ACTA ARITHMETICA
180.2 (2017)

Szemerédi-type theorems for subsets of locally compact abelian groups of positive upper Banach density

by

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1. Introduction. E. Szemerédi's theorem proved in 1975 says that if a set $E \subseteq \mathbb{Z}$ is of positive upper Banach density, i.e.,

$$\mathrm{BD}^*(E) := \limsup_{N-M\to\infty} \frac{|E\cap[M,N)|}{N-M} > 0,$$

then E contains arbitrarily long arithmetic progressions [8, 3]. By using their multiple recurrence theorem, H. Fürstenberg and Y. Katznelson proved in 1978 a multidimensional version of Szemerédi's theorem: If $E \subseteq \mathbb{Z}^n$ is of positive upper Banach density and F is a finite subset of $\mathbb{Z}^n$, then for some vector $u \in \mathbb{Z}^n$ and integer $d \geq 1$, $u + dF \subset E$ ([4] or [3, Theorem 7.16]).

Based on the above multidimensional version, H. Fürstenberg [3, Theorem 7.17] proved that if $E \subseteq \mathbb{R}^n$ is of positive upper Banach density with respect to the Lebesgue measure on $\mathbb{R}^n$ and F is a finite subset of $\mathbb{R}^n$, then for some vector $u \in \mathbb{R}^n$ and integer $d \geq 1$, $u + dF \subset E$.

In the more general case where $\mathbb{R}^n$ is replaced by an abelian discrete additive group $(G, +)$, using the Stone-Čech compactification βG of G and Fürstenberg's multiple recurrence theorem, Hindman and Strauss [5, Theorems 5.5 and 5.6] proved in 2006 that if $E \subset G$ is of positive upper density relative to some Følner net in G, then for any $a \in G$ and $l \in \mathbb{N}$, $\{u \in G \,|\, u + d\{a, 2a, \ldots, la\} \subseteq E\}$ has positive upper density relative to the same Følner net for some $d \in \mathbb{N}$.

In this note, we will extend the usual notion of upper Banach density, and then by using ergodic-theoretic techniques following H. Fürstenberg,

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2010 *Mathematics Subject Classification*: Primary 22B05; 37A15; Secondary 11B25; 37A45.

Key words and phrases: Szemerédi theorem, Fürstenberg correspondence principle, amenable group.
Received 17 March 2016; revised 30 March 2017.
Published online 1 August 2017.

DOI: 10.4064/aa8470-4-2017

we shall derive more generalizations of Szemerédi's theorem for any finite configurations not limited to the form $\{a, 2a, \ldots, la\}$ (Theorems 3.1 and 3.3 and Corollary 3.5), for any locally compact Hausdorff, not necessarily discrete, abelian group G with any fixed Haar measure.

2. Følner sequences and upper density.

In this section we introduce some basic notions and preliminary lemmas.

2.1. Basic notions. Let $(G, +)$ be a locally compact Hausdorff additive topological group. According to Haar's theorem [6, Theorem 29C], there exists a left-invariant Haar measure on $(G, +)$, which we denote by $|\cdot|$ or dg. When G is discrete, $|\cdot|$ is just the usual counting measure on G.

A sequence $(F_n)_{n=1}^{\infty}$ of compact subsets in G is called a *classical Følner sequence* in $(G, +, |\cdot|)$ if

$$\lim_{n\to\infty} \frac{|(g + F_n) \triangle F_n|}{|F_n|} = 0 \quad \forall g \in G.$$

It is well known that if G is a locally compact σ-compact Hausdorff abelian group, like $(\mathbb{Z}^m, +)$ with the discrete topology or $(\mathbb{R}^m, +)$ with the Euclidean metric topology, then it has classical Følner sequences. Although a discrete uncountable abelian group does not have any classical Følner sequences, it always has Følner nets [1, 5].

Since an uncountable abelian group has no classical Følner sequence under the discrete topology, we need to introduce a nonclassical Følner sequence in any locally compact Hausdorff group $(G, +, |\cdot|)$.

DEFINITION 2.1. Given any subset $F \subseteq G$, a sequence $\mathcal{F} = (F_n)_{n=1}^{\infty}$ of compact subsets of G is called an *F-Følner sequence* in $(G, +, |\cdot|)$ if

$$\lim_{n\to\infty} \frac{|(g + F_n) \triangle F_n|}{|F_n|} = 0 \quad \forall g \in F. \tag{2.1}$$

Notice that (2.1) holds only for $g \in F$ but not for any $g \in G$.

A classical Følner sequence in $(G, +, |\cdot|)$ is just a G-Følner sequence. Let us consider an example in order to show that an F-Følner sequence in $(G, +, |\cdot|)$ is not necessarily a classical Følner sequence for $F \neq G$. Let $(G, +) = (\mathbb{R}^2, +)$ with $|\cdot| = dxdy$ under the standard Euclidean topology and $\varepsilon > 0$. Define a sequence of thin rectangles $F_n = \{(x, y) : |x - n| \leq \varepsilon, |y| \leq n\}$. Clearly, $|F_n| = 4n\varepsilon \to \infty$ as $n \to \infty$. It is easy to see that for $F = \{0\} \times \mathbb{R}$, $(F_n)_{n=1}^{\infty}$ is an F-Følner sequence but not a classical Følner sequence in $(G, +, |\cdot|)$.

DEFINITION 2.2. Given any $F \subseteq G$, for any F-Følner sequence $\mathcal{F} = (F_n)_{n=1}^{\infty}$ in $(G, +, |\cdot|)$ and any $|\cdot|$-measurable subset $E \subseteq G$, we set

$$\mathrm{D}^*_{\mathcal{F}}(E) = \limsup_{n\to\infty} \frac{|E\cap F_n|}{|F_n|}, \tag{2.2}$$

which is called the *upper density of E relative to $\mathcal{F}$* in $(G,+,|\cdot|)$.

We shall say that E is of *positive upper Banach density* in $(G,+,|\cdot|)$, and write $\mathrm{BD}^*(E) > 0$, if $\mathrm{D}^*_{\mathcal{F}}(E) > 0$ for some classical Følner sequence $\mathcal{F} = (F_n)_{n=1}^\infty$ in $(G,+,|\cdot|)$. We shall be concerned with measurable sets $E \subseteq G$ having $\mathrm{BD}^*(E) > 0$.

A locally compact Hausdorff additive topological group $(G,+)$ is said to be *amenable* if the so-called *Følner condition* holds: for any compact set $K \subseteq G$ and any $\varepsilon > 0$, there exists a compact set $F \subseteq G$ such that

$$\frac{|(K+F)\,\triangle\, F|}{|F|} < \varepsilon$$

(see, e.g., [7]). Each locally compact Hausdorff abelian topological group is amenable. Of course, an amenable group need not have a classical Følner sequence if the group is not σ-compact [7]. However, for any compact subset K of an amenable group G there always exists a K-Følner sequence in the sense of Definition 2.1.

2.2. Preliminary lemmas. The following simple result follows from the definition of amenable group:

LEMMA 2.3. *If $(G,+)$ is an amenable and locally compact Hausdorff topological group, then for any compact set $K \subseteq G$ there exists a K-Følner sequence $(F_n)_{n=1}^\infty$ in $(G,+,|\cdot|)$.*

Proof. Since G is amenable and locally compact, [7, Theorem 4.10] shows that for any compact set $K \subseteq G$ and $\varepsilon_n > 0$, there exists some compact set $F_n \subseteq G$ such that

$$\frac{|(g+F_n)\,\triangle\, F_n|}{|F_n|} < \varepsilon_n \quad \forall g \in K.$$

Letting $\varepsilon_n \to 0$ implies the desired result. ■

This lemma enables us to choose an F-Følner sequence $\mathcal{F} = (F_n)_{n=1}^\infty$ in $(G,+,|\cdot|)$ for any compact set $F \subseteq G$ in Theorems 3.1 and 3.3 below.

It is well known that any discrete countable abelian group G has G-Følner sequences. Although this is not the case for any discrete abelian group, yet we can obtain the following

LEMMA 2.4. *Let $(G,+)$ be a discrete abelian group. Then for any finite subset $F \subseteq G$ and any Følner net $(F_\theta)_{\theta\in\Theta}$ in $(G,+,|\cdot|)$, there exists an F-Følner sequence $(F_n)_{n=1}^\infty$ in $(G,+,|\cdot|)$ such that $(F_n)_{n=1}^\infty \subseteq (F_\theta)_{\theta\in\Theta}$.*

Proof. Let $(F_\theta)_{\theta\in\Theta}$ be a Følner net in $(G,+,|\centerdot|)$ that is, $(\Theta,\geq)$ is a directed index set and each F_θ is a finite subset of G such that

$$\lim_{\theta\in\Theta}\frac{|(g+F_\theta)\vartriangle F_\theta|}{|F_\theta|}=0\quad \forall g\in G.$$

Then for any $\epsilon>0$, there is some $\theta_\epsilon\in\Theta$ such that

$$\frac{|(g+F_\theta)\vartriangle F_\theta|}{|F_\theta|}<\epsilon\quad \forall g\in F \text{ and } \theta\geq\theta_\epsilon.$$

Letting $\epsilon\to 0$ we get the desired F-Følner sequence $(F_n)_{n=1}^\infty$ in $(G,+,|\centerdot|)$. ∎

For any $K\subset G$, we denote by $\langle K\rangle$ the subgroup of $(G,+)$ spanned by K,

$$\langle K\rangle=\{k_1+\cdots+k_n \mid n\geq 1,\, k_i\in K\cup(-K)\},$$

where $-K=\{-k\,|\,k\in K\}$.

LEMMA 2.5. *Let $(G,+)$ be an amenable locally compact Hausdorff topological group and $K\subseteq G$ any compact set. If $(F_n)_{n=1}^\infty$ is a K-Følner sequence in $(G,+,|\centerdot|)$, then it is also a $\langle K\rangle$-Følner sequence in $(G,+,|\centerdot|)$.*

Proof. Given any $g,g_1,g_2\in K$, by (2.1) with K in place of F we have

$$\begin{aligned}\lim_{n\to\infty}&\frac{|(g_2+g_1+F_n)\vartriangle F_n|}{|F_n|}\\ &\leq\lim_{n\to\infty}\frac{|(g_2+g_1+F_n)\setminus F_n|+|F_n\setminus(g_2+g_1+F_n)|}{|F_n|}\\ &=\lim_{n\to\infty}\frac{|(g_2+g_1+F_n)\setminus(g_2+F_n)|+|(g_2+F_n)\setminus(g_2+g_1+F_n)|}{|F_n|}\\ &=0\end{aligned}$$

and

$$\lim_{n\to\infty}\frac{|(-g+F_n)\vartriangle F_n|}{|F_n|}=\lim_{n\to\infty}\frac{|F_n\vartriangle(g+F_n)|}{|F_n|}=0.$$

Thus $(F_n)_{n=1}^\infty$ is a $\langle K\rangle$-Følner sequence in $(G,+,|\centerdot|)$. ∎

We notice here that, as a σ-compact topological group, $(\langle K\rangle,+)$ itself need not be amenable, since $\langle K\rangle$ is not necessarily a closed subset of G. In addition, $(F_n\cap\langle K\rangle)_{n=1}^\infty$ need not be a Følner sequence in $\langle K\rangle$. In fact, $F_n\cap\langle K\rangle=\emptyset$ in the example constructed before. Moreover, it is possible that $E\cap\langle K\rangle=\emptyset$ in Definition 2.2.

3. Fürstenberg correspondence principle.

Let $(G,+)$ be a locally compact Hausdorff additive topological group with zero element $\boldsymbol{o}$. Given any compact Hausdorff space X, we shall say that $T\colon G\times X\to X$ is a *Borel G-action* on X (X is then called a *Borel G-space* and we simply write $G\curvearrowright_T X$) if

- $T_g\colon x \mapsto T(g,x)$ is a continuous selfmap of X for each $g \in G$;
- $T\colon (g,x) \mapsto T(g,x)$ is jointly Borel measurable;
- $T_{\boldsymbol{o}}x = x$ for each $x \in X$, and $T_{g+h} = T_g \circ T_h$ for all $g, h \in G$.

In this section, we will consider Fürstenberg correspondence principles between configurations in subsets of G and Borel G-space X associated to G.

The following Szemerédi-type theorem (Theorem 3.1) is one of our main results, in the proof of which there are two ingredients in our Fürstenberg correspondence principle: (1) the compact Hausdorff G-space X is not necessarily metrizable; and (2) although the topology of $(G, +, |\boldsymbol{\cdot}|)$ may not be discrete, yet to define the associated G-action we will employ the discrete topology that is not necessarily compatible with the fixed Haar measure $|\boldsymbol{\cdot}|$ on G.

THEOREM 3.1. *Let $(G,+)$ be a locally compact Hausdorff abelian topological group and $F \subseteq G$ a compact subset. If a measurable $E \subseteq G$ is of positive upper density corresponding to an F-Følner sequence $\mathcal{F} = (F_n)_{n=1}^\infty$ in $(G, +, |\boldsymbol{\cdot}|)$, then for any $g_1, \ldots, g_l \in \langle F \rangle$,*

$$\mathrm{BD}_*\big(\big\{d \in \mathbb{Z} \,\big|\, \mathrm{D}^*_{\mathcal{F}}(\{u \in E\colon u + d\{g_1, \ldots, g_l\} \subseteq E\}) > 0\big\}\big) > 0.$$

Here BD_* denotes the lower Banach density of sets in $(\mathbb{Z}, +, |\boldsymbol{\cdot}|_{\mathbb{Z}})$, and $\langle F \rangle$ stands for the subgroup of G generated by F.

Proof. By refining the F-Følner sequence $\mathcal{F} = (F_n)_{n=1}^\infty$ if necessary, we may assume

$$\mathrm{D}^*_{\mathcal{F}}(E) = \lim_{n\to\infty} \frac{|E \cap F_n|}{|F_n|}.$$

Let $X = \prod_{g\in G}\{0,1\}$ be the Cartesian product endowed with the standard pointwise convergence topology. Then X is a compact Hausdorff space. For any $x \in X$, we may identify it with the function $x(\boldsymbol{\cdot})\colon g \mapsto x(g)$ from G into the discrete space $\{0,1\}$. Note that $x(\boldsymbol{\cdot}) \in X$ is not necessarily a measurable function from G to $\{0,1\}$ under the locally compact Hausdorff topology of G and that the compact Hausdorff topology of X is independent of the topology of G.

Let $\chi \in X$ be given by $\chi(g) = 1$ if and only if $g \in E$. Since by hypothesis E is $|\boldsymbol{\cdot}|$-measurable, $\chi(\boldsymbol{\cdot})\colon G \to \{0,1\}$ is $|\boldsymbol{\cdot}|$-measurable under the locally compact Hausdorff topology of G.

Define the clopen cylinder set of X, $[1]_{\boldsymbol{o}} = \{x \in X \,|\, x(\boldsymbol{o}) = 1\}$, which is a compact G_δ-set and so is a Baire subset of X. Then the corresponding characteristic function $1_{[1]_{\boldsymbol{o}}}$ is a continuous function on X, i.e., $1_{[1]_{\boldsymbol{o}}} \in C(X)$.

As in the usual Fürstenberg correspondence principle, *under the discrete topology of G* we may now define a G-action on X as follows:

$$(g,x) \mapsto T_g x = x(\boldsymbol{\cdot} + g)$$

where $x(\boldsymbol{\cdot}+g)\in X$ is defined by

$$x(\boldsymbol{\cdot}+g)\colon t\mapsto x(t+g).$$

It should be noted that the continuity of $T_g\colon X\to X$ is obvious under the product topology of X. Thus, under the discrete topology of G, $G\curvearrowright_T X$ is a canonical G-action; in other words, X is a Borel G-space.

By the Riesz representation theorem we can identify Baire measures on X with positive functionals on $C(X)$. Using the refined F-Følner sequence $\mathcal{F}=(F_n)_{n=1}^\infty$ in $(G,+,|\boldsymbol{\cdot}|)$ we define a Baire probability on the product space X by

$$\mu_n(\varphi)=\frac{1}{|F_n|}\int\limits_{F_n}\varphi(T_g\chi)\,dg\qquad\forall\varphi\in C(X),$$

noting that $\varphi(T_{\boldsymbol{\cdot}}\chi)\colon g\mapsto\varphi(T_g\chi)$ is $|\boldsymbol{\cdot}|$-measurable and dg-integrable from F_n to $\{0,1\}$ under the topology of $(G,+,|\boldsymbol{\cdot}|)$ because of the measurability of E, and that the Haar measure $|\boldsymbol{\cdot}|$ is not defined under the newly introduced discrete topology of G.

Under the usual weak-$*$ topology of Baire probability measures on X, we can find a net $(\mu_\theta)_{\theta\in\Theta}$, which is a subnet of the Baire probability measure sequence $(\mu_n)_{n=1}^\infty$, such that

$$\lim_{\theta\in\Theta}\mu_\theta=\mu\qquad\text{weakly-}*$$

for some Baire probability measure μ on X. By Lemma 2.5, it is routine to check that μ is T_g-invariant for each $g\in\langle F\rangle$ (not for any $g\in G$).

Since $\mathrm{D}^*_{\mathcal{F}}(E)>0$, we have $\mu([1]_{\boldsymbol{o}})>0$. Indeed, as $1_{[1]_{\boldsymbol{o}}}(\boldsymbol{\cdot})\in C(X)$ and $1_{[1]_{\boldsymbol{o}}}(T_g\chi)=1_E(g)$, it follows that

$$\mu([1]_{\boldsymbol{o}})=\lim_{\theta\in\Theta}\frac{1}{|F_\theta|}\int\limits_{F_\theta}1_{[1]_{\boldsymbol{o}}}(T_g\chi)\,dg=\lim_{\theta\in\Theta}\frac{|E\cap F_\theta|}{|F_\theta|}=\mathrm{D}^*_{\mathcal{F}}(E).$$

Let $g_1,\dots,g_l\in\langle F\rangle$. Then by Fürstenberg's multiple recurrence theorem [4, 3], it follows that

$$D=\{d\in\mathbb{Z}\mid\mu([1]_{\boldsymbol{o}}\cap T_{g_1}^{-d}[1]_{\boldsymbol{o}}\cap\cdots\cap T_{g_l}^{-d}[1]_{\boldsymbol{o}})>0\}$$

is of positive lower Banach density in $(\mathbb{Z},+,|\boldsymbol{\cdot}|_{\mathbb{Z}})$. Set $K=\{g_1,\dots,g_l\}$. Next, given any $d\in D$ we set

$$A=\{u\in E\mid u+dK\subseteq E\}\quad\text{and}\quad U=[1]_{\boldsymbol{o}}\cap T_{g_1}^{-d}[1]_{\boldsymbol{o}}\cap\cdots\cap T_{g_l}^{-d}[1]_{\boldsymbol{o}}.$$

Then U is clopen and so is a Baire set in X for T_g is continuous of X to itself and hence

$$\mu(U)=\lim_{\theta\in\Theta}\frac{1}{|F_\theta|}\int\limits_{F_\theta}1_U(T_g\chi)\,dg\le\lim_{\theta\in\Theta}\frac{|A\cap F_\theta|}{|F_\theta|}\le\limsup_{n\to\infty}\frac{|A\cap F_n|}{|F_n|}=\mathrm{D}^*_{\mathcal{F}}(A).$$

This proves Theorem 3.1. ■

Let $\mathcal{B}a(X)$ be the σ-algebra of all Baire subsets of X. It should be noted that for $G \curvearrowright_T (X, \mathcal{B}a(X), \mu)$ associated to E in the proof of Theorem 3.1, X is never metrizable if G is uncountable. Further, since in our context there is no ergodic decomposition theorem because $(X, \mathcal{B}a(X), \mu)$ is not (isomorphic to) a Polish probability space and so has no quasi-generic point, the proof of $\mathrm{D}^*_{\mathcal{F}}(\{u \in E \,|\, u + d\{g_1, \ldots, g_l\} \subseteq E\}) > 0$ is of interest.

An interesting consequence of the above theorem is the following, which is a generalization of the classical Szemerédi theorem for $G = \mathbb{Z}$ due to E. Szemerédi [8], for $G = \mathbb{Z}^m$ due to Fürstenberg and Katznelson [4], and for $G = \mathbb{R}^m$ with the Euclidean metric topology due to H. Fürstenberg [3, Theorem 7.17].

COROLLARY 3.2. *Let $(G, +)$ be a second countable locally compact Hausdorff abelian topological group. If a measurable set $E \subseteq G$ is of positive upper Banach density, i.e.,* $\mathrm{BD}^*(E) > 0$, *then for any $g_1, \ldots, g_l \in G$,*

$$\mathrm{BD}_*\big(\big\{d \in \mathbb{Z} \;\big|\; \mathrm{BD}^*(\{u \in E \colon u + d\{g_1, \ldots, g_l\} \subseteq E\}) > 0\big\}\big) > 0.$$

Proof. Since every second countable locally compact Hausdorff group is σ-compact, G has a classical Følner sequence. Then $\mathrm{BD}^*(E)$ makes sense and the statement follows at once from Theorem 3.1 with $F = G$. ∎

If we now utilize the Bergelson–Leibman [2] polynomial multiple recurrence theorem instead of Fürstenberg's multiple recurrence theorem in the proof of Theorem 3.1, we can easily obtain the following.

THEOREM 3.3. *Let $(G, +)$ be a locally compact Hausdorff abelian group and $F \subseteq G$ a compact subset. If a measurable set $E \subseteq G$ is of positive upper density corresponding to an F-Følner sequence $\mathcal{F} = (F_n)_{n=1}^\infty$ in $(G, +, |\cdot|)$, then for any $g_1, \ldots, g_l \in \langle F \rangle$,*

$$\mathrm{BD}_*\big(\big\{d \in \mathbb{Z} \;\big|\; \mathrm{D}^*_{\mathcal{F}}(\{u \in E \colon u + \{p_1(d)g_1, \ldots, p_l(d)g_l\} \subseteq E\}) > 0\big\}\big) > 0$$

for any l polynomials $p_1(t), \ldots, p_l(t) \in \mathbb{Z}[t]$ with $p_i(0) = 0$ for $i = 1, \ldots, l$.

Recall that for any discrete abelian group $(G, +)$, $E \subseteq G$ is of positive upper Banach density if and only if there exists a Følner net $(F_\theta)_{\theta \in \Theta}$ in $(G, +, |\cdot|)$ such that

$$\mathrm{BD}^*(E) = \lim_{\theta \in \Theta} \frac{|E \cap F_\theta|}{|F_\theta|} > 0.$$

Then by Theorem 3.3 together with Lemma 2.4, we obtain

COROLLARY 3.4. *Let $(G, +)$ be a discrete abelian additive group and let $p_1(t), \ldots, p_l(t) \in \mathbb{Z}[t]$ with $p_i(0) = 0$ for $1 \le i \le l$. If $E \subseteq G$ is such that* $\mathrm{BD}^*(E) > 0$ *with Følner nets, then for any $\{g_1, \ldots, g_l\} \subseteq G$ we have*

$$\mathrm{BD}_*\big(\big\{d \in \mathbb{Z} \;\big|\; \mathrm{BD}^*(\{u \in E \colon u + p_i(d)g_i \in E \text{ for } i = 1, \ldots, l\}) > 0\big\}\big) > 0.$$

Proof. For any $\{g_1, \dots, g_l\} \subseteq G$, let $F = \{g_1, \dots, g_l\}$. By Lemma 2.4, we can find some F-Følner sequence, say $\mathcal{F} = (F_n)_{n=1}^{\infty}$, in $(G, +, |\cdot|)$ with $\mathrm{D}^*_{\mathcal{F}}(E) = \mathrm{BD}^*(E)$. Then the statement follows from Theorem 3.3. ∎

We notice here that G does not have any classical Følner sequence under the discrete topology when G is uncountable, and moreover the finite set $\{g_1 = a, g_2 = 2a, \dots, g_l = la\}$ in [5] is a special configuration in Corollary 3.2. A special case of Theorem 3.3 is the following

Corollary 3.5. *Let $(G, +)$ be an amenable group, $F \subset G$ a compact set and $E \subseteq G$ with $\mathrm{D}^*_{\mathcal{F}}(E) > 0$ for some F-Følner sequence $\mathcal{F} = (F_n)_{n=1}^{\infty}$ in G. Then for any $g \in \langle G\rangle$, $l \in \mathbb{N}$ and $p_1(t), \dots, p_l(t) \in \mathbb{Z}[t]$ with $p_i(0) = 0$ for $i = 1, \dots, l$, there is some $d \in \mathbb{N}$ such that*

$$\mathrm{D}^*_{\mathcal{F}}\big(\{u \in E : u + \{p_1(d)g, 2p_2(d)g, \dots, lp_l(d)g\} \subseteq E\}\big) > 0.$$

Proof. This follows from Theorem 3.3 with $g_1 = g, g_2 = 2g, \dots, g_l = lg$ by noting that $T_{g_1}, \dots, T_{g_l}$ are commuting. ∎

Finally, we note that in Theorems 3.1 and 3.3, we cannot consider $(\langle F\rangle, +)$ as an independent abelian group, because the F-Følner sequence $(F_n)_{n=1}^{\infty}$ only belongs to G, and not to $\langle F\rangle$, and moreover the E we consider here may have a void intersection with $\langle F\rangle$ and $\mathrm{D}^*_{\mathcal{F}}(E)$ is associated to $(F_n)_{n=1}^{\infty}$ in G.

Acknowledgements. The authors would like to thank the referee for his/her many constructive comments.

This work was partly supported by National Natural Science Foundation of China (Grant Nos. 11431012 and 11271183) and PAPD of Jiangsu Higher Education Institutions.

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