac{|h_{1r}(d_1)h_{2r}(d_2)|}{d_1d_2}+\sum_{\substack{d_j\le x+4-j\\ j=1,2\\ d_k>y}}|h_{1r}(d_1)h_{2r}(d_2)|$$

$$\ll xy^{-\gamma}\exp\biggl(c_2\frac{r^{\gamma}}{\log r}\biggr)(\log r)^{c_4}+x^{2\alpha}\exp\biggl(2c_2\frac{r^{1-\alpha}}{\log r}\biggr)$$

$$\ll xy^{-\gamma}\exp\biggl(c_5\frac{r^{\gamma}}{\log r}\biggr)+x^{2\alpha}\exp\biggl(2c_2\frac{r^{1-\alpha}}{\log r}\biggr).$$

Taking $1-\alpha=\gamma=2/3$, we have

$$T_{12}\ll xy^{-2/3}\exp\biggl(c_5\frac{r^{2/3}}{\log r}\biggr)+x^{2/3}\exp\biggl(2c_2\frac{r^{2/3}}{\log r}\biggr).$$

Setting $r=(\log\log x)^{3/2}$ we obtain

$$T_{12}\ll\frac{x}{y^{2/3}}(\log x)^{1/6}+xx^{-1/3}(\log x)^{1/6}\ll\frac{x}{(\log x)^{1/2}}.$$

So $T_1=o(1)$ as $x\to\infty$.

Estimation of T_2. Here we closely follow the method of R. Warlimont [WAR]. Now, let

$$N_r = \{n \le x \mid \exists j \in \{1,2\}\ \exists p > r \ni p^m \parallel n+4-j,\ |1-g_j(p^m)| > 1/2\}.$$

Decompose T_2 as

$$\begin{aligned} T_2 = \frac{1}{x} \sum_{n \in N_r} &|g^*_{1r}(n+3) g^*_{2r}(n+2) - R(r,x)| \\ &+ \frac{1}{x} \sum_{n \notin N_r} |g^*_{1r}(n+3) g^*_{2r}(n+2) - R(r,x)| =: T_5 + T_6. \end{aligned}$$

Now,

$$\begin{aligned} T_5 &\ll \frac{1}{x} \sum_{j=1}^{2} \sum_{r<p\le x+4-j} \frac{x+4-j}{p} |1-g_j(p)|^2 + \sum_{j=1}^{2} \sum_{p>r} \sum_{m\ge 2} \frac{1}{p^m} \\ &\ll S_2(r,x) + (r\log r)^{-1}. \end{aligned}$$

From (26) and (27), we have

$$\begin{aligned} T_6 \le{}& \frac{1}{x} \sum_{j=1}^{2} \sum_{n\le x} \Bigg| \sum_{\substack{p^m \parallel n+4-j \\ p>r}} (g_j(p^m)-1) - \sum_{\substack{p^m \le x \\ p>r}} \frac{g_j(p^m)-1}{p^m} \Bigg| \\ &+ \frac{1}{x} \Bigg| \sum_{\substack{p^m \le x \\ p>r}} \sum_{j=1}^{2} \frac{g_j(p^m)-1}{p^m} - \log R(r,x) \Bigg| + O\Bigg(\frac{1}{x} \sum_{n\le x} \sum_{\substack{p^m \parallel n+4-j \\ j=1,2,\, p>r}} |g_j(p^m)-1|^2 \Bigg) \\ &=: T_7 + T_8 + T_9. \end{aligned}$$

Now by the Cauchy–Schwarz inequality and the Turán–Kubilius inequality [ELL1, Lemma 4.4], we have

$$\begin{aligned} T_7 &\ll \Bigg(\sum_{j=1}^{2} \sum_{\substack{p^m \le x+4-j \\ p>r}} |g_j(p^m)-1|^2 \Bigg)^{1/2} + \frac{1}{x} \ll \Bigg(\sum_{j=1}^{2} \sum_{r<p\le x+4-j} |g_j(p)-1|^2 \Bigg)^{1/2} \\ &\quad + \Bigg(\sum_{p>r} \frac{1}{p^2} \Bigg)^{1/2} + \frac{1}{x} \ll (S_2(r,x))^{1/2} + (r\log r)^{-1/2} + x^{-1}. \end{aligned}$$

Now similar to estimation of E_{62}, we have

$$T_8 \ll \Bigg| \sum_{r<p\le x} \sum_{j=1}^{2} \frac{g_j(p^m)-1}{p^m} - \log R(r,x) \Bigg| \ll \sum_{p>r} \frac{1}{p^2} \ll (r\log r)^{-1},$$

$$T_9 \ll \frac{1}{x} \Bigg\{ \sum_{j=1}^{2} \sum_{\substack{p^m \le x+4-j \\ p>r}} \frac{|g_j(p^m)-1|^2}{p^m} \Bigg\} \ll S_2(r,x) + (r\log r)^{-1}.$$

Combining the above calculations yields

$$T_2 \ll (r\log r)^{-1/2} + (S_2(r,x))^{1/2} + x^{-1}.$$

By the above choice of r and from (10) we have $T_2 = o(1)$ as $x \to \infty$, completing the proof of Theorem 3.3.

Proof of Theorem 3.4. We set

$$\begin{aligned} R &:= M'_x(g_1,g_2,g_3) - M_x(g_3)P_3(r,x)Q_\tau(r) \\ &= P_3(r,x)(M'_x(g_{1r},g_{2r},g_3) - M_x(g_3)Q_\tau(r)) \\ &\quad + \frac{1}{x}\sum_{n\le x} g_{1r}(n+3)g_{2r}(n+2)g_3(n+1)(g^*_{1r}(n+3)g^*_{2r}(n+2) - P_3(r,x)). \end{aligned}$$

It is easy to see that $|P_3(r,x)| \le 1$. Therefore

$$\begin{aligned} R &\ll |M'_x(g_{1r},g_{2r},g_3) - M_x(g_3)Q_\tau(r)| \\ &\quad + \frac{1}{x}\sum_{n\le x} |g^*_{1r}(n+3)g^*_{2r}(n+2) - P_3(r,x)| =: U_1 + U_2. \end{aligned}$$

Estimation of U_1. We have

$$\begin{aligned} M'_x(g_{1r},g_{2r},g_3) &= \frac{1}{x}\sum_{n\le x}\sum_{d_1|n+3}\sum_{d_2|n+2} h_{1r}(d_1)h_{2r}(d_2)g_3(n+1) \\ &= \frac{1}{x}\sum_{d_1\le x+3} h_{1r}(d_1) \sum_{\substack{d_2\le x+2\\(d_1,d_2)=1}} h_{2r}(d_2) \sum_{\substack{2\le n\le x+1\\ n\equiv v\ (d_1d_2)}} g_3(n) \\ &= \frac{1}{x}\sum_{\substack{d_j\le y\\ j=1,2\\ (d_1,d_2)=1}} h_{1r}(d_1)h_{2r}(d_2) \sum_{\substack{2\le n\le x+1\\ n\equiv v\ (d_1d_2)}} g_3(n) \\ &\quad + \frac{1}{x}\sum_{k=1}^{2}\sum_{\substack{d_j\le x+4-j\\ j=1,2\\ d_k>y}} h_{1r}(d_1)h_{2r}(d_2)\left(\frac{x}{d_1d_2}+1\right) =: P_{2\tau} + U_{11}, \end{aligned}$$

where v is the unique solution of the system $n \equiv -2 \pmod{d_1}, n \equiv -1 \pmod{d_2}$ and $y := x^{1/4}(\log x)^{-\beta/2}$ with $\beta > 0$.

From (29) and (30), for $0 < \alpha, \gamma < 1$, we have

$$\begin{aligned} U_{11} &\ll y^{-\gamma}\exp\left(c_6\frac{r^\gamma}{\log r}\right) + x^{1-2\alpha}\exp\left(2c_2\frac{r^\alpha}{\log r}\right) \\ &\ll x^{-\gamma/4}(\log x)^{\gamma\beta/2}\exp\left(c_6\frac{r^\gamma}{\log r}\right) + x^{1-2\alpha}\exp\left(2c_2\frac{r^\alpha}{\log r}\right) \end{aligned}$$

and

$$P_{2\tau} = \frac{1}{x}\sum_{d_1\le y}\sum_{\substack{d_2\le y\\(d_1,d_2)=1}}\frac{h_{1r}(d_1)h_{2r}(d_2)}{\phi(d_1d_2)}\sum_{\substack{n\le x\\(n,d_1d_2)=1}} g_3(n) + O\biggl(\frac{1}{x}\sum_{\substack{d_j\le y\\j=1,2}}|h_{jr}(d_j)|\biggr)$$

$$+ O\biggl(\frac{1}{x}\sum_{\substack{d_j\le y\\j=1,2}}|h_{1r}(d_1)h_{2r}(d_2)|\biggl|\sum_{\substack{n\le x\\n\equiv v\,(d_1d_2)}} g_3(n) - \frac{1}{\phi(d_1d_2)}\sum_{\substack{n\le x\\(n,d_1d_2)=1}} g_3(n)\biggr|\biggr)$$

$$=: P_{3\tau} + U_{12} + U_{13}.$$

By Lemmas 6.3 and 6.4, we have

$$U_{13} \ll \frac{1}{x}\biggl(\sum_{l\le y^2}\biggl|\sum_{\substack{n\le x\\n\equiv v\,(l)}} g_3(n) - \frac{1}{\phi(l)}\sum_{\substack{n\le x\\(n,l)=1}} g_3(n)\biggr|\biggr)^{1/2}$$

$$\times\biggl(\sum_{\substack{d_j\le y\\j=1,2}}|h_{1r}(d_1)|^2|h_{2r}(d_2)|^2\biggl|\sum_{\substack{n\le x\\n\equiv v\,(d_1d_2)}} g_3(n) - \frac{1}{\phi(d_1d_2)}\sum_{\substack{n\le x\\(n,d_1d_2)=1}} g_3(n)\biggr|\biggr)^{1/2}$$

$$\ll \frac{1}{(\log x)^{A/2}}\biggl(\sum_{\substack{d_j\le y\\j=1,2}}\frac{|h_{1r}(d_1)|^2|h_{2r}(d_2)|^2}{\phi(d_1)\phi(d_2)}\biggr)^{1/2}.$$

Observe that

$$\sum_{d\le y}\frac{|h(d)|^2}{\phi(d)} \le \exp\biggl(c_7\sum_{p\le r}\frac{1}{p}\biggr) \le (\log r)^{c_8}.$$

Hence

$$U_{13} \ll \frac{(\log r)^{c_8}}{(\log x)^{A/2}}.$$

Now from (29) we get

$$U_{12} \ll \frac{y^{2(1-\alpha)}}{x}\sum_{\substack{d_j=1\\j=1,2}}^{\infty}\frac{|h_{1r}(d_1)h_{2r}(d_2)|}{d_1^{1-\alpha}d_2^{1-\alpha}} \ll x^{-\frac{1}{2}(1+\alpha)}(\log x)^{-\beta(1-\alpha)}\exp\biggl(\frac{2c_2r^\alpha}{\log r}\biggr).$$

Using Lemma 6.2, we obtain

$$P_{3\tau} = \frac{1}{x}\sum_{d_1\le y}\sum_{\substack{d_2\le y\\(d_1,d_2)=1}}\frac{h_{1r}(d_1)h_{2r}(d_2)}{\phi(d_1d_2)}\theta_\tau(d_1d_2)\sum_{n\le x} g_3(n)$$

$$+ O\biggl(\sum_{\substack{d_j\le y\\j=1,2\\(d_1,d_2)=1}}\frac{|h_{1r}(d_1)h_{2r}(d_2)|}{\phi(d_1)\phi(d_2)}\,\frac{(\log\log 3d_1d_2)^2}{(\log x)^{1/19}}\biggr) =: P_{4\tau} + U_{14}.$$

Now,

$$\sum_{\substack{d_1,d_2\le y\\(d_1,d_2)=1}}\frac{h_{1r}(d_1)h_{2r}(d_2)}{\phi(d_1)\phi(d_2)}\theta_\tau(d_1)\theta_\tau(d_2)=\sum_{\substack{d_1,d_2=1\\(d_1,d_2)=1}}^{\infty}\frac{h_{1r}(d_1)h_{2r}(d_2)}{\phi(d_1)\phi(d_2)}\theta_\tau(d_1)\theta_\tau(d_2)$$

$$+O\Bigg(\sum_{k=1}^{2}\sum_{\substack{d_j\le y\\j=1,2\\d_k>y}}\frac{h_{1r}(d_1)h_{2r}(d_2)}{\phi(d_1)\phi(d_2)}\theta_\tau(d_1)\theta_\tau(d_2)\Bigg)=:P_{5\tau}+U_{15}.$$

Now from the two estimates

$$\sum_{d>y}\frac{|h_{jr}(d)\theta_\tau(d)|}{\phi(d)}\le y^{-\alpha}\exp\Bigg(c_9\sum_{p\le r}\frac{1}{p^{1-\alpha}}\Bigg)$$
$$\ll x^{-(1-\alpha)/4}(\log x)^{\beta(1-\alpha)/2}\exp\Bigg(c_{10}\frac{r^\alpha}{\log r}\Bigg)$$

and

$$\sum_{d=1}^{\infty}\frac{|h_{jr}(d)\theta_\tau(d)|}{\phi(d)}\le\exp\Bigg(c_{11}\sum_{p\le r}\frac{1}{p}\Bigg)\le(\log r)^{c_{12}}.$$

We deduce that

$$U_{15}\ll x^{-(1-\alpha)/4}(\log x)^{\beta(1-\alpha)/2}\exp\left(c_{13}\frac{r^\alpha}{\log r}\right),$$

$$P_{5\tau}=\prod_{p\le r}\Bigg(1+\sum_{m=1}^{\infty}\frac{(h_1(p^m)+h_2(p^m))\theta_\tau(p^m)}{\phi(p^m)}\Bigg)$$
$$=\prod_{p\le r}\Bigg(1-\frac{2\theta_\tau(p)}{p-1}+\theta_\tau(p)\sum_{m=1}^{\infty}\frac{g_1(p^m)+g_2(p^m)}{p^m}\Bigg)=Q_\tau(r).$$

Observe that

$$\sum_{d=1}^{\infty}\frac{|h_{jr}(d)|\log\log d}{\phi(d)}=\prod_{p\le r}\Bigg(1+\sum_{\alpha=1}^{\infty}\frac{|h_{jr}(p^\alpha)|\alpha\log\log p}{\phi(p^\alpha)}\Bigg)$$
$$\ll\prod_{p\le r}\Bigg(1+\frac{2p\log\log p}{(p-1)^2}\Bigg)\ll\exp\Bigg(c_{14}\sum_{p\le r}\frac{\log\log p}{p}\Bigg)$$
$$\ll\exp(c_{15}(\log\log r)^2).$$

Hence

$$U_{14}\ll\frac{\exp(2c_{15}(\log\log r)^2)}{(\log x)^{1/19}}.$$

Using the same method as in estimation of T_2, we obtain

$$U_2\ll(S_2(r,x))^{1/2}+(r\log r)^{-1/2}.$$

Combining these results gives

$$\begin{aligned}R &\ll \big(x^{-\gamma/4}(\log x)^{\gamma\beta} + x^{(\alpha-1)/4}(\log x)^{\beta(1-\alpha)/2}\big)\exp\bigg(c_{16}\frac{r^{\gamma}}{\log r}\bigg)\\ &\quad + x^{1-2\alpha}\exp\bigg(c_{17}\frac{r^{\alpha}}{\log r}\bigg) + \frac{(\log r)^{c_8}}{(\log x)^{A/2}}\\ &\quad + x^{-(1+\alpha)/2}(\log x)^{\beta(1-\alpha)}\exp\bigg(c_{18}\frac{r^{\alpha}}{\log r}\bigg)\\ &\quad + \frac{\exp(2c_{15}(\log\log r)^2)}{(\log x)^{1/19}} + (S_2(r,x))^{1/2} + (r\log r)^{-1/2}.\end{aligned}$$

By choosing $\alpha = \gamma$, we get the conclusion of Theorem 3.4.

7. Proof of Theorem 3.6. Set

$$T(r,x) = \prod_{r<p\le x}\bigg\{1-\frac{1}{p}+\bigg(1-\frac{1}{p}\bigg)\sum_{m=1}^{\infty}\frac{g_1(p^m)}{p^m}\bigg\}.$$

Now,

$$\begin{aligned}\sum_{n\le x} g_1(n+3)\mu(n+2)\mu(n+1) &= T(r,x)\sum_{n\le x} g_{1r}(n+3)\mu(n+2)\mu(n+1)\\ &\quad + \sum_{n\le x} g_{1r}(n+3)\mu(n+2)\mu(n+1)(g_{1r}^*(n+3)-T(r,x)).\end{aligned}$$

It is easy to see that $|T(r,x)|\le 1$. Therefore,

$$\begin{aligned}\Big|\sum_{n\le x} g_1(n+3)\mu(n+2)\mu(n+1)\Big| &\le \Big|\sum_{n\le x} g_{1r}(n+3)\mu(n+2)\mu(n+1)\Big|\\ &\quad + \sum_{n\le x}|g_{1r}^*(n+3)-T(r,x)| =: V_1+V_2.\end{aligned}$$

Estimation of V_1. We have

$$\begin{aligned}V_1 &= \Big|\sum_{n\le x}\sum_{d\mid n+3} h_{1r}(d)\mu(n+2)\mu(n+1)\Big|\\ &\le \Big|\sum_{d\le y} h_{1r}(d)\sum_{\substack{n\le x\\ n\equiv -3\,(d)}}\mu(n+2)\mu(n+1)\Big|\\ &\quad + \Big|\sum_{y<d\le x+3} h_{1r}(d)\sum_{\substack{n\le x\\ n\equiv -3\,(d)}}\mu(n+2)\mu(n+1)\Big| =: V_{11}+V_{12},\end{aligned}$$

where $y := \log x$. Under Assumption 3.5, we have

$$\sum_{\substack{n\le x\\ n\equiv -3\,(d)}} \mu(n+2)\mu(n+1) = \sum_{n\le x}\mu(n+2)\mu(n+1)\frac{1}{d}\sum_{l=1}^{d}\exp\biggl(\frac{(n-3)l}{d}\biggr)$$

$$= \frac{1}{d}\sum_{l=1}^{d}\exp\biggl(\frac{-3l}{d}\biggr)\sum_{n\le x}\mu(n+2)\mu(n+1)\exp\biggl(\frac{nl}{d}\biggr) \ll \frac{x}{(\log x)^A}.$$

By choosing $A = 3$ we get

$$V_{11} \ll \frac{x}{(\log x)^3}\sum_{d\le y}|h_{1r}(d)| \ll \frac{xy^2}{(\log x)^3}\sum_{d=1}^{\infty}\frac{|h_{1r}(d)|}{d^2} \ll \frac{x}{\log x}.$$

Now using (29), for $0<\alpha<1$, we have

$$V_{12} \ll \sum_{y<d\le x+3}|h_{1r}(d)|\biggl(\frac{x}{d}+1\biggr) \ll (x+3)\sum_{d>y}\frac{|h_{1r}(d)|}{d}$$

$$\ll \frac{x+3}{y^\alpha}\sum_{d=1}^{\infty}\frac{|h_{1r}(d)|}{d^{1-\alpha}} \ll \frac{x}{y^\alpha}\exp\biggl(c_2\frac{r^\alpha}{\log r}\biggr).$$

By taking $r = (\log\log x)^{1/\alpha}$, we see that

$$V_{12} \ll xy^{-\alpha}y^{\alpha/2} = \frac{x}{(\log x)^{\alpha/2}}.$$

So as $x\to\infty$ we have $V_1 = o(x)$.

From a similar calculation to that in the estimation of T_2, we have

$$V_2 \ll x(r\log r)^{-1/2} + x\biggl(\sum_{r<p\le x+3}\frac{|g_1(p)-1|^2}{p}\biggr)^{1/2}.$$

From (12) and $r = (\log\log x)^{1/\alpha}$ we conclude that $V_2 = o(x)$ as $x\to\infty$, which completes the proof of Theorem.

8. Proofs of corollaries

Proof of Corollary 4.1. We need the following lemma:

LEMMA 8.1 ([TEN]). *Let* $\{u_n\}_{n=1}^{\infty}$ *and* $\{v_n\}_{n=1}^{\infty}$ *be two complex sequences such that*

$$\sum_{n=1}^{\infty}(|u_n|^2+|v_n|) < \infty.$$

Then

$$\prod_{n=1}^{\infty}(1+u_n+v_n)\ \textit{converges}\quad\textit{if and only if}\quad\sum_{n=1}^{\infty}u_n\ \textit{converges}.$$

Since $|g_j(p)-1|^2 \le 2(1-\Re(g_j(p)))$, from (5) we have

$$\sum_p \frac{|g_j(p)-1|^2}{p} < \infty. \tag{31}$$

So by setting $r = \log x$, the error term in Theorem 3.1 is $o(1)$ as $x \to \infty$. Now the infinite product $\prod_p w_p$ can be written as

$$\prod_p w_p = w_2 \prod_{p>2} w_p = w_2 \prod_p \left\{1 + \frac{g_1(p)+g_2(p)+g_3(p)-3}{p} + O\left(\frac{1}{p^2}\right)\right\}.$$

From (5) and Lemma 8.1 we can see that the above infinite product is convergent. Hence, by Theorem 3.1, the corollary is proved.

Proof of Corollary 4.2. We need Lemma 8.1 and the following lemma:

LEMMA 8.2 ([KAT, Lemma 6]). *Let $F(n)$ be a polynomial as above of degree $v \ge 2$. Then*

$$\mathbf{card}\{n \le x : F(n) \equiv 0 \pmod{p^{v-1}},\ y_1 < p < \infty\} = o(x)$$

when $y_1 = y_1(x)$ tends to infinity as $x \to \infty$.

Since $|g_j(p)-1|^2 \le 2(1-\Re(g_j(p)))$, from (13) we have

$$\sum_p \frac{|g_j(p)-1|^2 \varrho_j(p)}{p} < \infty, \qquad j = 1,2,3. \tag{32}$$

From (13) and Lemma 8.1, we find that $P_1(\gamma)$ and $P_2(\gamma)$ are convergent. From Lemma 8.2, it is easy to see that as $p \to \infty$,

$$(1-g_j(p^{v_j-1}))\varrho_j(p^{v_j-1}) \to 0, \qquad j = 1,2,3. \tag{33}$$

So by setting $r = \log x$ and from (14), (32), (33), the error term in Theorem 3.2 is $o(1)$ as $x \to \infty$. Hence, by Theorem 3.2, the corollary is proved.

Proof of Corollary 4.3. It is easy to see that $\phi(n)/\sigma(n)$ and $\phi(n)/n$ are close to 1. The remainder terms for both sums are estimated from the remainder term of Theorem 3.1 by choosing

$$\beta = 1-\alpha = \min\left(\frac{1}{2k}, \frac{2+c_{19}}{3}\right) \quad \text{and} \quad r = c_{20}(\log x \log\log x)^{1/\beta},$$

for sufficiently small $c_{19}, c_{20} > 0$. Hence, by Theorem 3.1, the corollary is proved.

Proof of Corollaries 4.4 and 4.5. Since $\phi(n)/n$ and $\phi(n)/\sigma(n)$ are close to 1, by applying Theorems 3.3 and 3.6 we get Corollaries 4.4 and 4.5 respectively.

Proof of Corollary 4.6. We need the following lemma:

Lemma 8.3. *Let p be an odd prime and $(a,p)=1$. Then the equation $x^2 \equiv a \pmod{p^k}$ has exactly two solutions if a is a quadratic residue of p, and no solution if a is quadratic nonresidue of p. Further, if a is odd, then the congruence $x^2 \equiv a \pmod 2$ is always solvable and has exactly one solution.*

We see that by Lemma 8.3, as $p \to \infty$,

$$\sum_p \frac{(g_j(p)-1)\varrho_j(p)}{p} = \sum_p \frac{-2}{p(p+1)} < \infty, \quad (g_j(p)-1)\varrho_j(p) = \frac{-2}{p+1} \to 0, \tag{34}$$

where $g_j(n) = \phi(n)/\sigma(n), j=1,2,3$, and

$$\sum_p \frac{(g_j(p)-1)\varrho_j(p)}{p} = \sum_p \frac{-2}{p^2} < \infty, \quad (g_j(p)-1)\varrho_j(p) = \frac{-2}{p} \to 0, \tag{35}$$

where $g_j(n) = \phi(n)/n, j=1,2,3$. The remainder terms for both sums are estimated from the remainder term of Theorem 3.2 by choosing

$$\alpha = (5+c_{21})/6 \quad \text{and} \quad r = c_{22}(\log x \log\log x)^{1/\alpha},$$

for sufficiently small $c_{21}, c_{22} > 0$. Hence, by Theorem 3.2, the corollary is proved.

Proof of Corollary 4.7. We need the following lemma:

Lemma 8.4 ([TEN]). *Let $\{F_n\}_{n=1}^\infty$ be a sequence of distribution functions and $\{\phi_n\}_{n=1}^\infty$ the corresponding sequence of characteristic functions. Then F_n converges weekly to a distribution function F if and only if ϕ_n converges pointwise on $\mathbb{R}$ to a function ϕ which is continuous at* 0. *In addition, in this case, ϕ is the characteristic function of F and the convergence of ϕ_n to ϕ is uniform on any compact subset.*

The characteristic function of the distribution (19) equals

$$\frac{1}{[x]} \sum_{n \le x} \exp\big(it(f_1(n+3) + f_2(n+2) + f_3(n+1))\big). \tag{36}$$

Since

$$\sum_p \sum_{j=1}^3 \frac{\exp(itf_j(p)) - 1}{p} = t \sum_{j=1}^3 \sum_{|f_j(p)|\le 1} \frac{f_j(p)}{p} + O\bigg(\sum_{j=1}^3 \bigg(t^2 \sum_{|f_j(p)|\le 1} \frac{f_j^2(p)}{p} + \sum_{|f_j(p)|>1} \frac{1}{p}\bigg)\bigg),$$

from the convergence of the series (16)–(18) and from Lemma 8.1 we infer that the infinite product (20) converges for every t. This product is continuous at $t=0$ because it converges uniformly for $|t| \le T$ where $T>0$ is arbitrary.

Since for $j=1,2,3$,

$$\sum_p \frac{|\exp(itf_j(p))-1|^2}{p} \ll t^2 \sum_{|f_j(p)|\le 1} \frac{|f_j(p)|^2}{p} + \sum_{|f_j(p)|>1} \frac{1}{p},$$

from the convergence of (16) and (17) it follows that $S(r,x)\to 0$ as $r,x\to\infty$. Choosing $r=\log x$ in our Theorem 3.1 we see that the remainder term vanishes as $x\to\infty$. Thus the characteristic function (36) has limit (20) for every real t, and this limit is continuous at $t=0$. Therefore, by Lemma 8.4, the corollary is proved.

Proof of Corollary 4.8. We will use Lemmas 8.1 and 8.4. The characteristic function of the distribution (25) equals

$$\frac{1}{[x]}\sum_{n\le x} \exp\bigl(it(f_1(F_1(n))+f_2(F_2(n))+f_3(F_3(n)))\bigr). \tag{37}$$

Since

$$\begin{aligned}\sum_p\sum_{j=1}^{3}\frac{(\exp(itf_j(p))-1)\varrho_j(p)}{p} &= t\sum_{j=1}^{3}\sum_{|f_j(p)|\le 1}\frac{f_j(p)\varrho_j(p)}{p}\\ &\quad + O\Bigl(t^2\sum_{j=1}^{3}\sum_{|f_j(p)|\le 1}\frac{f_j^2(p)\varrho_j(p)}{p}\Bigr) + O\Bigl(\sum_{j=1}^{3}\sum_{|f_j(p)|>1}\frac{\varrho_j(p)}{p}\Bigr),\end{aligned}$$

from the convergence of the series (21)–(23) and from Lemma 8.1 we deduce that $P_1(\gamma)$ and $P_2(\gamma)$ are convergent for every real t. Further, the infinite product $P_1(\gamma)P_2(\gamma)$ is continuous at $t=0$ because it converges uniformly for $|t|\le T$ where $T>0$ is arbitrary.

Since for $j=1,2,3$,

$$\sum_p \frac{|\exp(itf_j(p))-1|^2\varrho_j(p)}{p} \ll t^2\sum_{|f_j(p)|\le 1}\frac{|f_j(p)|^2\varrho_j(p)}{p} + \sum_{|f_j(p)|>1}\frac{\varrho_j(p)}{p},$$

from the convergence of (21) and (22) it follows that $S_1(r,x), T(x)\to 0$ as $r,x\to\infty$.

Now from (24) and Lemma 8.2 it is easy to see that

$$(\exp(itf_j(p^m)-1)\varrho_j(p^m))\to 0 \quad \text{as } p\to\infty, m<v_j, j=1,2,3.$$

Then $\frac{1}{x}C(r,x)\to 0$ as $r,x\to\infty$. Choosing $r=\log x$ in our Theorem 3.2 we see that the remainder term vanishes as $x\to\infty$.

Thus the characteristic function (37) has limit $P_1(\gamma)P_2(\gamma)$ for every real t, and this limit is continuous at $t=0$.

Therefore, by Lemma 8.4, the corollary is proved.

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