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Automorphism loci for the moduli space of rational maps

by

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1. Introduction and main results. Let k be an algebraically closed field and $\mathbb{P}^1$ the projective line over k. If one fixes homogeneous coordinates X, Y on $\mathbb{P}^1$, then any rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d > 1$ can be realized as a pair of homogeneous polynomials of degree d in X and Y:

$$\phi(X, Y) = [F(X, Y) : G(X, Y)]$$

where F, G have no common roots in k.

Fix a degree $d > 1$. The collection of all such pairs of homogeneous polynomials $[F : G]$ can be naturally parametrized as the projective space $\mathbb{P}^{2d+1}$ where

$$[F : G] = [a_0X^d + a_1X^{d-1}Y + \cdots + a_dY^d : b_0X^d + b_1X^{d-1}Y + \cdots + b_dY^d]$$

and

$$[F : G] \mapsto [a_0 : a_1 : \cdots : b_d].$$

It is clear that not every such pair determines a rational map of degree d on the projective line. There is a homogeneous polynomial of degree $2d$ defined over $\mathbb{Z}$ in the coefficients a_i, b_j, called the *resultant*, which vanishes precisely when F and G have a common root in k. We denote the resultant by Res and note that $\mathrm{Res} \in \Gamma(\mathbb{P}^{2d+1}, \mathcal{O}(2d))$. It follows that we can construct the parameter space of rational maps on $\mathbb{P}^1$ as the complement of the vanishing locus of Res.

Definition 1.1. The space of *rational maps of degree d on $\mathbb{P}^1$* is $\mathrm{Rat}_d = \mathbb{P}^{2d+1} \setminus V(\mathrm{Res})$.

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From elementary algebraic geometry, we can see that Rat_d is an affine variety over k of dimension $2d+1$. We will often identify rational maps ϕ of degree d with points of Rat_d.

The space of rational maps carries a natural PGL_2-action of conjugation. For any $f \in \mathrm{PGL}_2$ and any rational map $\phi \in \mathrm{Rat}_d$ we define the *conjugate* of ϕ as $\phi^f = f^{-1} \circ \phi \circ f$. In fact, PGL_2 acts on the entire projective space $\mathbb{P}^{2d+1}$. We also must consider the related action of SL_2 on the affine cone over this space. Since k is algebraically closed, we have $\mathrm{PSL}_2 \cong \mathrm{PGL}_2$, and so the action of SL_2 actually induces the action of PGL_2 on $\mathbb{P}^{2d+1}$. In particular, these actions have identical orbits when we descend to projective space.

DEFINITION 1.2. The moduli space of rational maps of degree d on $\mathbb{P}^1$ is defined to be the geometric quotient $\mathcal{M}_d = \mathrm{Rat}_d/\mathrm{SL}_2$. We use $\pi : \mathrm{Rat}_d \to \mathcal{M}_d$ to denote the quotient map.

If two rational maps $\phi, \psi \in \mathrm{Rat}_d$ are conjugate we say they are *isomorphic*. They have the same dynamics as rational maps on $\mathbb{P}^1$.

The moduli space $\mathcal{M}_d$ exists as an affine variety over k of dimension $2d-1$. This space is a coarse moduli space for the moduli problem of rational maps: its points parametrize conjugacy classes of rational maps. Milnor studied this space as a complex orbifold in dynamics over $\mathbb{C}$ (see [10, Appendix G]), and Silverman studied this space as a geometric quotient scheme over $\mathbb{Z}$ (see [17]). Moduli spaces for dynamics on higher dimension projective spaces were constructed and studied by Levy [8].

Little is known about the geometry of these spaces. Many, however, hope that by studying the geometry of these spaces the properties uniform for all dynamical systems can be determined, much in the same way that the modular curves $X_0(p), X_1(p)$, which are certain moduli of elliptic curves, were used to prove Mazur's Theorem on uniform boundedness of torsion on elliptic curves defined over $\mathbb{Q}$.

We give a brief synopsis of the known results. The *multiplier* of a fixed point of a rational map ϕ is defined to be the derivative evaluated at that fixed point. The (finite) *set* of multipliers is invariant under conjugation, and thus the symmetric functions in these multipliers are dynamical invariants and, in fact, are regular functions on $\mathcal{M}_d$ (more generally this is true for the set of n-periodic points for a given period n). For dynamical systems of degree 2, Milnor [11] showed that $\mathcal{M}_d(\mathbb{C}) \cong \mathbb{A}^2(\mathbb{C})$ by constructing an explicit isomorphism map in terms of symmetric functions of the multipliers of the fixed points. Silverman [17] proved that Milnor's isomorphism works over an arbitrary base, i.e. $\mathcal{M}_2 \cong \mathbb{A}^2$ as a scheme over $\mathbb{Z}$. Levy [8] has shown that all $\mathcal{M}_d$ are rational varieties. By geometric invariant theory, $\mathcal{M}_d$ is known to be normal, integral, and connected.

As mentioned previously, the moduli space $\mathcal{M}_d$ is not fine: families of dynamical systems parametrized by a base scheme T do not correspond to morphisms $T \to \mathcal{M}_d$. One cause of this phenomenon is the existence of rational maps with automorphisms.

DEFINITION 1.3. Let $\phi \in \mathrm{Rat}_d$ be a rational map of degree $d > 1$. We say that $f \in \mathrm{PGL}_2$ is an *automorphism of* ϕ if $\phi^f = \phi$.

The collection of automorphisms for fixed ϕ forms a group, denoted by $\mathrm{Aut}\,\phi$; this group is the stabilizer of the point ϕ for the PGL_2-action by conjugation. It is also clear that if $\phi, \psi \in \mathrm{Rat}_d$ are isomorphic, then so are their automorphism groups. It is known that these groups are finite and are bounded in terms of d (see [8]). For a point $[\phi] \in \mathcal{M}_d$ we can define $\mathrm{Aut}([\phi])$ as an abstract group using $\mathrm{Aut}\,\phi$ for any map in the conjugacy class.

A natural question to ask is when a rational map has a non-trivial automorphism and to describe the automorphism locus in $\mathcal{M}_d$. We denote the locus of automorphisms in Rat_d by A and the corresponding locus in $\mathcal{M}_d$ by $\mathcal{A}$. Specifically,

$$A = \{\phi \in \mathrm{Rat}_d \mid \mathrm{Aut}\,\phi \neq 1\}$$

and $\mathcal{A} = \pi(A)$. The only explicitly known case is for quadratic rational maps. When $d = 2$, via Milnor's isomorphism $\mathcal{M}_2 \cong \mathbb{A}^2$, the automorphism locus $\mathcal{A}$ is known to be a cuspidal cubic. Furthermore, the only possible automorphism groups are $\mathcal{S}_3$, which occurs precisely at the cusp, and $\mathbb{Z}/2\mathbb{Z}$, which occurs at all other points of this curve [16, Prop. 4.15]. For a general $d > 1$, it is known that the only possible automorphism groups are $\mathbb{Z}/m\mathbb{Z}$, the cyclic group of order m, D_m, the dihedral group of order $2m$, A_4, A_5, the alternating group on four or five letters, respectively, or S_4, the full symmetric group on four letters [16, Example 2.54].

A second special locus to consider is the singular locus. We denote the singular locus of $\mathcal{M}_d$ by $\mathcal{S}$, and $\mathcal{S}$ consists of all points in $\mathcal{M}_d$ which have tangent space of dimension strictly larger that $2d - 1$. Again, the only explicitly known case is for quadratic rational maps, where $\mathcal{S} = \emptyset$ since $\mathcal{M}_2$ is smooth. It would be interesting to determine if these "singular" rational maps have additional dynamical structure.

A third geometric question about the moduli space of rational maps, related to the previous two, consists in calculating the Picard group $\mathrm{Pic}(\mathcal{M}_d)$. When the action of an algebraic group on an affine variety is free, the Picard group of the quotient variety is particularly easy to calculate. Because of the existence of non-trivial automorphisms, the action of SL_2 on Rat_d is not free. However, after determining the automorphism loci $A \subset \mathrm{Rat}_d$ and $\mathcal{A} \subset \mathcal{M}_d$ to be of codimension greater than 1, we compute $\mathrm{Pic}(\mathcal{M}_d)$, $\mathrm{Pic}(\mathcal{M}_d^s)$, and $\mathrm{Pic}(\mathcal{M}_d^{ss})$ by looking at the locus where the action is free.

We now discuss the results of this paper.

We begin by studying the loci of points in $\mathcal{M}_d$ whose automorphism group contains a copy of a cyclic or dihedral group.

Definition 1.4. Let G be a finite subgroup of PGL_2. Then

$$\mathcal{A}(G) = \{[\phi] \mid \mathrm{Aut}([\phi]) \supseteq G\}.$$

In Section 2 we calculate the dimension of $\mathcal{A}(G)$ when G is cyclic or dihedral, in particular we show that $\mathcal{A}(\mathbb{Z}/m\mathbb{Z}) \neq \emptyset$ if and only if m divides one of d, $d \pm 1$.

In Section 3 we use the decomposition of L. West [19] for the $2d+2$-dimensional representation of SL_2 associated with conjugation of rational maps to obtain the following result which characterizes the remaining possible automorphism groups.

Proposition. *Let $d > 1$.*

(1) $\mathcal{A}(A_4) \neq \emptyset$ *if and only if d is odd.*
(2) $\mathcal{A}(S_4) \neq \emptyset$ *if and only if d is coprime to* 6.
(3) $\mathcal{A}(A_5) \neq \emptyset$ *if and only if d is congruent to one of* $1, 11, 19, 21$ *modulo* 30.

Let G be any of the groups A_4, A_5, S_4 and suppose d is such that $\mathcal{A}(G) \neq \emptyset$. Then $\mathcal{A}(G)$ is irreducible and

$$\dim \mathcal{A}(G) = [2d/|G|],$$

where $[n]$ denotes the greatest integer less than or equal to n.

In Section 5 we apply D. Luna's Etale Slice Theorem and our computations of dimensions in Section 2 to prove the following theorem inspired by Rauch–Popp–Oort's characterization of singular points on the moduli space for curves.

Theorem. *Let $d > 2$. Then the singular locus $\mathcal{S}$ and the automorphism locus $\mathcal{A}$ of $\mathcal{M}_d$ coincide.*

D. Luna's Etale Slice Theorem is an algebro-geometric tool, and for us $\mathcal{M}_d$ is a quotient *variety*. However, $\mathcal{M}_d(\mathbb{C})$ equipped with the analytic topology becomes a topological quotient of $\mathrm{Rat}_d(\mathbb{C})$ when the latter is equipped with the usual topology (see [14]).

Our strategy is as follows: We call a point ϕ in the automorphism locus *simple* if $\mathrm{Aut}\,\phi \cong \mathbb{Z}/p\mathbb{Z}$ with p prime. We then show the following:

1. Simple points in $\mathcal{M}_d$ are algebraically singular and in fact topologically singular, i.e. they have no Euclidean neighborhood in the analytic topology.
2. Simple points are dense in the automorphism locus of $\mathcal{M}_d$.
3. Finally, we use the fact that the algebraic singularities form a closed set.

In Section 6 we use the results of the previous sections to calculate the Picard and class groups of $\mathcal{M}_d$, its projective closure $\mathcal{M}_d^{ss}$ and the intermediate variety $\mathcal{M}_d^s$. Our main tool for the following theorem is Narasimhan–Drézet's Descent Lemma for vector bundles equipped with a group action (see [4]).

THEOREM. *Let $d > 1$ and $\mathcal{M}_d, \mathcal{M}_d^s, \mathcal{M}_d^{ss}$ be the moduli spaces of rational maps of degree d, stable points in $\mathbb{P}^{2d+1}$, and semistable points in $\mathbb{P}^{2d+1}$.*

(1) $\mathrm{Cl}(\mathcal{M}_d) = \mathbb{Z}/2d\mathbb{Z}$ *or* $\mathbb{Z}/d\mathbb{Z}$ *when d is odd or even, respectively.*
(2) $\mathrm{Cl}(\mathcal{M}_d^s) = \mathrm{Cl}(\mathcal{M}_d^{ss}) = \mathbb{Z}$.
(3) $\mathrm{Pic}(\mathcal{M}_d)$ *is trivial.*
(4) $\mathrm{Pic}(\mathcal{M}_d^s) = \mathrm{Pic}(\mathcal{M}_d^{ss}) = \mathbb{Z}$.

2. Dimension of the automorphism locus. In this section we work over $\mathbb{C}$, although the conclusions hold for any algebraically closed field of characteristic 0.

We recall the definition of an automorphism of a rational map from Section 1.

DEFINITION 2.1. Let $\phi \in \mathrm{Rat}_d$ with $d > 1$. We say that $f \in \mathrm{PGL}_2$ is an *automorphism* of ϕ if $\phi^f = \phi$.

For every rational function ϕ there is a canonical metric g on $\mathbb{P}^1(\mathbb{C})$ which makes $\mathbb{P}^1(\mathbb{C})$ isometric to the usual 2-dimensional sphere. Any automorphism $f \in \mathrm{Aut}\,\phi$ must fix the metric g, and therefore $\mathrm{Aut}\,\phi$ embeds into $\mathrm{SO}_3(\mathbb{R})$. It is well known that $\mathrm{Aut}(\phi)$ is always finite [15, Prop. 4.65] and the complete list of finite subgroups of $\mathrm{SO}_3(\mathbb{R})$ are: cyclic of order m, dihedral D_m (of order $2m$), and the groups of rotations of platonic solids: A_4 (of tetrahedron), S_4 (of cube and octahedron), and A_5 (of dodeca- and icosahedron) [15, Remark 4.66].

Thus the groups arising as $\mathrm{Aut}\,\phi$ are all from this list. In this section we examine $\mathcal{A}(G)$ and calculate its dimension for all G. We will use the shorter notation $\mathcal{A}_m = \mathcal{A}(\mathbb{Z}/m\mathbb{Z})$ for the case of a cyclic group of order m.

REMARK 2.2. In the more general setting of maps on $\mathbb{P}^n$, recent work by de Faria and Hutz [5] shows that every finite subgroup of PGL_{n+1} occurs as the automorphism group of a map on $\mathbb{P}^n$. The description of the possible finite subgroups of PGL_{n+1} is much more complicated for $n > 1$ than for $n = 1$, which means the exhaustive approach we take here probably would not be feasible for higher dimensions.

We start with a well-known lemma, and then proceed case-by-case depending on the subgroup $G \subset \mathrm{PGL}_2$.

Lemma 2.3. *Two finite subgroups of* PGL_2 *which are isomorphic are conjugate. Moreover,*

$$\langle \zeta_m z\rangle \cong \mathbb{Z}/m\mathbb{Z}, \quad \langle \zeta_m z, 1/z\rangle \cong D_m,$$
$$\left\langle -z, i\frac{z+1}{z-1}\right\rangle \cong A_4, \quad \left\langle iz, i\frac{z+1}{z-1}\right\rangle \cong S_4.$$

Proof. The first statement is well known and dates back to Klein [7], and the rest follows immediately. ∎

2.1. Cyclic groups. First we look at the case of a cyclic group. The presentation of the material in this section was suggested by John Milnor. Let ζ_m be a primitive mth root of unity with $m > 1$. By Lemma 2.3 every finite cyclic subgroup of $\mathrm{PGL}_2(\mathbb{C})$ of order m is conjugate to $\langle z \mapsto \zeta_m z\rangle$. If ϕ commutes with $\langle z \mapsto \zeta_m z\rangle$ then it must map the set $\{0,\infty\} = \mathrm{Fix}(\langle z \mapsto \zeta_m z\rangle)$ to itself. Hence, ϕ induces a set map $\{0,\infty\} \to \{0,\infty\}$, leading us to the following definition suggested again by John Milnor.

Definition 2.4. If $\sigma \in \mathrm{PGL}_2(\mathbb{C})$ is a non-trivial automorphism of a rational function ϕ, we say that σ is an automorphism of ϕ *of type* t if

$$t + 1 = |\mathrm{Fix}(\phi) \cap \mathrm{Fix}(\sigma)|.$$

Remark 2.5. Since every automorphism of $\mathbb{P}^1$ of finite order is conjugate to $z \mapsto \zeta_m z$, it has two fixed points and hence the type t can only be -1, 0, or 1. As we will see below, a rational map with $z \mapsto \zeta_m z$ must admit a particular form for which only certain coefficients can be non-zero. This is what will allow us to compute the dimension of the locus having $z \mapsto \zeta_m z$ as an automorphism. In particular, for type t, the coefficients for which the one-parameter subgroup $\langle z \mapsto \zeta_m z\rangle$ acts with weight congruent to t mod m may be non-zero, while all of the others must vanish.

Let $A^t_{\zeta_m} \subset \mathrm{Rat}_d$ (where t is one of $-, 0, +$ rather than $-1, 0, 1$) be the locus of all maps for which $z \mapsto \zeta_m z$ is an automorphism of type t. We will now characterize $A^t_{\zeta_m}$.

Lemma 2.6. *A rational map* ϕ *commutes with the linear map* $z \mapsto \zeta_m z$ *if and only if it has the form* $\phi(z) = z\psi(z^m)$ *for some rational function* ψ.

Proof. Setting $\eta(z) = \phi(z)/z$, note that

$$\eta(\zeta_m z) = \phi(\zeta_m z)/(\zeta_m z) = \phi(z)/z = \eta(z).$$

It follows easily that $\eta(z) = \psi(z^m)$ for a uniquely defined rational function ψ. The converse is a straightforward calculation. ∎

Let $d \geq 2$ be the degree of ϕ and let $d' \geq 1$ be the degree of ψ. Based on Lemma 2.6, set

$$\psi(u) = \frac{\alpha u^{d'} + \cdots + \beta}{\gamma u^{d'} + \cdots + \delta}$$

so that

$$\psi(\infty) = \alpha/\gamma \quad \text{and} \quad \psi(0) = \beta/\delta.$$

It follows easily that

$$\phi(0) = \infty \iff \psi(0) = \infty \iff \delta = 0,$$

with $\phi(0) = 0$ whenever $\delta \neq 0$. Similarly

$$\phi(\infty) = 0 \iff \psi(\infty) = 0 \iff \alpha = 0,$$

with $\phi(\infty) = \infty$ whenever $\alpha \neq 0$.

The discussion can now be divided into three cases according to the type of the map $z \mapsto \zeta_m z$ as an automorphism of ϕ.

Type 1. If $\alpha \neq 0$, and $\delta \neq 0$, so that 0 and ∞ are fixed points of ϕ, then it follows easily that

$$d = md' + 1, \qquad \dim A^+_{\zeta_m} = 2d' + 1.$$

Type -1. If $\alpha = \delta = 0$, so that $\{0, \infty\}$ is a period two orbit for ϕ, then canceling a factor of z from the numerator and denominator of ϕ we see that $d < md'$, and it follows easily that

$$d = md' - 1, \qquad \dim A^-_{\zeta_m} = 2d' - 1.$$

Type 0. In the intermediate case where just one of the two coefficients α and δ is zero, so that $\phi(0) = \phi(\infty) \in \{0, \infty\}$, the locus $A^0_{\zeta_m}$ has two irreducible components which are conjugate via $z \mapsto 1/z$:

$$\{\phi \mid \phi(\zeta_m z) = \zeta_m \phi(z),\ \phi(0) = \phi(\infty) = 0\} \subset \mathrm{Rat}_d,$$
$$\{\phi \mid \phi(\zeta_m z) = \zeta_m \phi(z),\ \phi(0) = \phi(\infty) = \infty\} \subset \mathrm{Rat}_d,$$

and a similar argument shows that for each component,

$$d = md', \qquad \dim A^0_{\zeta_m} = 2d'.$$

Let $\mathcal{A}^t_m = \pi(A^t_m)$ be the locus in $\mathcal{M}_d$ of points parametrizing conjugacy classes which admit an automorphism of order m and type t. Then

$$\dim \mathcal{A}^t_m = \dim A^t_{\zeta_m} - 1,$$

since the group $\{z \mapsto cz \mid c \in k\}$ acts on $A^t_{\zeta_m}$ and all fibers of the surjective map $A^t_{\zeta_m} \to \mathcal{A}^t_m$ are 1-dimensional, being finite unions of orbits under this action.

This discussion leads to the following proposition.

Proposition 2.7. *The space* Rat_d *admits a map with an automorphism of order* m *of type* t *if and only if* m *is a divisor of* $d - t$*. In such a case the resulting locus in* $\mathcal{M}_d$ *of all maps having an automorphism of order* m *and type* t *is closed and irreducible of dimension*

$$\dim \mathcal{A}^t_m = \frac{2(d-t)}{m} + t - 1.$$

Remark 2.8. Except for the case $m = 2$ and d odd, an automorphism of order m can occur only in one type. In this case $\mathcal{A}_m = \mathcal{A}_m^t$ depending on the type t that does occur. When d is odd, automorphisms of order 2 can occur in types $+1, -1$. Thus for d odd, $\mathcal{A}_2 = \mathcal{A}_2^- \cup \mathcal{A}_2^+$.

Proof of Proposition 2.7. Everything has been established above except that the locus is closed. For $m \neq 2$ only one type t occurs for a fixed degree d, and $\mathcal{A}_m^t$ is closed by our Corollary 4.11 to Luna's Etale Slice Theorem (cf. [3]). For $m = 2$ and d odd, both $\mathcal{A}_2^+$ and $\mathcal{A}_2^-$ are non-empty. Let us prove now that $\mathcal{A}_2^+$ is closed. Let $\phi \notin \mathcal{A}_2^+$. Since $\operatorname{Aut}\phi$ is finite, it is an open condition on $\psi \in \mathrm{Rat}_d$ that ψ does not have an automorphism of order 2 and type $+1$ inside $\operatorname{Aut}\phi$. Also the set of functions ψ that satisfy this condition is $\operatorname{Aut}\phi$-invariant. Now take en etale slice V at ϕ which is small enough so that its points satisfy the condition above. Then the image of the slice in $\mathcal{M}_d$ is an open set containing $[\phi]$ and disjoint from $\mathcal{A}_2^+$. We have shown that $\mathcal{A}_2^+$ is closed. The proof that $\mathcal{A}_2^-$ is closed is similar. ∎

Remark 2.9. It is easy to see that if $2 < m < n$, then

$$d - 1 = \dim \mathcal{A}_2 > \dim \mathcal{A}_m \geq \dim \mathcal{A}_n.$$

Corollary 2.10. *For $d \geq 1$, the codimension of the automorphism loci $A \subset \mathrm{Rat}_d$ and $\mathcal{A} \subset \mathcal{M}_d$ is $d - 1$.*

Proof. This follows immediately for $\mathcal{A} \subset \mathcal{M}_d$. For A, we observe that because A is SL_2-invariant, closed, and contained in the stable locus, $\mathcal{A}$ is a geometric quotient of A by the action of SL_2. It follows that $\operatorname{codim}(A, \mathrm{Rat}_d) = \operatorname{codim}(\mathcal{A}, \mathcal{M}_d)$. ∎

2.2. Dihedral groups. We now examine the locus, $\mathcal{A}(D_m)$, of those $[\phi]$ for which $\operatorname{Aut}\phi$ contains a copy of the dihedral group D_m of order $2m$. Recall that, by Lemma 2.3, every dihedral group D_m inside $\mathrm{PGL}_2(\mathbb{C})$ is conjugate to $\langle \zeta_m z, 1/z \rangle$.

Lemma 2.11. *A rational function ϕ commutes with both $\zeta_m z$ and $1/z$ if and only if it can be written in the form $\phi = z\psi(z^m)$ where ψ commutes with $1/z$.*

Proof. By Lemma 2.6, ϕ can be written in the form $\phi(z) = z\psi(z^m)$ if and only if it commutes with $\zeta_m z$. Now, $z\psi(z^m)$ commutes with $1/z$ if and only if

$$z\psi(z^m) = \frac{1}{1/z\psi(1/z^m)} \;\Leftrightarrow\; \phi(z^m) = \frac{1}{\psi(1/z^m)} \;\Leftrightarrow\; \psi(u) = \frac{1}{\psi(1/u)}$$

if and only if $1/u$ commutes with ψ. ∎

Proposition 2.12. *The dimension of $\mathcal{A}(D_m)$ is as follows:*

- $t = 1$: *if $m \mid d - 1$, then* $\dim \mathcal{A}(D_m) = (d - 1)/m$,

- $t = 0$: *if* $m \mid d$, *then* $\mathcal{A}(D_m) = \emptyset$,
- $t = -1$: *if* $m \mid d+1$, *then* $\dim \mathcal{A}(D_m) = (d+1)/m - 1$.

Proof. Suppose ϕ commutes with both ζ_m and $1/z$. Then we deduce from Lemma 2.11 that $\phi = z\psi(z^m)$ where ψ commutes with $1/z$. Let $[a_0, \ldots, a_{d'}, b_0, \ldots, b_{d'}]$ be the coefficients of ψ.

In the $t = 1$ case, the degree of ψ is $d' = (d-1)/m$, and this gives $2d' + 2 = 2(d-1)/m + 2$ coefficients. Because ψ commutes with $1/z$, the coefficients of ψ satisfy the additional $d' + 1$ homogeneous equations $a_0 = \lambda b_{d'}$, $a_1 = \lambda b_{d'-1}, \ldots, a_{d'} = \lambda b_0$ where $\lambda = \pm 1$. Each of these independent equations cuts the dimension down by one. Therefore the dimension of the locus in Rat_d in this case is

$$2d' + 2 - (d' + 1) - 1 = d' = \frac{d-1}{m}.$$

In the $t = 0$ case, exactly one of $a_0, b_{d'}$ vanishes, which is impossible for a map that commutes with $1/z$. Therefore, $\mathcal{A}(D_m) = \emptyset$.

In the $t = -1$ case, the degree of ψ is $d' = (d+1)/m$, and this gives $2d' + 2 = 2(d+1)/m + 2$ coefficients. That $1/z$ is an automorphism of ψ gives the same relations as in the $t = 1$ case; also we have $a_0 = 0$, $b_{d'} = 0$, so the first relation $a_0 = \pm b_{d'}$ is redundant. This gives a total of $d' + 2$ independent equations. Therefore the dimension of the locus in Rat_d in this case is

$$2d' + 2 - (d' + 2) - 1 = \frac{d+1}{m} - 1.$$

Since there is no infinite family in PGL_2 which commutes with both $\zeta_m z$ and $1/z$, the map from the locus $A(D_m)$ in Rat_d to $\mathcal{A}(D_m)$ is finite-to-one, and the result follows. ∎

Remark 2.13. Note that for $m = 2$ the cases ± 1 agree with each other.

2.3. Automorphism locus in $\mathbb{P}^{2d+1}$

Proposition 2.14. *For $d \geq 1$ the codimension of the automorphism locus in $\mathbb{P}^{2d+1}$ is $d - 1$.*

Proof. One can prove this directly in a way similar to the proof for Rat_d. However, here is another proof. We have already proved the corresponding statement for Rat_d (Corollary 2.10), so for any irreducible component Z which meets Rat_d, we have $\operatorname{codim} Z \geq d - 1$. Let Z be an irreducible component of $\mathrm{Aut}(\mathbb{P}^{2d+1})$ entirely inside $V(\mathrm{Res})$. Let

$$m = \min\{\deg \gcd(F, G) \mid (F : G) \in Z\}$$

and let $W = H^0(\mathbb{P}^1, O(m))$, the space of m-forms in two variables. Consider the morphism

$$f : \mathbb{P}(W) \times \mathbb{P}^{2(d-m)+1} \to V(\mathrm{Res}), \quad ([H], (F : G)) \mapsto (HF : HG).$$

It is not hard to see that $\mathbb{P}(W)\times\mathrm{Rat}_{d-m}$ carries a PGL_2-action and that the map f is equivariant for this action. Since the image of f restricted to $\mathbb{P}(W)\times\mathrm{Rat}_{d-m}$ consists of all pairs $(F:G)$ with $\deg\gcd(F,G)=m$ (i.e. common zeros of exact order m), it follows that $f^{-1}(f(\mathbb{P}(W)\times\mathrm{Rat}_{d-m}))=\mathbb{P}(W)\times\mathrm{Rat}_{d-m}$.

We claim the restriction of f to $\mathbb{P}(W)\times\mathrm{Rat}_{d-m}$ is injective. Suppose $\big([H_1],(F_1:G_1)\big),\big([H_2],(F_2:G_2)\big)$ both map to $(H_1F_1:H_1G_1)=(H_2F_2:H_2G_2)$. Then, for some $\lambda\in k^\times$, $H_1F_1=\lambda H_2F_2$ and $H_1G_1=\lambda H_2G_2$. These equalities imply that $H_1\mid H_2F_2$ and $H_1\mid H_2G_2$, but $(F_2:G_2)\in\mathrm{Rat}_{d-m}$ means they have no common zeros. It follows that $H_1\mid H_2$, and since their degrees are the same, $[H_1]=[H_2]$ in $\mathbb{P}(W)$. After canceling in the previous equations we deduce that $(F_1:G_1)=(F_2:G_2)$, and the restriction is injective.

Now, the image of f contains Z, and Z meets $f(\mathbb{P}(W)\times\mathrm{Rat}_{d-m})$. From injectivity it follows that

$$f(\mathrm{Aut}(\mathbb{P}(W)\times\mathrm{Rat}_{d-m}))=\mathrm{Aut}(f(\mathbb{P}(W)\times\mathrm{Rat}_{d-m})),$$

and we also have

$$\mathrm{Aut}(\mathbb{P}(W)\times\mathrm{Rat}_{d-m})\subseteq\mathbb{P}(W)\times\mathrm{Aut}(\mathrm{Rat}_{d-m}).$$

Thus we get

$$\begin{aligned}\dim Z&=\dim Z\cap f(\mathbb{P}(W)\times\mathrm{Rat}_{d-m})\le\dim\mathrm{Aut}\big(f(\mathbb{P}(W)\times\mathrm{Rat}_{d-m})\big)\\&\le\dim\mathrm{Aut}(\mathbb{P}(W)\times\mathrm{Rat}_{d-m})\\&\le\dim\mathbb{P}(W)\times\mathrm{Aut}\,\mathrm{Rat}_{d-m}\\&=\begin{cases}m+(d-m)+2=d+2 & \text{for } m<d,\\ d+1 & \text{for } m=d.\end{cases}\end{aligned}$$

We remark that the first equality arises because f is projective, and hence closed, and so $\overline{f(\mathbb{P}(W)\times\mathrm{Rat}_{d-m})}=f(\mathbb{P}(W)\times\mathbb{P}^{2(d-m)+1})$. ■

Given an element $g\in\mathrm{SL}_2$, a point $\phi\in\mathbb{P}^{2d+1}$ fixed by g comes from an eigenvector of multiplication by g in $\mathbb{A}^{2d+2}$. Specifically, g acts on the stalk of $\mathcal{O}(1)$ and $\mathcal{O}(-1)$ at ϕ via multiplication by the eigenvalue.

LEMMA 2.15. *Let $\eta\neq0$. Suppose $\Phi=(F,G)\in\mathbb{A}^{2d+2}$ is an eigenvector for the linear transformation given by the action of $\left(\begin{smallmatrix}\eta&0\\0&\eta^{-1}\end{smallmatrix}\right)$, and λ the associated eigenvalue. Write*

$$(F,G)=(a_0,\dots,a_d,b_0,\dots,b_d).$$

(1) *If k is such that $a_k\neq0$, then $\lambda=\eta^{d-2k-1}$.*
(2) *If k is such that $b_k\neq0$, then $\lambda=\eta^{d-2k+1}$.*

Proof. Under the hypothesis of the lemma it follows that

$$(\eta^{-1}F(\eta X,\eta^{-1}Y),\eta G(\eta X,\eta^{-1}Y))=\lambda(F,G),$$

and therefore

$$\left(a_0\eta^{d-1}, \ldots, \frac{a_d}{\eta^{d+1}}, b_0\eta^{d+1}, \ldots, \frac{b_d}{\eta^{d-1}}\right) = (\lambda a_0, \ldots, \lambda a_d, \lambda b_0, \ldots, \lambda b_d).$$

If $a_k \neq 0$, then one can equate the corresponding coordinates and solve for λ, which results in the first equation. In a similar fashion one obtains the equation when $b_k \neq 0$. ∎

COROLLARY 2.16. *Suppose $\phi \in \mathrm{Rat}_d$, and let $\sigma \in \mathrm{PGL}_2$ be an automorphism of ϕ of order $m > 1$ and type t. Let $d' = (d-t)/m$. Let $g \in \mathrm{SL}_2$ represent σ and assume that the order of g is $2m$. Then g acts on the stalk of $\mathcal{O}(1)$ at ϕ via multiplication by a root of unity whose order s is as follows:*

- Case $t = \pm 1$: *$s = 1$ if d' is even, and $s = 2$ if d' is odd.*
- Case $t = 0$: *$s = m$ if d is odd, and $s = 2m$ if d is even.*

Proof. After an appropriate conjugation we can assume that

$$\sigma(z) = \zeta z,$$

where ζ is an mth root of unity. Then we have the following: If $t = 1$, then $a_0 \neq 0$. If $t = -1$, then $b_0 \neq 0$. If $t = 0$, then either $a_0 \neq 0$ or $b_0 \neq 0$, with the same result. ∎

The above lemma and corollary will help us to compute the Picard and class groups of $\mathcal{M}_d$, $\mathcal{M}_d^s$, and $\mathcal{M}_d^{ss}$ in Section 6.

3. Automorphism loci for groups of rotations of platonic solids. In this section we use L. West's decomposition to calculate dimension formulas for automorphism loci for the tetrahedral group A_4, the octahedral group S_4 and the icosahedral group A_5.

Let V be the vector space k^2 regarded as the space of columns. Then SL_2 acts naturally on the left on V by multiplication. Let X, Y be the coordinate functions on k^2 corresponding to the standard basis, and let $R = k[X, Y]$. Let R_n denote the nth homogeneous component of R. Since R is the coordinate ring of V, the action of SL_2 on V induces a right SL_2-action on R by pre-composition, where if

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $F \in R$, then $F^g = F(aX + bY, cX + dY)$. This action preserves grading, hence there is a (right) action on each homogeneous component R_n. Let W be the space of morphisms $V \to V$ given by a pair of homogeneous polynomials of degree d in X and Y. Then SL_2 acts on the right on W by conjugation: $(\Phi^g)(v) = g^{-1}\Phi(gv)$. There is an isomorphism of SL_2-representations:

$$W \cong R_{d+1} \otimes V, \qquad (F, G) \mapsto F \otimes e_1 + G \otimes e_2,$$

where e_1, e_2 is the standard basis of V.

THEOREM 3.1 (L. West [18]). *The following is an isomorphism of* SL_2*-representations:*

$$w : W \to R_{d-1} \oplus R_{d+1}, \qquad w(F, G) = \frac{\partial F}{\partial X} + \frac{\partial G}{\partial Y} + YF - XG,$$

and its inverse is

$$w^{-1}(H, J) = \frac{1}{d+1}\left(XH + \frac{\partial J}{\partial Y}, YH - \frac{\partial J}{\partial X}\right).$$

Proof. Morphisms $V \to V$ can be canonically (independently of coordinates on V) identified with vector fields on V, by which we mean elements of $T := \mathrm{Hom}_R(\Omega^1_{R/k}, R)$. Namely for each vector field $F\partial/\partial X + G\partial/\partial Y \in T$, we get the morphism $V \to V$ given by $(X, Y) \mapsto (F, G)$. Let us now endow V with the volume form $\langle\, ,\rangle$ given by $\langle (x_1, y_1), (x_2, y_2)\rangle = x_1y_2 - y_1x_2$. This form is preserved by the SL_2-action. To prove that L. West's map w is an SL_2-morphism, we will define w and w^{-1} in terms of quantities preserved by the SL_2-action:

$$w(\Phi) = (\nabla \cdot \Phi, \langle \Phi, r\rangle), \qquad w^{-1}(H, J) = \frac{1}{d+1}(Hr + (\mathrm{d}J)^\sharp),$$

where $r = (X, Y)$, $v^\flat = \langle v, -\rangle$, $^\sharp$ is the inverse of $^\flat$, and we identify vector fields on k^2 with maps $k^2 \to k^2$. ∎

NOTATION 3.2. We denote by $\bar{w}$ the PGL_2-isomorphism

$$\bar{w} : \mathbb{P}(W) \to \mathbb{P}(R_{d-1} \oplus R_{d+1})$$

induced by w.

One wonders what L. West's map $\bar{w}$ does to the resultant.

LEMMA 3.3. *Let $(F, G) \in W$, and $(H, J) = w(F, G)$. For a point (a, b) to be a common zero of F and G it is necessary and sufficient that*

$$J(a, b) = 0 \quad \textit{and} \quad H(a, b) = J_1(a, b)$$

where $J_1 \in R_d$ is given by $J = (bX - aY)J_1$.

Proof. Let us first obtain a formula for (F, G) at a zero of the fixed-point polynomial $J = YF - XG$. So suppose for a moment that $J(a, b) = 0$ and write $J = (bX - aY)J_1$ where $J_1 \in R_d$. Taking advantage of the Leibniz rule, we get, at the point (a, b),

$$(F, G) = \frac{H - J_1}{d+1} \cdot (a, b).$$

From this, sufficiency follows at once, since either a or b is non-zero. Necessity follows if one observes that a common zero of F and G is also a zero of the fixed-point polynomial J. ∎

As an easy corollary we get the following

LEMMA 3.4. *Let $\mathbb{G}_{\mathrm{m}}$ act on $R_{d-1} \oplus R_{d+1}$ via*

$$t \cdot (H, J) = (tH, J).$$

This action induces a $\mathbb{G}_{\mathrm{m}}$-action on $\mathbb{P}(R_{d-1} \oplus R_{d+1})$. This action commutes with the PGL_2-action. Hence PGL_2-stabilizers are identical for all points within a $\mathbb{G}_{\mathrm{m}}$-orbit. The $\mathbb{G}_{\mathrm{m}}$-orbit of $[(H, J)]$ meets $\bar{w}(\mathrm{Rat}_d)$ if and only if no multiple zero of J is also a zero of H.

Proof. If (a, b) is a multiple zero of J and a zero of H, from Lemma 3.3 it is clear that $w^{-1}(tH, J)$ vanishes at (a, b) for any t, and thus $\mathbb{G}_{\mathrm{m}} \cdot [(H, J)]$ does not meet $\bar{w}(\mathrm{Rat}_d)$.

Conversely, suppose that $[w^{-1}(tH, J)] \notin \mathrm{Rat}_d$ for all t. Then the components of $w^{-1}(tH, J)$ have a common root for all t. But such a common root must also be a root of J. By the pigeon-hole principle, some root (a, b) of J will occur in this way for at least two values of t. Then Lemma 3.3 shows that such an (a, b) is in fact a multiple root of J and also a root of H. ∎

Lemma 3.4 implies the following interesting corollary, which we will not need in the remainder of the paper. We omit the proof.

COROLLARY 3.5. *Let G be non-cyclic and suppose that $\dim \mathcal{A}(G) = 0$. Then if $\mathrm{Aut}\,\phi$ contains a copy of G, then ϕ must have $\deg \phi + 1$ distinct fixed points with all multipliers equal to $-\deg \phi$.*

LEMMA 3.6. *Let m, k be positive integers, η a primitive $2m$th root of 1, and F a binary form of degree k for which 0 and ∞ are not multiple zeros, and suppose F is an eigenvector for the action of $\left(\begin{smallmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{smallmatrix}\right)$ with eigenvalue λ. Then λ is as in Table 1.*

Table 1. Eigenforms without 0 and ∞ as multiple zeros

Fixed zeros	$F(0) \neq 0$	$F(0) = 0$
$F(\infty) \neq 0$	$m \mid k$ $\lambda = (-1)^{k/m}$	$m \mid k-1$ $\lambda = (-1)^{(k-1)/m}\eta$
$F(\infty) = 0$	$m \mid k-1$ $\lambda = (-1)^{(k-1)/m}\eta^{-1}$	$m \mid k-2$ $\lambda = (-1)^{(k-2)/m}$

Proof. Write $F(X, Y) = a_0 X^k + a_1 X^{k-1} Y + \cdots + a_k Y^k$. Because F is an eigenvector, we have

$$\begin{aligned}F(\eta X, \eta^{-1} Y) &= a_0 \eta^k X^k + a_1 \eta^{k-2} X^{k-1} Y + \cdots + \frac{a_k}{\eta^k} Y^k \\ &= \lambda a_0 X^k + \lambda a_1 X^{k-1} Y + \cdots + \lambda a_k Y^k\end{aligned}$$

We write $F(0) := F(0,1)$ and $F(\infty) := F(1,0)$. If $F(0) = 0$, then $a_k = 0$, but because 0 is not a multiple zero, $a_{k-1} \neq 0$. Similarly, if $F(\infty) = 0$, then $a_0 = 0$, but since ∞ is not a multiple zero, $a_1 \neq 0$.

The four cases in the table now correspond to four systems of linear equations. If $F(0), F(\infty) \neq 0$, then $a_0, a_k \neq 0$. Equating the first and last coefficients and canceling a_0 and a_k gives the system $\lambda = \eta^k$ and $\lambda = 1/\eta^k$. Since η is a $2m$th root of unity, it follows that $m \mid k$ and $\lambda = (-1)^{k/m}$.

Similarly, if $F(0) = F(\infty) = 0$, then $a_0 = a_k = 0$ and we can equate the second and second to last coefficients, getting the system $\lambda = \eta^{k-2}$ and $\lambda = 1/\eta^{k-2}$. In this case $m \mid k-2$ and $\lambda = (-1)^{(k-2)/m}$.

The other two cases are symmetrical. Suppose $F(0) = 0$ and $F(\infty) \neq 0$. Then $a_k = 0$, $a_{k-1} \neq 0$, and $a_0 \neq 0$. Equating the first and second to last coefficients and canceling a_0 and a_{k-1} gives $\lambda = \eta^k$ and $\lambda = 1/\eta^{k-2}$. Solving for η gives $1 = \eta^{2k-2}$, hence $m \mid k-1$ and $\lambda = (-1)^{(k-1)/m}\eta$. If $F(0) \neq 0$, $F(\infty) = 0$, the calculation is identical, but $\lambda = (-1)^{(k-1)/m}\eta^{-1}$. ∎

As described at the beginning of this section, there is a natural SL_2-action on R_n which induces a $\mathrm{PSL}_2 = \mathrm{PGL}_2$-action on $\mathbb{P}(R_n)$, which we can consider as a PGL_2-action on the group of divisors: if $F \in R_n$ and $D = \mathrm{div}(F)$, then $D^g = \mathrm{div}(F^g)$. So if $D = \sum n_i(P_i)$, then $D^g = \sum n_i(g^{-1} \cdot P_i)$.

For a divisor D on $\mathbb{P}^1$ and a finite group $G \subset \mathrm{PGL}_2$, we can consider the orbit of D under G. If $p \in \mathbb{P}^1$, then we can consider p as a divisor and speak of its orbit under this action. This orbit consists of at most $|G|$ points. If the orbit has less than $|G|$ points, then D has non-trivial stabilizer under this action.

Lemma 3.7. *Let $G \subset \mathrm{PGL}_2$ be a finite subgroup of even order k. Let $P \in \mathbb{P}^1$. Let D be the divisor*

$$D = \sum_{\sigma \in G} P^\sigma.$$

Let $\sigma_0 \in G$, let g be a preimage of σ_0 in SL_2, and denote by m the order of σ_0. Let $F \in R_k$ be such that $\mathrm{div}(F) = D$. Then the action of g on F is multiplication by $(-1)^{k/m}$, and in particular if G is one of A_4, S_4, A_5, then the preimage of G in SL_2 fixes F.

Proof. Let n be the order of the stabilizer of P in G. Then for any $\sigma \in G$, the stabilizer of P^σ in G is also of order n, so $D = nD'$ for some divisor D' with no coefficients greater than 1. In fact, D' is the support of D. The degree of D' is k/n.

Now let F be a homogeneous form such that $\mathrm{div}(F) = D$ and let σ_0 and g be as in the statement of the lemma. We can write $F = H^n$ where H is a homogeneous form with no multiple zeros and $\mathrm{div}(H) = D'$. Since D is clearly fixed by G, the action of g on F must be multiplication by a scalar λ. The same is true for D' and H: indeed, g acts on H by a scalar λ' and we have $(\lambda')^n = \lambda$.

Now, with Lemma 3.6 in mind, let η be a primitive $2m$th root of unity and consider the matrix:

$$\begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} \in \mathrm{SL}_2.$$

Conjugating g to this matrix does not change the eigenvalues λ and λ', and so without loss of generality we may assume g has this form. Note also that the two lifts of σ_0 to $g \in \mathrm{SL}_2$ give the same eigenvalue λ for F because k is even. Now apply Lemma 3.6 to H. For the top left case of this lemma, the eigenvalue of H is $\lambda' = (-1)^{k/(nm)}$ so that $\lambda = (-1)^{k/m}$. For the bottom right case, first note that in this case 0 is a zero of H, and also a fixed point of σ_0 (after our conjugation). This means that σ_0 is in the stabilizer of 0, which, being a root of H, is of order n. Thus $m \mid n$. Then $(-1)^{(k/n-2)/m}$ raised to the nth power gives $\lambda = (-1)^{(k-2n)/m} = (-1)^{k/m}(-1)^{-2n/m} = (-1)^{k/m}$. The other two cases are similar, yielding $\lambda = (-1)^{k/m}$, and we omit the calculation. Note that k/m is an integer, as m is the order of the element σ_0 in the group G of order k.

If $G = A_4, S_4, A_5$, then $k = 12, 24, 60$, respectively. For A_4, the possible m are $m = 2, 3$, so $k/m = 6, 4$ and $(-1)^{k/m} = 1$. The justification for S_4, A_5 is entirely analogous, and we again omit the calculation. ∎

In the following definitions let $G \subset \mathrm{PGL}_2$ be a finite subgroup.

Definition 3.8. We say that an orbit of $P \in \mathbb{P}^1$ is *degenerate* if the length of the orbit, $|GP|$, is less than the order of the group, $|G|$.

Definition 3.9. We call a pair (D_1, D_2) of effective divisors on $\mathbb{P}^1$, where $D_1 = \sum_p s_p p$ and $D_2 = \sum_p t_p p$, a *relevant pair* if:

(1) $\deg D_2 - \deg D_1 \equiv 2 \bmod |G|$.
(2) If H, J are homogeneous forms corresponding to D_1, D_2 respectively, then for the action of every element of the preimage of G in SL_2, H and J are eigenvectors with equal eigenvalues.
(3) For all $p \in \mathbb{P}^1$, $t_p > 1 \Rightarrow s_p = 0$.
(4) For all $p \in \mathbb{P}^1$, $s_p, t_p < |G_p|$.

Definition 3.10. Similarly, an effective divisor $D = \sum t_p p$ is a *relevant divisor* if:

(1) D is fixed by G.

(2) All t_p are either 0 or 1.
(3) Every point in $\mathrm{supp}(D)$ has a degenerate orbit.

If D is any effective divisor and G a finite subgroup of PGL_2, we now define a new divisor D_G to be the effective divisor obtained D by subtracting as many divisors of the form

$$\sum_{\sigma\in G} P^\sigma$$

as possible. More precisely, let $O_1,\dots,O_n$ be the complete orbits that occur in $\mathrm{supp}(D)$. For each $i=1,\dots,n$, let $O_i=\{P_i^1,\dots,P_i^{m_i}\}$. Let $n_i=|\mathrm{Stab}_G(O_i)|$ (i.e. the order of the stabilizer of any point in the orbit O_i). Note that $|G|=m_in_i$, and for any $P\in O_i$,

$$\sum_{\sigma\in G} P^\sigma=\sum_{j=1}^{m_i} n_iP_i^j=n_i\sum_{j=1}^{m_i}P_i^j=:\mathcal{O}_i.$$

Now, let N_i be the maximum number such that $D-N_i\mathcal{O}_i$ is effective. Define

$$D_G=D-\sum_i N_i\mathcal{O}_i.$$

Note that since $|\mathcal{O}_i|=n_im_i=|G|$, $|D_G|\equiv|D| \bmod |G|$. Note also that if F is a homogeneous form, then $\mathrm{div}(F)$ is G-invariant, and therefore contains only complete orbits of points. However, $\mathrm{div}(F)_G$ may still be non-trivial if $\mathrm{div}(F)$ contains a degenerate orbit with multiplicity not divisible by the order of the stabilizer of that orbit.

Let $\bar{w}(\mathrm{Rat}_d)^G=\bar{w}(\mathrm{Rat}_d^G)$ be the fixed set for the action of G on $\bar{w}(\mathrm{Rat}_d)$.

Lemma 3.11. *Let G be a finite subgroup of* PGL_2*. Let $[(H,J)]$ be in $\bar{w}(\mathrm{Rat}_d)^G$. If $H\neq 0$, then*

$$(\mathrm{div}(H)_G,\mathrm{div}(J)_G)$$

is a relevant pair. Furthermore, if $H=0$ then $\mathrm{div}(J)_G$ *is a relevant divisor.*

Proof. For $H\neq 0$, condition (1) of Definition 3.9 follows from the fact that H is of degree $d-1$, J is of degree $d+1$, and any divisors of the form $\sum_{g\in G} gp$ that we subtract in passing to $(\mathrm{div}(H)_G,\mathrm{div}(J)_G)$ have degree $|G|$. For (2), being in $\bar{w}(\mathrm{Rat}_d)^G$ implies that H and J are eigenvectors with equal eigenvalues for the preimage of G in SL_2, and then by Lemma 3.7 this is unchanged when removing the linear factors corresponding to the divisors that were subtracted when passing to $(\mathrm{div}(H)_G,\mathrm{div}(J)_G)$. The requirement in (3) follows immediately for $(\mathrm{div}(H),\mathrm{div}(J))$ by Lemma 3.4, and remains true when passing to $(\mathrm{div}(H)_G,\mathrm{div}(J)_G)$ because any t_p can only get smaller in this process. For (4), first note that the divisors $\mathrm{div}(H)$ and $\mathrm{div}(J)$ are fixed by the action of G. Divisors of the form $\sum_{g\in G} gp$ are fixed by the action of G, and so when we remove them, the resulting divisor pair $(\mathrm{div}(H)_G,\mathrm{div}(J)_G)$

is also fixed by G. Note also that if p is in the support of a divisor fixed by G, then clearly the entire orbit of p is in that support as well, since otherwise the divisor would change under the action of G. It is also clear that the coefficients in a divisor must be equal for all points in the same orbit. Thus if any of the coefficients s_p or t_p of $(\mathrm{div}(H)_G, \mathrm{div}(J)_G)$ are greater than or equal to $|G_p|$, it would be possible to subtract a divisor of the form $\sum_{g\in G} gp$, contrary to the definition of D_G.

For $H = 0$, (1) of Definition 3.10 is clear and (2) follows as above from Lemma 3.4. For (3), if a given p does not have a degenerate orbit, then the divisor $\sum_{g\in G} gp$ has coefficients all equal to 1. Since, as above, the full orbit of p occurs in the divisor $\mathrm{div}(J)$ with equal coefficients, it follows that the entire orbit will be subtracted off in passing to $\mathrm{div}(J)_G$, leaving only degenerate orbits. ∎

We have therefore defined a map

$$S : \bar{w}(\mathrm{Rat}_d)^G \to \text{relevant pairs} \cup \text{relevant divisors}.$$

Proposition 3.12. *A relevant pair (D_1, D_2) is in the image of S if and only if*

(1) $\deg D_1 \le d - 1$,
(2) $\deg D_2 \le d + 1$, *and* $\deg D_2 \equiv d + 1 \bmod |G|$,

Proof. Let $(D_1, D_2) = (\mathrm{div}(H)_G, \mathrm{div}(J)_G)$ be in the image of S. Then $\deg D_1 = \deg \mathrm{div}(H)_G \le \deg \mathrm{div}(H) = d - 1$, and similarly $\deg(D_2) = \deg \mathrm{div}(J)_G \le \deg \mathrm{div}(J) = d+1$. Since D_2 is obtained by subtracting orbits of points from $\mathrm{div}(J)$, we infer that $d+1 = \deg \mathrm{div}(J) = \deg \mathrm{div}(J)_G + m|G|$, so $\deg D_2 \equiv d + 1 \bmod |G|$.

Conversely, let (D_1, D_2) be a relevant pair of divisors. We aim to construct a pair of forms (H, J) of degree $d - 1$ and $d + 1$, respectively, such that

$$(D_1, D_2) = (\mathrm{div}(H)_G, \mathrm{div}(J)_G).$$

Let H_0 and J_0 be forms associated to D_1 and D_2. Since (D_1, D_2) is a relevant pair, we have $\deg J_0 = \deg H_0 + 2 + m|G|$ for some m. We also know that, for every action of the preimage of G in SL_2, H_0 and J_0 are eigenvectors with equal eigenvalues, no multiple zero of J_0 is a zero of H_0, and finally every zero of H_0 or J_0 is degenerate. By condition (2) of this proposition, we see that $\deg J_0 + \ell|G| = d + 1$. Pick ℓ non-degenerate orbits of $\mathbb{P}^1$ and let J_1 be the product of these orbits as homogeneous forms. Set $J = J_0 J_1$. It is clear that $\mathrm{div}(J)_G = J_0$. Similarly, $\deg(H_0) + n|G| = d - 1$. Choose n non-degenerate orbits of $\mathbb{P}^1$ distinct from the previous ℓ ones. Let H_1 be the product of these orbits as homogeneous forms and set $H = H_0 H_1$. Again, $\mathrm{div}(H)_G = H_0$.

That H and J are G-invariant follows by condition (2) of a relevant pair and the construction of H_1 and J_1. By construction, any multiple zeros potentially added to J will not be zeros of H, so by Lemma 3.4, $[(tH, J)]$ is in $\bar{w}(\mathrm{Rat}_d)$ for all but finitely many t. ∎

We have a similar description of relevant divisors.

PROPOSITION 3.13. *A relevant divisor D is in the image of S if and only if*

(1) $\deg D \le d + 1$,
(2) $\deg D \equiv d + 1 \bmod |G|$.

Proof. This proof is entirely analogous to that of Proposition 3.12, and therefore we omit it. ∎

Propositions 3.12 and 3.13 now allow us to calculate the dimension of fibers over a relevant pair (D_1, D_2) or a relevant divisor D:

$$\begin{aligned}\dim S^{-1}(D_1, D_2) &= \frac{d-1-\deg D_1}{|G|} + \frac{d+1-\deg D_2}{|G|} + 1 \\ &= \frac{2d - (\deg D_1 + \deg D_2)}{|G|} + 1 \\ &= \left[\frac{2d}{|G|}\right] + 1 - \left[\frac{\deg D_1 + \deg D_2}{|G|}\right], \\ \dim S^{-1}(D) &= \frac{d+1-\deg D}{|G|}.\end{aligned}$$

PROPOSITION 3.14. *Fix a finite non-trivial subgroup $G \subset \mathrm{PGL}_2$. Then there are only finitely many relevant pairs and relevant divisors on $\mathbb{P}^1$ for that group.*

Proof. Condition (4) in the definition of a relevant pair implies that a relevant pair only involves degenerate orbits, since if $G_p = 1$, then $s_p = t_p = 0$. Degenerate orbits occur only for points that have non-trivial stabilizer, and there are clearly only finitely many such points in $\mathbb{P}^1$. We have a bound on the size of the coefficient of each point in our pair of divisors, and so the result follows. The proof for relevant divisors is similar. ∎

Since there are finitely many relevant pairs and relevant divisors, it is now clear how to calculate $\dim \mathrm{Rat}_d^G$.

LEMMA 3.15. *Let G be one of A_4, S_4, A_5. Then there are three degenerate orbits, $2^3 = 8$ relevant divisors and eight relevant pairs which are of the form $(D_1(D_2), D_2)$ where $D_2 = \sum_p t_p p$ is a relevant divisor and $D_1(D_2) = \sum_p s_p p$ is defined by*

$$s_p = \begin{cases} 0 & \text{if } t_p = 1, \\ |G_p| - 1 & \text{if } t_p = 0. \end{cases}$$

Proof. The group C of characters of G is cyclic of order $3,2,1$ for $G = A_4, S_4, A_5$ resp. Let G be one of these groups. Let χ denote a generator for the group of characters. Using geometry of a regular solid and Lemma 3.6, one sees that degenerate orbits of G and their corresponding characters are as follows:

Table 2.

A_4		S_4		A_5	
Size	Character	Size	Character	Size	Character
4	χ	8	1	12	1
6	1	12	χ	30	1
4	χ^{-1}	6	χ	20	1

Let H be the quotient of Div^G, the group of divisors on $\mathbb{P}^1$ fixed by G, by the subgroup generated by all divisors of the form $\sum_{\sigma\in G} P^\sigma$. Then H is a finite group. Each divisor fixed by G gives rise to a character, and this induces a surjective homomorphism

$$\alpha : H \to C.$$

Let $n = |G|$. The degree of the divisor gives rise to a homomorphism

$$\overline{\deg} : H \to 2\mathbb{Z}/n\mathbb{Z}.$$

Let u be the image in H of the divisor which is the sum of all points in $\mathbb{P}^1$ with non-trivial stabilizers in G, with all multiplicities equal to 1:

$$\sum_{|G_P|\neq\langle 1\rangle} P \mapsto u \in H.$$

From Table 2 we see that $\deg u = n+2$ and $\alpha(u) = 1$. The element u shows that the resulting homomorphism

$$H \to C \times 2\mathbb{Z}/n\mathbb{Z}$$

is surjective, while counting the orders of the groups shows that this homomorphism is in fact an isomorphism. Pairs of divisors satisfying conditions (1), (2) and (4) of the definition of relevant pairs (Definition 3.9) correspond bijectively to pairs of the form $(x, x+u)$ where $x \in H$. Imposing the remaining condition (3) on such pairs yields the desired result. ■

THEOREM 3.16. *Let $d > 1$.*

(1) $\mathcal{A}(A_4) \neq \emptyset$ *if and only if d is odd.*
(2) $\mathcal{A}(S_4) \neq \emptyset$ *if and only if d is coprime to* 6.
(3) $\mathcal{A}(A_5) \neq \emptyset$ *if and only if d is congruent to one of* $1, 11, 19, 29$ *modulo* 30.

Let G be any of these groups and suppose d is such that $\mathcal{A}(G) \neq \emptyset$. Then $\mathcal{A}(G)$ is irreducible and

$$\dim \mathcal{A}(G) = \left[\frac{2d}{|G|}\right].$$

Proof. In the notation of Lemma 3.15, $S^{-1}(D_2) \cup S^{-1}(D_1(D_2), D_2)$ is closed and irreducible in Rat_d. For S_4 and A_5 the eight relevant divisors occur for distinct residues of d modulo $|G|$, while for A_4 this is not the case, but the resulting multiple components are conjugate. The total number of points in the three degenerate orbits of G equals $|G| + 2$. With the aid of this fact, one deduces from the formula above that

$$\left[\frac{\deg D_1(D_2) + \deg D_2}{|G|}\right] = 1$$

for all relevant divisors. That proves the proposition for $d > |G|$. For $d \leq |G|$, the formula is easy to check. ∎

COROLLARY 3.17. *Let G be the group of rotations of a platonic solid, and $d > 1$ such that $\mathcal{A}(G) \neq \emptyset$. Then $\mathcal{A}(G)$ contains a function with distinct fixed points and all multipliers equal to $d + 1$.*

EXAMPLE 3.18. Let $d = 5$. Then $\mathcal{A}(A_4) = \mathcal{A}(S_4)$ is a single point in $\mathcal{M}_5$ given by

$$f(z) = \frac{z^5 - 5z}{1 - 5z^4}.$$

4. Luna's Etale Slice Theorem. In this section we state Luna's Etale Slice Theorem and related corollaries that we will be using. For the proofs, we refer to Luna's original paper [9] or to Drézet [3].

By an (*algebraic*) *variety* we shall mean a scheme of finite type over a field, which we assume to be algebraically closed and of characteristic 0. Throughout this section, G is a reductive algebraic group, and $\phi : X \to Y$ a G-morphism of affine algebraic varieties, $\pi_X : X \to X//G$ and $\pi_Y : Y \to Y//G$ the quotient morphisms, and finally $x \in X$ a point whose orbit Gx is closed.

DEFINITION 4.1 (Strongly etale morphism [3, Def. 4.14]). We say that $\phi : X \to Y$ is *strongly etale* if $\phi_{/G} : X//G \to Y//G$, the morphism induced by ϕ, is etale and $\phi_{/G} \circ \pi_X = \pi_Y \circ \phi$ is a cartesian square.

PROPOSITION 4.2 (Properties of strongly etale morphisms [3, Prop. 4.15]). *Let $\phi : X \to Y$ be strongly etale. Then:*

(1) *ϕ is etale.*
(2) *For every $u \in X//G$, ϕ induces an isomorphism $\pi_X^{-1}(u) \cong \pi_Y^{-1}(\phi_{/G}(u))$.*

(3) *For every $x \in X$, the restriction of ϕ to Gx is injective, and Gx is closed if and only if $G\phi(x)$ is closed.*

Let H be an algebraic subgroup of G, and assume that H is reductive and acts on the left on an affine variety V. Let H act on $G \times V$ via

$$H \times (G \times V) \to G \times V, \qquad (h, (g, v)) \mapsto (gh^{-1}, hv).$$

Under this action, $G \times V$ is a principal H-bundle which admits a *geometric* quotient.

Definition 4.3 (Extension of group action). We denote

$$G \times_H V := (G \times V)/H$$

and say that $G \times_H V$ was obtained from V by *extending the action of H to G.*

Proposition 4.4 ([3, Prop. 4.9]). *Let $X = G \times_H V$. Then:*

(1) *The projection $G \times V \to V$ induces an isomorphism $X//G \cong V//H$.*
(2) *For any H-morphism $\theta : V \to W$, if we denote by $i_V : V \to \overline{G \times_H V}$ and $i_W : \overline{G \times_H W}$ the closed immersions given by $v \mapsto \overline{(e, v)}$ and $w \mapsto \overline{(e, w)}$, the following is a cartesian square:*

$$i_W \circ \theta = (G \times_H \theta) \circ i_V.$$

(3) *Let $\phi : X \to Y$ be a G-morphism, W an H-variety and $\theta : W \to Y$ an H-morphism. Then the projections of the product $W \times_Y X$ induce an isomorphism*

$$G \times_H (W \times_Y X) \cong (G \times_H W) \times_Y X.$$

(4) *For every $u \in X//G = V//H$, we have a canonical isomorphism*

$$\pi_X^{-1}(u) \cong G \times_H \pi_V^{-1}(u).$$

(5) *Let $v \in V$, $g \in G$, $u = \overline{(g, v)} \in X$. Then $G_u = gG_v g^{-1}$.*
(6) *Let $X' \subset X$ be a closed G-invariant subvariety. Then there exists a closed H-invariant subvariety $V' \subset V$ such that $X' = G \times_H V'$.*
(7) *The closed orbits of X are the subvarieties $G \times_H Hv$ where Hv is a closed orbit of V.*
(8) *If $v \in V$ and $\bar{v}$ is the image of (e, v) in X, then we have a canonical isomorphism*

$$T(X_{\bar{v}}) \cong (T_e(G) \oplus T_v(V))/T_e(H),$$

where the inclusion $T_e(H) \subset T_e(G) \oplus T_v(V)$ comes from the tangent map at e of the morphism

$$H \to G \times V, \qquad h \mapsto (h^{-1}, hx).$$

Moreover, if V is smooth at v, then X is smooth at $\bar{v}$.

Definition 4.5 (Etale slice). Let $x \in X$ be a point whose orbit is closed. We call a locally closed subvariety V of X an *etale slice at* x if

(1) $x \in V$ and V is invariant under the action of the stabilizer G_x,
(2) the inclusion of V into X induces a strongly etale morphism $G \times_{G_x} V \to X$.

Theorem 4.6 (Luna's Etale Slice Theorem [3, Thm. 5.3]). *There exists an etale slice at every point x in X whose orbit Gx is closed.*

Theorem 4.7 (Smooth version [3, Thm. 5.4]). *Let $x \in X$ be a point whose orbit is closed and where X is smooth. Let $N_x = T_x(X)/T_x(Gx)$ be the normal space to the orbit of x. Consider the natural action of G_x on N_x. Then there exists an etale slice V at x which admits a strongly etale G_x-morphism $\phi : V \to N_x$ such that $\phi(x) = 0$, and the map of tangent spaces $T_x(V) \to T_0(N_x) \cong N_x$ induced by ϕ is the same as the one induced by the inclusion $V \hookrightarrow X$.*

Corollary 4.8 (to Theorem 4.6). *Let x be a point whose orbit is closed. Then there exists an open set W containing $\pi_X(x)$ such that the stabilizer of every point in $\pi_X^{-1}(W)$ is conjugate to a subgroup of G_x.*

Proof. Let V be an etale slice at x. Then the inclusion $V \hookrightarrow X$ induces a strongly etale G-morphism $\psi : G \times_{G_x} V \to X$, and we can take W to be the range of $\pi_X \circ \psi$. Indeed, Proposition 4.2(3) shows that ψ, as a strongly etale morphism, preserves stabilizers, and the assertion follows from Proposition 4.4(5). ∎

Definition 4.9. Let H be any group, and let $A_X(H)$ denote the locus in X of those points x for which G_x contains a subgroup isomorphic to H. Let $\mathcal{A}_X(H)$ be the image of $A_X(H)$ in $X//G$.

Definition 4.10. A set S in X is *saturated* if $S = \pi_X^{-1}(\pi_X(S))$.

Corollary 4.11. *For any H, $A_X(H)$ is closed and saturated. Let $x \in A_X(H)$ be a point with a closed orbit, V an etale slice at x. Then*

$$\dim_{\pi_X(x)} \mathcal{A}_X(H) = \dim_{\pi_V(x)} \mathcal{A}_V(H).$$

Moreover, if X is smooth at x, the above dimensions are also equal to

$$\dim \mathcal{A}_{N_x}(H), \qquad \text{where } N_x = T_x(X)/T_x(Gx).$$

5. Automorphism locus equals singular locus. In this section we combine D. Luna's Etale Slice Theorem and the dimension calculations of Section 2 to show that the singular locus of $\mathcal{M}_d$ coincides with its automorphism locus for $d > 2$. The impetus for the work presented in this section is an example of a family of singularities in $\mathcal{M}_3$ in John Milnor's dynamics book [10]. Effectively, in that example he identifies an etale slice and uses it to find singularities. The following is a variation on a definition suggested by John Milnor in personal correspondence [12].

DEFINITION 5.1. We call a point $[\phi] \in \mathcal{M}_d(k)$ *simple* if $\operatorname{Aut}\phi$ is cyclic of prime order.

LEMMA 5.2. *Let p be a prime number and let H be a cyclic group of order p, acting on a vector space V. Suppose* $\operatorname{codim}(V^H, V) > 1$. *Let $x \in V$, and let $\pi : V \to V/H$ be the quotient map. Then $\pi(x)$ in V/H is a singular point precisely when the stabilizer H_x is non-trivial.*

Proof. Let $x \in V$ have non-trivial stabilizer, i.e. $H_x = H$. If $h \in H$ and $v \in V$, then $h(v - x) = hv - x$, so translation by $-x$ commutes with the action of H. This implies that the affine isomorphism $v \mapsto v - x$ induces an isomorphism $V/H \to V/H$ sending the image of x to that of the origin. Therefore, without loss of generality, we may assume $x = 0$.

Let C denote the preimage in V of the singular locus of V/H. If $\lambda \in k$, $v \in V$, then $\pi(\lambda v)$ is singular iff $\pi(v)$ is singular. Because C is closed, this implies that the image of the origin is a singular point iff V/H is itself singular. We now apply the Chevalley–Shephard–Todd theorem [1]: V/H is non-singular iff H is generated by pseudo-reflections. Because of the assumption $\operatorname{codim}(V^H, V) > 1$, there are no pseudo-reflections, and thus V/H is singular, and hence 0 is singular.

The converse follows from the fact that, at the points where the stabilizer is trivial, the quotient map $V \to V/H$ is etale [3, Prop. 4.11]. ∎

REMARK 5.3. In fact, over $\mathbb{C}$, for such x, letting $\ell = 2\dim V^H + 2$, we have

$$H_\ell(V/H, V/H - \bar{x}) = \mathbb{Z}/p\mathbb{Z}.$$

This implies the singularities of $\mathcal{M}_d$ are topological, not merely algebraic.

LEMMA 5.4. *Let $d > 2$. Simple points of $\mathcal{M}_d(k)$ are singular.*

Proof. Let $\phi \in \operatorname{Rat}_d$, H be the stabilizer of ϕ, and assume $H = \langle\sigma\rangle$ is cyclic of prime order p. Let N be the normal space at ϕ to the orbit of ϕ:

$$N = T_\phi(\operatorname{Rat}_d)/T_\phi(\operatorname{SL}_2 \cdot \phi).$$

There is an induced action of H on N. Consequently, by Luna's Etale Slice Theorem for smooth varieties (Theorem 4.7), the two rings—the local ring at $[\phi]$ in $\mathcal{M}_d$ and the local ring at the image of 0 in the quotient N/H—have isomorphic completions. Thus we need only prove that the image of 0 in N/H is a singular point. It follows easily from Corollary 4.11 to Luna's Etale Slice Theorem for smooth varieties that

$$\operatorname{codim}(N^H, N) = \operatorname{codim}(\mathcal{A}_p, \mathcal{M}_d).$$

The latter codimension is at least 2 for $d > 2$, as shown in Section 2. Therefore, by Lemma 5.2, $[\phi]$ is singular in $\mathcal{M}_d$. ■

REMARK 5.5 (Simple points are topologically singular). If $\pi : \mathrm{Rat}_d \to \mathcal{M}_d$ is the quotient map, then $\pi^{an} : \mathrm{Rat}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ is a topological quotient with respect to the classical topology (cf. Neeman [14]). Moreover, isomorphism of completions of local rings in the proof above implies that in classical topology, there is a neighborhood of $[\phi]$ in $\mathcal{M}_d$ which is homeomorphic to a neighborhood of the image of 0 in N/H. The latter image is topologically singular according to Lemma 5.2.

LEMMA 5.6. *Simple points are dense in the automorphism locus $\mathcal{A}$.*

Proof. It suffices to show that simple points are dense in $\mathcal{A}_p$ for each prime divisor p of $(d-1)d(d+1)$. Start with $\mathcal{A}_2$. Recall that D_m denotes the dihedral group of order $2m$ and consider

$$\mathcal{A}_2 \setminus \Big(\bigcup_{m>2} \mathcal{A}_m \cup \mathcal{A}(D_2) \Big).$$

This is a dense subset of $\mathcal{A}_2$ because $\dim \mathcal{A}_m < \dim \mathcal{A}_2$ for $m > 2$ (see Remark 2.9) and $\dim \mathcal{A}(D_2) < \dim \mathcal{A}_2$ by Proposition 2.12. It contains only points with stabilizer isomorphic to $\mathbb{Z}/2\mathbb{Z}$, since every platonic solid has a symmetry of order 3.

Let now p be a prime greater than 2. If $\mathcal{A}_p \subseteq \mathcal{A}_2$, simple points with stabilizer isomorphic to $\mathbb{Z}/2\mathbb{Z}$ are already dense in A_p. Otherwise

$$\mathcal{A}_p \setminus (\mathcal{A}_2 \cup \mathcal{A}_{2p} \cup \mathcal{A}_{3p} \cup \cdots)$$

is non-empty, open, and consists of points with stabilizer isomorphic to $\mathbb{Z}/p\mathbb{Z}$. It is non-empty because $\dim \mathcal{A}_{kp} < \dim \mathcal{A}_p$ for $k > 1$, as seen from the formula in Proposition 2.7. It consists of simple points because dihedral groups and symmetry groups of platonic solids contain an element of order 2. ■

EXAMPLE 5.7. In the case of degree 2 maps, we have $\mathcal{A}_3 = \mathcal{A}(D_3)$, hence simple points are not dense in $\mathcal{A}_3$. This agrees with the well-known result that $\mathcal{M}_2$ is smooth, but has non-trivial automorphism locus.

THEOREM 5.8. *When $d > 2$, the singular locus of $\mathcal{M}_d$ coincides with the automorphism locus: $\mathcal{A} = \mathcal{S}$.*

Proof. The singular locus is closed and Lemma 5.6 shows that $\mathcal{S} \cap \mathcal{A}$ is dense in the automorphism locus, hence $\mathcal{A}$ is contained in the singular locus $\mathcal{S}$. The converse follows directly from Luna's Etale Slice Theorem for smooth varieties (Theorem 4.7). We conclude that $\mathcal{A} = \mathcal{S}$. ■

6. The Picard and class groups of $\mathcal{M}_d$, $\mathcal{M}_d^s$, and $\mathcal{M}_d^{ss}$. In this section we use the results of Sections 2 and 5 to compute the Picard and class groups of $\mathcal{M}_d$, $\mathcal{M}_d^s$, and $\mathcal{M}_d^{ss}$, where $\mathcal{M}_d^s$ and $\mathcal{M}_d^{ss}$ denote the moduli spaces of stable and semistable rational maps in the sense of Geometric Invariant Theory (GIT). In this section we work over an algebraically closed field k of characteristic 0. We will freely use terminology from GIT, for which we refer the reader to [13]. We begin by applying standard algebraic geometry results to compute the Picard groups of Rat_d, $(\mathbb{P}^{2d+1})^s$, and $(\mathbb{P}^{2d+1})^{ss}$.

LEMMA 6.1. *Let* $d > 1$. *Then* $\mathrm{Pic}(\mathrm{Rat}_d) \cong \mathbb{Z}/2d\mathbb{Z}$.

Proof. As $\mathrm{Rat}_d = \mathbb{P}^n \setminus V(\mathrm{Res})$ is an affine open subscheme of projective space, it immediately follows that $\mathrm{Pic}(\mathrm{Rat}_d) \cong \mathrm{Cl}(\mathrm{Rat}_d)$, and it suffices to show that $\mathrm{Cl}(\mathrm{Rat}_d) \cong \mathbb{Z}/2d\mathbb{Z}$.

The resultant Res is a homogeneous polynomial of degree $2d$ on $\mathbb{P}^{2d+1}$. It follows from [6, Proposition 6.5] that the following sequence is exact:

$$\mathbb{Z} \to \mathrm{Cl}(\mathbb{P}^{2d+1}) \to \mathrm{Cl}(\mathrm{Rat}_d) \to 0$$

where the first map is defined by $1 \mapsto 1 \cdot V(\mathrm{Res})$. Using the well-known fact that $\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$ for any n, we conclude that $\mathrm{Cl}(\mathrm{Rat}_d) \cong \mathbb{Z}/\deg(\mathrm{Res})\mathbb{Z} = \mathbb{Z}/2d\mathbb{Z}$. ∎

In fact, this lemma proves something stronger: $\mathrm{Pic}(\mathrm{Rat}_d)$ is cyclic of order $2d$ generated by $\mathcal{O}(1)|_{\mathrm{Rat}_d}$.

We must now define some terms from GIT.

DEFINITION 6.2. For a group variety G acting on a variety X via the action $\sigma : G \times X \to X$ and for any invertible sheaf $\mathcal{L} \in \mathrm{Pic}(X)$, a *$G$-linearization* of $\mathcal{L}$ is an isomorphism $\sigma^*\mathcal{L} \to p_2^*\mathcal{L}$, where p_2 is the projection $G \times X \to X$, subject to a certain cocycle condition which is equivalent to requiring that the corresponding line bundle is equipped with a G-action which is compatible with the G-action on X.

We refer to [13, Section 1.3] for further details. The tensor product of two G-linearized line sheaves and the inverse of a G-linearized line sheaf carry natural G-linearizations. The collection of G-linearized line sheaves, modulo isomorphism, forms an abelian group denoted by $\mathrm{Pic}^G(X)$. For a G-linear morphism $f : X \to Y$ of varieties with G-actions there is an induced pull-back homomorphism $f^* : \mathrm{Pic}^G(Y) \to \mathrm{Pic}^G(X)$. There is a natural morphism $\mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$ given by forgetting the G-linearization. If we fix a G-linearized line sheaf on the variety X, then we can describe the largest subset of X which has a geometric quotient.

DEFINITION 6.3. Let G be an algebraic group acting on a variety X, all over k, and let $\mathcal{L}$ be a G-linearized line sheaf. Let $x \in X$ be a geometric point. We call x *semistable* if for some n there exists a section $s \in H^0(X, \mathcal{L}^n)$ such that $s(x) \neq 0$, the fiber X_s is affine, and s is invariant. If furthermore the action of G on X_s is closed, then we call x *stable*.

We let X^s denote the stable locus for the action of G on X and fixed linearization, and X^{ss} the semistable locus. It is clear from the definition that these are open subsets. In this paper, $(\mathbb{P}^{2d+1})^s$ and $(\mathbb{P}^{2d+1})^{ss}$ refer to the stable and semistable loci for the SL_2-linearization of $\mathcal{O}(1)$ on $\mathbb{P}^{2d+1}$ for the SL_2-action by conjugation.

LEMMA 6.4. *Let $d > 1$ and let $(\mathbb{P}^{2d+1})^s$ and $(\mathbb{P}^{2d+1})^{ss}$ denote the stable and semistable loci for the SL_2-action by conjugation on $\mathbb{P}^{2d+1}$. Then the open immersions $(\mathbb{P}^{2d+1})^s \subset (\mathbb{P}^{2d+1})^{ss} \subset \mathbb{P}^{2d+1}$ induce isomorphisms $\mathrm{Pic}((\mathbb{P}^{2d+1})^s) \cong \mathrm{Pic}((\mathbb{P}^{2d+1})^{ss}) \cong \mathrm{Pic}(\mathbb{P}^{2d+1}) \cong \mathbb{Z}$.*

Proof. Let $Z^s = \mathbb{P}^{2d+1} \setminus (\mathbb{P}^{2d+1})^s$ and $Z^{ss} = \mathbb{P}^{2d+1} \setminus (\mathbb{P}^{2d+1})^{ss}$ be the unstable and non-semistable loci, respectively. As the stable and semistable loci are open, it follows that Z^s and Z^{ss} are closed subsets of $\mathbb{P}^{2d+1}$. Furthermore, Z^s, Z^{ss} are proper closed subsets of the irreducible closed subset $V(\mathrm{Res})$. Consequently, $\mathrm{codim}(Z^s, \mathbb{P}^{2d+1}) > 1$ and $\mathrm{codim}(Z^{ss}, \mathbb{P}^{2d+1}) > 1$. It follows from [6, Prop. 6.5] that $\mathrm{Pic}((\mathbb{P}^{2d+1})^s) \cong \mathrm{Pic}(\mathbb{P}^{2d+1}) \cong \mathbb{Z}$, and similarly $\mathrm{Pic}((\mathbb{P}^{2d+1})^{ss}) \cong \mathbb{Z}$. ∎

Let $A \subset \mathbb{P}^{2d+1}$ be the subset of points of $\mathbb{P}^{2d+1}$ which have a non-trivial stabilizer for the PGL_2-action by conjugation. Then A is closed in $(\mathbb{P}^{2d+1})^{ss}$ as a consequence of Corollary 4.8 to Luna's Etale Slice Theorem and the fact that $(\mathbb{P}^{2d+1})^{ss}$ can be covered by PGL_2-invariant affine open sets.

DEFINITION 6.5. The set $\mathbb{P}^{2d+1}_* = \mathbb{P}^{2d+1} - A$ is the *free locus* of $\mathbb{P}^{2d+1}$ for the PGL_2-action by conjugation. For any subvariety of $X \subset \mathbb{P}^{2d+1}$ to which the PGL_2-action restricts, we use the lower star to indicate the intersection with the free locus: $X_* = X \cap \mathbb{P}^{2d+1}_*$.

It immediately follows from the definition that for any subvariety of the semistable locus, $X \subseteq (\mathbb{P}^{2d+1})^{ss}$, X_* is an open subvariety of X and the restricted action of PGL_2 on X_* is free.

PROPOSITION 6.6. *Let $d > 1$ be an integer. Then*

$$\mathrm{Pic}(\mathrm{Rat}_{d*}) \cong \mathrm{Pic}(\mathrm{Rat}_d),$$
$$\mathrm{Pic}((\mathbb{P}^{2d+1})^s_*) \cong \mathrm{Pic}((\mathbb{P}^{2d+1})^s),$$
$$\mathrm{Pic}((\mathbb{P}^{2d+1})^{ss}_*) \cong \mathrm{Pic}((\mathbb{P}^{2d+1})^{ss}).$$

Proof. Since the varieties involved are non-singular, class group is isomorphic to Picard group, and the proof follows immediately by using Proposition 2.14 on the codimension of the automorphism locus in $\mathbb{P}^{2d+1}$ and [6, Prop. 6.5]. ∎

We will use the following Descent Lemma [4, Théorème 2.3].

THEOREM 6.7 (Descent Lemma). *Let $\pi : X \to Y$ be a good quotient. Let F be a G-vector bundle on X. Then F descends to Y if and only if for* *all closed points x of X such that the orbit Gx is closed, the stabilizer of x in G acts trivially on the fiber F_x.*

Note: Theorem 6.7 is true for a good quotient, which is a more general notion than a geometric quotient, but we will not need this in what follows.

COROLLARY 6.8. *There is a natural injection*

$$\pi^\star : \mathrm{Pic}(Y) \to \mathrm{Pic}^G(X),$$

whose image consists of all isomorphism classes of G-linearized line bundles L on X with the following property: For every closed point x in X for which the orbit Gx is closed, the stabilizer of x in G acts trivially on the fiber L_x.

Proof. It is easy to show that the map $\pi^\star : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is injective when π is a good quotient. Furthermore, $\mathrm{Pic}(X) \to \mathrm{Pic}^G(X)$ is injective, proving the corollary. ■

PROPOSITION 6.9. *Let X be a normal irreducible variety over k with an* SL_2*-action. Then any $\mathcal{L} \in \mathrm{Pic}(X)$ admits a unique* SL_2*-linearization.*

Proof. By [2, Thm. 7.2], the natural map $\alpha : \mathrm{Pic}^{\mathrm{SL}_2}(X) \to \mathrm{Pic}(X)$ fits into an exact sequence

$$0 \to \mathrm{Ker}(\alpha) \to \mathrm{Pic}^{\mathrm{SL}_2}(X) \overset{\alpha}{\to} \mathrm{Pic}(X) \to \mathrm{Pic}(\mathrm{SL}_2).$$

$\mathrm{Pic}(\mathrm{SL}_2)$ is known to be trivial and consequently α is surjective. $\mathrm{Ker}(\alpha)$ consists of the isomorphism classes of SL_2-linearizations on the trivial bundle $X \times \mathbb{A}^1$. In fact, by [2, Cor. 7.1], $\mathrm{Ker}(\alpha) \cong \mathrm{Hom}^{\mathrm{n}}_{\mathrm{d}}(\mathrm{SL}_2, \mathbb{G}_{\mathrm{m}})$, i.e. $\mathrm{Ker}(\alpha)$ is isomorphic to the character group of SL_2. The character group of a connected affine algebraic group which is complete is trivial, so $\mathrm{Ker}(\alpha) = 1$ and the proof is finished. ■

We note that Proposition 6.9 implies that there is a natural homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}^{\mathrm{SL}_2}(X)$ when X is normal and irreducible.

THEOREM 6.10. *In the notation above, there is a natural isomorphism between* $\mathrm{Rat}_{d*}/\mathrm{SL}_2$ *and* $\mathcal{M}_{d,*} = \mathcal{M}_d \setminus \mathcal{A}$.

Proof. This follows from basic facts of GIT. First, $\mathcal{M}_d$ is a geometric quotient. Next, Rat_{d*} is an open, SL_2-invariant subset of Rat_d, so it also has a geometric quotient.

In general, if G acts on X and $\mathcal{L} \in \mathrm{Pic}^G(X)$, then if $U, V \subset X$ are open sets such that $V \subset U \subset X^s(\mathcal{L})$, then $V/G \subset U/G$ is an open subset of the quotient, both being geometric.

It follows that $\mathrm{Rat}_{d*}/\mathrm{SL}_2 \subset \mathcal{M}_d$ is an open subvariety where all points have trivial stabilizer. Moreover, it is the largest such open subvariety. It therefore coincides with $\mathcal{M}_{d,*} = \mathcal{M}_d \setminus \mathcal{A}$. ■

Proposition 6.11.

(1) $\mathrm{Cl}(\mathcal{M}_d)$ *is* $\mathbb{Z}/2d\mathbb{Z}$ *when* d *is odd, and* $\mathbb{Z}/d\mathbb{Z}$ *when* d *is even.*
(2) $\mathrm{Cl}(\mathcal{M}_d^s) = \mathrm{Cl}(\mathcal{M}_d^{ss}) \cong \mathbb{Z}$.

Proof. (1) Note first that because Rat_d is a normal scheme, $\mathcal{M}_d$ is also normal (see [2, Prop. 3.1, p. 45]), and so [6, Prop. 6.5] applies. We can thus conclude that $\mathrm{Cl}(\mathcal{M}_d) = \mathrm{Cl}(\mathcal{M}_{d*})$. It follows from the previous theorems that $\mathcal{M}_{d*}$ is a good (in fact geometric) quotient of Rat_{d*} by SL_2, and by the Descent Lemma (Theorem 6.7) the image of $\mathrm{Pic}(\mathcal{M}_d)$ in $\mathrm{Pic}^{\mathrm{SL}_2}(\mathrm{Rat}_{d*})$ consists of the isomorphism classes of linearized line bundles L for which $\langle \pm I\rangle$ acts trivially on the fiber of L at every closed point of Rat_{d*}. By Lemma 2.15 the action of $-I$ on $\mathcal{O}_{\mathbb{P}^{2d+1}}(1)$ is trivial for d odd, and is via multiplication by -1 throughout when d is even.

Rat_{d*} is an open subscheme of projective space and is therefore normal. By Proposition 6.9 it follows that $\mathrm{Pic}^{\mathrm{SL}_2}(\mathrm{Rat}_{d*}) = \mathrm{Pic}(\mathrm{Rat}_{d*}) = \mathbb{Z}/2d\mathbb{Z}$. And our calculation shows that the image of $\mathrm{Pic}(\mathcal{M}_d)$ in $\mathrm{Pic}(\mathrm{Rat}_{d*})$ is generated by $\mathcal{O}(1)$ when d is odd, and by $\mathcal{O}(2)$ when d is even. We conclude that $\mathrm{Pic}(\mathcal{M}_{d*}) = \mathbb{Z}/2d\mathbb{Z}$ for d odd, and $\mathrm{Pic}(\mathcal{M}_{d*}) = \mathbb{Z}/d\mathbb{Z}$ for d even. Finally, since $\mathcal{M}_{d*}$ is non-singular (Proposition 5.8), $\mathrm{Pic}(\mathcal{M}_{d*}) = \mathrm{Cl}(\mathcal{M}_{d*})$, and since codimension of its complement in $\mathcal{M}_d$ is greater than 1, $\mathrm{Cl}(\mathcal{M}_{d*}) = \mathrm{Cl}(\mathcal{M}_d)$.

(2) The proof is nearly identical for $\mathcal{M}_d^s$. This approach requires more care for $\mathcal{M}_d^{ss}$ as $\mathcal{M}_d^{ss}$ is not a geometric quotient. Regardless, $\mathcal{M}_d^s$ is a dense open subset of $\mathcal{M}_d^{ss}$, and therefore $\mathcal{M}_d^s = \mathcal{M}_d^{ss} \setminus V$ for some closed subset V of codimension $\mathrm{codim}(V, \mathcal{M}_d^{ss}) > 1$. It follows that $\mathrm{Cl}(\mathcal{M}_d^{ss}) = \mathrm{Cl}(\mathcal{M}_d^s) = \mathbb{Z}$. ∎

Theorem 6.12. *Let* $d > 1$ *be an integer. Then* $\mathrm{Pic}(\mathcal{M}_d)$ *is trivial.*

Proof. By the Descent Lemma (Theorem 6.7), $\mathrm{Pic}(\mathcal{M}_d)$ can be identified with those elements L of

$$\mathrm{Pic}^{\mathrm{SL}_2}(\mathrm{Rat}_d) = \mathrm{Pic}(\mathrm{Rat}_d) = \mathbb{Z}\mathcal{O}(1)/\mathbb{Z}\mathcal{O}(2d)$$

for which the stabilizer of every function acts trivially on the corresponding fiber of L. We know that Rat_d contains functions ϕ with automorphisms $\sigma \in \mathrm{PGL}_2$ of order m as long as $m \mid d$ or $m \mid d \pm 1$. Let g represent σ in SL_2. Then g acts on the stalk of $\mathcal{O}(1)$ at ϕ by multiplication by an sth root of unity, where s is described in Corollary 2.16.

If d is even, take $m = d$. Then $s = 2d$, hence only line bundles $\mathcal{O}(2dk)$ descend to $\mathcal{M}_d$, but these are all trivial.

If d is odd, take $m = d$ again. Then $s = d$. Next take $m = d + 1$ or $m = d - 1$. Then $s = 2$. Again, only line bundles $\mathcal{O}(2dk)$ descend to $\mathcal{M}_d$, and these are all trivial. ∎

LEMMA 6.13. *Let $x \in (\mathbb{P}^{2d+1})^{ss}$ be a closed point. Then $x \in (\mathbb{P}^{2d+1})^{s}$ if and only if its orbit is closed in $(\mathbb{P}^{2d+1})^{ss}$ and its stabilizer is of finite order. Moreover, the order of the stabilizer of x is bounded as x runs through $(\mathbb{P}^{2d+1})^{s}$.*

Proof. For the first part, see [13, Amplification 1.11]. For the second, consider the quasi-finite maps

$$\mathrm{SL}_2 \times (\mathbb{P}^{2d+1})^{s} \to (\mathbb{P}^{2d+1})^{s} \times (\mathbb{P}^{2d+1})^{s}, \qquad (g,x) \mapsto (x,gx).$$

The stabilizers then correspond to the fibers over the diagonal, and quasi-finite maps of varieties have bounded fiber cardinality. ∎

THEOREM 6.14. *Let $d>1$ be an integer. The open immersion $\mathcal{M}_d^s \subset \mathcal{M}_d^{ss}$ induces an isomorphism $\mathrm{Pic}(\mathcal{M}_d^{ss}) \cong \mathrm{Pic}(\mathcal{M}_d^{s}) \cong \mathbb{Z}$.*

Proof. Recall that $\mathrm{Pic}((\mathbb{P}^{2d+1})^{s}) = \mathrm{Pic}((\mathbb{P}^{2d+1})^{ss})$ (cf. Lemma 6.4) is infinite cyclic generated by $\mathcal{O}(1)$.

Let us look at $\mathrm{Pic}(\mathcal{M}_d^{s})$ first. Denote by M a common multiple of orders of stabilizers of points in $(\mathbb{P}^{2d+1})^{s}$. Such an M exists by Lemma 6.13. Then $\mathcal{O}(M)$ descends to the quotient by the Descent Lemma (Theorem 6.7), and since the Picard subgroup of the quotient can be identified with line bundles which descend to the quotient, the conclusion follows.

Now let us look at $\mathrm{Pic}(\mathcal{M}_d^{ss})$ and assume that $d = 2m+1$, since for d even, $\mathcal{M}_d^{ss} = \mathcal{M}_d^{s}$. Let $x \in \mathrm{Pic}((\mathbb{P}^{2d+1})^{ss}) \setminus \mathrm{Pic}((\mathbb{P}^{2d+1})^{s})$ have a closed orbit. Then the stabilizer of x is infinite by Lemma 6.13. Direct computation using the numerical criterion [15, Thm. 4.40] shows that x is conjugate to either $(X^{m+1}Y^m : 0)$ or $(cX^{m+1}Y^m : X^mY^{m+1})$ with $\{f(z) = cz \mid c \neq 0\}$ as the stabilizer within PGL_2 in both cases. Using Lemma 2.15 one can see that in both cases the stabilizer within SL_2 acts trivially on the fiber of the line bundle $\mathcal{O}(1)$ at x. Hence such a point x does not obstruct the descent of line bundles, and thus a line bundle over $(\mathbb{P}^{2d+1})^{ss}$ descends to the quotient if and only if its restriction to $(\mathbb{P}^{2d+1})^{s}$ does. ∎

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