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Essential norm of generalized composition operators on Zygmund type spaces and Bloch type spaces

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Abstract. The essential norm of generalized composition operators on Zygmund type spaces and Bloch type spaces with normal weight is investigated.

1. Introduction. Let k be a positive continuous function on $[0,1)$. It is called *normal* if there exist numbers $0 < a < b$ and $\delta \in [0,1)$ such that (see [SW])

$$\frac{k(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1) \quad \text{and} \quad \lim_{r\to 1}\frac{k(r)}{(1-r)^a}=0; \tag{1.1}$$

$$\frac{k(r)}{(1-r)^b} \text{ is increasing on } [\delta,1) \quad \text{and} \quad \lim_{r\to 1}\frac{k(r)}{(1-r)^b}=\infty. \tag{1.2}$$

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. Let ω be normal on $[0,1)$. A function $f \in H(\mathbb{D})$ is said to belong to the *Bloch type space* $\mathcal{B}_\omega$ if

$$\|f\|_{\mathcal{B}_\omega} = |f(0)| + \sup_{z\in\mathbb{D}} \omega(|z|)|f'(z)| < \infty.$$

It is easy to see that $\mathcal{B}_\omega$ is a Banach space with the norm $\|\cdot\|_{\mathcal{B}_\omega}$. When $\omega(t) = 1-t^2$, we get the *Bloch space*, denoted by $\mathcal{B} = \mathcal{B}(\mathbb{D})$. See [Z2] for more information on Bloch type spaces.

Suppose μ is normal on $[0,1)$. The *Zygmund type space* $\mathscr{Z}_\mu$ is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|f''(z)| < \infty.$$

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It is also easy to see that $\mathscr{Z}_\mu$ is a Banach space with the norm $\|\cdot\|_{\mathscr{Z}_\mu}$. When $\mu(t) = 1 - t^2$, we get the *Zygmund space* (see [D, LS1]).

The weighted space $H_\mu^\infty = H_\mu^\infty(\mathbb{D})$ is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H_\mu^\infty} = \sup_{z\in\mathbb{D}} \mu(|z|)\,|f(z)| < \infty.$$

H_μ^∞ is a Banach space with the norm $\|\cdot\|_{H_\mu^\infty}$ (see [M2]).

Throughout the paper, $S(\mathbb{D})$ denotes the set of all analytic self-maps of $\mathbb{D}$. Associated with $\varphi \in S(\mathbb{D})$ is the composition operator C_φ, which is defined by

$$(C_\varphi f)(z) = f(\varphi(z)), \qquad f \in H(\mathbb{D}).$$

We refer to the books [CM, Z2] for the theory of composition operators. Composition operators mapping into the Bloch space on $\mathbb{D}$ were studied in [HW, MM, T, WZZ, Z1]. The essential norm of composition operators on the Bloch space was studied in [HL, MCZ, MZ, M1, Z1]. See [CKS, HY, LF, LS2] for the study of composition operators mapping into the Zygmund space.

Motivated by the fact that weighted composition operators naturally come from isometries of some function spaces, for $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, Li and Stević in [LS2] defined a generalized composition operator, denoted by C_φ^g, as follows:

$$C_\varphi^g f(z) = \int_0^z f'(\varphi(\xi))g(\xi)\,d\xi, \qquad f \in H(\mathbb{D}),\ z \in \mathbb{D}.$$

They characterized the boundedness and compactness of C_φ^g on the Zygmund space and the Bloch space. See [L, S1, S2, YM, Z3, Z4] for a further study of the operator C_φ^g and its generalizations.

Du and Zhu [DZ] investigated the boundedness and compactness of C_φ^g on Zygmund type spaces and Bloch type spaces with normal weight. In this paper, motivated by [DZ, LS2], we investigate the essential norm of C_φ^g acting on such spaces.

Suppose X, Y are Banach spaces. Recall that the *essential norm* of $C_\varphi^g : X \to Y$, denoted by $\|C_\varphi^g\|_{e,X\to Y}$, is defined by

$$\|C_\varphi^g\|_{e,X\to Y} = \inf\{\|C_\varphi^g - K\|_{X\to Y} : K \text{ is a compact operator from } X \text{ to } Y\}.$$

In this paper, constants are denoted by C, they are positive and may differ from one occurrence to the next. We write $A \lesssim B$ if there exists a constant C such that $A \le CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Auxiliary results. In this section, we give some auxiliary results which will be used in proving the main results of this paper. They are incorporated in the lemmas below.

LEMMA 2.1 ([H]). *Suppose that $\mu(t)$ is normal on $[0,1)$. Then there exists $\mu_* \in H(\mathbb{D})$ such that*

(i) *for any $t \in [0,1)$, $\mu_*(t) \in \mathbb{R}^+$, and $\mu_*(t)$ is increasing on $[0,1)$;*
(ii) *for any $z \in \mathbb{D}$, $|\mu_*(z)| \le \mu_*(|z|)$, and*

$$\inf_{t\in[0,1)} \mu(t)\mu_*(t) > 0, \qquad \sup_{z\in\mathbb{D}} \mu(|z|)|\mu_*(z)| < \infty.$$

In the rest of this paper, we will always use μ_* to denote the analytic function related to μ as in Lemma 2.1.

LEMMA 2.2 ([DZ]). *Suppose that μ is normal on $[0,1)$. Then the following statements hold.*

(i) *There exists a $\delta \in (0,1)$ such that μ is decreasing on $[\delta,1)$ and* $\lim_{t\to 1}\mu(t) = 0$.
(ii) *For all $\alpha > 1$ and $\beta \in (0,1)$, when $t \in (0,1)$ and $s \in (\beta,1)$,*

$$\mu(t) \approx \mu(t^\alpha) \approx \frac{1}{\mu_*(t)}, \qquad \int_0^{s^\alpha} \frac{1}{\mu(t)}\,dt \approx \int_0^{s} \frac{1}{\mu(t)}\,dt.$$

(iii) *For any $z \in \mathbb{D}$,*

$$\Big|\int_0^z \mu_*(\eta)\,d\eta\Big| \lesssim \int_0^{|z|} \mu_*(t)\,dt.$$

If $|\eta| \le |z|$, then $\mu(|z|)|\mu_*(\eta)| < C$.

LEMMA 2.3 ([ZX]). *Suppose that μ is normal on $[0,1)$. Then for every $f \in \mathscr{Z}_\mu$,*

$$|f'(z)| < \Big(1 + \int_0^{|z|} \frac{1}{\mu(t)}\,dt\Big)\|f\|_{\mathscr{Z}_\mu}, \qquad z \in \mathbb{D}.$$

LEMMA 2.4 ([ZX]). *Suppose μ is normal on $[0,1)$ and*

$$\int_0^1 \frac{1}{\mu(t)}\,dt < \infty.$$

If $\{f_n\}$ is bounded in $\mathcal{B}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$, then

$$\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f_n(z)| = 0.$$

The relationship between Zygmund type spaces and Bloch type spaces is well known—see [DZ], for example.

LEMMA 2.5 ([DZ]). *Suppose that μ is normal on $[0,1)$. Let $\mu_+(t) = (1-t)\mu(t)$. Then:*

(i) μ_+ *is normal on* $[0,1)$, *and*

$$\lim_{|z|\to 1} \int_0^{|z|} \frac{1}{\mu_+(t)}\,dt = \infty.$$

(ii) $\mathcal{B}_\mu = \mathscr{Z}_{\mu_+}$ *and* $\|f\|_{\mathcal{B}_\mu} \approx \|f\|_{\mathscr{Z}_{\mu_+}}$ *for any* $f \in \mathcal{B}_\mu$.

To study compactness, we need the following lemma, which can be deduced from [T, Lemma 2.10].

LEMMA 2.6. *Suppose that* ω, μ *are normal on* $[0,1)$. *Suppose that a linear operator* $T : \mathcal{B}_\mu$ (*or* $\mathscr{Z}_\mu$) $\to \mathcal{B}_\omega$ (*or* $\mathscr{Z}_\omega$) *is bounded. Then* T *is compact if and only if whenever* $\{f_n\}$ *is bounded in* $\mathcal{B}_\mu$ (*or* $\mathscr{Z}_\mu$) *and* $f_n \to 0$ *uniformly on compact subsets of* $\mathbb{D}$,

$$\lim_{n\to\infty} \|Tf_n\|_{\mathcal{B}_\omega} = 0 \quad \Big(\text{or } \lim_{n\to\infty} \|Tf_n\|_{\mathscr{Z}_\omega} = 0\Big).$$

REMARK 2.7. Assume that $\sup_{z\in\mathbb{D}} |\varphi(z)| < 1$ and $C_\varphi^g : \mathcal{B}_\mu$ (or $\mathscr{Z}_\mu$) $\to \mathcal{B}_\omega$ (or $\mathscr{Z}_\omega$) is bounded. Then it is automatically compact. In fact, suppose $\{f_n\}$ is bounded in $\mathcal{B}_\mu$ (or $\mathscr{Z}_\mu$) and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Since $z, z^2 \in \mathcal{B}_\mu \cap \mathscr{Z}_\mu$ and $C_\varphi^g : \mathcal{B}_\mu$ (or $\mathscr{Z}_\mu$) $\to \mathscr{Z}_\omega$ is bounded, we have

$$\sup_{z\in\mathbb{D}} \omega(|z|)|g'(z)| < \infty, \qquad \sup_{z\in\mathbb{D}} \omega(|z|)|\varphi'(z)g(z)| < \infty.$$

Since

$$\begin{aligned}\|C_\varphi^g f_n\|_{\mathscr{Z}_\omega} &\le \sup_{z\in\mathbb{D}} \omega(|z|)|g'(z)| \sup_{z\in\mathbb{D}} |f_n'(\varphi(z))| \\ &\quad + \sup_{z\in\mathbb{D}} \omega(|z|)|\varphi'(z)g(z)| \sup_{z\in\mathbb{D}} |f_n''(\varphi(z))|,\end{aligned}$$

by Cauchy's estimate and Lemma 2.6, we see that $C_\varphi^g : \mathcal{B}_\mu$ (resp. $\mathscr{Z}_\mu$) $\to \mathscr{Z}_\omega$ is compact. Similarly, we see that $C_\varphi^g : \mathcal{B}_\mu$ (resp. $\mathscr{Z}_\mu$) $\to \mathcal{B}_\omega$ is compact.

REMARK 2.8. Let $\rho \in (0,1)$. For all $f \in H(\mathbb{D})$, let $f_\rho = f(\rho z)$. Since

$$\|f_\rho\|_{\mathcal{B}_\mu} \le \|f\|_{\mathcal{B}_\mu} \quad (\|f_\rho\|_{\mathscr{Z}_\mu} \le \|f\|_{\mathscr{Z}_\mu}) \quad \text{and} \quad C_{\rho\varphi}^g f = \frac{1}{\rho} C_\varphi^g f_\rho,$$

we see that $C_{\rho\varphi}^g : \mathcal{B}_\mu$ (or $\mathscr{Z}_\mu$) $\to \mathcal{B}_\omega$ (or $\mathscr{Z}_\omega$) is bounded when $C_\varphi^g : \mathcal{B}_\mu$ (or $\mathscr{Z}_\mu$) $\to \mathcal{B}_\omega$ (or $\mathscr{Z}_\omega$) is bounded. By Remark 2.7, we see that $C_{\rho\varphi}^g : \mathcal{B}_\mu$ (or $\mathscr{Z}_\mu$) $\to \mathcal{B}_\omega$ (or $\mathscr{Z}_\omega$) is compact when $C_\varphi^g : \mathcal{B}_\mu$ (or $\mathscr{Z}_\mu$) $\to \mathcal{B}_\omega$ (or $\mathscr{Z}_\omega$) is bounded.

3. Main results. Suppose μ is normal on $[0,1)$. Set $\mu_+(t) = (1-t)\mu(t)$.

THEOREM 3.1. *Suppose* ω, μ *are normal on* $[0,1)$, $g \in H(\mathbb{D})$, *and* $\varphi \in S(\mathbb{D})$ *with* $\sup_{z\in\mathbb{D}} |\varphi(z)| = 1$. *Suppose* $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ *is bounded. Then the following statements hold.*

(i) *If* $\int_0^1 \frac{1}{\mu(t)}\,dt < \infty$, *then*

$$\|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \approx \limsup_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)}.$$

(ii) *If* $\int_0^1 \frac{1}{\mu(t)}\,dt = \infty$, *then*

$$\|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \approx \limsup_{|\varphi(z)|\to 1} \omega(|z|)|g'(z)| \int_0^{|\varphi(z)|} \frac{1}{\mu(t)}\,dt + \limsup_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)}.$$

Proof. Since $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded and $z, z^2 \in \mathscr{Z}_\mu$, we have

$$\sup_{z\in\mathbb{D}} \omega(|z|)|g'(z)| < \infty, \quad \sup_{z\in\mathbb{D}} \omega(|z|)|\varphi'(z)g(z)| < \infty. \tag{3.1}$$

Let $s \in (0,1), \rho_n = 1 - \frac{1}{n+1},\ n \in \mathbb{N}$. By Remark 2.8, $C_{\rho_n\varphi}^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact. Since

$$\|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \le \|C_\varphi^g - C_{\rho_n\varphi}^g\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega} = \sup_{\|f\|_{\mathscr{Z}_\mu}\le 1} \|(C_\varphi^g - C_{\rho_n\varphi}^g)f\|_{\mathscr{Z}_\omega},$$

we have

$$\begin{aligned} \|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \le &\sup_{\|f\|_{\mathscr{Z}_\mu}\le 1} \sup_{|\varphi(z)|\le s} (I_{n,f}(z) + J_{n,f}(z)) \\ &+ \sup_{\|f\|_{\mathscr{Z}_\mu}\le 1} \sup_{|\varphi(z)|> s} (I_{n,f}(z) + J_{n,f}(z)), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} I_{n,f}(z) &= \omega(|z|)|g'(z)|\,|f'(\varphi(z)) - f'(\rho_n\varphi(z))|, \\ J_{n,f}(z) &= \omega(|z|)|\varphi'(z)g(z)|\,|f''(\varphi(z)) - \rho_n f''(\rho_n\varphi(z))|. \end{aligned}$$

Suppose $\|f\|_{\mathscr{Z}_\mu} \le 1$. By the triangle inequality, we have

$$\sup_{|\varphi(z)|>s} J_{n,f}(z) \le 2 \sup_{|\varphi(z)|>s} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)}. \tag{3.3}$$

By the Maximum Modulus Principle and

$$|f'(\varphi(z)) - f'(\rho_n\varphi(z))| = \Big|\int_{\rho_n\varphi(z)}^{\varphi(z)} f''(\eta)\,d\eta\Big|, \tag{3.4}$$

we have

$$\sup_{|\varphi(z)|\le s} I_{n,f}(z) \le \sup_{|\varphi(z)|\le s} (1-\rho_n)\frac{\omega(|z|)|g'(z)|}{\mu(|\varphi(z)|)}, \tag{3.5}$$

$$\sup_{|\varphi(z)|>s} I_{n,f}(z) \le \sup_{|\varphi(z)|>s} \omega(|z|)|g'(z)| \int_{\rho_n|\varphi(z)|}^{|\varphi(z)|} \frac{1}{\mu(t)}\,dt \tag{3.6}$$

and

$$\sup_{|\varphi(z)|>s} I_{n,f}(z) \leq 2 \sup_{|\varphi(z)|>s} \omega(|z|)|g'(z)| \int_0^{|\varphi(z)|} \frac{1}{\mu(t)}\, dt. \tag{3.7}$$

When $|\eta| \leq s$, by Cauchy's estimate we have

$$\begin{aligned} |f'''(\eta)| &\leq \frac{2}{1-|\eta|} \max_{|\xi-\eta|\leq(1-|\eta|)/2} |f''(\xi)| \leq \frac{2}{1-s} \max_{|\xi|=(1+s)/2} |f''(\xi)| \\ &\leq \frac{2\|f\|_{\mathscr{Z}_\mu}}{(1-s)\mu\left(\frac{1+s}{2}\right)}. \end{aligned} \tag{3.8}$$

So when $|\varphi(z)| \leq s$, we have

$$\begin{aligned} |f''(\varphi(z)) - \rho_n f''(\rho_n\varphi(z))| &\leq |f''(\varphi(z)) - f''(\rho_n\varphi(z))| \\ &\quad + |f''(\rho_n\varphi(z)) - \rho_n f''(\rho_n\varphi(z))| \\ &\leq \left| \int_{\rho_n\varphi(z)}^{\varphi(z)} f'''(\eta)\, d\eta \right| + \frac{2(1-\rho_n)}{\mu(\rho_n|\varphi(z)|)} \\ &\leq \frac{2(1-\rho_n)}{(1-s)\mu(\frac{1+s}{2})} + \frac{2(1-\rho_n)}{\mu(\rho_n|\varphi(z)|)}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sup_{|\varphi(z)|\leq s} J_{n,f}(z) \\ &\quad \leq 2(1-\rho_n)\Big(\sup_{|\varphi(z)|\leq s} \frac{\omega(|z|)|\varphi'(z)g(z)|}{(1-s)\mu\left(\frac{1+s}{2}\right)} + \sup_{|\varphi(z)|\leq s} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(\rho_n|\varphi(z)|)} \Big). \end{aligned} \tag{3.9}$$

Let $n \to \infty$ and $s \to 1$. By (3.1), (3.3), (3.5) and (3.9), we have

$$\begin{aligned} &\lim_{s\to 1} \lim_{n\to\infty} \sup_{\|f\|_{\mathscr{Z}_\mu}\leq 1} \sup_{|\varphi(z)|>s} J_{n,f}(z) \lesssim \limsup_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)}, \\ &\lim_{s\to 1} \lim_{n\to\infty} \sup_{\|f\|_{\mathscr{Z}_\mu}\leq 1} \sup_{|\varphi(z)|\leq s} I_{n,f}(z) = 0, \\ &\lim_{s\to 1} \lim_{n\to\infty} \sup_{\|f\|_{\mathscr{Z}_\mu}\leq 1} \sup_{|\varphi(z)|\leq s} J_{n,f}(z) = 0. \end{aligned}$$

By (3.2), we get

$$\|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \lesssim \lim_{s\to 1} \lim_{n\to\infty} \sup_{\|f\|_{\mathscr{Z}_\mu}\leq 1} \sup_{|\varphi(z)|>s} I_{n,f}(z) + \limsup_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)}. \tag{3.10}$$

(i) Suppose $\int_0^1 \frac{1}{\mu(t)}\, dt < \infty$. By (3.1), (3.6) and (3.10), we have

$$\|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \lesssim \limsup_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)}.$$

Next we prove that

$$\|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \gtrsim \limsup_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)}.$$

Assume $\{z_n\}\subset\mathbb{D}$ with $\lim_{n\to\infty}|\varphi(z_n)|=1$. Let $a_n=\varphi(z_n)$ and

$$r_n(z)=\mu(|a_n|)\int_0^{\overline{a_n}z}\int_0^t \mu_*^2(\eta)\,d\eta\,dt.$$

Then $\{r_n\}$ is bounded in $\mathscr{Z}_\mu$ and $r_n\to 0$ uniformly on compact subsets of $\mathbb{D}$. If $K:\mathscr{Z}_\mu\to\mathscr{Z}_\omega$ is compact, by Lemmas 2.4 and 2.6, we have

$$\lim_{n\to\infty}\sup_{z\in\mathbb{D}}|r_n'(z)|=0,\qquad \lim_{n\to\infty}\|Kr_n\|_{\mathscr{Z}_\omega}=0. \tag{3.11}$$

Therefore, by (3.1), (3.11) and Lemma 2.2, we have

$$\begin{aligned}
&\|C_\varphi^g-K\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega}\\
&\quad\gtrsim \limsup_{n\to\infty}\|C_\varphi^g r_n\|_{\mathscr{Z}_\omega}-\lim_{n\to\infty}\|Kr_n\|_{\mathscr{Z}_\omega}\ge \limsup_{n\to\infty}\omega(|z_n|)|(C_\varphi^g r_n)''(z_n)|\\
&\quad\ge \limsup_{n\to\infty}\omega(|z_n|)|\varphi'(z_n)g(z_n)r_n''(\varphi(z_n))|-\lim_{n\to\infty}\omega(|z_n|)|g'(z_n)r_n'(\varphi(z_n))|\\
&\quad\approx \limsup_{n\to\infty}\frac{\omega(|z_n|)|\varphi'(z_n)g(z_n)|}{\mu(|\varphi(z_n)|)},
\end{aligned}$$

which implies the desired result.

(ii) Suppose $\int_0^1\frac{1}{\mu(t)}\,dt=\infty$. By (3.1), (3.7) and (3.10), we have

$$\|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega}\lesssim \limsup_{|\varphi(z)|\to1}\omega(|z|)|g'(z)|\int_0^{|\varphi(z)|}\frac{1}{\mu(t)}\,dt+\limsup_{|\varphi(z)|\to1}\frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)}.$$

Next we prove that

$$\|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega}\gtrsim \limsup_{|\varphi(z)|\to1}\omega(|z|)|g'(z)|\int_0^{|\varphi(z)|}\frac{1}{\mu(t)}\,dt+\limsup_{|\varphi(z)|\to1}\frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)}.$$

Assume $\{z_n\}\subset\mathbb{D}$ with $\lim_{n\to\infty}|\varphi(z_n)|=1$. Let $a_n=\varphi(z_n)$ and

$$k_n(z)=\frac{3p_{n,0}(z)-2p_{n,1}(z)}{\int_0^{|a_n|}\mu_*(\eta)\,d\eta},\qquad q_n(z)=\frac{p_{n,0}(z)-p_{n,1}(z)}{\int_0^{|a_n|}\mu_*(\eta)\,d\eta},$$

where

$$p_{n,\tau}(z)=\int_0^{\overline{a_n}z}\Big(\int_0^{t^{2+\tau}/|a_n|^{2\tau}}\mu_*(\eta)\,d\eta\Big)^2\,dt,\qquad \tau=0,1.$$

A calculation shows that

$$p'_{n,\tau}(z) = \overline{a_n}\Big(\int\limits_0^{(\overline{a_n}z)^{2+\tau}/|a_n|^{2\tau}} \mu_*(\eta)\,d\eta\Big)^2,$$

$$p''_{n,\tau}(z) = \frac{2(2+\tau)\overline{a_n}^{3+\tau}z^{1+\tau}}{|a_n|^{2\tau}}\mu_*\Big(\frac{(\overline{a_n}z)^{2+\tau}}{|a_n|^{2\tau}}\Big)\int\limits_0^{(\overline{a_n}z)^{2+\tau}/|a_n|^{2\tau}} \mu_*(\eta)\,d\eta.$$

When $\tau = 0$ or 1, we have $|\overline{a_n}^{2+\tau}/|a_n|^{2\tau}| \le |a_n|$. By Lemma 2.1,

$$|p''_{n,\tau}(z)| \le 6\mu_*(|a_n z|)\int\limits_0^{|a_n z|} \mu_*(\eta)\,d\eta.$$

If $0 < r < 1$, for all $|z| \le r$ we get

$$\begin{aligned}|p_{n,\tau}(z)| &\le \int\limits_0^{|a_n r|}\Big|\int\limits_0^{t^{2+\tau}/|a_n|^{2\tau}} \mu_*(\eta)\,d\eta\Big|^2 dt \le \int\limits_0^{|a_n r|}\Big(\int\limits_0^{|a_n r|}\sup_{|\eta|\le r}|\mu_*(\eta)|\,d\eta\Big)^2 dt\\ &\le \mu_*^2(r).\end{aligned}$$

By Lemma 2.2, $\{k_n\}$ and $\{q_n\}$ are bounded in $\mathscr{Z}_\mu$ and converge to 0 uniformly on compact subsets of $\mathbb{D}$. Since

$$p'_{n,\tau}(a_n) = \overline{a_n}\Big(\int\limits_0^{|a_n|^4} \mu_*(\eta)\,d\eta\Big)^2,$$

$$p''_{n,\tau}(a_n) = 2(2+\tau)\overline{a_n}^2|a_n|^2\mu_*(|a_n|^4)\int\limits_0^{|a_n|^4} \mu_*(\eta)\,d\eta,$$

by Lemmas 2.1 and 2.2 we obtain

$$|k'_n(a_n)| \approx \int\limits_0^{|a_n|}\frac{1}{\mu(t)}\,dt,\quad |k''_n(a_n)| = 0,\quad q'_n(a_n) = 0,\quad |q''_n(a_n)| \approx \frac{1}{\mu(|a_n|)}.$$

Suppose that $K : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact. By Lemma 2.6, we have

$$\lim_{n\to\infty}\|Kk_n\|_{\mathscr{Z}_\omega} = 0,\qquad \lim_{n\to\infty}\|Kq_n\|_{\mathscr{Z}_\omega} = 0.$$

So, from the fact that

$$\begin{aligned}\|C^g_\varphi - K\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega} &\gtrsim \|(C^g_\varphi - K)k_n\|_{\mathscr{Z}_\omega}\\ &\ge \omega(|z_n|)|(C^g_\varphi k_n)''(z_n)| - \|Kk_n\|_{\mathscr{Z}_\omega},\end{aligned}\tag{3.12}$$

we deduce that

$$\|C^g_\varphi - K\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \gtrsim \limsup_{n\to\infty}\omega(|z_n|)|g'(z_n)|\int\limits_0^{|\varphi(z_n)|}\frac{1}{\mu(t)}\,dt.\tag{3.13}$$

Similarly to (3.12) and (3.13), by using $\{q_n\}$ we find that

$$\|C_\varphi^g - K\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \gtrsim \limsup_{n\to\infty} \frac{\omega(|z_n|)|\varphi'(z_n)g(z_n)|}{\mu(|\varphi(z_n)|)}. \tag{3.14}$$

From (3.13) and (3.14), we get the desired result. ■

THEOREM 3.2. *Suppose ω,μ are normal on $[0,1)$, $g\in H(\mathbb{D})$, and $\varphi\in S(\mathbb{D})$ with $\sup_{z\in\mathbb{D}}|\varphi(z)|=1$. Suppose $C_\varphi^g:\mathscr{Z}_\mu\to\mathcal{B}_\omega$ is bounded. Then the following statements hold.*

(i) *If $\int_0^1 \frac{1}{\mu(t)}\,dt<\infty$, then*

$$\|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathcal{B}_\omega}=0.$$

(ii) *If $\int_0^1 \frac{1}{\mu(t)}\,dt=\infty$, then*

$$\begin{aligned}\|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathcal{B}_\omega} &\approx \limsup_{|\varphi(z)|\to1} \omega(|z|)|g(z)| \int_0^{|\varphi(z)|}\frac{1}{\mu(t)}\,dt\\ &\approx \limsup_{|\varphi(z)|\to1} \omega_+(|z|)|g'(z)| \int_0^{|\varphi(z)|}\frac{1}{\mu(t)}\,dt + \limsup_{|\varphi(z)|\to1}\frac{\omega_+(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)}.\end{aligned}$$

Proof. (i) When $\int_0^1 \frac{1}{\mu(t)}\,dt<\infty$, by [DZ, Theorem 3.4] we see that $C_\varphi^g:\mathscr{Z}_\mu\to\mathcal{B}_\omega$ is compact. So $\|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathcal{B}_\omega}=0$.

(ii) When $\int_0^1 \frac{1}{\mu(t)}\,dt=\infty$, by Theorem 3.1 and Lemma 2.5 we have

$$\|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathcal{B}_\omega} \approx \limsup_{|\varphi(z)|\to1} \omega_+(|z|)|g'(z)| \int_0^{|\varphi(z)|}\frac{1}{\mu(t)}\,dt + \limsup_{|\varphi(z)|\to1}\frac{\omega_+(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)}.$$

Hence, we only need to prove that

$$\|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathcal{B}_\omega} \approx \limsup_{|\varphi(z)|\to1} \omega(|z|)|g(z)| \int_0^{|\varphi(z)|}\frac{1}{\mu(t)}\,dt.$$

Since $C_\varphi^g:\mathscr{Z}_\mu\to\mathcal{B}_\omega$ is bounded and $z\in\mathscr{Z}_\mu$, we have

$$\sup_{z\in\mathbb{D}} \omega(|z|)|g(z)|<\infty. \tag{3.15}$$

Let $s\in(0,1)$ and $\rho_n=1-\frac{1}{n+1}$, $n\in\mathbb{N}$. By Remark 2.8, $C_{\rho_n\varphi}^g$ is compact. So,

$$\begin{aligned}\|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathcal{B}_\omega} &\le \|C_\varphi^g - C_{\rho_n\varphi}^g\|_{\mathscr{Z}_\mu\to\mathcal{B}_\omega} = \sup_{\|f\|_{\mathscr{Z}_\mu}\le1}\|(C_\varphi^g - C_{\rho_n\varphi}^g)f\|_{\mathcal{B}_\omega}\\ &\le \sup_{\|f\|_{\mathscr{Z}_\mu}\le1}\sup_{|\varphi(z)|\le s} K_{n,f}(z) + \sup_{\|f\|_{\mathscr{Z}_\mu}\le1}\sup_{|\varphi(z)|>s} K_{n,f}(z),\end{aligned}$$

where

$$K_{n,f}(z) = \omega(|z|)|g(z)|\,|f'(\varphi(z)) - f'(\rho_n\varphi(z))| = \omega(|z|)|g(z)|\Big|\int\limits_{\rho_n\varphi(z)}^{\varphi(z)} f''(\eta)\,d\eta\Big|.$$

By (3.15) and $\|f\|_{\mathscr{Z}_\mu} \le 1$, we have

$$\sup_{|\varphi(z)|\le s} K_{n,f}(z) \lesssim \Big|\int\limits_{\rho_n\varphi(z)}^{\varphi(z)} f''(\eta)\,d\eta\Big| \le (1-\rho_n)|\varphi(z)| \sup_{|\eta|=s}|f''(\eta)| \le \frac{1-\rho_n}{\mu(s)},$$

$$\sup_{|\varphi(z)|>s} K_{n,f}(z) \lesssim \sup_{|\varphi(z)|>s} \omega(|z|)|g(z)| \int\limits_0^{|\varphi(z)|} \frac{1}{\mu(\eta)}\,d\eta.$$

Therefore,

$$\begin{aligned}\|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathcal{B}_\omega} &\le \lim_{s\to1}\lim_{n\to\infty}\Big(\sup_{\|f\|_{\mathscr{Z}_\mu}\le1}\sup_{|\varphi(z)|\le s} K_{n,f}(z) + \sup_{\|f\|_{\mathscr{Z}_\mu}\le1}\sup_{|\varphi(z)|>s} K_{n,f}(z)\Big)\\ &\lesssim \limsup_{|\varphi(z)|\to1} \omega(|z|)|g(z)| \int\limits_0^{|\varphi(z)|} \frac{1}{\mu(t)}\,dt.\end{aligned}$$

Next, we prove that

$$\|C_\varphi^g\|_{e,\mathscr{Z}_\mu\to\mathcal{B}_\omega} \gtrsim \limsup_{|\varphi(z)|\to1} \omega(|z|)|g(z)| \int\limits_0^{|\varphi(z)|} \frac{1}{\mu(t)}\,dt.$$

Suppose $\{z_n\}\subset\mathbb{D}$ with $\lim_{n\to\infty}|\varphi(z_n)| = 1$. From the proof of Theorem 3.1, we see that $\{k_n\}$ is bounded in $\mathscr{Z}_\mu$ and $k_n\to0$ uniformly on compact subsets of $\mathbb{D}$. Suppose $K:\mathscr{Z}_\mu\to\mathcal{B}_\omega$ is compact. By Lemma 2.6, we have $\lim_{n\to\infty}\|Kk_n\|_{\mathcal{B}_\omega} = 0$. By Lemma 2.2, (3.15) and $\int_0^1 \frac{1}{\mu(t)}\,dt = \infty$, we get

$$\begin{aligned}\|C_\varphi^g - K\|_{\mathscr{Z}_\mu\to\mathcal{B}_\omega} &\gtrsim \limsup_{n\to\infty}\|(C_\varphi^g - K)k_n\|_{\mathcal{B}_\omega}\\ &\ge \limsup_{n\to\infty}\omega(|z_n|)|(C_\varphi^g k_n)'(z_n)| - \limsup_{n\to\infty}\|Kk_n\|_{\mathcal{B}_\omega}\\ &\gtrsim \limsup_{n\to\infty}\omega(|z_n|)|g(z_n)|\int\limits_0^{|\varphi(z_n)|}\frac{1}{\mu(t)}\,dt,\end{aligned}$$

which implies the desired result. ∎

By Theorem 3.1 and Lemma 2.5, the following theorem clearly holds.

THEOREM 3.3. *Suppose* ω,μ *are normal on* $[0,1)$, $g\in H(\mathbb{D})$, *and* $\varphi\in S(\mathbb{D})$ *with* $\sup_{z\in\mathbb{D}}|\varphi(z)| = 1$. *If* $C_\varphi^g:\mathcal{B}_\mu\to\mathscr{Z}_\omega$ *is bounded, then*

$$\|C_\varphi^g\|_{e,\mathcal{B}_\mu\to\mathscr{Z}_\omega} \approx \limsup_{|\varphi(z)|\to1}\omega(|z|)|g'(z)|\int\limits_0^{|\varphi(z)|}\frac{1}{\mu_+(t)}\,dt + \limsup_{|\varphi(z)|\to1}\frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu_+(|\varphi(z)|)}.$$

Finally, we consider the essential norm of $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$.

THEOREM 3.4. *Suppose* ω, μ *are normal on* $[0,1)$, $g \in H(\mathbb{D})$, *and* $\varphi \in S(\mathbb{D})$ *with* $\sup_{z\in\mathbb{D}} |\varphi(z)| = 1$. *If* $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ *is bounded, then*

$$\begin{aligned}\|C_\varphi^g\|_{e,\mathcal{B}_\mu\to\mathcal{B}_\omega} &\approx \limsup_{|\varphi(z)|\to 1} \omega_+(|z|)|g'(z)| \int_0^{|\varphi(z)|} \frac{1}{\mu_+(t)}\,dt \\ &\quad + \limsup_{|\varphi(z)|\to 1} \frac{\omega_+(|z|)|\varphi'(z)g(z)|}{\mu_+(|\varphi(z)|)} \\ &\approx \limsup_{|\varphi(z)|\to 1} \omega(|z|)|g(z)| \int_0^{|\varphi(z)|} \frac{1}{\mu_+(t)}\,dt \\ &\approx \limsup_{|\varphi(z)|\to 1} \frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)}.\end{aligned}$$

Proof. By Lemma 2.5, and Theorems 3.1 and 3.2, we have

$$\begin{aligned}\|C_\varphi^g\|_{e,\mathcal{B}_\mu\to\mathcal{B}_\omega} &\approx \limsup_{|\varphi(z)|\to 1} \omega_+(|z|)|g'(z)| \int_0^{|\varphi(z)|} \frac{1}{\mu_+(t)}\,dt \\ &\quad + \limsup_{|\varphi(z)|\to 1} \frac{\omega_+(|z|)|\varphi'(z)g(z)|}{\mu_+(|\varphi(z)|)} \\ &\approx \limsup_{|\varphi(z)|\to 1} \omega(|z|)|g(z)| \int_0^{|\varphi(z)|} \frac{1}{\mu_+(t)}\,dt.\end{aligned}$$

Next, we prove that

$$\|C_\varphi^g\|_{e,\mathcal{B}_\mu\to\mathcal{B}_\omega} \approx \limsup_{|\varphi(z)|\to 1} \frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)}.$$

For the codimension 1 complemented subspace $\widetilde{\mathcal{B}}_\mu$ of functions in $\mathcal{B}_\mu$ vanishing at 0, the derivative operator $D_\mu : \widetilde{\mathcal{B}}_\mu \to H_\mu^\infty$ is a linear isometry. On $\widetilde{\mathcal{B}}_\mu$ the operator C_φ^g coincides with $D_\omega^{-1} W_{\varphi,g} D_\mu$, where $W_{\varphi,g} : H_\mu^\infty \to H_\omega^\infty$ is the weighted composition operator defined by $W_{\varphi,g} f(z) = g(z) f(\varphi(z))$. Therefore $\|C_\varphi^g\|_{e,\mathcal{B}_\mu\to\mathcal{B}_\omega}$ and $\|W_{\varphi,g}\|_{e,H_\mu^\infty\to H_\omega^\infty}$ are equivalent. From [M2, Theorem 2.1], we immediately get the desired result. ■

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