ACTA ARITHMETICA
180.3 (2017)

Double shuffle relations for multiple Dedekind zeta values

by

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0. Introduction. Multiple zeta values (MZVs) were defined as infinite sums by Euler [Eu]:

$$\zeta(m_1,\ldots,m_d)=\sum_{0<k_1<\cdots<k_d}\frac{1}{k_1^{m_1}\ldots k_d^{m_d}}. \tag{0.1}$$

Using the infinite sum representation (0.1), we can express a product of two MZVs as a sum of such. The MZVs can also be expressed as iterated path integrals (see [G1]). This integral representation leads to an integral shuffle relation, which is a formula expressing a product of multiple zeta values as a sum of such. However, the infinite sum representation gives different formulas for the product compared to the shuffle coming from the integral representation. Expressing a product of multiple zeta values as a sum of such in two different ways leads to linear relations among multiple zeta values (see for example [G1], [G2], [GKZ], [IKZ]).

Multiple Dedekind zeta values (MDZV) are number field analogues of multiple zeta values. Similarly to multiple zeta values, they are expected to have two types of shuffle relations. Constructions of both shuffle relations are the central theme of this paper. We also examine one example in detail.

Multiple Dedekind zeta values are defined in [H3] using a higher-dimensional analogue of iterated path integrals, which we call iterated integrals over membranes. The idea of such integrals was introduced in [H1], and further developed in [H2], for the purpose of generalizing Manin's noncommutative modular symbol [M] to Hilbert modular surfaces.

Structure of the paper. In Section 1, we give the definitions of multiple Dedekind zeta values needed for the two types of shuffles. We ex-

2010 *Mathematics Subject Classification*: 11M32, 11M41, 11R42, 11R47.
Key words and phrases: multiple zeta values, Dedekind zeta values.
Received 28 September 2014; revised 29 June 2017.
Published online 30 August 2017.

DOI: 10.4064/aa7945-8-2016

tend the set of MDZVs defined in [H3] to ordinary and permutational MDZVs.

In Section 2, we recall the infinite sum shuffle for multiple zeta values. We construct an analogous shuffle for ordinary multiple Dedekind zeta values. In particular, we obtain an infinite sum shuffle for a product of two Dedekind zeta values at $s = 2$. We also give examples of other infinite sum shuffle relations, and we construct the infinite sum shuffle of ordinary MDZVs in the general case.

In Section 3, we present an integral shuffle relation for ordinary MDZVs. In order to do that, we first give examples of classical polylogarithms and multiple zeta values in terms of iterated integrals. We also give examples of integral shuffles of multiple zeta values. Then we express polylogarithms as integrals involving exponent variables in order to generalize them to multiple Dedekind polylogarithms in the form of iterated integrals over membranes. We give an example of an integral shuffle relation for Dedekind polylogarithms, and for the convenience of the reader we represent integrals as diagrams.

We are able to obtain an integral shuffle relation for two Dedekind zeta values at $s = 2$. This relation expresses the product as a sum of 36 permutational MDZVs. To visualize these shuffles it is best to represent each of the 36 integrals as a diagram. The value of each integral can be written as an infinite sum and we write the corresponding summand below each diagram.

At the end of Section 3, we construct the integral shuffle of ordinary MDZVs in the general case.

In Section 4, we give an explicit example of a linear relation among several ordinary MDZVs.

1. Definitions of multiple Dedekind zeta values. In order to introduce the reader to the MDZVs we first recall classical multiple zeta values.

First, recall the double zeta value

$$\zeta(a,b) = \sum_{0<m<n} \frac{1}{m^a n^b},$$

which converges for $a \geq 1$ and $b \geq 2$. It can also be written as

$$\zeta(a,b) = \sum_{m,n=1}^{\infty} \frac{1}{m^a (m+n)^b}.$$

Let $\mathcal{O}_K$ be the ring of integers in an imaginary quadratic field K. If $\alpha \in \mathcal{O}_K$, we define $\alpha_1 = \alpha$ and let α_2 be the complex conjugate of α_1. We will consider the norm of α defined as $N(\alpha) = \alpha_1\alpha_2$. Following Gangl, Kaneko and Zagier [GKZ], we define a cone C_+ by

$$C_+ = \mathbb{N} \cup \{\alpha \in \mathcal{O}_K \mid \mathrm{Im}(\alpha_1) > 0\}.$$

Let also

$$C_- = \{-\alpha \mid \alpha \in C_+\}.$$

Then we have a decomposition of the ring of integers into a disjoint union

$$\mathcal{O}_K = C_+ \cup \{0\} \cup C_-.$$

We define a *Dedekind zeta value* to be

$$\zeta_K^1(a_1; b_2) = \sum_{\alpha \in C_+} \frac{1}{\alpha_1^{a_1} \alpha_2^{b_2}},$$

which converges for $a_1 \geq 1$, $b_2 \geq 1$ and $a_1 + b_2 \geq 3$. The proof is essentially the same as for multiple zeta values. Note that $\zeta_K^1(a; a)$ is equal to the classical Dedekind zeta value at $s = a$, where the summation is over all non-zero principal ideals in $\mathcal{O}_K$.

Definition 1.1. For an imaginary quadratic field K, with the above definition of the cone C_+, we define the following *multiple Dedekind zeta values*:

$$\zeta_K^1(a_1, b_1; c_2, d_2) = \sum_{\alpha, \beta \in C_+} \frac{1}{\alpha_1^{a_1}(\alpha_1 + \beta_1)^{b_1} \alpha_2^{c_2} (\alpha_2 + \beta_2)^{d_2}}$$

and

$$\zeta_K^\rho(a_1, b_1; c_2, d_2) = \sum_{\alpha, \beta \in C_+} \frac{1}{\beta_1^{a_1}(\alpha_1 + \beta_1)^{b_1} \alpha_2^{c_2} (\alpha_2 + \beta_2)^{d_2}}.$$

Note that ζ_K^ρ is obtained from ζ_K^1 by interchanging α_1 and β_1.

The series $\zeta_K^1(a_1, b_1; c_2, d_2)$ and $\zeta_K^\rho(a_1, b_1; c_2, d_2)$ converge when all the parameters are ≥ 1 and $b_1 + d_2 \geq 3$.

The infinite sum shuffle between two Dedekind zeta values of the type $\zeta_K^1(a; b)$ will be represented as a double Dedekind zeta value of the type ζ^1. And the integral shuffle relation will represent the product of two values of the type $\zeta_K^1(a, b)$ as a sum of double Dedekind zeta values of the types ζ^1 and ζ^ρ.

Let us also define

$$\zeta_K^1(a_1, b_1, c_1; d_2, e_2, f_2)$$
$$= \sum_{\alpha, \beta, \gamma \in C_+} \frac{1}{\alpha_1^{a_1}(\alpha_1 + \beta_1)^{b_1}(\alpha_1 + \beta_1 + \gamma_1)^{c_1} \alpha_2^{d_2}(\alpha_2 + \beta_2)^{e_2}(\alpha_2 + \beta_2 + \gamma_2)^{f_2}}.$$

Again we have convergence when all the parameters are ≥ 1 and $c_1 + f_2 \geq 3$.

Definition 1.2 (ordinary multiple Dedekind zeta values). Let $f : C_+^d \to C_+^m$ be an $\mathbb{N}$-linear map with coefficients in $\{0\} \cup \mathbb{N}$. Let $f = (f_1, \ldots, f_m)$ and let

$$\delta_1 = f_1(\alpha_1, \ldots, \alpha_d), \ \ldots, \ \delta_m = f_m(\alpha_1, \ldots, \alpha_d).$$

Then we define an *ordinary multiple Dedekind zeta value* as

$$\zeta_f(a_1, \ldots, a_m; b_1 \ldots, b_m) = \sum_{\alpha_1, \ldots, \alpha_d \in C_+;\, \delta_1 < \cdots < \delta_m} \frac{1}{\delta_1^{a_1} \ldots \delta_m^{a_m} \bar{\delta}_1^{b_1} \ldots \bar{\delta}_m^{b_m}}.$$

Definition 1.3 (permutational multiple Dedekind zeta values). Let $f = (f_1, \ldots, f_m) : C_+^d \to C_+^m$ be an $\mathbb{N}$-linear map with coefficients in $\{0\} \cup \mathbb{N}$ and $\delta_1 = f_1(\alpha_1, \ldots, \alpha_d), \ldots, \delta_m = f_m(\alpha_1, \ldots, \alpha_d)$, and let ρ be a permutation of m elements. Then we define a *permutational multiple Dedekind zeta value* as

$$\zeta_f^{\rho}(a_1, \ldots, a_m; b_1 \ldots, b_m) = \sum_{\alpha_1, \ldots, \alpha_d \in C_+} \frac{1}{\delta_1^{a_1} (\delta_1 + \delta_2)^{a_2} \ldots (\delta_1 + \cdots + \delta_m)^{a_m}} \times \frac{1}{\bar{\delta}_{\rho(1)}^{b_1} (\bar{\delta}_{\rho(1)} + \bar{\delta}_{\rho(2)})^{b_2} \ldots (\bar{\delta}_{\rho(1)} + \cdots + \bar{\delta}_{\rho(m)})^{b_m}}.$$

Remark 1.4. Both the ordinary multiple Dedekind zeta values and the permutational multiple Dedekind zeta values converge when all the parameters are ≥ 1 and $a_m + b_m \geq 3$. Also, $\zeta_f^{\mathrm{id}} = \zeta_f$, where id signifies the identity permutation.

2. Infinite sum shuffles. Shuffle relations are well known properties of multiple zeta values. Let us recall them as they are a prototype of shuffle relations for MDZVs.

The *zeta values* are defined by Euler as

$$\zeta(m) = \sum_{k=1}^{\infty} \frac{1}{k^m}.$$

Multiple zeta values are defined as

$$\zeta(m_1, \ldots, m_d) = \sum_{0 < k_1 < \cdots < k_d} \frac{1}{k_1^{m_1} \ldots k_d^{m_d}}.$$

For multiple zeta values we have the following sum shuffle relation:

$$\zeta(m_1)\zeta(m_2) = \sum_{k_1=1}^{\infty} \frac{1}{k_1^{m_1}} \sum_{k_2=1}^{\infty} \frac{1}{k_2^{m_2}} = \Big(\sum_{0<k_1<k_2} + \sum_{0<k_1=k_2} + \sum_{0<k_2<k_1} \Big) \frac{1}{k_1^{m_1} k_2^{m_2}}$$
$$= \zeta(m_1, m_2) + \zeta(m_1 + m_2) + \zeta(m_2, m_1).$$

Similarly, a product of a zeta value and a double zeta value can be expressed as a sum of multiple zeta values:

$$\begin{aligned}\zeta(m_1)\zeta(m_2,m_3) &= \sum_{k_1=1}^{\infty}\frac{1}{k_1^{m_1}}\sum_{k_2<k_3}^{\infty}\frac{1}{k_2^{m_2}k_3^{m_3}}\\ &= \Big(\sum_{0<k_1<k_2<k_3}+\sum_{0<k_1=k_2<k_3}+\sum_{0<k_2<k_1<k_3}\\ &\quad+\sum_{0<k_2<k_1=k_3}+\sum_{0<k_2<k_3<k_1}\Big)\frac{1}{k_1^{m_1}k_2^{m_2}k_3^{m_3}}\\ &= \zeta(m_1,m_2,m_3)+\zeta(m_1+m_2,m_3)+\zeta(m_2,m_1,m_3)\\ &\quad+\zeta(m_2,m_1+m_3)+\zeta(m_2,m_3,m_1).\end{aligned}$$

Now we are going to give examples of infinite sum shuffles for (multiple) DZVs.

For α and β in C_+, we write $\alpha<\beta$ if $\beta-\alpha$ is in C_+.

THEOREM 2.1. *With the above notation, we have the following infinite sum shuffle for Dedekind zeta values:*

$$\begin{aligned}\zeta_K^1(a_1;c_2)\zeta_K^1(b_1;d_2) &= \zeta_K^1(a_1,b_1;c_2,d_2)+\zeta_K^1(a_1+b_1;c_2+d_2)\\ &\quad+\zeta_K^1(b_1,a_1;d_2,c_2).\end{aligned}$$

The equality is valid when all of the parameters are ≥ 1, *and* $a_1+c_2\geq 3$ *and* $b_1+d_2\geq 3$. *In that case all of the series converge absolutely.*

Proof. We have

$$\begin{aligned}\zeta_K^1(a_1;c_2)\zeta_K^1(b_1;d_2) &= \sum_{\alpha,\beta\in C_+}\frac{1}{\alpha_1^{a_1}\alpha_2^{c_2}\beta_1^{b_1}\beta_2^{d_2}}\\ &= \Big(\sum_{\alpha<\beta}+\sum_{\alpha=\beta}+\sum_{\beta<\alpha}\Big)\frac{1}{\alpha_1^{a_1}\beta_1^{b_1}\alpha_2^{c_2}\beta_2^{d_2}}\\ &= \zeta_K^1(a_1,b_1;c_2,d_2)+\zeta_K^1(a_1+b_1;c_2+d_2)+\zeta_K^1(b_1,a_1;d_2,c_2).\ \blacksquare\end{aligned}$$

In particular, we obtain:

COROLLARY 2.2.

$$\zeta_K^1(2;2)\zeta_K^1(2;2)=\zeta_K^1(4;4)+2\zeta_K^1(2,2;2,2) \tag{2.1}$$

Similarly, we define the following infinite sum shuffle for MDZVs.

THEOREM 2.3.

$$\begin{aligned}\zeta_K^1(a_1,b_1;d_2,e_2)\zeta_K^1(c_1;f_2) &= \zeta_K^1(a_1,b_1,c_1;d_2,e_2,f_2)\\ &\quad+\zeta_K^1(a_1,b_1+c_1;d_2,e_2+f_2)+\zeta_K^1(a_1,c_1,b_1;d_2,f_2,e_2)\\ &\quad+\zeta_K^1(a_1+c_1,b_1;d_2+f_2,e_2)+\zeta_K^1(c_1,a_1,b_1;f_2,d_2,e_2).\end{aligned}$$

Proof. We have

$$\begin{aligned}
\zeta_K^1(a_1,b_1;d_2,e_2)\zeta_K^1(c_1;f_2) &= \sum_{\alpha,\beta,\gamma\in C_+;\,\alpha<\beta} \frac{1}{\alpha_1^{a_1}\beta_1^{b_1}\gamma_1^{c_1}\alpha_2^{d_2}\beta_2^{e_2}\gamma_2^{f_2}} \\
&= \Big(\sum_{\alpha<\beta<\gamma}+\sum_{\alpha<\beta=\gamma}+\sum_{\alpha<\gamma<\beta}+\sum_{\gamma=\alpha<\beta}+\sum_{\gamma<\alpha<\beta}\Big)\frac{1}{\alpha_1^{a_1}\beta_1^{b_1}\gamma_1^{c_1}\alpha_2^{d_2}\beta_2^{e_2}\gamma_2^{f_2}} \\
&= \zeta_K^1(a_1,b_1,c_1;d_2,e_2,f_2)+\zeta_K^1(a_1,b_1+c_1;d_2,e_2+f_2) \\
&\quad+\zeta_K^1(a_1,c_1,b_1;d_2,f_2,e_2)+\zeta_K^1(a_1+c_1,b_1;d_2+f_2,e_2) \\
&\quad+\zeta_K^1(c_1,a_1,b_1;f_2,d_2,e_2). \ \blacksquare
\end{aligned}$$

THEOREM 2.4 (Infinite sum shuffle for multiple Dedekind zeta values). *Given two ordinary MDZVs, we can express their product as a finite sum of ordinary MDZVs.*

Proof. Let $\zeta_f^1(a_1,\ldots,a_{d_1};b_1,\ldots,b_{d_1})$ and $\zeta_F^1(A_1,\ldots,A_{d_2};B_1,\ldots,B_{d_2})$ be two ordinary multiple Dedekind zeta values of depths d_1 and d_2, respectively, where

$$f: C_+^{e_1}\to C_+^{d_1} \quad\text{and}\quad F: C_+^{e_2}\to C_+^{d_2}$$

are $\mathbb{N}$-linear maps from Definition 1.2. Then

$$\zeta_f^1(a_1,\ldots,a_{d_1};b_1,\ldots,b_{d_1}) = \sum_{\alpha_1<\cdots<\alpha_{d_1}} \frac{1}{\alpha_1^{a_1}\ldots\alpha_{d_1}^{a_{d_1}}\bar{\alpha}_1^{b_1}\ldots\bar{\alpha}_{d_1}^{b_{d_1}}},$$

$$\zeta_F^1(A_1,\ldots,A_{d_2};B_1,\ldots,B_{d_1}) = \sum_{\alpha_{d_1+1}<\cdots<\alpha_{d_1+d_2}} \frac{1}{\alpha_{d_1+1}^{a_1}\ldots\alpha_{d_1+d_2}^{a_{d_1}}\bar{\alpha}_{d_1+1}^{b_1}\ldots\bar{\alpha}_{d_1+d_2}^{b_{d_1}}}.$$

The product of $\zeta_f^1(a_1,\ldots,a_{d_1};b_1,\ldots,b_{d_1})$ and $\zeta_F^1(A_1,\ldots,A_{d_2};B_1,\ldots,B_{d_2})$ can be written as a sum of ordinary multiple Dedekind zeta values, where the sum is over all possible ways of arranging the two chains of inequalities $\alpha_1<\cdots<\alpha_{d_1}$ and $\alpha_{d_1+1}<\cdots<\alpha_{d_1+d_2}$ into a single chain of inequalities. We call such an arrangement a *shuffle*. Note that such shuffles need not be strict, in the sense that the inequality signs between the numbers might not be strict; they could be "=" or "<". Let d, called *depth*, be the number of strict inequalities plus 1. Then the resulting ordinary multiple Dedekind zeta values can be of depth $\le d_1+d_2$. Note that the indices of the alphas are in the image of the maps f and F.

Consider one of the chains of inequalities. Suppose it has length $d < d_1+d_2$. Then certain pairs of indices, one from $C_+^{d_1}$ and the other from $C_+^{d_2}$, must coincide. In such instances we will consider them as an image coming from a common element of the source C_+^d. In this way we define a map $j: C_+^d \to C_+^{d_1}\times C_+^{d_2}$. We define $g: C_+^e \to C_+^d$ as the pull-back of the map

$(f,F) : C_+^{e_1} \times C_+^{e_2} \to C_+^{d_1} \times C_+^{d_2}$ with respect to the map j. Then ζ_g^1 has arguments a_k and A_l if $\alpha_k \neq \alpha_{d_1+l}$ and with $a_k + A_l$ if $\alpha_k = \alpha_{d_1+l}$ (and similarly for the indices b_k and B_l). The shuffle consists of all possible ways of obtaining a single chain of inequalities. There are a finite number of them. Moreover, for each of them we constructed an ordinary MDZV. That proves the theorem. ■

3. Integral shuffles

3.1. Classical polylogarithms. Let us recall the notion of *polylogarithm* and its relation to Riemann zeta values.

The first polylogarithm is defined as

$$\mathrm{Li}_1(x_1) = \int_0^{x_1} \frac{dx_0}{1-x_0} = \int_0^{x_1} (1 + x_0 + x_0^2 + \cdots)\, dx_0 = x_1 + \frac{x_1^2}{2} + \frac{x_1^3}{3} + \cdots,$$

and the mth polylogarithm is defined by iteration:

$$\mathrm{Li}_m(x_m) = \int_0^{x_m} \mathrm{Li}_{m-1}(x_{m-1}) \frac{dx_{m-1}}{x_{m-1}}. \tag{3.1}$$

Equation (3.1) is a representation of the mth polylogarithm as an iterated integral. By a direct computation it follows that

$$\mathrm{Li}_m(x) = x + \frac{x^2}{2^m} + \frac{x^3}{3^m} + \cdots,$$

and the relation

$$\zeta(m) = \mathrm{Li}_m(1)$$

is straightforward. Using (3.1), we can express the mth polylogarithm as

$$\mathrm{Li}_m(x) = \int_{0<x_1<\cdots<x_m<x} \frac{dx_1}{1-x_1} \wedge \frac{dx_2}{x_2} \wedge \cdots \wedge \frac{dx_m}{x_m}.$$

Let $x_i = e^{-t_i}$. Then the mth polylogarithm can be written in terms of the variables $t_1, \ldots, t_m$ in the following way:

$$\mathrm{Li}_m(e^{-t}) = \int_{t_1>t_2>\cdots>t_m>t} \frac{dt_1 \wedge \cdots \wedge dt_m}{e^{t_1}-1}. \tag{3.2}$$

This can be achieved, first, by changing variables in the differential forms

$$\frac{dx_1}{1-x_1} = \frac{d(-t_1)}{e^{t_1}-1} \quad \text{and} \quad \frac{dx_i}{x_i} = d(-t_i),$$

and second, by reversing the bounds of integration $0 < x_1 < \cdots < x_m < x$

to $t_1 > \cdots > t_m > t$, which absorbs the sign. We have the infinite sum expression

$$\mathrm{Li}_m(e^{-t}) = \sum_{n>0} \frac{e^{-nt}}{n^m}. \tag{3.3}$$

Formulas (3.2) and (3.3) will be generalized to Dedekind polylogarithms (see (3.14) and (3.15)).

Below we present similar formulas for multiple polylogarithms with exponential variables. We will construct their generalizations to Dedekind multiple polylogarithms (see Subsection 3.2, Lemma 3.2).

Let us recall the definition of the *double logarithm*:

$$\mathrm{Li}_{1,1}(1,x) = \int_0^x \mathrm{Li}_1(x_1)\,\frac{dx_1}{1-x_1} = \int_0^x \Big(\sum_{n_1=1}^{\infty} \frac{x_1^{n_1}}{n_1}\Big)\Big(\sum_{n_2=1}^{\infty} x_1^{n_2}\Big)\,\frac{dx_1}{x_1}$$
$$= \sum_{n_1,n_2=1}^{\infty} \frac{x^{n_1+n_2}}{n_1(n_1+n_2)}.$$

Let $x_1 = e^{-t_1}$ and $x = e^{-t}$. Then $\mathrm{Li}_{1,1}(1,e^{-t})$ can be written as an iterated integral in terms of the variables t_0, t_1, t in the following way:

$$\mathrm{Li}_{1,1}(1,e^{-t}) = \int_{t_0>t_1>t} \frac{dt_0 \wedge dt_1}{(e^{t_0}-1)(e^{t_1}-1)}.$$

We also have an infinite sum expression:

$$\mathrm{Li}_{1,1}(1,e^{-t}) = \sum_{n_1,n_2=1}^{\infty} \frac{e^{-(n_1+n_2)t}}{n_1(n_1+n_2)}. \tag{3.4}$$

An example of a multiple zeta value is

$$\zeta(1,2) = \sum_{n_1,n_2=1}^{\infty} \frac{1}{n_1(n_1+n_2)^2} = \int_0^1 \mathrm{Li}_{1,1}(x)\,\frac{dx}{x}.$$

Thus, an integral representation of $\zeta(1,2)$ is

$$\zeta(1,2) = \int_{t_1>t_2>t_3>0} \frac{dt_1}{(e^{t_1}-1)} \wedge \frac{dt_2}{(e^{t_2}-1)} \wedge dt_3. \tag{3.5}$$

Similarly,

$$\zeta(1,3) = \int_{t_1>t_2>t_3>t_4>0} \frac{dt_1}{(e^{t_1}-1)} \wedge \frac{dt_2}{(e^{t_2}-1)} \wedge dt_3 \wedge dt_4, \tag{3.6}$$

$$\zeta(2,2) = \int_{t_1>t_2>t_3>t_4>0} \frac{dt_1}{(e^{t_1}-1)} \wedge dt_2 \wedge \frac{dt_3}{(e^{t_3}-1)} \wedge dt_4. \tag{3.7}$$

An example of an integral shuffle relation is

$$
\begin{aligned}
(3.8)\qquad \zeta(2)\zeta(2) &= \int_{u_1>u_2>0} \frac{du_1}{(e^{u_1}-1)} \wedge du_2 \int_{t_1>t_2>0} \frac{dt_1}{(e^{t_1}-1)} \wedge dt_2 \\
&= \Bigg(\int_{u_1>u_2>t_1>t_2>0} + \int_{u_1>t_1>u_2>t_2>0} + \int_{u_1>t_1>t_2>u_2>0} + \int_{t_1>u_1>u_2>t_2>0} \\
&\quad + \int_{t_1>u_1>t_2>u_2>0} + \int_{t_1>t_2>u_1>u_2>0} \Bigg) \frac{du_1}{(e^{u_1}-1)} \wedge du_2 \wedge \frac{dt_1}{(e^{t_1}-1)} \wedge dt_2 \\
&= \zeta(2,2) + \zeta(1,3) + \zeta(1,3) + \zeta(1,3) + \zeta(1,3) + \zeta(2,2) \\
&= 2\zeta(2,2) + 4\zeta(1,3).
\end{aligned}
$$

From the infinite sum shuffle, we have

$$(3.9)\qquad \zeta(2)\zeta(2) = \zeta(4) + 2\zeta(2,2).$$

Therefore, from (3.8) and (3.9), we obtain the following relation between multiple zeta values:

$$(3.10)\qquad \zeta(1,3) = \tfrac{1}{4}\zeta(4).$$

3.2. Dedekind polylogarithms. A key part of this subsection is to find formulas analogous to (3.8) that provide an integral shuffle for Dedekind zeta values. In order to do so, we need to express (multiple) Dedekind zeta values as iterated integrals over membranes (see [H2] and [H3]). First, we recall analogues of (multiple) polylogarithms, which we call Dedekind (multiple) polylogarithms over imaginary quadratic fields (see (3.14), (3.15) and Lemma 3.2). We will denote by f_m the mth Dedekind polylogarithm, which will be an analogue of the mth polylogarithm $\mathrm{Li}_m(e^{-t})$ with an exponential variable. Each f_m will have an integral representation, resembling an iterated integral, and an infinite sum representation, resembling the classical Dedekind zeta values over a quadratic number field. We also draw diagrams in order to give a geometric image of the iterated integrals over membranes in dimension 2. We will give examples of multiple Dedekind zeta values (MDZV) over a quadratic number field, using the Dedekind (multiple) polylogarithms.

We will generalize equations (3.3) and (3.4) for (multiple) polylogarithms to their analogues over a quadratic number field. We first recall some properties and definitions related to quadratic extensions. For more information about quadratic fields, one may consider [IR].

In order to make the examples easier to follow, we use two sequences of inequalities

$$t_1 > u_1 > v_1 > w_1 \quad \text{and} \quad t_2 > u_2 > v_2 > w_2,$$

when we deal with a small number of iterations.

Let $K = \mathbb{Q}(\sqrt{-d})$. The elements of K are of the form $p + q\sqrt{-d}$, where p and q are rational numbers. Let $\mathcal{O}_K$ be the ring of algebraic integers in K. If d is square free and $d \equiv 1$ or $2 \bmod 4$, then $\mathcal{O}_K$ consists of elements of the form $a + b\sqrt{-d}$, where a and b are integers. If $d \equiv -1 \bmod 4$, then $\mathcal{O}_K$ consists of elements of the form $a + b(1 + \sqrt{-d})/2$, where a and b are integers.

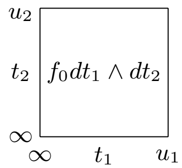
We are going to define a function f_1, which will be an analogue of $\mathrm{Li}_1(e^{-t})$. Recall that $C_+ = \mathbb{N} \cup \{\alpha \in \mathcal{O}_K \mid \mathrm{Im}(\alpha) > 0\}$. Let

$$f_0(C_+; t_1, t_2) = \sum_{\alpha \in C_+} \exp(-\alpha_1 t_1 - \alpha_2 t_2). \tag{3.11}$$

We define

$$f_1(C_+; u_1, u_2) = \int_\infty^{u_1}\int_\infty^{u_2} f_0(C_+; t_1, t_2)\, dt_1 \wedge dt_2. \tag{3.12}$$

For this integral we can draw the following diagram:



The diagram represents that the integrand is $f_0(C_+; t_1, t_2)\, dt_1 \wedge dt_2$, depending on the variables t_1 and t_2, subject to the restrictions $\infty > t_1 > u_1$ and $\infty > t_2 > u_2$.

Remark. Here the diagram may seem superfluous, but further we have much more complicated integrals that are easily explained by similar diagrams.

In order to obtain explicit formulas for iterated integrals, we need:

Lemma 3.1.

(a)
$$\int_\infty^{u} e^{-kt}\, dt = \frac{e^{-ku}}{k}.$$

(b) *Let* $N(\alpha) = \alpha_1\alpha_2$. *Then*

$$\int_\infty^{u_1}\int_\infty^{u_2} \exp(-\alpha_1 t_1 - \alpha_2 t_2)\, dt_1 \wedge dt_2 = \frac{\exp(-\alpha_1 u_1 - \alpha_2 u_2)}{N(\alpha)}.$$

The proof is straightforward.

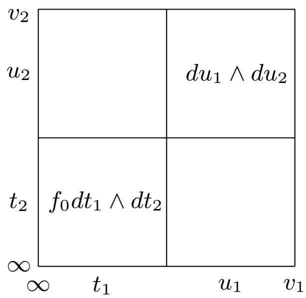
Using the above lemma, we obtain

$$f_1(C_+; u_1, u_2) = \sum_{\alpha \in C_+} \frac{\exp(-\alpha_1 u_1 - \alpha_2 u_2)}{N(\alpha)}.$$

We define a *Dedekind dilogarithm* f_2 by

$$f_2(C_+; v_1, v_2) = \int_{\infty}^{v_1}\int_{\infty}^{v_2} f_1(C_+; u_1, u_2)\, du_1 \wedge du_2 \tag{3.13}$$
$$= \int_{t_1>u_1>v_1;\, t_2>u_2>v_2} f_0(C_+; t_1, t_2)\, dt_1 \wedge dt_2 \wedge du_1 \wedge du_2.$$

We can also associate a diagram to this integral representation:



The diagram represents that the variables under the integral are t_1, t_2, u_1, u_2, subject to the conditions $\infty > t_1 > u_1 > v_1$ and $\infty > t_2 > u_2 > v_2$. Also, the function f_0 in the diagram depends on the variables t_1 and t_2.

Similarly to (3.1), we inductively define the mth Dedekind polylogarithm over a quadratic number field by

$$f_m(C_+; u_1, u_2) = \int_{\infty}^{u_1}\int_{\infty}^{u_1} f_{m-1}(C_+; t_1, t_2)\, dt_1 \wedge dt_2. \tag{3.14}$$

The above integral is the key example of an iterated integral over a membrane. For more examples and properties, see [H3].

From (3.14), we can derive an analogue of the infinite sum representation (3.3):

$$f_m(C_+; u_1, u_2) = \sum_{\alpha \in C_+} \frac{\exp(-\alpha u_1 - \overline{\alpha} u_2)}{N(\alpha)^m}. \tag{3.15}$$

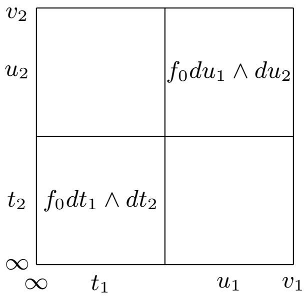
Now we can define an analogue of the double logarithm $\mathrm{Li}_{1,1}(1, e^{-t})$ over an imaginary quadratic field:

$$f_{1,1}(C_+; v_1, v_2) = \int_{\infty}^{v_1}\int_{\infty}^{v_2} f_1(C_+; u_1, u_2) f_0(C_+; u_1, u_2)\, du_1 \wedge du_2, \tag{3.16}$$

called a *Dedekind double logarithm.* As an analogue for (3.13), we can express $f_{1,1}$ only in terms of f_0 by

$$f_{1,1}(C_+;v_1,v_2) = \int\limits_{t_1>u_1>v_1;\, t_2>u_2>v_2} (f_0(C_+;t_1,t_2)\,dt_1\wedge dt_2)\wedge(f_0(C_+;u_1,u_2)\,du_1\wedge du_2).$$

This allows us to associate a diagram to $f_{1,1}$:



The variables t_1, t_2, u_1, u_2 in the diagram are the variables in the integrand. They are subject to the conditions $t_1 > u_1 > v_1$ and $t_2 > u_2 > v_2$. Also, the lower left function f_0 in the diagram depends on the variables t_1 and t_2, and the upper right f_0 depends on u_1 and u_2.

The similarity between $f_{1,1}(C_+;v_1,v_2)$ and $\mathrm{Li}_{1,1}(1,e^{-t})$ can be noticed in the infinite sum representation:

Lemma 3.2.

$$f_{1,1}(C_+;v_1,v_2) = \sum_{\alpha,\beta\in C_+} \frac{\exp(-(\alpha+\beta)v_1 - (\overline{\alpha}+\overline{\beta})v_2)}{N(\alpha)N(\alpha+\beta)}.$$

Proof. We have

$$\begin{aligned} f_{1,1}(C_+;v_1,v_2) &= \int_\infty^{v_1}\int_\infty^{v_2} f_1(C_+;u_1,u_2)f_0(C_+;u_1,u_2)\,du_1\wedge du_2 \\ &= \int_\infty^{v_1}\int_\infty^{v_2} \sum_{\alpha\in C_+}\frac{\exp(-\alpha_1u_1-\alpha_2u_2)}{N(\alpha)} \sum_{\beta\in C_+}\exp(-\beta_1u_1-\beta_2u_2)\,du_1\wedge du_2 \\ &= \int_\infty^{v_1}\int_\infty^{v_2} \sum_{\alpha,\beta\in C_+}\frac{\exp(-(\alpha_1+\beta_1)u_1-(\alpha_2+\beta_2)u_2)}{N(\alpha)}\,du_1\wedge du_2 \\ &= \sum_{\alpha,\beta\in C_+}\frac{\exp(-(\alpha_1+\beta_1)v_1-(\alpha_2+\beta_2)v_2)}{N(\alpha)N(\alpha+\beta)}. \ \blacksquare \end{aligned}$$

3.3. Examples of integral shuffles. In this subsection, we consider several examples of shuffle relations based on iterated integrals over membranes.

Let C_1 and C_2 be two cones. For example,

$$C_1 = \mathbb{N}\{1, 1+i\} = \{a + b(1+i) \mid a, b \in \mathbb{N}\},$$
$$C_2 = \mathbb{N}\{1, 1-i\} = \{a + b(1-i) \mid a, b \in \mathbb{N}\}.$$

Similarly to (3.11) and (3.12), we define

$$f_0(C_1; t_1, t_2) = \sum_{\alpha \in C_1} \exp(-\alpha_1 t_1 - \alpha_2 t_2),$$
$$f_1(C_1; u_1, u_2) = \int_{\infty}^{u_1} \int_{\infty}^{u_2} f_0(C_1; t_1, t_2)\, dt_1 \wedge dt_2.$$

We define two types of Dedekind double logarithms in order to present an integral shuffle relation in a simpler setting:

$$f_{1,1}(C_1, C_2; v_1, v_2) = \int_{\boldsymbol{t_1} > \boldsymbol{u_1} > v_1} \int_{t_2 > u_2 > v_2} f_0(C_1; t_1, t_2)\, dt_1 \wedge dt_2 \wedge f_0(C_2; u_1, u_2)\, du_1 \wedge du_2,$$
$$f^{\rho}_{1,1}(C_1, C_2; v_1, v_2) = \int_{\boldsymbol{u_1} > \boldsymbol{t_1} > v_1} \int_{t_2 > u_2 > v_2} f_0(C_1; t_1, u_2)\, dt_1 \wedge dt_2 \wedge f_0(C_2; u_1, t_2)\, du_1 \wedge du_2.$$

In particular, $f_{1,1}(C_+, C_+; v_1, v_2) = f_{1,1}(C_+; v_1, v_2)$, where the second $f_{1,1}$ is from (3.16).

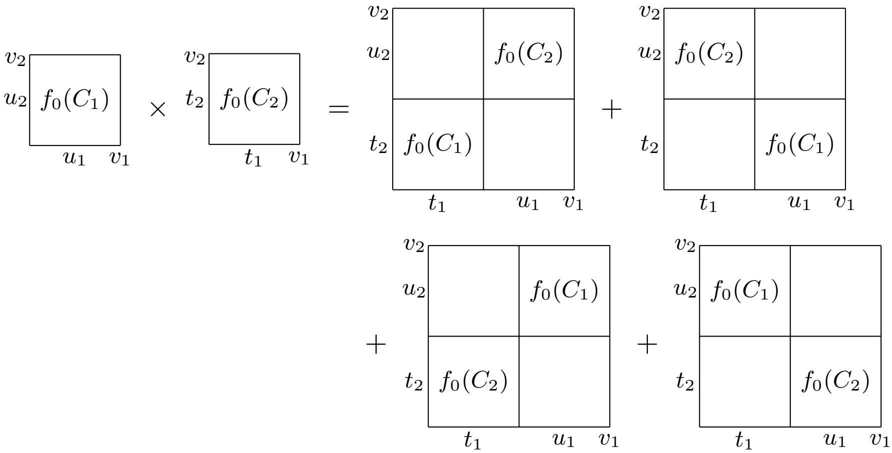
THEOREM 3.3. (Shuffle relation for the Dedekind (poly-)logarithms $f_1(C_1; v_1, v_2)$ and $f_1(C_2; v_1, v_2)$).

$$f_1(C_1; v_1, v_2) f_1(C_2; v_1, v_2) = f_{1,1}(C_1, C_2; v_1, v_2) + f^{\rho}_{1,1}(C_1, C_2; v_1, v_2) + f_{1,1}(C_2, C_1; v_1, v_2) + f^{\rho}_{1,1}(C_2, C_1; v_1, v_2). \tag{3.17}$$

Proof. We have

$$\begin{aligned} f_1(C_1; v_1, v_2) f_1(C_2; v_1, v_2) &= \int_{\infty}^{v_1} \int_{\infty}^{v_2} f_0(C_1; u_1, u_2)\, du_1 \wedge du_2 \int_{\infty}^{v_1} \int_{\infty}^{v_2} f_0(C_2; t_1, t_2)\, dt_1 \wedge dt_2 \\ &= \Big(\int_{u_1 > t_1 > v_1} + \int_{t_1 > u_1 > v_1} \Big) \Big(\int_{u_2 > t_2 > v_2} + \int_{t_2 > u_2 > v_2} \Big) \\ &\qquad f_0(C_1; u_1, u_2)\, du_1 \wedge du_2 \wedge f_0(C_2; t_1, t_2)\, dt_1 \wedge dt_2 \\ &= f_{1,1}(C_1, C_2; v_1, v_2) + f^{\rho}_{1,1}(C_1, C_2; v_1, v_2) + f_{1,1}(C_2, C_1; v_1, v_2) \\ &\quad + f^{\rho}_{1,1}(C_2, C_1; v_1, v_2). \ \blacksquare \end{aligned}$$

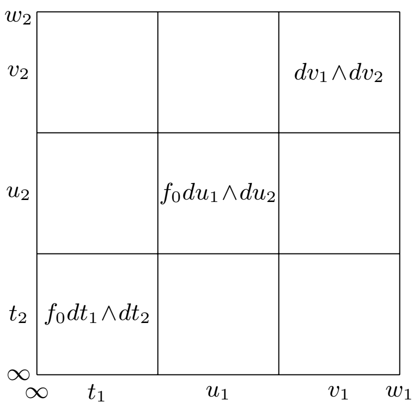
In terms of diagrams, the shuffle relation can be expressed as



Similarly to $f_{1,1}$, we define a *multiple Dedekind polylogarithm*

$$f_{1,2}(C;w_1,w_2) = \int_{\infty}^{w_1}\int_{\infty}^{w_2} f_{1,1}(C;v_1,v_2)\,dv_1\wedge dv_2.$$

We can associate the following diagram to it:



The diagram represents the following: The variables of the integrand are t_1, t_2, u_1, u_2, v_1, v_2, subject to the conditions $t_1 > u_1 > v_1 > w_1$ and $t_2 > u_2 > v_2 > w_2$. The lower left f_0 depends on t_1 and t_2, and the middle f_0 depends on u_1 and u_2. The upper right 2-form is $dv_1 \wedge dv_2$. Thus, the diagram represents the following integral:

$$(3.18)\qquad f_{1,2}(C_+;w_1,w_2) = \int_{D_{w_1,w_2}} (f_0(C_+;t_1,t_2)\,dt_1\wedge dt_2)\wedge(f_0(C_+;u_1,u_2)\,du_1\wedge du_2)\wedge(dv_1\wedge dv_2),$$

where the domain of integration is

$$D_{w_1,w_2} = \{(t_1,t_2,u_1,u_2,v_1,v_2) \in \mathbb{R}^6 \mid t_1 > u_1 > v_1 > w_1 \text{ and } t_2 > u_2 > v_2 > w_2\}.$$

A direct computation leads to

$$f_{1,2}(C_+;w_1,w_2) = \sum_{\alpha,\beta\in C_+} \frac{\exp(-(\alpha+\beta)w_1 - (\overline{\alpha}+\overline{\beta})w_2)}{N(\alpha)N(\alpha+\beta)^2}. \tag{3.19}$$

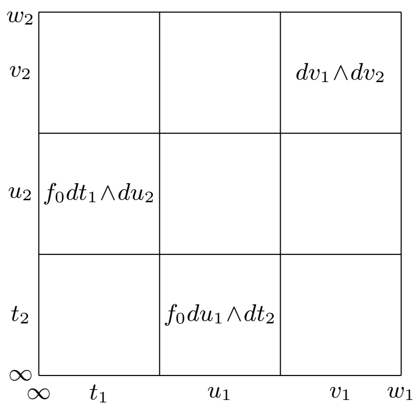
Using (3.19), we define the corresponding multiple Dedekind zeta value as

$$\zeta_K^1(1,2;1,2) = f_{1,2}(C_+;0,0) = \sum_{\alpha,\beta\in C_+} \frac{1}{N(\alpha)N(\alpha+\beta)^2}.$$

Similarly to $f_{1,2}$, we define a multiple Dedekind polylogarithm

$$f_{1,2}^{(1,2)(1)}(C_+;w_1,w_2) = \int_{\infty}^{w_1}\int_{\infty}^{w_2} f_{1,1}^{(1,2),(1)}(C_+,C_+;v_1,v_2)\,dv_1\wedge dv_2. \tag{3.20}$$

This definition is needed for the integral shuffle relation for multiple Dedekind zeta values. We can associate the following diagram to (3.20):



The diagram represents the following: The variables of the integrand are t_1, t_2, u_1, u_2, v_1, v_2, subject to the conditions $t_1 > u_1 > v_1 > w_1$ and $t_2 > u_2 > v_2 > w_2$. One of the functions f_0 depends on u_1 and t_2, and the other on t_1 and u_2.

From the diagram we obtain the following integral, which is more useful for writing explicit formulas:

$$\begin{aligned} &f_{1,2}^{(1,2),(1)}(C_+;w_1,w_2) \\ &= \int_{D_{w_1,w_2}} (f_0(C_+;t_1,t_2)\,dt_1\wedge dt_2)\wedge(f_0(C_+;u_1,u_2)\,du_1\wedge du_2)\wedge(dv_1\wedge dv_2), \end{aligned} \tag{3.21}$$

where the domain of integration is

$D_{w_1,w_2} =$

$$\{(t_1, t_2, u_1, u_2, v_1, v_2) \in \mathbb{R}^6 \mid \boldsymbol{u_1} > \boldsymbol{t_1} > v_1 > w_1 \text{ and } t_2 > u_2 > v_2 > w_2\}.$$

A direct computation leads to

$$f_{1,2}^{(1,2),(1)}(C_+; w_1, w_2) = \sum_{\alpha,\beta \in C_+} \frac{\exp(-(\alpha+\beta)w_1 - (\overline{\alpha}+\overline{\beta})w_2)}{\beta_1 \alpha_2 N(\alpha+\beta)^2}.$$

We use it to define the corresponding permutational multiple Dedekind zeta value

$$\zeta_K^{(1,2)}(1,2;1,2) = f_{1,2}^{(1,2),(1)}(C_+; 0, 0) = \sum_{\alpha,\beta \in C_+} \frac{1}{\beta_1(\alpha_1+\beta_1)^2 \alpha_2(\alpha_2+\beta_2)^2}.$$

On pages 218–219, we will consider 36 similar diagrams. Then the above diagram, with $w_1 = 0$ and $w_2 = 0$, associated to $\zeta_K^{(1,2)}(1,2;1,2)$, will be denoted by

	t_1	u_1	v_1
v_2			0
t_2	1		
u_2		1	

or simply

		0
1		
	1	

The 1's signify $f_0(t_1, t_2)dt_1 \wedge dt_2$ or $f_0(u_1, u_2)du_1 \wedge du_2$. The 0 signifies the differential form $dv_1 \wedge dv_2$. Also, for the variables t_1, u_1, v_1 we have $t_1 > u_1 > v_1 > 0$, and similarly $u_2 > t_2 > v_2 > 0$ for the variables left of the box.

Compare the above diagram with the previous one. They represent the same integrals, when $w_1 = 0$ and $w_2 = 0$.

Now, we are going to consider the particular example of an integral shuffle relation for the product $\zeta_{K,C_+}(2)\zeta_{K,C_+}(2)$.

Theorem 3.4 (Example of integral shuffle). *We can express the above product of Dedekind zeta values as a sum of permutational MDZVs:*

$$\begin{aligned}(3.22)\qquad \zeta_{K;C_+}(2)\zeta_{K;C_+}(2) &= 2\zeta^1_{K;C_+}(2,2;2,2) + 8\zeta^1_{K;C_+}(1,3;1,3)\\ &+ 4\zeta^1_{K;C_+}(1,3;2,2) + 4\zeta^1_{K;C_+}(2,2;1,3) + 2\zeta^\rho_{K;C_+}(2,2;2,2)\\ &+ 8\zeta^\rho_{K;C_+}(1,3;1,3) + 4\zeta^\rho_{K;C_+}(1,3;2,2) + 4\zeta^\rho_{K;C_+}(2,2;1,3).\end{aligned}$$

Proof. We recall the definition $C_+ = \mathbb{N} \cup \{\alpha \mid \mathrm{Im}(\alpha_1) > 0\}$. Due to convergence issues, first we consider the exponents $\exp(-\alpha_1 t_1 - \alpha_2 t_2) \times \exp(-\beta_1 u_1 - \beta_2 u_2)$. Then we use iterated integrals over membranes. And finally we sum over elements α of the cone C_+. The sums of such integrals converge for imaginary quadratic fields.

The shuffle is done in the following way. The variables $t_1 > u_1 > 0$ are shuffled with $v_1 > w_1 > 0$. For this, there are six possibilities. Then the variables $t_2 > u_2 > 0$ are shuffled with $v_2 > w_2 > 0$. Shuffling simultaneously among the variables t_1, u_1, v_1, w_1 with index 1 and among the variables t_2, u_2, v_2, w_2 with index 2, we obtain 6^2 possibilities. When the variables with index 1 are shuffled, we have two summations of the type $\frac{1}{\alpha_1^2(\alpha_1+\beta_1)^2} \times \cdots$ and four summations of the type $\frac{1}{\alpha_1(\alpha_1+\beta_1)^3} \times \cdots$. We obtain similar terms when we shuffle the variables with index 2, namely, two summations of the type $\frac{1}{\alpha_2^2(\alpha_2+\beta_2)^2} \times \cdots$ and four summations of the type $\frac{1}{\alpha_2(\alpha_2+\beta_2)^3} \times \cdots$. We obtain

$$
\begin{aligned}
(3.23)\quad & \zeta_{K,C_+}(2)\zeta_{K,C_+}(2) \\
&= \int_{t_1>u_1>0;\, t_2>u_2>0} f_0(C_+; t_1, t_2)\, dt_1 \wedge dt_2 \wedge du_1 \wedge du_2 \\
&\qquad \times \int_{v_1>w_1>0;\, v_2>w_2>0} f_0(C_+; v_1, v_2)\, dv_1 \wedge dv_2 \wedge dw_1 \wedge dw_2 \\
&= 2 \sum_{\alpha,\beta\in C_+} \frac{1}{N(\alpha)^2 N(\alpha+\beta)^2} + 8 \sum_{\alpha,\beta\in C_+} \frac{1}{N(\alpha) N(\alpha+\beta)^3} \\
&\quad + 4 \sum_{\alpha,\beta\in C_+} \frac{1}{\alpha_1^1(\alpha_1+\beta_1)^3 \alpha_2^2(\alpha_2+\beta_2)^2} + 4 \sum_{\alpha,\beta\in C_+} \frac{1}{\alpha_1^2(\alpha_1+\beta_1)^2 \alpha_2^1(\alpha_2+\beta_2)^3} \\
&\quad + 2 \sum_{\alpha,\beta\in C_+} \frac{1}{\alpha_1^2 \beta_2^2 N(\alpha+\beta)^2} + 8 \sum_{\alpha,\beta\in C_+} \frac{1}{\alpha_1 \beta_2 N(\alpha+\beta)^3} \\
&\quad + 4 \sum_{\alpha,\beta\in C_+} \frac{1}{\alpha_1^2(\alpha_1+\beta_1)^2 \beta_2^1(\alpha_2+\beta_2)^3} + 4 \sum_{\alpha,\beta\in C_+} \frac{1}{\alpha_1^1(\alpha_1+\beta_1)^3 \beta_2^2(\alpha_2+\beta_2)^2} \\
&= 2\zeta^1_{K;C_+}(2,2;2,2) + 8\zeta^1_{K;C_+}(1,3;1,3) + 4\zeta^1_{K;C_+}(1,3;2,2) + 4\zeta^1_{K;C_+}(2,2;1,3) \\
&\quad + 2\zeta^\rho_{K;C_+}(2,2;2,2) + 8\zeta^\rho_{K;C_+}(1,3;1,3) + 4\zeta^\rho_{K;C_+}(1,3;2,2) \\
&\quad + 4\zeta^\rho_{K;C_+}(2,2;1,3).
\end{aligned}
$$

The following 36 diagrams reprepresent all the possible shuffles among the variables $t_1 > u_1 > 0$ and $v_1 > w_1 > 0$, and among $t_2 > u_2 > 0$ and $v_2 > w_2 > 0$. The diagrams show all possibilities of how these variables can be ordered respecting the inequalities $t_1 > u_1 > 0$, $v_1 > w_1 > 0$, $t_2 > u_2 > 0$ and $v_2 > w_2 > 0$. In the boxes, the number 1 with coordinates (t_1, t_2) and (v_1, v_2) corresponds to the 2-forms $f_0(t_1, t_2)dt_1 \wedge dt_2$ and $f_0(v_1, v_2)dv_1 \wedge dv_2$ under the integral. Also, the occurrence of 0 in the boxes with coordinates (u_1, u_2) and (w_1, w_2) signifies that under the integral we have the forms $du_1 \wedge du_2$ and $dw_1 \wedge dw_2$, respectively. In each diagram, the variables in horizontal and in vertical direction are arranged in decreasing order.

Note that a diagram itself stands for an integral representation of one summand of a permutational multiple Dedekind zeta value. The summand is given below each diagram.

Table $(1, 1)$

w_2				0
v_2			1	
u_2		0		
t_2	1			
	t_1	u_1	v_1	w_1

$\frac{1}{N(\alpha)^2 N(\alpha+\beta)^2}$

w_2				0
v_2		1		
u_2			0	
t_2	1			
	t_1	v_1	u_1	w_1

$\frac{1}{\alpha_1^1(\alpha_1+\beta_1)^3\alpha_2^2(\alpha_2+\beta_2)^2}$

w_2			0	
v_2		1		
u_2				0
t_2	1			
	t_1	v_1	w_1	u_1

$\frac{1}{\alpha_1^1(\alpha_1+\beta_1)^3\alpha_2^2(\alpha_2+\beta_2)^2}$

w_2				0
u_2		0		
v_2			1	
t_2	1			
	t_1	u_1	v_1	w_1

$\frac{1}{\alpha_1^2(\alpha_1+\beta_1)^2\alpha_2^1(\alpha_2+\beta_2)^3}$

w_2				0
u_2			0	
v_2		1		
t_2	1			
	t_1	v_1	u_1	w_1

$\frac{1}{N(\alpha)^1 N(\alpha+\beta)^3}$

w_2			0	
u_2				0
v_2		1		
t_2	1			
	t_1	v_1	w_1	u_1

$\frac{1}{N(\alpha)^1 N(\alpha+\beta)^3}$

u_2		0		
w_2				0
v_2			1	
t_2	1			
	t_1	u_1	v_1	w_1

$\frac{1}{\alpha_1^2(\alpha_1+\beta_1)^2\alpha_2^1(\alpha_2+\beta_2)^3}$

u_2			0	
w_2				0
v_2		1		
t_2	1			
	t_1	v_1	u_1	w_1

$\frac{1}{N(\alpha)^1 N(\alpha+\beta)^3}$

u_2				0
w_2			0	
v_2		1		
t_2	1			
	t_1	v_1	w_1	u_1

$\frac{1}{N(\alpha)^1 N(\alpha+\beta)^3}$

Table $(1, 2)$

w_2				0
v_2	1			
u_2			0	
t_2		1		
	v_1	t_1	u_1	w_1

$\frac{1}{\beta_1^1(\alpha_1+\beta_1)^3\alpha_2^2(\alpha_2+\beta_2)^2}$

w_2			0	
v_2	1			
u_2				0
t_2		1		
	v_1	t_1	w_1	u_1

$\frac{1}{\beta_1^1(\alpha_1+\beta_1)^3\alpha_2^2(\alpha_2+\beta_2)^2}$

w_2		0		
v_2	1			
u_2				0
t_2			1	
	v_1	w_1	t_1	u_1

$\frac{1}{\beta_1^2(\alpha_1+\beta_1)^2\alpha_2^2(\alpha_2+\beta_2)^2}$

w_2				0
u_2			0	
v_2	1			
t_2		1		
	v_1	t_1	u_1	w_1

$\frac{1}{\beta_1^1(\alpha_1+\beta_1)^3\alpha_2^1(\alpha_2+\beta_2)^3}$

w_2			0	
u_2				0
v_2	1			
t_2		1		
	v_1	t_1	w_1	u_1

$\frac{1}{\beta_1^1(\alpha_1+\beta_1)^3\alpha_2^1(\alpha_2+\beta_2)^3}$

w_2		0		
u_2				0
v_2	1			
t_2			1	
	v_1	w_1	t_1	u_1

$\frac{1}{\beta_1^2(\alpha_1+\beta_1)^2\alpha_2^1(\alpha_2+\beta_2)^3}$

u_2			0	
w_2				0
v_2	1			
t_2		1		
	v_1	t_1	u_1	w_1

$\frac{1}{\beta_1^1(\alpha_1+\beta_1)^3\alpha_2^1(\alpha_2+\beta_2)^3}$

u_2				0
w_2			0	
v_2	1			
t_2		1		
	v_1	t_1	w_1	u_1

$\frac{1}{\beta_1^1(\alpha_1+\beta_1)^3\alpha_2^1(\alpha_2+\beta_2)^3}$

u_2				0
w_2		0		
v_2	1			
t_2			1	
	v_1	w_1	t_1	u_1

$\frac{1}{\beta_1^2(\alpha_1+\beta_1)^2\alpha_2^1(\alpha_2+\beta_2)^3}$

Table (2, 1)

w_2				0
u_2		0		
t_2	1			
v_2			1	
	t_1	u_1	v_1	w_1

$$\frac{1}{\beta_1^2(\alpha_1+\beta_1)^2\alpha_2^1(\alpha_2+\beta_2)^3}$$

w_2				0
u_2			0	
t_2	1			
v_2		1		
	t_1	v_1	u_1	w_1

$$\frac{1}{\beta_1^1(\alpha_1+\beta_1)^3\alpha_2^1(\alpha_2+\beta_2)^3}$$

w_2			0	
u_2				0
t_2	1			
v_2		1		
	t_1	v_1	w_1	u_1

$$\frac{1}{\beta_1^1(\alpha_1+\beta_1)^3\alpha_2^1(\alpha_2+\beta_2)^3}$$

u_2		0		
w_2				0
t_2	1			
v_2			1	
	t_1	u_1	v_1	w_1

$$\frac{1}{\alpha_1^2(\alpha_1+\beta_1)^2\beta_2^1(\alpha_2+\beta_2)^3}$$

u_2			0	
w_2				0
t_2	1			
v_2		1		
	t_1	v_1	u_1	w_1

$$\frac{1}{\beta_1^1(\alpha_1+\beta_1)^3\alpha_2^1(\alpha_2+\beta_2)^3}$$

u_2				0
w_2			0	
t_2	1			
v_2		1		
	t_1	v_1	w_1	u_1

$$\frac{1}{\beta_1^1(\alpha_1+\beta_1)^3\alpha_2^1(\alpha_2+\beta_2)^3}$$

u_2		0		
t_2	1			
w_2				0
v_2			1	
	t_1	u_1	v_1	w_1

$$\frac{1}{\alpha_1^2\beta_2^2N(\alpha+\beta)^2}$$

u_2			0	
t_2	1			
w_2				0
v_2		1		
	t_1	v_1	u_1	w_1

$$\frac{1}{\alpha_1^1(\alpha_1+\beta_1)^3\beta_2^2(\alpha_2+\beta_2)^2}$$

u_2				0
t_2	1			
w_2			0	
v_2		1		
	t_1	v_1	w_1	u_1

$$\frac{1}{\alpha_1^1(\alpha_1+\beta_1)^3\beta_2^2(\alpha_2+\beta_2)^2}$$

Table (2, 2)

w_2				0
u_2			0	
t_2		1		
v_2	1			
	v_1	t_1	u_1	w_1

$$\frac{1}{N(\beta)^1N(\alpha+\beta)^3}$$

w_2			0	
u_2				0
t_2		1		
v_2	1			
	v_1	t_1	w_1	u_1

$$\frac{1}{N(\beta)^1N(\alpha+\beta)^3}$$

w_2		0		
u_2				0
t_2			1	
v_2	1			
	v_1	w_1	t_1	u_1

$$\frac{1}{\alpha_1^2(\alpha_1+\beta_1)^2\alpha_2^1(\alpha_2+\beta_2)^3}$$

u_2			0	
w_2				0
t_2		1		
v_2	1			
	v_1	t_1	u_1	w_1

$$\frac{1}{N(\beta)^1N(\alpha+\beta)^3}$$

u_2				0
w_2			0	
t_2		1		
v_2	1			
	v_1	t_1	w_1	u_1

$$\frac{1}{N(\beta)^1N(\alpha+\beta)^3}$$

u_2				0
w_2		0		
t_2			1	
v_2	1			
	v_1	w_1	t_1	u_1

$$\frac{1}{\alpha_1^2(\alpha_1+\beta_1)^2\alpha_2^1(\alpha_2+\beta_2)^3}$$

u_2			0	
t_2		1		
w_2				0
v_2	1			
	v_1	t_1	u_1	w_1

$$\frac{1}{\alpha_1^1(\alpha_1+\beta_1)^3\alpha_2^2(\alpha_2+\beta_2)^2}$$

u_2				0
t_2		1		
w_2			0	
v_2	1			
	v_1	t_1	w_1	u_1

$$\frac{1}{\alpha_1^1(\alpha_1+\beta_1)^3\alpha_2^2(\alpha_2+\beta_2)^2}$$

u_2				0
t_2			1	
w_2		0		
v_2	1			
	v_1	w_1	t_1	u_1

$$\frac{1}{N(\beta)^2N(\alpha+\beta)^2} \qquad \blacksquare$$

3.4. General integral shuffles. Recall that an ordinary multiple Dedekind zeta value is

$$\zeta(a_1,\ldots,a_d;b_1,\ldots,b_d)=\sum_{\alpha_1<\cdots<\alpha_d}\frac{1}{\alpha_1^{a_1}\ldots\alpha_d^{a_d}\bar{\alpha}_1^{b_1}\ldots\bar{\alpha}_d^{b_d}}.$$

In order to represent such a value as an iterated integral over a membrane we consider a square block diagram with non-zero blocks on the diagonal and zero blocks off the diagonal. For now assume that all $a_1,\ldots,a_d,b_1,\ldots,b_d$ are non-zero positive integers. In that case consider a diagonal consisting of d blocks, so that the ith block is of size $a_i\times b_i$ and it has the following entries:

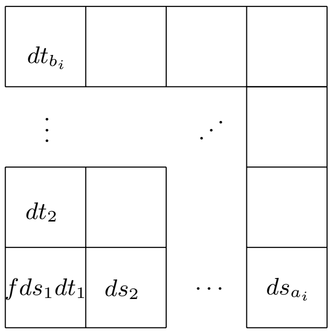


Fig. 1

Here the box is of size $a_i\times b_i$ and $f=\exp(-\beta_i s_1-\bar{\beta}_i t_1)$. Now we describe the degenerate cases.

Let $a_i>0$ and $b_i=0$. Then the box is degenerate of size $a_i\times 0$. The corresponding diagram is

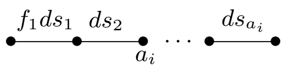


Fig. 2

Here $f_1=\exp(-\beta_i s_1)$.

Let $a_i=0$ and $b_i>0$. Then the box is degenerate of size $0\times b_i$. The corresponding diagram is

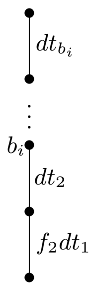


Fig. 3

Here $f_2=\exp(-\bar{\beta}_i t_2)$.

When we consider the integral associated to the diagram consisting of the above d boxes on the diagonal, we obtain

$$\frac{1}{\beta_1^{a_1}(\beta_1+\beta_2)^{a_2}\dots(\beta_1+\dots+\beta_d)^{a_d}\bar{\beta}_1^{b_1}(\bar{\beta}_1+\bar{\beta}_2)^{b_2}\dots(\bar{\beta}_1+\dots+\bar{\beta}_d)^{b_d}}.$$

If we substitute

$$\begin{aligned}\alpha_1 &= \beta_1,\\ \alpha_2 &= \beta_1+\beta_2,\\ &\vdots\\ \alpha_d &= \beta_1+\dots+\beta_d,\end{aligned}$$

we find that the integral is equal to

$$\frac{1}{\alpha_1^{a_1}\alpha_2^{a_2}\dots\alpha_d^{a_d}\bar{\alpha}_1^{b_1}\bar{\alpha}_2^{b_2}\dots\bar{\alpha}_d^{b_d}}.$$

Note that Fig. 1 and Fig. 4 below give the same summand of a multiple Dedekind zeta value. The invariance of a multiple Dedekind zeta value with respect to the choice of a diagram is the following: we are allowed to move each of the differential forms $ds_1,\dots,ds_{a_i}$ vertically to another box. Similarly, we are allowed to move the differential forms $dt_1,\dots,dt_{b_i}$ horizontally to another box. Such moves do not change the value of the corresponding iterated integral.

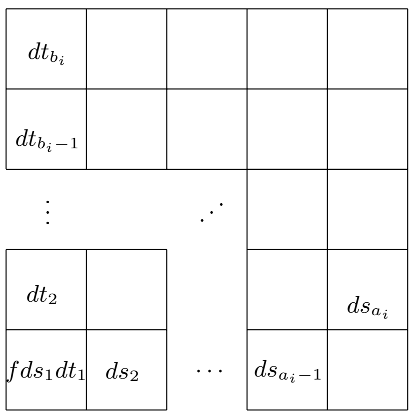


Fig. 4

If we multiply two multiple Dedekind zeta values of the above type then we can consider the corresponding diagrams D_1 and D_2 in a block-diagonal form, with blocks of the type given in Fig. 1. The product is equal to the sum of the integrals of the following diagrams: One has to shuffle the rows of the diagrams D_1 and D_2 as if each row of a diagram were a card and each diagram were a deck of cards. Besides shuffling the rows, one has to shuffle the columns as if each column were a card and the diagrams D_1 and D_2 were two decks of cards. If needed, we can move the forms ds horizontally and the forms dt vertically. The resulting diagrams will not necessarily be in a block-diagonal form with blocks of the type of Fig. 1; however, they will be very close to that form, which we will describe in Proposition 3.5 and in its proof.

Let d_1 and d_2 be the number of blocks in the diagrams D_1 and D_2, respectively. We call d_1 and d_2 depths. Then we have the following:

PROPOSITION 3.5. *In the product of the multiple Dedekind zeta values corresponding to the diagrams D_1 and D_2, we obtain "block-permutational" diagrams, where each block is of the type of Fig.* 1 *and the blocks are arranged in the same way as the entries* 1 *in a permutational matrix. Explicitly, the values represented by the block-permutational diagrams whose blocks are of the type of Fig.* 1 *are permutational multiple Dedekind zeta values from Definition* 1.3.

Proof. After the shuffle of D_1 and D_2, no two boxes containing an f are in the same row or the same column. Put 1 in each box where the function f appears and ignore all rows and columns where no f appears. Put 0 in all the remaining boxes. That produces a permutational matrix.

Now, consider the diagram representing a particular shuffle of the diagrams D_1 and D_2. Since we are allowed to move the forms ds_i vertically and dt_j horizontally, we can arrange the resulting diagram so that near each square containing f we have a subdiagram that is of the type of Fig. 1. By a direct computation it follows that the iterated integrals over membranes associated to block-permutational diagrams give permutational multiple Dedekind zeta values. ∎

Let us give an example of a permutational MDZV. The corresponding diagram is a block-permutational diagram of the form

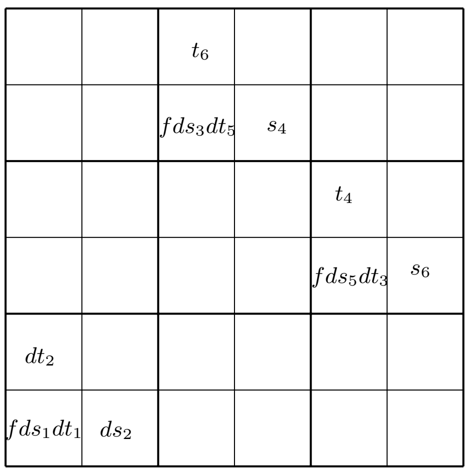


Fig. 5

Then the resulting permutational multiple Dedekind zeta value is $\zeta_g^{\rho}(2,2,2;2,2,2)$ with permutation $\rho = (1)(32)$ and a map $g = \text{identity}$.

PROPOSITION 3.6. *We can express a permutational multiple Dedekind zeta value as a finite sum of ordinary multiple Dedekind zeta values.*

Proof. Recall that a permutational multiple Dedekind zeta value is defined in the following way: Let $f = (f_1, \ldots, f_m) : C_+^a \to C_+^d$ be an $\mathbb{N}$-linear map with coefficients in $\{0\} \cup \mathbb{N}$, let

$$\delta_1 = f_1(\alpha_1, \ldots, \alpha_a), \ \ldots, \ \delta_d = f_d(\alpha_1, \ldots, \alpha_a), \tag{3.24}$$

and let ρ be a permutation of m elements. Then we define

$$\begin{aligned}\zeta_f(a_1, \ldots, a_d; b_1, \ldots, b_d) = \sum_{\alpha_1,\ldots,\alpha_d \in C_+} & \frac{1}{\delta_1^{a_1}(\delta_1+\delta_2)^{a_2}\ldots(\delta_1+\cdots+\delta_d)^{a_d}} \\ & \times \frac{1}{\bar{\delta}_{\rho(1)}^{b_1}(\bar{\delta}_{\rho(1)}+\bar{\delta}_{\rho(2)})^{b_2}\ldots(\bar{\delta}_{\rho(1)}+\cdots+\bar{\delta}_{\rho(d)})^{b_d}}.\end{aligned}$$

For $i = 1, \ldots, d$, let

$$\beta_i = \delta_1 + \cdots + \delta_i, \tag{3.25}$$

$$\gamma_i = \delta_{\rho(1)} + \cdots + \delta_{\rho(i)}. \tag{3.26}$$

If β and γ are in C_+, there are three cases: either $\beta - \gamma \in C_+$, or $\beta - \gamma = 0$, or $\gamma - \beta \in C_+$. Following Gangl, Kaneko and Zagier [GKZ] we define $\beta < \gamma$ if $\gamma - \beta \in C_+$. With respect to this order we can shuffle the two chains of inequalities $\beta_1 < \cdots < \beta_d$ and $\gamma_1 < \cdots < \gamma_d$ in order to obtain a single chain of inequalities (if we have the "=" sign among two of the elements β_i or γ_j, we consider them as one element in a chain of inequalities). Denote any such single chain of inequalities by

$$\mu_1 < \cdots < \mu_m. \tag{3.27}$$

The possible values for m are integers in the interval $[d, 2d]$. We claim that for each shuffle, the resulting zeta value is an ordinary multiple Dedekind zeta value. Indeed, all the terms μ_i in the denominator and their conjugates are raised to a non-negative integer power. Also, for each shuffle the sum is taken over $\mu_1 < \cdots < \mu_m$, subject to extra linear conditions inherited from $\alpha_1, \ldots, \alpha_d$. More specifically, the sum of the ordinary multiple Dedekind zeta values is of the form

$$\zeta_G(A_1, \ldots, A_m; B_1 \ldots, B_m) = \sum_{\alpha_1,\ldots,\alpha_d;\, \mu_1<\cdots<\mu_m} \frac{1}{\mu_1^{a_1}\ldots\mu_d^{a_m}\bar{\mu}_1^{b_1}\ldots\bar{\mu}_d^{b_m}}.$$

Here $G = h \circ g \circ f$, where: $f : C_+^a \to C_+^d$ is defined by (3.24); using (3.25) and (3.26), we define

$$g : C_+^d \to C_+^d \oplus C_+^d, \qquad (\delta_{(1)}, \ldots, \delta_{(d)}) \mapsto (\beta_1, \ldots, \beta_d) \oplus (\gamma_1, \ldots, \gamma_d);$$

and using any of the chains (3.27), we define

$$h : C_+^d \oplus C_+^d \to C_+^m, \qquad (\beta_1, \ldots, \beta_d) \oplus (\gamma_1, \ldots, \gamma_d) \mapsto (\mu_1, \ldots, \mu_m).$$

The integers $A_1, \ldots, A_m, B_1 \ldots, B_m$ are defined in the following way: Fix a chain (3.27). If $\mu_k = \beta_i$ for some k and i, but $\mu_k \neq \gamma_j$ for any j, then we define $A_k = a_i$. Otherwise $A_k = 0$. If $\mu_k = \gamma_j$ for some k and j, but $\mu_k \neq \beta_i$ for any i, then we define $B_k = b_j$. Otherwise $B_k = 0$.

That expresses the permutational MDZV as a finite sum of ordinary MDZVs. ∎

THEOREM 3.7. *A product of two ordinary MDZVs can be expressed as a finite sum of such by using the integral shuffle.*

Proof. An integral shuffle of two ordinary MDZVs can be expressed as a sum of permutational MDZVs by Proposition 3.5. And by Proposition 3.6 we can express a permutational MDZV as a finite sum of ordinary MDZVs. That proves the theorem. ∎

4. Relations among ordinary multiple Dedekind zeta values. In this section, we find a linear relation among ordinary MDZVs, using both types of shuffles of ordinary MDZVs. We use the infinite sum shuffle for the product of two Dedekind zeta values for imaginary quadratic fields at $s = 2$ from Corollary 2.2. We also express the same product as a sum of permutational MDZVs using the integral shuffle from Theorem 3.4. Then we express the permutational MDZVs as a finite sum of ordinary MDZVs using the construction from the proof of Proposition 3.6.

From the two types of shuffles for $\zeta_{K;C_+}(2)\zeta_{K;C_+}(2)$ from Corollary 2.2 and Theorem 3.4, we obtain the following linear relation among permutational MDZVs:

THEOREM 4.1. *Permutational multiple Dedekind zeta values associated to an imaginary quadratic number field K satisfy the following linear relation:*

$$\begin{aligned}(4.1)\qquad \zeta^1_{K;C_+}(4;4) - 8\zeta^1_{K,C_+}(1,3;1,3) &= 4\zeta^1(1,3;2,2) + 4\zeta^1(2,2;1,3)\\ &+ 2\zeta^\rho(2,2;2,2) + 8\zeta^\rho(1,3;1,3) + 4\zeta^\rho(1,3;2,2) + 4\zeta^\rho(2,2;1,3).\end{aligned}$$

If we express the permutational MDZV in terms of ordinary MDZV, using Proposition 3.6, we obtain:

THEOREM 4.2. *Ordinary multiple Dedekind zeta values associated to an imaginary quadratic number field K satisfy the following linear relation:*

$$\begin{aligned}\tfrac{3}{8}\zeta^1(4;4) - 8\zeta^1(1,3;1,3) = {}& 4\zeta^1(1,3;2,2) + 4\zeta^1(2,2;1,3)\\ &+ 2\zeta_f^1(2,0,2;0,2,2) + 2\zeta_f^1(0,2,2;2,0,2)\\ &+ 8\zeta_f^1(1,0,3;0,1,3) + 8\zeta_f^1(0,1,3;1,0,3)\\ &+ 4\zeta_f^1(1,0,3;0,2,2) + 4\zeta_f^1(0,1,3;2,0,2)\\ &+ 4\zeta_f^1(2,0,2;0,1,3) + 4\zeta_f^1(0,2,2;1,0,3),\end{aligned} \tag{4.2}$$

where $f : C_+^2 \to C_+^3$ is defined by $f(\alpha,\beta) = (\alpha,\beta,\alpha+\beta)$. Here the corresponding ordinary MDZV is

$$\zeta_f^1(a_1,a_2,a_3;b_1,b_2,b_3) = \sum_{0<\alpha<\beta} \frac{1}{\alpha^{a_1}\beta^{a_2}(\alpha+\beta)^{a_3}\bar{\alpha}^{b_1}\bar{\beta}^{b_2}(\bar{\alpha}+\bar{\beta})^{b_3}}.$$

Proof. We recall that

$$\zeta^\rho(a_1,a_2;b_1,b_2) = \sum_{\alpha,\beta\in C_+} \frac{1}{\alpha^{a_1}(\alpha+\beta)^{a_2}\bar{\beta}^{b_1}(\bar{\alpha}+\bar{\beta})^{b_2}}.$$

Using the method in the proof of Proposition 3.6, we obtain

$$\begin{aligned}2\zeta^\rho(2,2;2,2) &= \sum_{\alpha,\beta\in C_+} \frac{1}{\alpha^2(\alpha+\beta)^2\bar{\beta}^2(\bar{\alpha}+\bar{\beta})^2}\\ &= \sum_{0<\alpha<\beta} \frac{1}{\alpha^2\beta^0(\alpha+\beta)^2\bar{\alpha}^0\bar{\beta}^2(\bar{\alpha}+\bar{\beta})^2}\\ &\quad + \sum_{0<\beta<\alpha} \frac{1}{\alpha^2\beta^0(\alpha+\beta)^2\bar{\alpha}^0\bar{\beta}^2(\bar{\alpha}+\bar{\beta})^2} + \sum_{0<\alpha=\beta} \frac{1}{\alpha^2\beta^0(\alpha+\beta)^2\bar{\alpha}^0\bar{\beta}^2(\bar{\alpha}+\bar{\beta})^2}\\ &= 2\zeta_f^1(2,0,2;0,2,2) + 2\zeta_f^1(0,2,2;2,0,2) + 2\sum_{0<\alpha} \frac{1}{\alpha^2(2\alpha)^2\bar{\alpha}^2(2\bar{\alpha})^2}\\ &= 2\zeta_f^1(2,0,2;0,2,2) + 2\zeta_f^1(0,2,2;2,0,2) + \tfrac{1}{8}\zeta^1(4;4).\end{aligned}$$

Similarly,

$$\begin{aligned}8\zeta^\rho(1,3;1,3) &= 8\zeta_f^1(1,0,3;0,1,3) + 8\zeta_f^1(0,1,3;1,0,3) + 8\sum_{0<\alpha} \frac{1}{\alpha(2\alpha)^3\bar{\alpha}^(2\bar{\alpha})^3}\\ &= 8\zeta_f^1(1,0,3;0,1,3) + 8\zeta_f^1(0,1,3;1,0,3) + \tfrac{1}{8}\zeta^1(4;4),\\ 4\zeta^\rho(1,3;2,2) &= 4\zeta_f^1(1,0,3;0,2,2) + 4\zeta_f^1(0,1,3;2,0,2) + 4\sum_{0<\alpha} \frac{1}{\alpha(2\alpha)^3\bar{\alpha}^2(2\bar{\alpha})^2}\\ &= 4\zeta_f^1(1,0,3;0,2,2) + 4\zeta_f^1(0,1,3;2,0,2) + \tfrac{1}{8}\zeta^1(4;4)\end{aligned}$$

and

$$4\zeta^{\rho}(2,2;1,3) = 4\zeta^{1}_{f}(2,0,2;0,1,3)+4\zeta^{1}_{f}(0,2,2;1,0,3)+4\sum_{0<\alpha}\frac{1}{\alpha^2(2\alpha)^2\bar{\alpha}^2(2\bar{\alpha})^2}$$
$$= 4\zeta^{1}_{f}(2,0,2;0,1,3)+4\zeta^{1}_{f}(0,2,2;1,0,3)+\tfrac{1}{4}\zeta^{1}(4;4). \blacksquare$$

Using the two types of shuffle relations for multiple zeta values, one obtains linear relations among multiple zeta values. It is known that the dimension of the space of certain (motivic) multiple zeta values of small depth can be expressed as a dimension of a certain cohomology of arithmetic groups (see [GKZ], [G1], [G2]). We expect that a similar analogy holds for ordinary MDZVs, namely, from the two types of shuffles we obtain linear relations. We expect that the dimensions of the spaces of certain ordinary MDZVs can be expressed by the dimensions of the cohomology of arithmetic groups.

Acknowledgements. I would like to thank Don Zagier for the many questions he asked during my talk at Isaac Newton Institute for Mathematical Sciences, which greatly improved the presentation, and moreover led to further results described in this paper. I am also indebted to Richard Hain, whose interest and questions greatly accelerated the preparation of this paper. I would also like to thank Herbert Gangl for his questions and his interest.

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