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On the Cesàro average of the "Linnik numbers"

by

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1. Introduction. We continue the recent work of Languasco and Zaccagnini on additive problems with prime summands. In [7] and [8] they study the Cesàro weighted explicit formula for the Goldbach numbers (the integers that can be written as the sum of two primes) and for the Hardy–Littlewood numbers (the integers that can be written as the sum of a prime and a square). In a similar manner, we will study a Cesàro weighted explicit formula for the integers that can be written as the sum of a prime and two squares. We will obtain an asymptotic formula with a main term and more terms depending explicitly on the zeros of the Riemann zeta function. The study of these numbers is classical. For example, Hardy and Littlewood [5] studied the number of solutions of the equation

$$n = p + a^2 + b^2,$$

and Linnik [10] derived an asymptotic formula for the number of representations of these numbers. Similar averages of arithmetical functions are common in the literature: see, e.g., Chandrasekharan–Narasimhan [2] and Berndt [1] who built on earlier classical work. For our work we will need the Bessel functions $J_v(u)$ of complex order v and real argument u. For their definition and main properties we refer to Watson [12], but we recall that they were introduced by Daniel Bernoulli and they are the canonical solution of the differential equation

$$u^2 \frac{d^2 J}{du^2} + u \frac{dJ}{du} + (u^2 - v^2) J = 0$$

for any complex number v. In particular, [12, equation (8), p. 177] gives the Sonine representation

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$$J_\nu(u) = \frac{(u/2)^\nu}{2\pi i} \int_{(a)} e^s s^{-\nu-1} e^{-u^2/(4s)}\, ds \tag{1.1}$$

where $\int_{(a)}$ means $\int_{a-i\infty}^{a+i\infty}$. The method we will use in this additive problem is based on a formula due to Laplace [9], namely

$$\frac{1}{2\pi i} \int_{(a)} v^{-s} e^v\, dv = \frac{1}{\Gamma(s)} \tag{1.2}$$

with $\mathrm{Re}(s) > 0$ and $a > 0$ (see, e.g., [3, formula 5.4(1), p. 238]). As in [8], we combine this approach with line integrals with the classical methods dealing with infinite sums over primes and integers. Similarly to [8], the problem naturally involves the modular relation for the complex Jacobi θ_3 function; the presence of the Bessel functions in our statement strictly depends on that relation.

2. Preliminary definitions and lemmas.

Let

$$r_Q(n) = \sum_{m_1+m_2^2+m_3^2=n} \Lambda(m_1)$$

and let $J_v(u)$ be the Bessel function of complex order v and real argument u. Let $z = a + iy$, $a > 0$, and set

$$\theta_3(z) = \sum_{m\in\mathbb{Z}} e^{-m^2 z}, \tag{2.1}$$

$$\widetilde{S}(z) = \sum_{m\geq 1} \Lambda(m) e^{-mz}, \tag{2.2}$$

$$\omega_2(z) = \sum_{m\geq 1} e^{-m^2 z}; \tag{2.3}$$

we can see that

$$\theta_3(z) = 1 + 2\omega_2(z). \tag{2.4}$$

Furthermore, we have the functional equation (see, for example, Freitag–Busam [4, Proposition VI.4.3, p. 340])

$$\theta_3(z) = \left(\frac{\pi}{z}\right)^{1/2} \theta_3\left(\frac{\pi^2}{z}\right), \qquad \mathrm{Re}(z) > 0, \tag{2.5}$$

and so

$$\omega_2^2(z) = \left(\frac{1}{2}\left(\frac{\pi}{z}\right)^{1/2} - \frac{1}{2}\right)^2 + \frac{\pi}{z}\omega_2^2\left(\frac{\pi^2}{z}\right) + \left(\left(\frac{\pi}{z}\right)^{1/2} - 1\right)\left(\left(\frac{\pi}{z}\right)^{1/2} \omega_2\left(\frac{\pi^2}{z}\right)\right). \tag{2.6}$$

A trivial but important estimate is

$$|\omega_2(z)| \le \omega_2(a) \le \int_0^\infty e^{-at^2}\,dt = \frac{\sqrt{\pi}}{2\sqrt{a}} \ll a^{-1/2}. \tag{2.7}$$

LEMMA 2.1. *Let $z = a + iy$, $a > 0$ and $y \in \mathbb{R}$. Then*

$$\widetilde{S}(z) = \frac{1}{z} - \sum_\rho z^{-\rho}\Gamma(\rho) + E(a,y) \tag{2.8}$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$ and

$$E(a,y) \ll |z|^{1/2}\begin{cases} 1, & |y| \le a, \\ 1 + \log^2(|y|/a), & |y| > a. \end{cases}$$

(For a proof see [7, Lemma 1]. The bound for $E(a,y)$ has been corrected in [6].) So in particular, taking $z = 1/N + iy$ we have

$$\Big|\sum_\rho z^{-\rho}\Gamma(\rho)\Big| = \left|\frac{1}{z} - \widetilde{S}(z) + E\Big(\frac{1}{N}, y\Big)\right| \ll N + \frac{1}{|z|} + \left|E\Big(\frac{1}{N}, y\Big)\right| \tag{2.9}$$
$$\ll \begin{cases} N, & |y| \le 1/N, \\ N + |z|^{1/2}\log^2(2N|y|), & |y| > 1/N. \end{cases}$$

Now we have to recall that the Prime Number Theorem (PNT) is equivalent, via Lemma 2.1, to the statement

$$\widetilde{S}(a) \sim a^{-1} \quad \text{when } a \to 0^+$$

(see [5, Lemma 9]). For our purposes it is important to introduce the Stirling approximation

$$|\Gamma(x+iy)| \sim \sqrt{2\pi}\, e^{-\pi|y|/2}|y|^{x-1/2} \tag{2.10}$$

(see, for example, [11, §4.42]) uniformly for $x \in [x_1, x_2]$, x_1 and x_2 fixed, and the identity

$$|z^{-w}| = |z|^{-\operatorname{Re}(w)}\exp\big(\operatorname{Im}(w)\arctan(y/a)\big). \tag{2.11}$$

We now quote Lemmas 2 and 3 from [7]:

LEMMA 2.2. *Let $\beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta function and let $\alpha > 1$ be a parameter. The series*

$$\sum_{\rho,\,\gamma>0} \gamma^{\beta-1/2}\int_1^\infty \exp(-\gamma\arctan(1/u))\,\frac{du}{u^{\alpha+\beta}}$$

converges provided that $\alpha > 3/2$. For $\alpha \le 3/2$ the series does not converge. The result remains true if we insert in the integral a factor $\log^c(u)$ for any fixed $c \ge 0$.

Lemma 2.3. *Let $\beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta function, and let $z = a + iy$, $a \in (0,1)$, $y \in \mathbb{R}$ and $\alpha > 1$. Then*

$$\sum_{\rho} |\gamma|^{\beta-1/2} \int_{\mathbb{Y}_1 \cup \mathbb{Y}_2} \exp\left(\gamma \arctan\left(\frac{y}{a}\right) - \frac{\pi}{2}|\gamma|\right) \frac{dy}{|z|^{\alpha+\beta}} \ll_{\alpha} a^{-\alpha}$$

where $\mathbb{Y}_1 = \{y \in \mathbb{R} : \gamma y \le 0\}$ and $\mathbb{Y}_2 = \{y \in [-a,a] : y\gamma > 0\}$. The result remains true if we insert in the integral a factor $\log^c(|y|/a)$ for any fixed $c \ge 0$.

We now establish an important lemma. We will use it to prove that there is a limitation in our technique. Essentially the lower bound of k is linked to the number of squares in the problem. We have

Lemma 2.4. *Let $\beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta-function, and let N, d be positive integers, $\|\cdot\|$ the euclidean norm in $\mathbb{R}^d$ and $k > 0$ a real number. Then the series*

$$\sum_{\bar{l} \in (0,\infty)^d} \sum_{\gamma > 0} \gamma^{-k-3/2} \int_0^{\gamma} e^{-N\|\bar{l}\|^2 v^2/\gamma^2} e^{-v} v^{k+\beta}\, dv,$$

where

$$\sum_{\bar{l} \in (0,\infty)^d} = \sum_{l_1 \ge 1} \cdots \sum_{l_d \ge 1},$$

converges if $k > d - 1/2$, and this result is optimal.

Proof. From (2.4) we have

$$\omega_2^d(z) = \frac{1}{2^d} \sum_{m=0}^{d} \binom{d}{m} (-1)^{d-m} \theta_3^m(z).$$

Hence

$$\begin{aligned} I &= \sum_{\bar{l} \in (0,\infty)^d} \sum_{\gamma > 0} \gamma^{-k-3/2} \int_0^{\gamma} e^{-N\|\bar{l}\|^2 v^2/\gamma^2} e^{-v} v^{k+\beta}\, dv \\ &= \sum_{\gamma > 0} \gamma^{-k-3/2} \int_0^{\gamma} \omega_2^d\left(\frac{Nv^2}{\gamma^2}\right) e^{-v} v^{k+\beta}\, dv \\ &= \frac{1}{2^d} \sum_{m=0}^{d} \binom{d}{m} (-1)^{d-m} \sum_{\gamma > 0} \gamma^{-k-3/2} \int_0^{\gamma} \theta_3^m\left(\frac{Nv^2}{\gamma^2}\right) e^{-v} v^{k+\beta}\, dv. \end{aligned}$$

Now, using the functional equation (2.5) we find that

$$I = \frac{1}{2^d}\sum_{m=0}^{d}\binom{d}{m}(-1)^{d-m}\frac{\pi^{m/2}}{N^{m/2}}\sum_{\gamma>0}\gamma^{m-k-3/2}\int_0^{\gamma}\theta_3^m\left(\frac{\pi^2\gamma^2}{Nv^2}\right)e^{-v}v^{k+\beta-m}\,dv$$

$$=: \frac{1}{2^d}\sum_{m=0}^{d}\binom{d}{m}(-1)^{d-m}\frac{\pi^{m/2}}{N^{m/2}}\sum_{\gamma>0}\gamma^{m-k-3/2}I_{\gamma,m}.$$

Now we claim that

$$\theta_3\left(\frac{\pi^2\gamma^2}{Nv^2}\right)\asymp 1,$$

where $f(x)\asymp g(x)$ means $g(x)\ll f(x)\ll g(x)$. Indeed, $\theta_3(x)$ is a continuous function in the interval $[\pi^2/N,\infty)$ (i.e. the range of $1/v^2$) and

$$\lim_{x\to\infty}\theta_3(x)=1.$$

So we have

$$I_{\gamma,m}\asymp\sum_{\gamma>0}\gamma^{m-k-3/2}\int_0^{\gamma}e^{-v}v^{k+\beta-m}\,dv,$$

and now, assuming $k+\beta-m+1>0$, we get

$$\int_0^{\gamma}e^{-v}v^{k+\beta-m}\,dv\asymp 1.$$

Hence

$$I_{\gamma,m}\asymp_k\sum_{\gamma>0}\gamma^{m-k-3/2}$$

and the series converges if $k>m-1/2$. Since $m=0,\dots,d$, for global convergence we must have $k>d-1/2$, and this result is optimal. ∎

Lemma 2.5. *Let $\rho=\beta+i\gamma$ run over the non-trivial zeros of the Riemann zeta function, and let $z=1/N+iy$, $N>1$ a natural number, $y\in\mathbb{R}$ and $\alpha>3/2$. Then*

$$\sum_{\rho}|\Gamma(\rho)|\int_{(1/N)}|e^{Nz}|\,|z^{-\rho}|\,|z|^{-\alpha}\,|dz|\ll_\alpha N^\alpha.$$

Proof. Set $a=1/N$. Using the identity (2.11) and (2.10) we see that the left hand side above is

$$\sum_{\rho}|\gamma|^{\beta-1/2}\int_{\mathbb{R}}\exp\left(\gamma\arctan\left(\frac{y}{a}\right)-\frac{\pi}{2}|\gamma|\right)\frac{dy}{|z|^{\alpha+\beta}},\tag{2.12}$$

which by Lemma 2.3 is $\ll_\alpha a^{-\alpha}$ in $\mathbb{Y}_1\cup\mathbb{Y}_2$. For the outside of this set, we can see that

$$\sum_{\rho} \gamma^{\beta-1/2} \int_{a}^{\infty} \exp\Big(-\gamma \arctan\Big(\frac{a}{y}\Big)\Big) \frac{dy}{|z|^{\alpha+\beta}} = a^{-\alpha-\beta+1} \sum_{\rho} \gamma^{\beta-1/2} \int_{1}^{\infty} \exp\Big(-\gamma \arctan\Big(\frac{1}{u}\Big)\Big) \frac{dy}{u^{\alpha+\beta}}$$

since

$$|z|^{-1} \asymp \begin{cases} a^{-1}, & |y| \le a, \\ |y|^{-1}, & |y| \ge a, \end{cases} \tag{2.13}$$

and so by Lemma 2.2 we have convergence if $\alpha > 3/2$. ∎

3. Settings for the Main Theorem. Using (2.1)–(2.3) it is not hard to see that

$$\widetilde{S}(z)\omega_2^2(z) = \sum_{m_1\ge1}\sum_{m_2\ge1}\sum_{m_3\ge1} \Lambda(m_1)e^{-(m_1+m_2^2+m_3^2)z} = \sum_{n\ge1} r_Q(n)e^{-nz}.$$

Let $z = a+iy$, $a>0$, and consider

$$\frac{1}{2\pi i}\int_{(a)} e^{Nz}z^{-k-1}\widetilde{S}(z)\omega_2^2(z)\,dz = \frac{1}{2\pi i}\int_{(a)} e^{Nz}z^{-k-1}\sum_{n\ge1} r_Q(n)e^{-nz}\,dz.$$

Now we prove that we can exchange the integral with the series. From (2.7) and the Prime Number Theorem in the form quoted above we have

$$\sum_{n\ge1} |r_Q(n)e^{-nz}| = \widetilde{S}(a)\omega_2^2(a) \ll a^{-2},$$

hence

$$\int_{(a)} |e^{Nz}z^{-k-1}|\,|\widetilde{S}(z)\omega_2^2(z)|\,|dz| \ll a^{-2}e^{Na}\Big(\int_{-a}^{a} a^{-k-1}\,dy + 2\int_{a}^{\infty} y^{-k-1}\,dy\Big) \ll_k a^{-2-k}e^{Na}$$

assuming $k>0$. So finally,

$$\sum_{n\le N} r_Q(n)\frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_{(a)} e^{Nz}z^{-k-1}\widetilde{S}(z)\omega_2^2(z)\,dz. \tag{3.1}$$

Now, using (2.8), we can write (3.1) as

$$\sum_{n\le N} r_Q(n)\frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_{(a)} e^{Nz}z^{-k-1}\Big(\frac{1}{z} - \sum_{\rho} z^{-\rho}\Gamma(\rho)\Big)\omega_2^2(z)\,dz + O\Big(\int_{(a)} |e^{Nz}|\,|z|^{-k-1}|\omega_2^2(z)|\,|E(a,y)|\,|dz|\Big), \tag{3.2}$$

and the error term can be estimated, using Lemma 2.1, (2.7) and (2.13), by

$$a^{-1}e^{Na}\Big(\int_{-a}^{a} a^{-k-1}dy + \int_{a}^{\infty} y^{-k-1/2}(1+\log^2(y/a))\,dy\Big) \ll_k e^{Na}a^{-k-1}$$

assuming $k > 1/2$. Hereafter we will consider $a = 1/N$. We have

$$\sum_{n\le N} r_Q(n)\frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\Big(\frac{1}{z} - \sum_{\rho} z^{-\rho}\Gamma(\rho)\Big)\omega_2^2(z)\,dz + O(N^{k+1})$$

and using the functional equation (2.6), we get

$$\begin{aligned}
&\sum_{n\le N} r_Q(n)\frac{(N-n)^k}{\Gamma(k+1)}\\
&= \frac{1}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\Big(\frac{1}{z} - \sum_{\rho} z^{-\rho}\Gamma(\rho)\Big)\Big(\Big(\frac{\pi}{z}\Big)^{1/2} - 1\Big)^2 dz\\
&+ \frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\Big(\frac{1}{z} - \sum_{\rho} z^{-\rho}\Gamma(\rho)\Big)\frac{\pi}{z}\omega_2^2\Big(\frac{\pi^2}{z}\Big)\,dz\\
&+ \frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\Big(\frac{1}{z} - \sum_{\rho} z^{-\rho}\Gamma(\rho)\Big)\Big(\Big(\frac{\pi}{z}\Big)^{1/2} - 1\Big)\Big(\Big(\frac{\pi}{z}\Big)^{1/2}\omega_2\Big(\frac{\pi^2}{z}\Big)\Big)\,dz\\
&+ O(N^{k+1})\\
&=: I_1 + I_2 + I_3 + O(N^{k+1}).
\end{aligned}$$

4. Evaluation of I_1. From I_1 we will find the main terms $M_1(N,k)$ and $M_2(N,k)$ of our asymptotic formulae. We have

$$\begin{aligned}
I_1 &= \frac{1}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-2}\Big(\Big(\frac{\pi}{z}\Big)^{1/2} - 1\Big)^2 dz\\
&\quad - \frac{1}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\sum_{\rho} z^{-\rho}\Gamma(\rho)\Big(\Big(\frac{\pi}{z}\Big)^{1/2} - 1\Big)^2 dz\\
&=: I_{1,1} - I_{1,2}.
\end{aligned}$$

For $I_{1,1}$ we observe that

$$\begin{aligned}
I_{1,1} &= \frac{\pi}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-3}\,dz + \frac{1}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-2}\,dz\\
&\quad - \frac{\pi^{1/2}}{4\pi i}\int_{(1/N)} e^{Nz}z^{-k-5/2}\,dz,
\end{aligned}$$

so if we set $Nz = s$, $ds = N\,dz$ and use (1.2), we immediately get

$$I_{1,1} = \frac{\pi}{4}\frac{N^{k+2}}{2\pi i}\int_{(1)} e^s s^{-k-3}\,ds + \frac{N^{k+1}}{4}\frac{1}{2\pi i}\int_{(1)} e^s s^{-k-2}\,ds - \frac{\pi}{2}\frac{N^{k+3/2}}{2\pi i}\int_{(1)} e^s s^{-k-5/2}\,ds = M_1(N,k)$$

(see (6.2) below). For $I_{1,2}$ we have

$$I_{1,2} = \frac{\pi}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-2}\sum_\rho z^{-\rho}\Gamma(\rho)\,dz + \frac{1}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\sum_\rho z^{-\rho}\Gamma(\rho)\,dz - \frac{\pi^{1/2}}{4\pi i}\int_{(1/N)} e^{Nz}z^{-k-3/2}\sum_\rho z^{-\rho}\Gamma(\rho)\,dz =: \mathcal{I}_1 + \mathcal{I}_2 - \mathcal{I}_3.$$

We observe that by Lemma 2.5 we have the absolute convergence of the integrals if, respectively, $k > -1/2$, $k > 1/2$ and $k > 0$. Hence for $k > 1/2$ we have

$$\mathcal{I}_1 = \frac{\pi}{4}\sum_\rho \Gamma(\rho)\frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-2-\rho}\,dz = \frac{\pi}{4}\sum_\rho \frac{\Gamma(\rho)}{\Gamma(k+2+\rho)}N^{k+1+\rho},$$

$$\mathcal{I}_2 = \frac{1}{4}\sum_\rho \Gamma(\rho)\frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-1-\rho}\,dz = \frac{1}{4}\sum_\rho \frac{\Gamma(\rho)}{\Gamma(k+1+\rho)}N^{k+\rho},$$

$$\mathcal{I}_3 = \frac{\pi^{1/2}}{2}\sum_\rho \Gamma(\rho)\frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-3/2-\rho}\,dz = \frac{\pi^{1/2}}{2}\sum_\rho \frac{\Gamma(\rho)}{\Gamma(k+3/2+\rho)}N^{k+1/2+\rho}.$$

5. Evaluation of I_2. We have

$$I_2 = \frac{\pi}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-3}\omega_2^2\left(\frac{\pi^2}{z}\right)dz - \frac{\pi}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-2}\sum_\rho z^{-\rho}\Gamma(\rho)\omega_2^2\left(\frac{\pi^2}{z}\right)dz =: I_{2,1} - I_{2,2}.$$

Evaluation of $I_{2,1}$. We have

$$I_{2,1} = \frac{\pi}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-3} \omega_2^2\Big(\frac{\pi^2}{z}\Big)\, dz$$
$$= \frac{\pi}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-3} \Big(\sum_{l_1\ge 1} e^{-l_1^2\pi^2/z}\Big)\Big(\sum_{l_2\ge 1} e^{-l_2^2\pi^2/z}\Big)\, dz;$$

let us prove that we can exchange the integral with the series. Set

$$A_1 := \sum_{l_1\ge 1} \int_{(1/N)} |e^{Nz}|\, |z|^{-k-3} e^{-l_1^2\pi^2 \operatorname{Re}(1/z)} \Big|\omega_2\Big(\frac{\pi^2}{z}\Big)\Big|\, |dz|.$$

From

$$\operatorname{Re}(1/z) = \frac{N}{1+N^2y^2} \gg \begin{cases} N, & |y| \le 1/N, \\ 1/(Ny^2), & |y| > 1/N, \end{cases} \tag{5.1}$$

we have

$$A_1 \ll \sum_{l_1\ge 1} \int_0^{1/N} \frac{e^{-l_1^2 N}}{|z|^{k+3}} \omega_2(N)\, dy + N^{1/2} \sum_{l_1\ge 1} \int_{1/N}^{\infty} \frac{y e^{-l_1^2/(Ny^2)}}{|z|^{k+3}}\, dy =: U_1 + U_2.$$

Hence, recalling (2.7) and (2.13), we obtain

$$U_1 \ll N^{k+2}\omega_2^2(N) \ll N^{k+1},$$

and from (2.13) (with $a = 1/N$) we get

$$U_2 \ll N^{1/2} \sum_{l_1\ge 1} \int_{1/N}^{\infty} \frac{e^{-l_1^2/(Ny^2)}}{y^{k+2}}\, dy \ll N^{k/2+1} \sum_{l_1\ge 1} \frac{1}{l_1^{k+1}} \int_0^{l_1^2 N} u^{k/2-1/2} e^{-u}\, du$$
$$\le \Gamma\Big(\frac{k+1}{2}\Big) N^{k/2+1} \sum_{l_1\ge 1} \frac{1}{l_1^{k+1}} \ll_k N^{k/2+1}$$

assuming $k > 0$. Now we have to study the convergence of

$$A_2 := \sum_{l_1\ge 1}\sum_{l_2\ge 1} \int_{(1/N)} |e^{Nz}|\, |z|^{-k-3} e^{-l_1^2\pi^2 \operatorname{Re}(1/z)} e^{-l_2^2\pi^2 \operatorname{Re}(1/z)}\, |dz|.$$

Again from (2.13) we have

$$A_2 \ll \sum_{l_1\ge 1}\sum_{l_2\ge 1} \int_0^{1/N} \frac{e^{-(l_1^2+l_2^2)N}}{|z|^{k+3}}\, dy + \sum_{l_1\ge 1}\sum_{l_2\ge 1} \int_{1/N}^{\infty} \frac{e^{-(l_1^2+l_2^2)/(Ny^2)}}{|z|^{k+3}}\, dy$$
$$=: V_1 + V_2.$$

For V_1 we can repeat the same reasoning as for U_1 to get

$$V_1 \ll N^{k+2}\omega_2^2(N) \ll N^{k+1},$$

and for V_2, assuming $k > 1$, we have

$$V_2 \ll \sum_{l_1\ge 1}\sum_{l_2\ge 1}\int_{1/N}^{\infty} \frac{e^{-(l_1^2+l_2^2)/(Ny^2)}}{y^{k+3}}\,dy \ll_k N^{k/2+1/2}.$$

Finally,

$$\begin{aligned} I_{2,1} &= \frac{\pi}{2\pi i}\sum_{l_1\ge 1}\sum_{l_2\ge 1}\int_{(1/N)} e^{Nz}z^{-k-3}e^{-(l_1^2+l_2^2)\pi^2/z}\,dz \\ &= N^{k+2}\pi\sum_{l_1\ge 1}\sum_{l_2\ge 1}\frac{1}{2\pi i}\int_{(1)} e^{s}s^{-k-3}e^{-(l_1^2+l_2^2)\pi^2 N/s}\,ds, \end{aligned}$$

from which, recalling the definition of the Bessel functions (1.1), we have, taking $u = 2\pi(l_1^2+l_2^2)^{1/2}N^{1/2}$ and assuming $k > 1$,

$$I_{2,1} = \frac{N^{k/2+1}}{\pi^{k+1}}\sum_{l_1\ge 1}\sum_{l_2\ge 1}\frac{J_{k+2}(2\pi(l_1^2+l_2^2)^{1/2}N^{1/2})}{(l_1^2+l_2^2)^{k/2+1}}.$$

Evaluation of $I_{2,2}$. We have to calculate

$$I_{2,2} = \frac{\pi}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-2}\sum_{\rho} z^{-\rho}\Gamma(\rho)\Big(\sum_{l_1\ge 1} e^{-l_1^2\pi^2/z}\Big)\Big(\sum_{l_2\ge 1} e^{-l_2^2\pi^2/z}\Big)\,dz.$$

Again we have to prove that is possible to exchange the integral with the series, so set

$$A_3 := \sum_{l_1\ge 1}\int_{(1/N)} |e^{Nz}|\,|z^{-k-2}|\,\Big|\sum_{\rho} z^{-\rho}\Gamma(\rho)\Big|e^{-l_1^2\pi^2\,\mathrm{Re}(1/z)}\Big|\omega_2\Big(\frac{\pi^2}{z}\Big)\Big|\,|dz|.$$

Using (2.9) and (2.7) we obtain

$$\begin{aligned} A_3 &\ll N^{1/2}\sum_{l_1\ge 1}\int_{0}^{1/N}\frac{e^{-l_1^2 N}}{|z|^{k+2}}\,dy + N^{3/2}\sum_{l_1\ge 1}\int_{1/N}^{\infty}\frac{ye^{-l_1^2/(Ny^2)}}{|z|^{k+2}}\,dy \\ &\quad + N^{1/2}\sum_{l_1\ge 1}\int_{1/N}^{\infty} y\log^2(2Ny)\frac{e^{-l_1^2/(Ny^2)}}{|z|^{k+3/2}}\,dy \\ &=: W_1 + W_2 + W_3. \end{aligned}$$

We can easily see that

$$W_1 \ll N^{k+3/2}\omega_2(N) \ll N^{k+1}$$

and, taking $u = l_1^2/(Ny^2)$,

$$W_2 \ll N^{3/2} \sum_{l_1 \geq 1} \int_{1/N}^{\infty} \frac{e^{-l_1^2/(Ny^2)}}{y^{k+1}}\, dy$$
$$\ll N^{k/2+3/2} \sum_{l_1 \geq 1} \frac{1}{l_1^k} \int_0^{l_1^2 N} e^{-u} u^{k/2-1}\, du \ll_k N^{k/2+3/2}$$

assuming $k > 1$. We have now to check W_3. Taking again $u = l_1^2/(Ny^2)$ we deduce, assuming $k > 3/2$, that

$$W_3 \ll N^{k/2-1/4} \sum_{l_1 \geq 1} \frac{1}{l_1^{k-1/2}} \int_0^{l_1^2 N} \log^2\left(\frac{4Nl_1^2}{u}\right) e^{-u} u^{k/2-5/4}\, du$$
$$\ll N^{k/2-1/4} \sum_{l_1 \geq 1} \frac{1}{l_1^{k-1/2}} \ll_k N^{k/2}.$$

Consider

$$A_4 := \sum_{l_1 \geq 1} \sum_{l_2 \geq 2} \int_{(1/N)} |e^{Nz}|\, |z^{-k-2}| \left| \sum_{\rho} z^{-\rho} \Gamma(\rho) \right| e^{-l_1^2 \pi^2 \operatorname{Re}(1/z)} e^{-l_2^2 \pi^2 \operatorname{Re}(1/z)}\, |dz|.$$

By (2.9) we get

$$A_4 \ll N \sum_{l_1 \geq 1} \sum_{l_2 \geq 2} \int_0^{1/N} \frac{e^{-(l_1^2+l_2^2)N}}{|z|^{k+2}}\, dy + \sum_{l_1 \geq 1} \sum_{l_2 \geq 2} \int_{1/N}^{\infty} \frac{e^{-(l_1^2+l_2^2)/(Ny^2)}}{|z|^{k+2}}\, dy$$
$$+ \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{1/N}^{\infty} \log^2(2Ny) \frac{e^{-(l_1^2+l_2^2)/(Ny^2)}}{|z|^{k+3/2}}\, dy$$
$$=: R_1 + R_2 + R_3.$$

We have immediately

$$R_1 \ll N^{k+2} \omega^2(N) \ll N^{k+1},$$

and if we take $u = (l_1^2 + l_2^2)/(Ny^2)$, we obtain

$$R_2 \ll \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{1/N}^{\infty} \frac{e^{-(l_1^2+l_2^2)/(Ny^2)}}{y^{k+2}}\, dy \ll_k N^{(k+1)/2}$$

for $k > 1$. So it remains to evaluate R_3. Again we take $u = (l_1^2 + l_2^2)/(Ny^2)$ to get

$$R_3 \ll N^{k/2+1/4} \sum_{l_1\geq 1}\sum_{l_2\geq 1} \frac{\log^2(4N(l_1^2+l_2^2))}{(l_1^2+l_2^2)^{k/2+1/4}} \int_0^{(l_1^2+l_2^2)^{1/2}N} e^{-u}u^{k/2-3/4}\,du$$
$$- N^{k/2+1/4} \sum_{l_1\geq 1}\sum_{l_2\geq 1} \frac{1}{(l_1^2+l_2^2)^{k/2+1/4}} \int_0^{(l_1^2+l_2^2)^{1/2}N} \log^2(u)e^{-u}u^{k/2-3/4}\,du,$$

with convergence if $k > 3/2$. Note that the estimation of R_3 is optimal. To see this, take $c = (l_1^2+l_2^2)/N$, and assume $k \leq 3/2$ and $y > 1$. Then

$$S := \sum_{l_1\geq 1}\sum_{l_2\geq 1} \int_{1/N}^{\infty} \log^2(2Ny)\frac{e^{-c/y^2}}{y^{k+3/2}}\,dy \geq \sum_{l_1\geq 1}\sum_{l_2\geq 1} \int_1^{\infty} \log^2(2Ny)\frac{e^{-c/y^2}}{y^{k+3/2}}\,dy.$$

Since $y \geq 1$ we have $\log^2(2Ny) \geq \log^2(2N)$, and since $k \leq 3/2$ we get

$$S \geq \log(2N) \sum_{l_1\geq 1}\sum_{l_2\geq 1} \int_1^{\infty} \frac{e^{-c/y^2}}{y^{k+3/2}}\,dy \geq \log(2N) \sum_{l_1\geq 1}\sum_{l_2\geq 1} \int_1^{\infty} \frac{e^{-c/y^2}}{y^3}\,dy$$
$$= \log(2N) \sum_{l_1\geq 1}\sum_{l_2\geq 1} \frac{1}{2c}(1-e^{-c}) \geq \frac{N\log(2N)(1-e^{-2/N})}{2} \sum_{l_1\geq 1}\sum_{l_2\geq 1} \frac{1}{l_1^2+l_2^2}.$$

The last double series diverges since

$$\sum_{l_1\geq 1}\sum_{l_2\geq 1} \frac{1}{l_1^2+l_2^2} \geq \sum_{l_1\geq 1}\sum_{1\leq l_2\leq l_1} \frac{1}{l_1^2+l_2^2} \geq \frac{1}{2}\sum_{l_1\geq 1}\frac{1}{l_1}.$$

Now we have to estimate

$$A_5 := \sum_{l_1\geq 1}\sum_{l_2\geq 1}\sum_{\rho} |\Gamma(\rho)| \int_{(1/N)} |e^{Nz}|\,|z^{-k-2}|\,|z^{-\rho}| e^{-l_1^2\pi^2\operatorname{Re}(1/z)} e^{-l_2^2\pi^2\operatorname{Re}(1/z)}\,|dz|.$$

Using (2.10) and (2.11) we have

$$A_5 \ll \sum_{l_1\geq 1}\sum_{l_2\geq 1}\sum_{\rho,\gamma>0} e^{-\pi\gamma/2}\gamma^{\beta-1/2}$$
$$\times \int_{1/N}^{\infty} |z|^{-k-2}|z|^{-\beta}\exp(\gamma\arctan(Ny)) e^{-l_1^2\pi^2\operatorname{Re}(1/z)} e^{-l_2^2\pi^2\operatorname{Re}(1/z)}\,|dz|.$$

Let $Q_k = \sup_\beta \Gamma(k/2+\beta/2+1/2)$ and assume $y < 0$. Using the trivial bound $\gamma\arctan(Ny) - \gamma\pi/2 \leq -\gamma\pi/2$, we get

$$A_5 \ll N^{k+1} \sum_{l_1\geq 1} e^{-l_1^2N} \sum_{l_2\geq 1} e^{-l_2^2N} \sum_{\rho,\gamma>0} N^{\beta} e^{-\pi\gamma/2}\gamma^{\beta-1/2}$$
$$+ N^{(k+1)/2}Q_k \sum_{l_1\geq 1}\sum_{l_2\geq 1} \frac{1}{(l_1^2+l_2^2)^{(k+1)/2}} \sum_{\rho,\gamma>0} N^{\beta}\frac{e^{-\pi\gamma/2}\gamma^{\beta-1/2}}{(l_1^2+l_2^2)^{\beta}} \ll_k N^k. \tag{5.2}$$

If $y > 0$ we have

$$A_5 \ll \sum_{l_1\ge 1}\sum_{l_2\ge 1}\sum_{\rho:\gamma>0} e^{-\pi\gamma/2}\gamma^{\beta-1/2}\int_0^{1/N} N^{k+2+\beta}e^{-(l_1^2+l_2^2)N}\,dy$$

$$+\sum_{l_1\ge 1}\sum_{l_2\ge 1}\sum_{\rho:\gamma>0}\gamma^{\beta-1/2}\int_{1/N}^{\infty}\exp\bigl(\gamma(\arctan(Ny)-\pi/2)\bigr)\frac{e^{-(l_1^2+l_2^2)/(Ny^2)}}{y^{k+2+\beta}}\,dy,$$

and by a well-known trigonometric identity it follows that

$$A_5 \ll N^{k+1} + \sum_{l_1\ge 1}\sum_{l_2\ge 1}\sum_{\rho:\gamma>0}\gamma^{\beta-1/2}\int_{1/N}^{\infty}\exp\biggl(-\gamma\arctan\Bigl(\frac{1}{Ny}\Bigr)\biggr)\frac{e^{-(l_1^2+l_2^2)/(Ny^2)}}{y^{k+2+\beta}}\,dy$$

$$\ll N^{k+1} + \sum_{l_1\ge 1}\sum_{l_2\ge 1}\sum_{\rho:\gamma>0}\gamma^{\beta-1/2}\int_{1/N}^{\infty}\exp\Bigl(-\frac{\gamma}{Ny}-\frac{l_1^2+l_2^2}{Ny^2}\Bigr)y^{-k-2-\beta}\,dy.$$

Setting $\gamma/(Ny) = v$, we get

$$A_5 \ll N^{k+1} + \sum_{l_1\ge 1}\sum_{l_2\ge 1}\sum_{\rho:\gamma>0}\gamma^{\beta-1/2}\int_0^{\gamma} e^{-v}e^{-Nv^2(l_1^2+l_2^2)/\gamma^2}\Bigl(\frac{\gamma}{Nv}\Bigr)^{-k-2-\beta}\frac{\gamma}{Nv^2}\,dv \tag{5.3}$$

$$\ll N^{k+1} + \sum_{l_1\ge 1}\sum_{l_2\ge 1}\sum_{\rho:\gamma>0}\gamma^{-k-3/2}\int_0^{\infty} e^{-v}e^{-Nv^2(l_1^2+l_2^2)/\gamma^2}v^{k+\beta}\,dv.$$

Now we can observe that we are in the situation of Lemma 2.4 with $d = 2$, and so we can conclude immediately that we have convergence for $k > 3/2$, and this result is optimal.

Having the convergence, we finally obtain, using again the identity (1.1),

$$I_{2,2} = \pi^{-k}N^{k/2+1/2}\sum_{\rho}\frac{\Gamma(\rho)}{\pi^{\rho}}N^{\rho/2}\sum_{l_1\ge 1}\sum_{l_2\ge 1}\frac{J_{k+1+\rho}(2\pi(l_1^2+l_2^2)^{1/2}N^{1/2})}{(l_1^2+l_2^2)^{(k+1+\rho)/2}}.$$

6. Evaluation of I_3. We have

$$I_3 = \frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\biggl(\frac{\pi^{1/2}}{z^{3/2}} - \Bigl(\frac{\pi}{z}\Bigr)^{1/2}\sum_{\rho} z^{-\rho}\Gamma(\rho) - \frac{1}{z} + \sum_{\rho} z^{-\rho}\Gamma(\rho)\biggr)$$
$$\times\biggl(\Bigl(\frac{\pi}{z}\Bigr)^{1/2}\omega_2\Bigl(\frac{\pi^2}{z}\Bigr)\biggr)\,dz$$

$$= \frac{1}{2i}\int_{(1/N)} e^{Nz}z^{-k-3}\omega_2\Bigl(\frac{\pi^2}{z}\Bigr)\,dz - \frac{1}{2i}\int_{(1/N)} e^{Nz}z^{-k-2}\sum_{\rho} z^{-\rho}\Gamma(\rho)\omega_2\Bigl(\frac{\pi^2}{z}\Bigr)\,dz$$

$$-\frac{1}{2\pi^{1/2}i}\int_{(1/N)} e^{Nz}z^{-k-5/2}\omega_2\Big(\frac{\pi^2}{z}\Big)$$
$$+\frac{1}{2\pi^{1/2}i}\int_{(1/N)} e^{Nz}z^{-k-3/2}\sum_{\rho} z^{-\rho}\Gamma(\rho)\omega_2\Big(\frac{\pi^2}{z}\Big)\,dz$$
$$=: I_{3,1}-I_{3,2}-I_{3,3}+I_{3,4}.$$

Evaluation of $I_{3,1}$. We have

$$I_{3,1}=\frac{1}{2i}\int_{(1/N)} e^{Nz}z^{-k-3}\omega_2\Big(\frac{\pi^2}{z}\Big)\,dz=\frac{1}{2i}\int_{(1/N)} e^{Nz}z^{-k-3}\sum_{m\ge1}e^{-m^2\pi^2/z}\,dz,$$

hence we have to establish the convergence of

$$A_6:=\sum_{m\ge1}\int_{(1/N)} |e^{Nz}|\,|z|^{-k-3}e^{-m^2\operatorname{Re}(1/z)}\,|dz|.$$

Using (2.7), (2.13) and (5.1) we have

$$A_6\ll N^{k+3/2}+\sum_{m\ge1}\int_0^\infty y^{-k-3}e^{-m^2/(Ny^2)}\,dy\ll_k N^{k+3/2}\tag{6.1}$$

for $k>-1$. So, recalling (1.1), we obtain

$$J_{3,1}=\frac{N^{k/2+1}}{\pi^{k+1}}\sum_{m\ge1}\frac{J_{k+2}(2m\pi N^{1/2})}{m^{k+2}}.$$

Evaluation of $I_{3,3}$. We have

$$I_{3,3}=\frac{1}{2\pi^{1/2}i}\int_{(1/N)} e^{Nz}z^{-k-5/2}\sum_{m\ge1}e^{-m^2\pi^2/z}\,dz,$$

so we have to establish the convergence of

$$\sum_{m\ge1}\int_{(1/N)} |e^{Nz}|\,|z|^{-k-5/2}e^{-m^2\operatorname{Re}(1/z)}\,|dz|.$$

Arguing as for $I_{3,1}$, we have the convergence for $k>-1/2$. Summing up,

$$I_{3,3}=\frac{N^{k/2+3/4}}{\pi^{k+1}}\sum_{m\ge1}\frac{J_{k+3/2}(2m\pi N^{1/2})}{m^{k+3/2}}.$$

Evaluation of $I_{3,2}$. We have to establish the convergence of

$$A_7:=\sum_{m\ge1}\int_{(1/N)} |e^{Nz}|\,|z^{-k-2}|\,\Big|\sum_{\rho} z^{-\rho}\Gamma(\rho)\Big|\,|e^{-m^2\pi^2/z}|\,|dz|.$$

Using (2.7), (2.13), (5.1) and (2.9) we get

$$A_7 \ll N^{k+1/2} + N \sum_{m\geq 1} \int_{1/N}^{\infty} y^{-k-2} e^{-m^2/(Ny^2)}\, dy$$
$$+ \log^2(2N) \sum_{m\geq 1} \int_{1/N}^{\infty} y^{-k-3/2} e^{-m^2/(Ny^2)}\, dy$$
$$+ \sum_{m\geq 1} \int_{1/N}^{\infty} \log^2(y) y^{-k-3/2} e^{-m^2/(Ny^2)}\, dy.$$

Now if we set $m^2/(Ny^2) = u$ we have

$$N \sum_{m\geq 1} \int_{1/N}^{\infty} y^{-k-2} e^{-m^2/(Ny^2)}\, dy \ll N^{k/2+3/2} \Gamma\Big(\frac{k+1}{2}\Big) \sum_{m\geq 1} m^{-k-1},$$

which converges if $k > 0$. With the same substitution we get

$$\log^2(2N) \sum_{m\geq 1} \int_{1/N}^{\infty} y^{-k-3/2} e^{-m^2/(Ny^2)}\, dy$$
$$\ll \log^2(2N) N^{k/2+1/4} \Gamma\Big(\frac{k}{2} + \frac{1}{4}\Big) \sum_{m\geq 1} m^{-k-1/2},$$

which converges for $k > 1/2$. To estimate the last integral in the bound of A_7 we observe that if we take $\epsilon > 0$ then

$$\sum_{m\geq 1} \int_{1/N}^{\infty} \log^2(y) y^{-k-3/2} e^{-m^2/(Ny^2)}\, dy \ll \sum_{m\geq 1} \int_{1/N}^{\infty} y^{-k-3/2+\epsilon} e^{-m^2/(Ny^2)}\, dy,$$

and so, arguing as we did for (6.1), we get

$$\ll N^{k/2+1/4-\epsilon/2} \Gamma\Big(\frac{k}{2} + \frac{1}{4} - \frac{\epsilon}{2}\Big) \sum_{m\geq 1} m^{-k-1/2+\epsilon},$$

and by the arbitrariness of ϵ we have convergence for $k > 1/2$. We have now to study

$$A_8 := \sum_{m\geq 1} \sum_{\rho} |\Gamma(\rho)| \int_{(1/N)} |e^{Nz}|\, |z^{-k-2}|\, |z^{-\rho}|\, |e^{-m^2\pi^2/z}|\, |dz|.$$

By symmetry we may assume $\gamma > 0$. If $y \leq 0$ we have $\gamma \arctan(y/a) - \pi\gamma/2 \leq -\pi\gamma/2$, and so using (2.10) and (2.11) we get

$$A_8 \ll \sum_{m\geq 1} \sum_{\gamma>0} \gamma^{\beta-1/2} \exp\Big(-\frac{\pi}{2}\gamma\Big)$$
$$\times \bigg(\int_{-1/N}^{0} N^{k+2+\beta} e^{-m^2 N}\, dy + \int_{-\infty}^{-1/N} \frac{e^{-m^2/(Ny^2)}}{|y|^{k+2+\beta}}\, dy \bigg)$$

$$\ll_k N^{k+3/2} + N^{k/2+1/2} Q_k \sum_{m\ge 1} \frac{1}{m^{k+1}} \sum_{\gamma>0} N^{\beta/2} \frac{\gamma^{\beta-1/2}}{m^\beta} \exp\Big(-\frac{\pi}{2}\gamma\Big)$$
$$\ll_k N^{k+3/2}$$

provided that $k > 0$ and $Q_k = \sup_\beta \Gamma(k/2 + 1/2 + \beta/2)$. Let $y > 0$. Then

$$\begin{aligned} A_8 &\ll \sum_{m\ge 1}\sum_{\gamma>0} \gamma^{\beta-1/2} \exp\Big(-\frac{\pi}{4}\gamma\Big) \int_0^{1/N} N^{k+2+\beta} e^{-m^2 N}\, dy \\ &\quad + \sum_{m\ge 1}\sum_{\gamma>0} \gamma^{\beta-1/2} \int_{1/N}^{\infty} \exp\Big(\gamma \arctan(Ny) - \frac{\pi}{2}\gamma\Big) \frac{e^{-m^2/(Ny^2)}}{y^{k+2+\beta}}\, dy \\ &=: L_1 + L_2. \end{aligned}$$

From (2.7) and (2.13) we have

$$L_1 \ll N^{k+1} \sum_{m\ge 1} e^{-m^2 N} \sum_{\gamma>0} N^\beta \gamma^{\beta-1/2} \exp(-\frac{\pi}{4}\gamma) \ll_k N^{k+3/2},$$

and again by a well-known trigonometric identity and taking $v = m/(N^{1/2} y)$ we obtain

$$\begin{aligned} L_2 &\ll \sum_{m\ge 1}\sum_{\gamma>0} \gamma^{\beta-1/2} \int_{1/N}^{\infty} \exp\Big(-\frac{\gamma}{Ny} - \frac{m^2}{Ny^2}\Big) \frac{dy}{y^{k+2+\beta}} \\ &= N^{(k+1)/2} \sum_{m\ge 1} \frac{1}{m^{k+1}} \sum_{\gamma>0} \frac{N^{\beta/2}}{m^\beta} \gamma^{\beta-1/2} \int_0^{m\sqrt{N}} \exp\Big(-\frac{\gamma v}{N^{1/2} m} - v^2\Big) v^{k+\beta}\, dv. \end{aligned}$$

Using $e^{-v^2} v^k = O_k(1)$ if $k > 0$, we get, taking $s = \gamma v/(N^{1/2} m)$,

$$L_2 \ll N^{k/2+1} \sum_{m\ge 1} \frac{1}{m^k} \sum_{\gamma>0} N^\beta \gamma^{-3/2} \int_0^\infty \exp(-s) s^\beta\, ds \ll_k N^{k/2+2}$$

for $k > 1$. Now we can exchange the series with the integral to deduce

$$I_{3,2} = \pi^{-k} N^{(k+1)/2} \sum_\rho \pi^{-\rho} N^{\rho/2} \Gamma(\rho) \sum_{m\ge 1} \frac{J_{k+1+\rho}(2m\pi N^{1/2})}{m^{k+1+\rho}}.$$

Evaluation of $I_{3,4}$. We have to establish the convergence of

$$I_{3,4} = \frac{1}{2\pi^{1/2} i} \int_{(1/N)} e^{Nz} z^{-k-3/2} \sum_\rho z^{-\rho} \Gamma(\rho) \omega_2\Big(\frac{\pi^2}{z}\Big)\, dz.$$

Arguing as we did for $I_{3,2}$, we obtain the condition $k > 1$. We can exchange the series with the integral and obtain

$$I_{3,4} = \pi^{-k} N^{k/2+1/4} \sum_{\rho} \pi^{-\rho} N^{\rho} \Gamma(\rho) \sum_{m\geq 1} \frac{J_{k+1/2+\rho}(2m\pi N^{1/2})}{m^{k+1/2+\rho}}.$$

Defining

$$M_1(N,k) = \frac{\pi N^{k+2}}{4\Gamma(k+3)} + \frac{N^{k+1}}{4\Gamma(k+2)} - \frac{\pi^{1/2} N^{k+3/2}}{2\Gamma(k+5/2)}, \tag{6.2}$$

$$M_2(N,k) = -\frac{\pi}{4} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+2+\rho)} N^{k+1+\rho} - \frac{1}{4} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+1+\rho)} N^{k+\rho} + \frac{\pi^{1/2}}{2} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+3/2+\rho)} N^{k+1/2+\rho}, \tag{6.3}$$

$$M_3(N,k) = \frac{N^{k/2+1}}{\pi^{k+1}} \sum_{l_1\geq 1} \sum_{l_2\geq 1} \frac{J_{k+2}(2\pi(l_1^2+l_2^2)^{1/2} N^{1/2})}{(l_1^2+l_2^2)^{k/2+1}} - \pi^{-k} N^{k/2+1/2} \sum_{\rho} \frac{\Gamma(\rho)}{\pi^{\rho}} N^{\rho/2} \sum_{l_1\geq 1} \sum_{l_2\geq 1} \frac{J_{k+1+\rho}(2\pi(l_1^2+l_2^2)^{1/2} N^{1/2})}{(l_1^2+l_2^2)^{(k+1+\rho)/2}}, \tag{6.4}$$

$$M_4(N,k) = \frac{N^{k/2+1}}{\pi^{k+1}} \sum_{m\geq 1} \frac{J_{k+2}(2m\pi N^{1/2})}{m^{k+2}} - \frac{N^{k/2+3/4}}{\pi^{k+1}} \sum_{m\geq 1} \frac{J_{k+3/2}(2m\pi N^{1/2})}{m^{k+3/2}} - \pi^{-k} N^{(k+1)/2} \sum_{\rho} \pi^{-\rho} N^{\rho/2} \Gamma(\rho) \sum_{m\geq 1} \frac{J_{k+1+\rho}(2m\pi N^{1/2})}{m^{k+1+\rho}} + \pi^{-k} N^{k/2+1/4} \sum_{\rho} \pi^{-\rho} N^{\rho/2} \Gamma(\rho) \sum_{m\geq 1} \frac{J_{k+1/2+\rho}(2m\pi N^{1/2})}{m^{k+1/2+\rho}}, \tag{6.5}$$

we have proved the following

Main Theorem 6.1. *Let N be a sufficiently large integer. Then*

$$\sum_{n\leq N} r_Q(n) \frac{(N-n)^k}{\Gamma(k+1)} = M_1(N,k) + M_2(N,k) + M_3(N,k) + M_4(N,k) + O(N^{k+1})$$

for $k > 3/2$, where ρ runs over the non-trivial zeros of the Riemann zeta function $\zeta(s)$, and $J_v(u)$ is the Bessel function of complex order v and real argument u. Furthermore, the bound $k > 3/2$ is optimal using this technique.

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References

[1] B. C. Berndt, *Identities involving the coefficients of a class of Dirichlet series, VIII*, Trans. Amer. Math. Soc. 201 (1975), 247–261.

[2] K. Chandrasekharan and R. Narasimhan, *Hecke's functional equation and arithmetical identities*, Ann. of Math. 74 (1961), 1–23.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms*, Vol. 1, McGraw-Hill, 1954.

[4] E. Freitag and R. Busam, *Complex Analysis*, 2nd ed., Springer, 2009.

[5] G. H. Hardy and J. E. Littlewood, *Some problems of 'Partitio Numerorum'; III: On the expression of a number as a sum of primes*, Acta Math. 44 (1923), 1–70.

[6] A. Languasco, *Applications of some exponential sums on prime powers: a survey*, Riv. Mat. Univ. Parma 7 (2016), 19–37.

[7] A. Languasco and A. Zaccagnini, *A Cesàro average of Goldbach numbers*, Forum Math. 27 (2015), 1945–1960.

[8] A. Languasco and A. Zaccagnini, *A Cesàro average of Hardy–Littlewood numbers*, J. Math. Anal. Appl. 401 (2013), 568–577.

[9] P. S. Laplace, *Théorie analytique des probabilités*, Courcier, 1812.

[10] Yu. V. Linnik, *An asymptotic formula in an additive problem of Hardy–Littlewood*, Izv. Akad. Nauk SSSR Ser. Mat. 24 (1960), 629–706 (in Russian).

[11] E. C. Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford Univ. Press, 1988.

[12] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge Univ. Press, 1966.

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