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NULL PSEUDO-ISOTROPIC LAGRANGIAN SURFACES

BY

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Dedicated to the memory of Witold Roter

Abstract. We show that a Lagrangian, Lorentzian surface M_1^2 in a complex space form $\widetilde{M}_1^2(4c)$ is pseudo-isotropic if and only if M is minimal. Next we obtain a complete classification of all Lagrangian, Lorentzian surfaces which are lightlike pseudo-isotropic but not pseudo-isotropic.

1. Introduction. The notion of isotropic submanifold was first introduced in [7] by O'Neill for immersions of Riemannian manifolds and recently extended by Cabrerizo, Fernández and Gómez [2] to the pseudo-Riemannian case. A submanifold is called *pseudo-isotropic* if, for any point p and any tangent vector v at p, we have

$$\langle h(v,v), h(v,v)\rangle = \widetilde{\lambda}(p)\langle v,v\rangle^2, \tag{1.1}$$

where h denotes the second fundamental form of the immersion and $\widetilde{\lambda}$ is a smooth function on the submanifold.

Note that since the induced metric is pseudo-Riemannian, it is natural to distinguish between timelike, spacelike and lightlike (or null) vectors. This leads in a natural way to the notions of

(i) *timelike pseudo-isotropic submanifold* if, for any point p and any timelike tangent vector v at p, equation (1.1) is satisfied,
(ii) *spacelike pseudo-isotropic submanifold* if, for any point p and any spacelike tangent vector v at p, equation (1.1) is satisfied,
(iii) *lightlike isotropic submanifold* if, for every lightlike vector v at p, $h(v,v)$ is again a lightlike vector.

It was shown in [2] that the notions of pseudo-isotropic, timelike pseudo-isotropic and spacelike pseudo-isotropic submanifold are equivalent. In the

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same paper they also included an example of an immersion which is lightlike pseudo-isotropic but not pseudo-isotropic.

Here we are particularly interested in Lagrangian immersions in complex space forms. In the positive definite case, isotropic Lagrangian immersions have been studied in [3]–[6] and [9]. In this paper we will consider pseudo-isotropic and lightlike pseudo-isotropic Lagrangian, Lorentzian surfaces M_1^2 in a complex space form $\widetilde{M}_1^2(4c)$. We will assume that the metric of the space form is not definite and hence has real signature 2. By changing the sign of the metric if necessary, it is sufficient to deal with the cases $c = 0$ or $c = 1$.

In Section 3 we will first show:

THEOREM 1.1. *Let M be a Lagrangian, Lorentzian surface in a complex space form. Then M is pseudo-isotropic if and only if M is minimal.*

Next we will obtain a complete classification of all Lagrangian, Lorentzian surfaces which are lightlike pseudo-isotropic but not pseudo-isotropic. To do so, we will first show in Section 4 that such a surface can be treated as the union of two surfaces (one of them possibly empty) that we will call of Type 1 and of Type 2. These will be classified, case by case, in Sections 5 and 6, respectively.

2. Preliminaries. Throughout this paper we will assume that M is a Lagrangian, Lorentzian submanifold M of a complex space form $\widetilde{M}$. We use the standard formulas of Gauss and Weingarten for a submanifold, introducing the second fundamental form h and the shape operators A by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi,$$

where X and Y are tangent vector fields and ξ is normal. Here, as usual, $\widetilde{\nabla}$ denotes the Levi-Civita connection on the ambient space and, if no confusion is possible, we will always identify M with its image in $\widetilde{M}$.

Since M is Lagrangian, the complex structure J interchanges the tangent and the normal spaces. In view of the formulas of Gauss and Codazzi this implies that

$$\nabla^\perp_X JY = J\nabla_X Y, \qquad A_{JX}Y = -Jh(X,Y) = A_{JY}X.$$

The latter formula implies that the cubic form $\langle h(X,Y), JZ\rangle$ is totally symmetric in all components.

We denote the curvature tensors of ∇ and $\nabla^\perp$ by R and $R^\perp$, respectively. The first covariant derivative of h is defined by

$$(\nabla h)(X,Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y), \tag{2.1}$$

where X, Y, Z and W are tangent vector fields.

The equations of Gauss, Codazzi and Ricci for a Lagrangian submanifold of $\widetilde{M}^n(4c)$ are

$$\langle R(X,Y)Z,W\rangle = \langle h(Y,Z),h(X,W)\rangle - \langle h(X,Z),h(Y,W)\rangle + c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle), \tag{2.2}$$

$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z), \tag{2.3}$$

$$\langle R^{\perp}(X,Y)JZ,JW\rangle = \langle [A_{JZ},A_{JW}]X,Y\rangle + c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle), \tag{2.4}$$

where X, Y Z and W are tangent vector fields. Note that for a Lagrangian submanifold the equations of Gauss and Ricci are mutually equivalent.

We refer to [1] for the construction of the standard models of indefinite complex space forms $\mathbb{C}P^n_s(4c)$ when $c>0$, $\mathbb{C}H^n_s(4c)$ when $c<0$ and $\mathbb{C}^n_s$. For our purposes, it is sufficient to know that there exist pseudo-Riemannian submersions, called Hopf fibrations, given by

$$\Pi : S^{2n+1}_{2s}(c) \to \mathbb{C}P^n_s(4c) : z \mapsto z\cdot\mathbb{C}^{\star}$$

if $c>0$, and by

$$\Pi : H^{2n+1}_{2s+1}(c) \to \mathbb{C}H^n_s(4c) : z \mapsto z\cdot\mathbb{C}^{\star},$$

if $c<0$, where

$$S^{2n+1}_{2s}(c) = \{z\in\mathbb{C}^{n+1} \mid b_{s,n+1}(z,z) = 1/c\},$$
$$H^{2n+1}_{s+1}(c) = \{z\in\mathbb{C}^{n+1} \mid b_{s+1,n+1}(z,z) = 1/c\}$$

and $b_{s,q}$ is the standard Hermitian form with index s on $\mathbb{C}^q$. For convenience, we will assume that we have chosen an orthonormal basis such that the first s odd terms appear with a minus sign.

In [1] it is shown that locally any indefinite complex space form is holomorphically isometric to either $\mathbb{C}^n_s$, $\mathbb{C}P^n_s(4c)$, or $\mathbb{C}H^n_s(4c)$. Note that, by replacing the metric $\langle\cdot,\cdot\rangle$ by $-\langle\cdot,\cdot\rangle$, we find that $\mathbb{C}H^n_s(4c)$ is holomorphically anti-isometric to $\mathbb{C}P^n_{n-s}(-4c)$. For this reason, as in our case $n=2$ and $s=1$, we only need to consider $\mathbb{C}^2_1$ and $\mathbb{C}P^2_1(4)$.

In order to study or explicitly obtain examples of Lagrangian submanifolds, it is usually more convenient to work with horizontal submanifolds. In that regard, we first recall some basic facts from [8] which relate Lagrangian submanifolds of $\mathbb{C}P^n_s(4c)$ to horizontal immersions in $S^{2n+1}_{2s}(c)$. Here, a *horizontal immersion* $f\colon M^n_s \to S^{2n+1}_{2s}(c)$ is an immersion which satisfies $if(p)\perp f_*(T_pM^n_s)$ for all $p\in M^n_s$, where $i=\sqrt{-1}$.

THEOREM 2.1 ([8]). *Let $\Pi : S^{2n+1}_{2s}(1)\to\mathbb{C}P^n_s(4)$ be the Hopf fibration. If $f : M^n_s \to S^{2n+1}_{2s}(c)$ is a horizontal immersion, then $F = \Pi\circ f : M^n_s \to \mathbb{C}P^n_s(4c)$ is a Lagrangian immersion.*

Conversely, let M_s^n be a simply connected manifold and let $F : M_s^n \to \mathbb{C}P_s^n(4)$ be a Lagrangian immersion. Then there exists a 1-parameter family of horizontal lifts $f : M_s^n \to S_{2s}^{2n+1}(1)$ such that $F = \Pi \circ f$. Any two such lifts f_1 and f_2 are related by $f_1 = e^{i\theta} f_2$, where θ is a constant.

Note that both immersions have the same induced metric and that the second fundamental forms of both immersions are also closely related. For more details, see [8].

3. Minimality and pseudo-isotropy. In this section we will prove Theorem 1.1. Suppose that M_1^2 is a Lagrangian, Lorentzian surface of a complex space form $\widetilde{M}$. We will assume that either $\widetilde{M} = \mathbb{C}_1^2$ or $\widetilde{M} = \mathbb{C}P_1^2(4)$.

Let $p \in M$. We say that $\{e_1, e_2\}$ is a *null frame* at a point p if

$$\langle e_i, e_j \rangle = 1 - \delta_{ij}, \quad i, j \in \{1, 2\}.$$

In terms of a null frame, it is clear that a Lagrangian immersion is minimal if and only if

$$h(e_1, e_2) = 0.$$

In view of the symmetries of the second fundamental form, this implies that there exist numbers λ and μ such that

$$h(e_1, e_1) = \lambda J e_2, \quad h(e_2, e_2) = \mu J e_1.$$

If we now write $v = v_1 e_1 + v_2 e_2$, it follows that

$$h(v, v) = \mu v_2^2 J e_1 + \lambda v_1^2 J e_2,$$
$$\langle h(v, v), h(v, v) \rangle = 2\lambda\mu v_1^2 v_2^2 = \tfrac{1}{2}\lambda\mu \langle v, v \rangle^2,$$

which shows that a minimal surface is indeed pseudo-isotropic.

In order to show the converse, we will use the following lemma of [2]:

LEMMA 3.1. *Let $F : M \to \widetilde{M}$ be an isometric pseudo-Riemannian immersion. The immersion is (pseudo-)isotropic if and only if for any $x, y, z, w \in T_pM$, we have*

$$\langle h(x, y), h(z, w) \rangle + \langle h(y, z), h(x, w) \rangle + \langle h(z, x), h(y, w) \rangle$$
$$= \widetilde{\lambda}(p)\{\langle x, y \rangle \langle z, w \rangle + \langle y, z \rangle \langle x, w \rangle + \langle z, x \rangle \langle y, w \rangle\}.$$

Note that in [2] the above lemma was formulated only for immersions into pseudo-Euclidean spaces. However, it is clear that it remains valid for arbitrary immersions in pseudo-Riemannian spaces.

Assume now that M is a pseudo-isotropic surface. Then it follows from the previous lemma that

(i) $h(e_1, e_1)$ is a lightlike vector, by taking $x = y = z = w = e_1$,
(ii) $h(e_2, e_2)$ is a lightlike vector, by taking $x = y = z = w = e_2$,

(iii) $h(e_1, e_2)$ is orthogonal to $h(e_1, e_1)$, by taking $x = y = z = e_1$ and $w = e_2$,

(iv) $h(e_1, e_2)$ is orthogonal to $h(e_2, e_2)$, by taking $x = y = z = e_2$ and $w = e_1$.

We now write

$$h(e_1, e_2) = v_1 Je_1 + v_2 Je_2.$$

The fact that the immersion is Lagrangian then implies that

$$h(e_1, e_1) = v_2 Je_1 + v_3 Je_2, \quad h(e_2, e_2) = v_4 Je_1 + v_1 Je_2.$$

Now assume that M is not minimal. Then, by interchanging e_1 and e_2 if necessary, we may assume that $v_2 \neq 0$. As $h(e_1, e_1)$ is lightlike by (i), this implies that $v_3 = 0$. It then follows from (iii) that $v_2 = 0$, which is a contradiction. This completes the proof of Theorem 1.

4. Lightlike isotropic Lagrangian, Lorentzian surfaces. In this section we will assume that M_1^2 is a Lagrangian, Lorentzian lightlike pseudo-isotropic surface of a complex space form $\widetilde{M}$. We will assume that either $\widetilde{M} = \mathbb{C}_1^2$ or $\widetilde{M} = \mathbb{C}P_1^2(4)$. We will also assume that M is not pseudo-isotropic, i.e. in view of the previous section the immersion is not minimal. We call a surface which is lightlike pseudo-isotropic without minimal points a *proper lightlike pseudo-isotropic surface.*

We again take a null frame at a point p, i.e. a frame $\{e_1, e_2\}$ such that

$$\langle e_i, e_j \rangle = 1 - \delta_{ij}, \quad i, j \in \{1, 2\}.$$

Note that if both $h(e_1, e_1) = h(e_2, e_2) = 0$, then it follows from the fact that M is Lagrangian that also $h(e_1, e_2) = 0$, and therefore M is pseudo-isotropic.

We say that M is *proper lightlike pseudo-isotropic of type* 1 at the point p if there exists a lightlike vector v such that $h(v, v)$ and Jv are independent. This means, after changing e_1 and e_2 if necessary, that we may assume that

$$h(e_1, e_1) = Je_2, \quad h(e_2, e_2) = \mu Je_1 + \lambda Je_2.$$

Since M is Lagrangian, we deduce from this that

$$h(e_1, e_2) = \lambda Je_1.$$

Since M is proper, we have $\lambda \neq 0$. Hence, since $h(e_2, e_2)$ is lightlike, we deduce that $\mu = 0$ and so $h(e_2, e_2) = \lambda Je_2$.

We say that M is *proper lightlike pseudo-isotropic of type* 2 if, for every lightlike vector v, $h(v, v)$ and Jv are dependent. This means, after changing e_1 and e_2 if necessary, that we may assume that

$$h(e_1, e_1) = Je_1, \quad h(e_2, e_2) = \lambda Je_2.$$

As M is Lagrangian, we deduce from this that

$$h(e_1, e_2) = \lambda Je_1 + Je_2.$$

We see that M is indeed not minimal and the immersion is therefore proper lightlike pseudo-isotropic.

Note that the points of type 1 form an open set $\mathcal{V}_1$ of M, whereas the points of type 2 form a closed set $\mathcal{V}_2$. It is clear that the null frame can be extended differentiably to a neighborhood of a point p belonging either to $\mathcal{V}_1$ or to the interior of $\mathcal{V}_2$. Given the analytic nature of the surfaces obtained in the next sections it then follows that the classification theorems remain valid on the whole of M.

5. Proper lightlike isotropic Lagrangian, Lorentzian surfaces of type 1. We take a null frame in a neighborhood of the point p as constructed in the previous section. So we have a frame $\{E_1, E_2\}$ such that

$$\langle E_i, E_j \rangle = 1 - \delta_{ij}, \quad i, j \in \{1, 2\},$$

and

$$h(E_1, E_1) = JE_2, \quad h(E_2, E_2) = \lambda JE_2, \quad h(E_1, E_2) = \lambda JE_1,$$

where λ is a nowhere vanishing function. We write

$$\begin{aligned} \nabla_{E_1} E_1 &= \alpha E_1, & \nabla_{E_1} E_2 &= -\alpha E_2, \\ \nabla_{E_2} E_1 &= -\beta E_1, & \nabla_{E_2} E_2 &= \beta E_2, \end{aligned}$$

where α and β are some functions.

LEMMA 5.1. *We have $\beta = 0$ and λ satisfies the following system of differential equations:*

$$E_1(\lambda) = -\alpha\lambda, \quad E_2(\lambda) = 0.$$

Proof. Clearly,

$$(\nabla h)(E_2, E_1, E_1) = \nabla^{\perp}_{E_2} JE_2 - 2h(\nabla_{E_2} E_1, E_1) = \beta JE_2 + 2\beta JE_2 = 3\beta JE_2.$$

On the other hand, we see that

$$\begin{aligned} (\nabla h)(E_1, E_2, E_1) &= \nabla^{\perp}_{E_1} \lambda JE_1 - h(\nabla_{E_1} E_2, E_1) - h(E_2, \nabla_{E_1} E_1) \\ &= (E_1(\lambda) + \alpha\lambda) JE_1 + \alpha h(E_2, E_1) - \alpha h(E_2, E_1) \\ &= (E_1(\lambda) + \alpha\lambda) JE_1. \end{aligned}$$

From the Codazzi equation, we therefore find that $\beta = 0$ and $E_1(\lambda) = -\alpha\lambda$. Similarly from the Codazzi equation $(\nabla h)(E_1, E_2, E_2) = (\nabla h)(E_2, E_1, E_2)$, we now deduce that $E_2(\lambda) = 0$. ∎

LEMMA 5.2. *We have $c = 0$ and α satisfies*

$$E_2(\alpha) = 0.$$

Proof. We compute $[E_1, E_2](\lambda)$ in two different ways. First,

$$[E_1, E_2](\lambda) = E_1(E_2(\lambda)) - E_2(E_1(\lambda)) = E_2(\alpha\lambda) = E_2(\alpha)\lambda,$$

and at the same time

$$[E_1, E_2](\lambda) = (\nabla_{E_1} E_2 - \nabla_{E_2} E_1)(\lambda)) = -\alpha E_2(\lambda) = 0.$$

Since $\lambda \neq 0$, we deduce that $E_2(\alpha) = 0$.

A direct computation then yields

$$\begin{aligned} R(E_1, E_2)E_1 &= -\nabla_{E_2}\nabla_{E_1}E_1 - \nabla_{\nabla_{E_1}E_2}E_1 = -\nabla_{E_2}(\alpha E_1) + \alpha\nabla_{E_2}E_1 \\ &= -E_2(\alpha)E_1 = 0. \end{aligned}$$

So from the Gauss equation we obtain

$$0 = cE_1 + A_{h(E_1,E_2)}e_1 - A_{h(E_1,E_1)}E_2 = cE_1 + \lambda A_{JE_1}E_1 - A_{JE_2}E_2 = cE_1.$$

Hence the ambient space must be flat. ∎

The previous lemma immediately implies:

THEOREM 5.3. *There does not exist a proper lightlike isotropic Lagrangian, Lorentzian surface of type* 1 *in* $\mathbb{C}P^2_1(4)$.

Moreover, we can also prove:

THEOREM 5.4. *Let M be a proper lightlike isotropic Lagrangian, Lorentzian surface of type* 1 *in* $\mathbb{C}^2_1$. *Then M is locally congruent with the surface parametrized by*

$$\left(\alpha(x)\frac{1}{\sqrt{2}}(-i,-i) + \beta(x)\frac{1}{\sqrt{2}}(1,-1)\right)e^{iv},$$

where x, v are the parameters and $\alpha'(x)\beta(x) - \alpha(x)\beta'(x) \neq 0$.

Proof. We introduce vector fields λE_1 and $\frac{1}{\lambda}E_2$. We have

$$\left[\lambda E_1, \frac{1}{\lambda}E_2\right] = -\frac{E_1(\lambda)}{\lambda}E_2 + [E_1, E_2] = \alpha E_2 - \alpha E_2 = 0.$$

Therefore, there exist coordinates u and v such that $\partial u = \lambda E_1$ and $\partial v = \frac{1}{\lambda}E_2$. If we denote the immersion by f, it follows that

$$\begin{aligned} &f_{vv} = if_v, \quad f_{uv} = if_u, \\ &f_{uu} = \lambda E_1(\lambda)E_1 + \lambda^2\nabla_{E_1}E_1 + \lambda^2 h(E_1, E_1) \\ &\quad\;\; = -\lambda^2\alpha + \lambda^2\alpha + \lambda^2 iE_2 = \lambda^3 if_v, \end{aligned}$$

where λ is a function depending only on u. Integrating the first two equations yields

$$f(u, v) = A_1(u)e^{iv} + A_2,$$

where A_1 is a vector valued function and A_2 is a constant. Of course, we may assume that A_2 vanishes by applying a translation of $\mathbb{C}^2_1$. The third equation then tells us that

$$A_1'' = -\lambda^3 A_1.$$

Note that this is precisely the expression of a curve lying in the plane spanned by $A_1(0)$ and $A_1'(0)$ parametrized in such a way that $|A_1 A_1'|$ is constant. As M is a Lagrangian surface A_1 and A_1' are linearly independent (over $\mathbb{C}$) and the plane spanned by A_1 and iA_1' is real. Therefore, the constant is not zero. Since $f_u = \lambda E_1$ and $f_v = \frac{1}{\lambda}E_2$, by choosing the initial conditions we may assume that $A_1(0) = \frac{1}{\sqrt{2}}(-i,-i)$ and $A_1'(0) = \frac{1}{\sqrt{2}}(1,-1)$.

Conversely, if we define a surface by

$$f(x,v) = \left(\alpha(x)\frac{1}{\sqrt{2}}(-i,-i) + \beta(x)\frac{1}{\sqrt{2}}(1,-1)\right)e^{iv},$$

where $\alpha'(x)\beta(x) - \alpha(x)\beta'(x) \neq 0$, we see that just as for the Euclidean arc length of a planar curve, it is possible to construct a parameter u for the curve (α,β) such that $\alpha'(u)\beta(u) - \alpha(u)\beta'(u) = 1$. A straightforward computation then shows that the surface $f(u,v)$ has the desired properties. ∎

6. Proper lightlike isotropic Lagrangian, Lorentzian surfaces of type 2. We take a null frame in a neighborhood of the point p as constructed in the previous section. Then we have a frame $\{E_1, E_2\}$ such that

$$\langle E_i, E_j\rangle = 1 - \delta_{ij}, \quad i,j \in \{1,2\},$$

and

$$h(E_1,E_1) = JE_1, \quad h(E_1,E_2) = \lambda JE_1 + JE_2, \quad h(E_2,E_2) = \lambda JE_2,$$

where λ is a function on M. We write

$$\begin{aligned} \nabla_{E_1}E_1 &= \alpha E_1, & \nabla_{E_1}E_2 &= -\alpha E_2,\\ \nabla_{E_2}E_1 &= -\beta E_1, & \nabla_{E_2}E_2 &= \beta E_2, \end{aligned}$$

where α and β are some functions.

LEMMA 6.1. *We have $\alpha = 0$ and λ satisfies the following system of differential equations:*

$$E_1(\lambda) = \beta, \quad E_2(\lambda) = \lambda\beta.$$

Proof. As one easily sees,

$$(\nabla h)(E_2,E_1,E_1) = \nabla^{\perp}_{E_2} JE_1 - 2h(\nabla_{E_2}E_1, E_1) = -\beta JE_1 + 2\beta JE_1 = \beta JE_1.$$

On the other hand,

$$\begin{aligned} (\nabla h)(E_1,E_2,E_1) &= \nabla^{\perp}_{E_1}(\lambda JE_1 + JE_2) - h(\nabla_{E_1}E_2, E_1) - h(E_2, \nabla_{E_1}E_1\\ &= (E_1(\lambda) + \alpha\lambda)JE_1 - \alpha JE_2 + \alpha h(E_2,E_1) - \alpha h(E_2,E_1)\\ &= (E_1(\lambda) + \alpha\lambda)JE_1 - \alpha JE_2. \end{aligned}$$

From the Codazzi equation, we therefore obtain $\alpha = 0$ and $E_1(\lambda) = \beta$. Similarly from the Codazzi equation $(\nabla h)(E_1,E_2,E_2) = (\nabla h)(E_2,E_1,E_2)$, we now deduce that $E_2(\lambda) = \lambda\beta$. ∎

Lemma 6.2. *The function β satisfies*

$$E_1(\beta) = -c - \lambda, \qquad E_2(\beta) = \lambda(-c - \lambda).$$

Proof. A direct computation yields

$$R(E_1, E_2)E_1 = \nabla_{E_1}\nabla_{E_2}E_1 + \nabla_{\nabla_{E_2}E_1}E_1 = -\nabla_{E_1}(\beta E_1) - \beta\nabla_{E_1}E_1 = -E_1(\beta)E_1.$$

So from the Gauss equation we obtain

$$-E_1(\beta)E_1 = cE_1 + A_{h(E_1,E_2)}E_1 - A_{h(E_1,E_1)}E_2 = cE_1 + \lambda A_{JE_1}E_1 + A_{JE_2}E_1 - A_{JE_1}E_2 = cE_1 + \lambda E_1,$$

which reduces to $E_1(\beta) = -(c + \lambda)$. In order to obtain the E_2 derivative of β, we compute $[E_1, E_2](\lambda)$ in two different ways. We have

$$[E_1, E_2](\lambda) = E_1(E_2(\lambda)) - E_2(E_1(\lambda)) = E_1(\lambda\beta) - E_2(\beta) = \beta^2 + \lambda(-c - \lambda) - E_2(\beta)$$

and

$$[E_1, E_2](\lambda) = (\nabla_{E_1}E_2 - \nabla_{E_2}E_1)(\lambda)) = \beta E_1(\lambda) = \beta^2,$$

which clearly concludes the proof. ∎

By a direct computation we obtain:

Corollary 6.3. *There exists a constant r such that*

$$(\lambda + c)^2 + \beta^2 = r^2.$$

Lemma 6.4. *There exist local coordinates u and v such that*

$$\frac{\partial}{\partial u} = E_1, \qquad \frac{\partial}{\partial v} = E_2 - \lambda E_1.$$

Proof. We define vector fields

$$U = E_1, \qquad V = E_2 - \lambda E_1,$$

and compute

$$[U, V] = [E_1, E_2 - \lambda E_1] = [E_1, E_2] - E_1(\lambda)E_1 = \beta E_1 - E_1(\lambda)E_1 = 0,$$

which proves the result. ∎

It then follows immediately from the previous systems of differential equations that β and λ do not depend on the variable v and are determined by

$$\frac{\partial \lambda}{\partial u} = \beta, \qquad \frac{\partial \beta}{\partial u} = -(c + \lambda).$$

Therefore, after a translation of the u coordinate if necessary, we may suppose that

$$\lambda = -c + r\sin u, \qquad \beta = r\cos u.$$

In the above equations, the constant r is allowed to be zero. In that case, we get the special solution $\beta = 0$ and $\lambda = -c$.

For dimensional reasons, changing if necessary the sign of the metric on the ambient space, we only have to consider the cases $c = 0$ or $c = 1$.

6.1. Case $c = 0$: Lightlike isotropic Lagrangian, Lorentzian surfaces in $\mathbb{C}^2_1$. We denote the immersion by f. It follows from the previous equations that f is determined by the system of differential equations

$$\begin{aligned} f_{uu} &= i f_u, \\ f_{uv} &= -re^{-iu} f_u + i f_v, \\ f_{vv} &= re^{-iu} f_v + ir^2(e^{-2iu} - 1) f_u. \end{aligned}$$

It follows from the first equation that there exist vector valued functions g_1 and g_2 such that

$$f(u, v) = g_1(v)e^{iu} + g_2(v).$$

Substituting this into the second equation gives

$$g_2'(v) = rg_1(v),$$

and the final equation now reduces to

$$g_1''(v)e^{iu} + g_2''(v) = re^{-iu}(g_1'(v)e^{iu} + g_2'(v)) - r^2(e^{-2iu} - 1)e^{iu} g_1(v).$$

Looking at the different powers of e^{iu}, we deduce that

$$g_1''(v) = r^2 g_1(v), \quad g_2''(v) = rg_1'(v), \quad 0 = rg_2'(v) - r^2 g_1(v).$$

So the remaining equations are

$$g_2'(v) = rg_1(v), \quad g_1''(v) = r^2 g_1(v).$$

The solution of the above system depends on the value of r.

6.1.1. *Case $r = 0$.* If $r = 0$ then $g_2(v)$ is a constant vector. Hence by applying a translation we may assume that this vector vanishes. Therefore,

$$f(u, v) = (vA_1 + A_2)e^{iu}$$

for some constant vectors A_1 and A_2. We take an initial point $p = (0, 0)$. Since $\lambda(0, 0) = 0$, it follows that

$$E_1(0, 0) = \frac{\partial f}{\partial u}(0, 0) = iA_2, \quad E_2(0, 0) = \frac{\partial f}{\partial v}(0, 0) = A_1.$$

It then follows from the choice of E_1 and E_2, together with the Lagrangian condition, that $\frac{1}{\sqrt{2}}(A_1 - iA_2)$, $\frac{i}{\sqrt{2}}(A_1 - iA_2)$, $\frac{1}{\sqrt{2}}(A_1 + iA_2)$, $\frac{i}{\sqrt{2}}(A_1 + iA_2)$ can be identified with $(1, 0)$, $(i, 0)$, $(0, 1)$, $(0, i)$. This implies that

$$A_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad A_2 = \left(\frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2}i\right).$$

6.1.2. *Case $r \neq 0$.* In this case we have

$$g_1(v) = A_1 e^{rv} + A_2 e^{-rv}.$$

Therefore

$$g_2'(v) = rA_1 e^{rv} + rA_2 e^{-rv},$$

which implies that, after applying a suitable translation,

$$g_2(v) = A_1 e^{rv} - A_2 e^{-rv}.$$

So we find that

$$f(u,v) = (A_1 e^{rv} + A_2 e^{-rv})e^{iu} + (A_1 e^{rv} - A_2 e^{-rv})$$

for some constant vectors A_1 and A_2. We again take $p = (0,0)$ as initial point. Since $\lambda(0,0) = 0$, it follows that

$$E_1(0,0) = \frac{\partial f}{\partial u}(0,0) = i(A_1 + A_2), \qquad E_2(0,0) = \frac{\partial f}{\partial v}(0,0) = 2rA_1,$$

or equivalently

$$A_1 = \frac{1}{2r}E_2, \qquad A_2 = -iE_1 - \frac{1}{2r}E_2.$$

It then follows from the choice of E_1 and E_2, together with the Lagrangian condition, that we may assume that $E_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $E_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, which implies that

$$A_1 = \left(-\frac{\sqrt{2}}{4r}, \frac{\sqrt{2}}{4r}\right), \qquad A_2 = \left(\frac{1-2ir}{2\sqrt{2}\,r}, \frac{-1-2ir}{\sqrt{2}\,2r}\right).$$

6.1.3. *Summary.* Combining the previous results, we get

THEOREM 6.5. *Let M be a proper lightlike isotropic Lagrangian, Lorentzian surface of type 2 in $\mathbb{C}_1^2$. Then M is congruent to one of the following surfaces, u, v being the parameters:*

(1) *the surface*

$$f(u,v) = (vA_1 + A_2)e^{iu}, \qquad A_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \; A_2 = \left(-\frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2}i\right),$$

(2) *the surface*

$$f(u,v) = (A_1 e^{rv} + A_2 e^{-rv})e^{iu} + (A_1 e^{rv} - A_2 e^{-rv}),$$

where r is a positive constant and

$$A_1 = \left(-\frac{\sqrt{2}}{4r}, \frac{\sqrt{2}}{4r}\right), \qquad A_2 = \left(\frac{1-2ir}{2\sqrt{2}\,r}, \frac{-1-2ir}{\sqrt{2}\,2r}\right).$$

6.2. Case $c = 1$: Lightlike isotropic Lagrangian, Lorentzian surfaces in $\mathbb{C}P^2_1(4)$. We denote the horizontal lift of the immersion into $S^5_2(1)$ by f. It follows from the previous equations that f is determined by the system of differential equations

$$\begin{aligned} f_{uu} &= if_u, \\ f_{uv} &= -(i + re^{-iu})f_u + if_v - f, \\ f_{vv} &= (i + re^{-iu})(-2f_u + f_v + 2f_u r\sin(u)) - 2(1 - r\sin(u))f. \end{aligned}$$

It follows from the first equation that there exist vector valued functions a_1 and a_2 such that

$$f(u, v) = a_1(v)e^{iu} + a_2(v).$$

Substituting this into the second equation gives

$$a_2'(v) = -ia_2(v) + ra_1(v).$$

The final equation now reduces to

$$a_1''(v) = a_1(v)r^2 + i(a_1'(v) - a_2(v)r).$$

The solution of this differential equation depends on the value of r.

6.2.1. *Case* $0 \le r < 1$. In this case we can write $r = \cos(t)$, where $t \in]0, \frac{\pi}{2}]$. We find that

$$\begin{aligned} a_1(v) = &-\csc^2(t)\Big(-c_3\sin(t)\sin(v\sin(t)). \\ &+ \big(c_1\cos^2(t) + ic_3\big)\cos(v\sin(t)) \\ &+ 2ic_2\cos(t)\sin^2\big(\tfrac{1}{2}v\sin(t)\big) - c_1 - ic_3\Big) \end{aligned} \tag{6.1}$$

$$\begin{aligned} a_2(v) = &-\csc^2(t)\Big(\cos(t)\big(-c_1\sin(t)\sin(v\sin(t)) \\ &+ (c_3 - ic_1)\big(\cos(v\sin(t)) - 1\big)\big) \\ &- c_2\big(\cos(v\sin(t)) - i\sin(t)\sin(v\sin(t))\big) + c_2\cos^2(t)\Big). \end{aligned} \tag{6.2}$$

We again take $p = (0, 0)$ as initial point. We have $\lambda(0, 0) = -1$. It follows that

$$\begin{aligned} f(0, 0) &= c_1 + c_2, \\ E_1(0, 0) &= \frac{\partial f}{\partial u}(0, 0) = ic_1, \\ E_2(0, 0) &= \frac{\partial f}{\partial v}(0, 0) + \lambda\frac{\partial f}{\partial u}(0, 0) = -i(c_1 + c_2) + c_3 + c_1\cos(t). \end{aligned}$$

So if we pick the initial conditions $f(0,0)=(0,0,1)$, $E_1(0,0)=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$ and $E_2(0,0)=\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$, we find that

$$c_1=\left(-\frac{i}{\sqrt{2}},-\frac{i}{\sqrt{2}},0\right), \tag{6.3}$$

$$c_2=\left(\frac{i}{\sqrt{2}},\frac{i}{\sqrt{2}},1\right), \tag{6.4}$$

$$c_3=\left(i\left(\frac{\cos(t)}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right),i\left(\frac{\cos(t)}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right),i\right). \tag{6.5}$$

6.2.2. *Case* $r=1$. We obtain the solution

$$a_1(v)=\tfrac{1}{2}\big(c_1(v^2+2)+v(2c_3-i(c_2-c_3)v)\big), \tag{6.6}$$

$$a_2(v)=\tfrac{1}{2}\big(2c_2+v(c_3v+(c_1-ic_2)(2-iv))\big). \tag{6.7}$$

We take $p=(0,0)$ as initial point. We have $\lambda(0,0)=-1$. It follows that

$$\begin{aligned} f(0,0)&=c_1+c_2,\\ E_1(0,0)&=\frac{\partial f}{\partial u}(0,0)=ic_1,\\ E_2(0,0)&=(1-i)c_1-ic_2+c_3. \end{aligned}$$

So, if we pick the initial conditions $f(0,0)=(0,0,1)$, $E_1(0,0)=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$ and $E_2(0,0)=\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$, we find that

$$c_1=\left(-\frac{i}{\sqrt{2}},-\frac{i}{\sqrt{2}},0\right), \tag{6.8}$$

$$c_2=\left(\frac{i}{\sqrt{2}},\frac{i}{\sqrt{2}},1\right), \tag{6.9}$$

$$c_3=\left(\frac{i-1}{\sqrt{2}},\frac{1+i}{\sqrt{2}},i\right). \tag{6.10}$$

6.2.3. *Case* $r>1$. We obtain the solution

$$\begin{aligned} a_1(v)=\frac{1}{r^2-1}\Big(&c_3\sqrt{r^2-1}\sinh\big(\sqrt{r^2-1}\,v\big)\\ &+(c_1r^2-ic_2r+ic_3)\cosh\big(\sqrt{r^2-1}\,v\big)+ic_2r-c_1-ic_3\Big), \end{aligned} \tag{6.11}$$

$$\begin{aligned} a_2(v)=\frac{1}{r^2-1}\Big(&\sqrt{r^2-1}(c_1r-ic_2)\sinh\big(\sqrt{r^2-1}\,v\big)\\ &+(-c_2+(c_3-ic_1)r)\cosh\big(\sqrt{r^2-1}\,v\big)+r(c_2r+ic_1-c_3)\Big). \end{aligned} \tag{6.12}$$

We take $p = (0, 0)$ as initial point. We have $\lambda(0, 0) = -1$. It follows that

$$f(0,0) = c_1 + c_2,$$
$$E_1(0,0) = \frac{\partial f}{\partial u}(0,0) = ic_1,$$
$$E_2(0,0) = -i(c_1 + c_2) + c_3 + c_1 r.$$

So, if we pick the initial conditions $f(0,0) = (0,0,1)$, $E_1(0,0) = \big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\big)$ and $E_2(0,0) = \big(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\big)$, we find that

$$c_1 = \left(-\frac{i}{\sqrt{2}}, -\frac{i}{\sqrt{2}}, 0\right), \tag{6.13}$$

$$c_2 = \left(\frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 1\right), \tag{6.14}$$

$$c_3 = \left(\frac{i(i+r)}{\sqrt{2}}, \frac{(1+ir)}{\sqrt{2}}, i\right). \tag{6.15}$$

6.2.4. *Summary.* Combining the previous results we get the following theorem which finishes our classification.

THEOREM 6.6. *Let M be a proper lightlike isotropic Lagrangian, Lorentzian surface in $\mathbb{C}P_1^2(4)$. Then the Hopf lift of M is congruent to one of the following immersions into $S_1^5(1)$ given by $f(u,v) = a_1(v)e^{iu} + a_2(v)$, where either*

(1) a_1, a_2, c_1, c_2, c_3 *are as described in* (6.1)–(6.5), *or*
(2) a_1, a_2, c_1, c_2, c_3 *are as described in* (6.6)–(6.10), *or*
(3) a_1, a_2, c_1, c_2, c_3 *are as described in* (6.11)–(6.15).

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