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ON A CLASS OF SUBMANIFOLDS IN A TANGENT BUNDLE WITH A g-NATURAL METRIC

BY

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Dedicated to the memory of Professor Witold Roter

Abstract. An isometric immersion of a Riemannian manifold M into a Riemannian manifold N gives rise in a natural way to the immersion of the tangent bundle TM into the tangent bundle TN with a non-degenerate g-natural metric G. It turns out that the normal bundle of the image of TM is completely determined by the normal bundle of M. The cases of M being either totally geodesic or pseudo-umbilical are discussed.

1. Introduction. Let $\pi : TN \to N$ be the tangent bundle of a Riemannian manifold N with the Levi-Civita connection ∇ on N. Then at each point $(x,u) \in TN$ the tangent space $T_{(x,u)}TN$ splits into the direct sum of two isomorphic spaces $V_{(x,u)}TN$ and $H_{(x,u)}TN$, where

$$V_{(x,u)}TN = \mathrm{Ker}(d\pi|_{(x,u)}), \quad H_{(x,u)}TN = \mathrm{Ker}(K|_{(x,u)})$$

and K is the connection map [7] (see also [13]).

More precisely, if $Z = \big(Z^r \frac{\partial}{\partial x^r} + \overline{Z}^r \frac{\partial}{\partial u^r}\big)\big|_{(x,u)} \in T_{(x,u)}TN$, $r = 1, \dots, n$, then the vertical and horizontal projections of Z on T_xN are given by

$$(d\pi)_{(x,u)}Z = Z^r \frac{\partial}{\partial x^r}\bigg|_x, \qquad K_{(x,u)}(Z) = (\overline{Z}^r + u^s Z^t \Gamma^r_{st}) \frac{\partial}{\partial x^r}\bigg|_x,$$

where Γ^r_{st} are the components of the Levi-Civita connection on N.

On the other hand, to each vector field X on N there correspond uniquely determined vector fields X^v and X^h on TN such that

$$d\pi|_{(x,u)}(X^v) = 0, \quad K|_{(x,u)}(X^v) = X,$$
$$K|_{(x,u)}(X^h) = 0, \quad d\pi|_{(x,u)}(X^h) = X.$$

X^v and X^h are called the *vertical lift* and the *horizontal lift* of X to TN respectively.

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In local coordinates (x^r, u^r) on TN, the horizontal and vertical lifts of a vector field $X = X^r \frac{\partial}{\partial x^r}$ are given by

$$X^h = X^r \frac{\partial}{\partial x^r} - u^s X^t \Gamma^r_{st} \frac{\partial}{\partial u^r}, \quad X^v = X^r \frac{\partial}{\partial u^r}.$$

In this paper we shall frequently use the frame $(\partial^h_k, \partial^v_l) = \big(\big(\frac{\partial}{\partial x^k}\big)^h, \big(\frac{\partial}{\partial x^l}\big)^v \big)$ known as the *adapted frame*.

Given an isometric immersion $f : M \to N$, we have two tangent bundles $\pi_N : TN \to N$ and $\pi_M : TM \to M$, where the latter is a subbundle of the former. Suppose M, N are Riemannian manifolds with metrics g_M and g_N and Levi-Civita connections ∇_M and ∇_N. Then T_pTM and T_pTN have at a common point p their own decompositions into vertical and horizontal parts, i.e.

$$T_pTM = V_pTM \oplus H_pTM = V_M \oplus H_M$$

and

$$T_pTN = V_pTN \oplus H_pTN = V_N \oplus H_N,$$

but neither $V_M \subset V_N$ nor $H_M \subset H_N$ need to hold along TM.

So, for a vector X tangent to M we define two vertical lifts X^{v_M}, X^{v_N} and two horizontal lifts X^{h_M}, X^{h_N} with respect to the bundles over M and N respectively and find relationships between them. These allow us to compute the shape operator of the immersion under consideration (see (4.1) below) and obtain some conclusions about the resulting submanifold of TN.

Note that totally geodesic submanifolds of the tangent bundle with g-natural metric were also studied in [4] and [10].

Throughout the paper all manifolds under consideration are Hausdorff and smooth. The metrics on base manifolds are Riemannian and the metrics on tangent spaces are non-degenerate. We adopt the Einstein summation convention.

2. Preliminaries on submanifolds. Throughout the paper we assume that the indices h, i, j, k, l, r, s, t run through the range $1, \dots, n$, while a, b, c, d, e run through $1, \dots, m$, and $m < n$. Moreover, $x, y, z = m+1, \dots, n$.

Let $(N, g), \dim N = n$, be a Riemannian manifold with metric g, covered by coordinate neighbourhoods $(U, (x^j))$, $j = 1, \dots, n$. Let $(M, \widetilde{g})$ be a Riemannian manifold covered by coordinate neighbourhoods $(V, (y^a))$, $a = 1, \dots, m$, isometrically immersed in (N, g), and let the local expression for this immersion be $x^r = x^r(y^a)$, $r = 1, \dots, n$, $a = 1, \dots, m$. Set $\partial_r = \frac{\partial}{\partial x^r}$ and $B^r_a = \frac{\partial x^r}{\partial y^a}$.

For the local immersion $x^r = x^r(y^a)$ the components of the Levi-Civita connection ∇ of the induced metric $g_{ab} = g(B^r_a \partial_r, B^s_b \partial_s) = g_{rs} B^r_a B^s_b$ are

$$\Gamma^c_{ab} = [B^r_{a.b} + \Gamma^r_{st} B^s_a B^t_b] B^c_r, \quad B^c_r = g^{cd} B^t_d g_{tr},$$

where the dot denotes partial derivative with respect to y^b. The van der Waerden–Bertolotti covariant derivative of B^r_a is defined by

$$\nabla_b B^r_a = B^r_{a.b} + \Gamma^r_{st} B^s_a B^t_b - \Gamma^c_{ab} B^r_c. \tag{2.1}$$

The operator ∇_b is the covariant differentiation on M with respect to Γ^c_{ab} and can be extended to tensor fields on M of mixed type. For example,

$$\nabla_c \nabla_b B^r_a = \partial_c(\nabla_b B^r_a) + \Gamma^r_{st} B^s_c \nabla_b B^t_a - \Gamma^d_{cb} \nabla_d B^r_a - \Gamma^d_{ca} \nabla_b B^r_d$$

(see [14]).

For any fixed indices a and b, the vector $\nabla_b B^r_a \partial_r$ is orthogonal to the submanifold. Hence

$$\nabla_b B^r_a \partial_r = h^x_{ab} N^r_x \partial_r,$$

where $N^r_x \partial_r$, $x = m+1, \dots, n$, are unit vectors normal to the submanifold. For fixed x, the h^x_{ab} are components of a symmetric $(0,2)$ tensor h on M, called the *second fundamental form*. Consequently, we have the decomposition

$$\nabla_c \nabla_b B^r_a = \nabla_c h^x_{ba} N^r_x + h^x_{ba} \nabla_c N^r_x,$$

where $(\nabla_c N^r_x)\partial_r$ is tangent to the submanifold for all c and x.

The *Gauss formula* is

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

for all vector fields X, Y tangent to M, where $\widetilde{\nabla}$ and ∇ denote the Levi-Civita connections on N and M respectively.

The *Weingarten formula* is

$$\widetilde{\nabla}_X \eta = -\widetilde{A}_\eta X + \widetilde{D}_X \eta,$$

where X is a tangent vector field and η is a normal one. Here $\widetilde{A}$ is called the *shape operator*, while $\widetilde{D}$ is the Levi-Civita connection induced in the normal bundle over M. We have

$$g(\widetilde{A}_\eta X, Y) = g(h(X,Y), \eta).$$

A submanifold M is said to be *totally geodesic* if the second fundamental form h vanishes identically, or equivalently if the shape operator $\widetilde{A}$ vanishes identically. For more details see [14] or [12].

3. Preliminaries on g-natural metrics. In [11] the class of g-natural metrics was defined. We have

LEMMA 1 ([11], [2], [3]). *Let (M,g) be a Riemannian manifold and G be a g-natural metric on TM. There exist functions a_j, $b_j : [0,\infty) \to \mathbb{R}$,*

$j = 1, 2, 3$, *such that for all* X, Y *and* $u \in T_xM$,

$G_{(x,u)}(X^h, Y^h) = (a_1 + a_3)(r^2)g_x(X, Y) + (b_1 + b_3)(r^2)g_x(X, u)g_x(Y, u),$

$G_{(x,u)}(X^h, Y^v) = G_{(x,u)}(X^v, Y^h) = a_2(r^2)g_x(X, Y) + b_2(r^2)g_x(X, u)g_x(Y, u),$

$G_{(x,u)}(X^v, Y^v) = a_1(r^2)g_x(X, Y) + b_1(r^2)g_x(X, u)g_x(Y, u),$

where $r^2 = g_x(u, u)$. *For* $\dim M = 1$ *the same holds for* $b_j = 0$, $j = 1, 2, 3$.

Setting $a_1 = 1$, $a_2 = a_3 = b_j = 0$ we obtain the Sasaki metric, while setting $a_1 = b_1 = \frac{1}{1+r^2}$, $a_2 = b_2 = 0 = 0$, $a_1 + a_3 = 1$, $b_1 + b_3 = 1$ we get the Cheeger–Gromoll metric.

Following [2] we set

$$\begin{aligned} a(t) &= a_1(t)(a_1(t) + a_3(t)) - a_2^2(t), \\ F_j(t) &= a_j(t) + tb_j(t), \\ F(t) &= F_1(t)[F_1(t) + F_3(t)] - F_2^2(t) \quad \text{for all } t \in [0, \infty). \end{aligned}$$

We shall often abbreviate $A = a_1 + a_3$, $B = b_1 + b_3$.

Lemma 2 ([2, Proposition 2.7]). *A g-natural metric* G *on the tangent bundle of a Riemannian manifold* (M, g) *is non-degenerate if and only if* $a(t) \neq 0$ *and* $F(t) \neq 0$ *for all* $t \in [0, \infty)$. *If* $\dim M = 1$ *this is equivalent to* $a(t) \neq 0$ *for all* $t \in [0, \infty)$.

Moreover, (TM, G) *is Riemannian if and only if*

$$a(t) > 0, \quad F(t) > 0, \quad a_1(t) > 0, \quad F_1(t) > 0$$

for all $t \in [0, \infty)$.

Proposition 3 ([1], [5]). *Let* (N, g) *be a Riemannian manifold,* ∇ *its Levi-Civita connection and* R *its Riemann curvature tensor. If* G *is a non-degenerate g-natural metric on* TN, *then the Levi-Civita connection* $\widetilde{\nabla}$ *of* (TN, G) *at* $(x, u) \in TN$ *is given by*

$$\begin{aligned} (\widetilde{\nabla}_{X^h} Y^h)_{(x,u)} &= (\nabla_X Y)^h_{(x,u)} + h\{\mathbf{A}(u, X_x, Y_x)\} + v\{\mathbf{B}(u, X_x, Y_x)\}, \\ (\widetilde{\nabla}_{X^h} Y^v)_{(x,u)} &= (\nabla_X Y)^v_{(x,u)} + h\{\mathbf{C}(u, X_x, Y_x)\} + v\{\mathbf{D}(u, X_x, Y_x)\}, \\ (\widetilde{\nabla}_{X^v} Y^h)_{(x,u)} &= h\{\mathbf{C}(u, Y_x, X_x)\} + v\{\mathbf{D}(u, Y_x, X_x)\}, \\ (\widetilde{\nabla}_{X^v} Y^v)_{(x,u)} &= h\{\mathbf{E}(u, X_x, Y_x)\} + v\{\mathbf{F}(u, X_x, Y_x)\}, \end{aligned}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$ *are some F-tensors defined on the tensor product* $TN \otimes TN \otimes TN$.

Remark 1. Expressions for $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$ were presented for the first time in the original papers [2], [3]. Unfortunately, they contain some misprints and omissions. For the correct form, we refer the reader to [1], [5] (see also [8], [9]).

4. Lifts of vector fields. Let $f : M \to N$ be an isometric immersion of a Riemannian manifold M into a Riemannian manifold N. Suppose that the following diagram commutes:

$$\begin{array}{ccc} TN \supset (\pi^{-1}(U), (x^r, u^r)) & \xleftarrow{\;\widetilde{f}\;} & (\pi^{-1}(V), (y^a, v^a)) \subset TM \\ \Big\downarrow{\scriptstyle \pi_N} & & \Big\downarrow{\scriptstyle \pi_M} \\ N \supset (U, (x^r)) & \xleftarrow{\quad f \quad} & (V, (y^a)) \subset M \end{array}$$

where $(U, (x^r))$ and $(V, (y^a))$ are coordinate neighbourhoods on N and M respectively, while the local expression for f is $x^r = x^r(y^a)$, $r = 1, \ldots, n$, $a = 1, \ldots, m$ and $m < n$. Then the local coordinate vector fields on M are given by

$$\frac{\delta}{\delta y^a} = \frac{\partial x^r}{\partial y^a} \frac{\partial}{\partial x^r} = B^r_a \frac{\partial}{\partial x^r}.$$

Define the map

$$\widetilde{f} : x^r = x^r(y^a), \qquad u^r = v^a B^r_a. \tag{4.1}$$

This is an immersion of rank $2m$ since the Jacobi matrix J is of the form

$$J = \begin{bmatrix} \partial x^r/\partial y^a & \partial x^r/\partial v^a \\ \partial u^r/\partial y^a & \partial u^r/\partial v^a \end{bmatrix} = \begin{bmatrix} B^r_a & 0 \\ v^b \partial_b B^r_a & B^r_a \end{bmatrix}.$$

Since $\frac{\delta}{\delta y^a} = B^r_a \frac{\partial}{\partial x^r}$, the coordinate vectors tangent to TM are

$$\frac{\partial}{\partial y^a} = \frac{\delta}{\delta y^a} + v^b B^r_{a.b} \frac{\partial}{\partial u^r}, \qquad \frac{\partial}{\partial v^a} = B^r_a \frac{\partial}{\partial u^r}.$$

Definition 1. Let $f : M \to N$ be an isometric immersion. Then the map $\widetilde{f} : TM \to TN$, locally given by (4.1), will be called the *lift of the immersion* f and its image LM will be called the *lift of the submanifold* M.

Remark 2. The lift of an immersion defined above seems to be the most natural since $B^r_a \partial_r|_{(x^r(y^a))}$ are coordinate vectors at $(x^r(y^a)) \in M$, and (v^a) are components of tangent vectors. This kind of lift appears quite naturally (see [6]).

Lemma 4. *Let $\widetilde{f} : TM \to TN$ be the lift of the isometric immersion $f : M \to N$ in the sense of Definition 1. Then the vertical and horizontal lifts of the coordinate vector field $\frac{\delta}{\delta y^a}$ on M with respect to TN and TM are related by*

$$\left(\frac{\delta}{\delta y^a}\right)^{v_N} = \left(\frac{\delta}{\delta y^a}\right)^{v_M},$$
$$\left(\frac{\delta}{\delta y^a}\right)^{h_N} = \left(\frac{\delta}{\delta y^a}\right)^{h_M} - v^b \nabla_b B^r_a \left(\frac{\partial}{\partial x^r}\right)^{v_N}.$$

Proof. By the use of the definitions of v_N and v_M we have

$$\left(\frac{\delta}{\delta y^a}\right)^{v_N} = B^r_a\left(\frac{\partial}{\partial x^r}\right)^{v_N} = B^r_a\frac{\partial}{\partial u^r} = \frac{\partial}{\partial v^a} = \left(\frac{\delta}{\delta y^a}\right)^{v_M}.$$

For horizontal lifts we have

$$\left(\frac{\delta}{\delta y^a}\right)^{h_M} = \frac{\partial}{\partial y^a} - \Gamma^c_{ab}v^b\frac{\partial}{\partial v^c}$$

and, along M,

$$\begin{aligned}\left(\frac{\delta}{\delta y^a}\right)^{h_N} &= B^r_a\left(\frac{\partial}{\partial x^r}\right)^{h_N} = B^r_a\left(\frac{\partial}{\partial x^r} - u^s\Gamma^t_{sr}\frac{\partial}{\partial u^t}\right)\\ &= \frac{\delta}{\delta y^a} - v^bB^r_bB^s_a\Gamma^t_{rs}\frac{\partial}{\partial u^t} \overset{(1)}{=} \frac{\delta}{\delta y^a} - v^b[\nabla_bB^r_a - B^r_{a.b} + \Gamma^c_{ab}B^r_c]\frac{\partial}{\partial u^r}\\ &= \left(\frac{\delta}{\delta y^a} + v^bB^r_{a.b}\frac{\partial}{\partial u^r}\right) - v^b\Gamma^c_{ab}\frac{\partial}{\partial v^c} - v^b\nabla_bB^r_a\frac{\partial}{\partial u^r}\\ &= \left(\frac{\delta}{\delta y^a}\right)^{h_M} - v^b\nabla_bB^r_a\frac{\partial}{\partial u^r} = \left(\frac{\delta}{\delta y^a}\right)^{h_M} - v^b\nabla_bB^r_a\left(\frac{\partial}{\partial x^r}\right)^{v_N}. \ \blacksquare\end{aligned}$$

We define the vertical vector field

$$K^{v_N}_a = v^b\nabla_bB^r_a\left(\frac{\partial}{\partial x^r}\right)^{v_N} = K^r_a\left(\frac{\partial}{\partial x^r}\right)^{v_N}.$$

COROLLARY 5. *If M is a totally geodesic submanifold in N, then the horizontal lifts h_N and h_M coincide on TM.*

4.1. Projections

LEMMA 6. *Let $\widetilde{f} : TM \to TN$ be the lift of the isometric immersion $f : M \to N$ in the sense of Definition 1. Then the projections $\pi_N : TN \to N$ and $\pi_M : TM \to M$ satisfy*

$$d(\pi_N)|_{TTM} = d(\pi_M).$$

Proof. The components of the projections: $\pi_N : TN \to N$ and $\pi_M : TM \to M$ can be written as

$$(\pi_N)^s(x^r, u^r) = x^s, \qquad (\pi_M)^b(y^a, v^a) = y^b,$$

where $r = 1, \ldots, n$, $a = 1, \ldots, m$. Thus for any fixed indices s, b we obtain

$$\frac{\partial(\pi_N)^r}{\partial x^t} = \delta^r_t, \quad \frac{\partial(\pi_N)^r}{\partial u^t} = 0, \quad \frac{\partial(\pi_M)^a}{\partial y^b} = \delta^a_b, \quad \frac{\partial(\pi_M)^a}{\partial v^b} = 0.$$

Then for the vectors $\frac{\delta}{\delta y^c}$, $c = 1, \ldots, m$, tangent to M, and $\frac{\partial}{\partial y^c}$ tangent to TM we find

$$d\pi_M|_{(x^r(y^a), v^bB^r_b)}\left(\frac{\partial}{\partial y^c}\right) = \frac{\partial(\pi_M)^a}{\partial y^c}\frac{\delta}{\delta y^a} = \delta^a_c\frac{\delta}{\delta y^a}\bigg|_{(x^r(y^a))} = \frac{\delta}{\delta y^c}\bigg|_{(x^r(y^a))},$$

and

$$d\pi_N|_{(x^r(y^a),v^bB^r_b)}\left(\frac{\partial}{\partial y^c}\right)=d\pi_N|_{(x^r(y^a),v^bB^r_b)}\left(B^r_c\frac{\partial}{\partial x^r}+v^bB^r_b\frac{\partial}{\partial u^r}\right)$$
$$=B^r_c\left[d\pi_N|_{(x^r(y^a),v^bB^r_b)}\left(\frac{\partial}{\partial x^r}\right)\right]+v^bB^r_b\left[d\pi_N|_{(x^r(y^a),v^bB^r_b)}\left(\frac{\partial}{\partial u^r}\right)\right]$$
$$=B^r_c\left(\frac{\partial(\pi_N)^s}{\partial x^r}\frac{\partial}{\partial x^s}\right)\bigg|_{(x^r(y^a))}+v^bB^r_b\left(\frac{\partial(\pi_N)^s}{\partial u^r}\frac{\partial}{\partial x^s}\right)\bigg|_{(x^r(y^a))}$$
$$=B^r_c\left(\delta^s_r\frac{\partial}{\partial x^s}\right)\bigg|_{(x^r(y^a))}=B^r_c\frac{\partial}{\partial x^s}\bigg|_{(x^r(y^a))}=\frac{\delta}{\delta y^c}\bigg|_{(x^r(y^a))}.$$

Similarly, for $\frac{\partial}{\partial v^c}$ tangent to TM, we get

$$d\pi_M|_{(x^r(y^a),v^bB^r_b)}\left(\frac{\partial}{\partial v^c}\right)=\frac{\partial(\pi_M)^a}{\partial v^c}\frac{\delta}{\delta y^a}=0,$$

and

$$d\pi_N|_{(x^r(y^a),v^bB^r_b)}\left(\frac{\partial}{\partial v^c}\right)=d\pi_N|_{(x^r(y^a),v^bB^r_b)}\left(B^r_c\frac{\partial}{\partial u^r}\right)$$
$$=B^r_c\left[d\pi_N|_{(x^r(y^a),v^bB^r_b)}\left(\frac{\partial}{\partial u^r}\right)\right]=B^r_c\left(\frac{\partial(\pi_N)^s}{\partial u^r}\frac{\partial}{\partial x^s}\right)\bigg|_{(x^r(y^a))}=0. \blacksquare$$

4.2. Connection map

LEMMA 7. *Let $\widetilde{f}:TM\to TN$ be the lift of the immersion $f:M\to N$ in the sense of Definition 1. Then the connection maps K_N and K_M with respect to the connections ∇_N and ∇_M respectively satisfy*

$$K_N\left(\frac{\partial}{\partial v^a}\right)=\frac{\delta}{\delta y^a}=K_M\left(\frac{\partial}{\partial v^a}\right),$$
$$K_N\left(\frac{\partial}{\partial y^a}\right)=v^b\nabla_bB^r_a\frac{\partial}{\partial x^r}+K_M\left(\frac{\partial}{\partial y^a}\right).$$

Proof. By definition we have

$$K_N:T_{(x,u)}TN\to T_xN,\qquad K_N\left(X^r\frac{\partial}{\partial x^r}+\overline{X}^r\frac{\partial}{\partial u^r}\right)=(\overline{X}^r+\Gamma^r_{st}u^sX^t)\frac{\partial}{\partial x^r},$$
$$K_M:T_{(y,v)}TM\to T_yM,\qquad K_M\left(Z^a\frac{\partial}{\partial y^a}+\overline{Z}^a\frac{\partial}{\partial v^a}\right)=(\overline{Z}^a+\Gamma^a_{bc}v^bZ^c)\frac{\delta}{\delta y^a}.$$

Hence,

$$K_N\left(\frac{\partial}{\partial v^a}\right)=K_N\left(B^r_a\frac{\partial}{\partial u^r}\right)=B^r_a\frac{\partial}{\partial x^r}=\frac{\delta}{\delta y^a}=K_M\left(\frac{\partial}{\partial v^a}\right).$$

Moreover,

$$K_M\left(\frac{\partial}{\partial y^a}\right)=\Gamma^c_{db}v^b\delta^d_a\frac{\delta}{\delta y^c}=\Gamma^c_{ab}v^bB^t_c\frac{\partial}{\partial x^t},$$

and, by the use of (2.1) and (4.1), we find

$$K_N\left(\frac{\partial}{\partial y^a}\right) = K_N\left(B^r_a\frac{\partial}{\partial x^r} + v^b\partial_b B^r_a\frac{\partial}{\partial u^r}\right) = (v^b\partial_b B^r_a + \Gamma^r_{st}B^s_a u^t)\frac{\partial}{\partial x^r}$$
$$= v^b(\partial_b B^r_a + \Gamma^r_{st}B^s_a B^t_b)\frac{\partial}{\partial x^r} = v^b(\nabla_b B^r_a + \Gamma^c_{ab}B^r_c)\frac{\partial}{\partial x^r}$$
$$= v^b\nabla_b B^r_a\frac{\partial}{\partial x^r} + v^b\Gamma^c_{ab}\frac{\delta}{\delta y^c} = v^b\nabla_b B^r_a\frac{\partial}{\partial x^r} + K_M\left(\frac{\partial}{\partial y^a}\right). \ \blacksquare$$

5. Vector fields normal to LM**.** In the case where M is totally geodesic in N, the unit vector fields normal to the lift of M can be chosen in the form $\alpha\eta_x^{h_N} + \beta\eta_x^{v_N}$, $x = m+1,\ldots,n$, where α, β are functions depending on generators of the g-natural metric G along the lift. The next lemma explains the structure of the vector fields normal to LM in the general case.

THEOREM 8. *Suppose that M is not necessarily totally geodesic in N and $\eta = H_\top^{h_N} + H_\perp^{h_N} + V_\top^{v_N} + V_\perp^{v_N}$ is a vector field normal to the lifted submanifold LM, where $H_\top$, $V_\top$ are tangent to M and $H_\perp$, $V_\perp$ are normal to M in TN. Then*

$$H_\top = V_\top = 0$$

for all $x \in M$ and

$$g(K_{ac}, a_2H_\perp + a_1V_\perp) = 0 \tag{5.1}$$

and

$$g(K_a, a_2'H_\perp + a_1'V_\perp) = 0.$$

Proof. The relation $G(\delta_a^{v_N}, \eta) = 0$ yields

$$a_2H_\top + b_2g(H_\top, u)u + a_1V_\top + b_1g(V_\top, u)u = 0, \tag{5.2}$$

while $G(\delta_a^{h_M}, \eta) = 0$, in virtue of Lemma 4, gives

$$g(\delta_a,\ AH_\top + Bg(H_\top, u)u + a_2V_\top + b_2g(V_\top, u)u) + g(K_a,\ a_2H_\perp + a_1V_\perp) = 0, \tag{5.3}$$

where $K_a = v^c\nabla_c B^r_a\partial_r$. Substituting $u = v^cB^r_c\partial_r = 0$ into (5.2) and (5.3) we get

$$AH_\top + a_2V_\top = 0, \qquad a_2H_\top + a_1V_\top = 0$$

for all $(x,0) \in TM$. Since G is non-degenerate and $H_\top$ and $V_\top$ do not depend on v^c, we get $H_\top = V_\top = 0$ for all $x \in M$. Consequently, by (5.3),

$$g(K_a, a_2H_\perp + a_1V_\perp) = 0 \tag{5.4}$$

for all $(x,u) \in TM$. Differentiating (5.4) with respect to v^c we obtain

$$g(K_{ac}, a_2H_\perp + a_1V_\perp) + 2g(K_a, a_2'H_\perp + a_1'V_\perp)g(\delta_c, u) = 0.$$

Transvecting with v^c, in view of (5.4), we get $g(K_a, a_2' H_\perp + a_1' V_\perp) r^2 = 0$ for all $r^2 = g(u,u) > 0$. Hence, by continuity, $g(K_a, a_2' H_\perp + a_1' V_\perp) = 0$ for all $u \in T_x M$. ∎

Corollary 9. *There exist submanifolds M isometrically immersed in a Riemannian manifold (N, g) such that for a given vector field $\eta = H_\perp^{h_N} + V_\perp^{v_N}$ normal to LM immersed in TN with non-degenerate g-natural metric G the relation $a_2 H_\perp + a_1 V_\perp = 0$ holds along M.*

Proof. It is enough to take as M a non-totally geodesic hypersurface in a Riemannian manifold N. ∎

6. Lift of a totally geodesic submanifold

6.1. Normal bundle. Suppose that M is a totally geodesic submanifold isometrically immersed in N and η_x, $x = m+1, \ldots, n$, are vector fields normal to M in TN. Then, by Lemma 4, the lifts of the vector fields from M to TM coincide with those to TN. The lifts $(\eta_x)^h = (\eta_x)^{h_N}$ and $(\eta_x)^v = (\eta_x)^{v_N}$ are orthogonal to $\left(\frac{\delta}{\delta y^a}\right)^{h_M}$ and $\left(\frac{\delta}{\delta y^b}\right)^{v_M}$ but are orthogonal to each other if and only if $a_2 = 0$ since $G((\eta_x)^h, (\eta_y)^v) = a_2 g(\eta_x, \eta_y)$ for all $x, y = m+1, \ldots, n$.

Proposition 10. *Let M, $\dim M = m$, be a totally geodesic submanifold isometrically immersed in a Riemannian manifold (N, g), $\dim N = n$ and η_x, $x = m+1, \ldots, n$ be a set of vector fields normal to M in TN. Suppose, moreover, that TN is endowed with a g-natural metric G and $a = a_1 A - a_2^2 \neq 0$.*

If LM denotes the lift (4.1) of M to TN, then the normal bundle of LM in TTN is spanned by the vector fields S_x, T_x, $x = m+1, \ldots, n$, below, and the following six cases can occur:

(1) *$Aa_1 \neq 0$, a_2 arbitrary. Let $\varepsilon = \operatorname{sgn}(a_1 A - a_2^2)$, $\delta = \operatorname{sgn} a_1$ and*

$$S_x = \frac{\varepsilon\delta\sqrt{|a_1|}}{\sqrt{|a|}} (\eta_x)^h - \frac{\varepsilon\delta a_2 \sqrt{|a_1|}}{a_1\sqrt{|a|}} (\eta_x)^v, \qquad T_x = \frac{\delta}{\sqrt{|a_1|}} (\eta_x)^v.$$

Then

$$G(S_x, S_y) = \varepsilon\delta g(\eta_x, \eta_y), \quad G(S_x, T_y) = 0, \quad G(T_x, T_y) = \delta g(\eta_x, \eta_y).$$

(2) *$a_2 \neq 0$, $A = 0$, $a_1 = 0, \varepsilon = \operatorname{sgn} a_2$ and*

$$S_x = \frac{\varepsilon}{2a_2} \eta_x^h - \eta_x^v, \qquad T_x = \frac{\varepsilon}{2a_2} \eta_x^h + \eta_x^v.$$

Then

$$G(S_x, S_y) = -\varepsilon g(\eta_x, \eta_y), \quad G(S_x, T_y) = 0, \quad G(T_x, T_y) = \varepsilon g(\eta_x, \eta_y).$$

(3) $a_2 \neq 0,\ A = 0,\ a_1 \neq 0,\ \varepsilon = \operatorname{sgn} a_1 = -1$ *and*

$$S_x = \frac{a_1}{\sqrt{3}\sqrt{|a_1|}a_2}\eta_x^h + \frac{1}{\sqrt{3}\sqrt{|a_1|}}\eta_x^v, \quad T_x = \frac{2a_1}{\sqrt{3}\sqrt{|a_1|}a_2}\eta_x^h - \frac{1}{\sqrt{3}\sqrt{|a_1|}}\eta_x^v.$$

Then

$$G(S_x, S_y) = -g(\eta_x, \eta_y), \quad G(S_x, T_y) = 0, \quad G(T_x, T_y) = g(\eta_x, \eta_y).$$

(4) $a_2 \neq 0,\ A = 0,\ a_1 \neq 0,\ \varepsilon = \operatorname{sgn} a_1 = 1$ *and*

$$S_x = \frac{a_1}{\sqrt{a_1}a_2}\eta_x^h - \frac{1}{\sqrt{a_1}}\eta_x^v, \qquad T_x = \frac{1}{\sqrt{a_1}}\eta_x^v.$$

Then

$$G(S_x, S_y) = -g(\eta_x, \eta_y), \quad G(S_x, T_y) = 0, \quad G(T_x, T_y) = g(\eta_x, \eta_y).$$

(5) $a_2 \neq 0,\ A \neq 0,\ a_1 = 0,\ \varepsilon = \operatorname{sgn} A = -1$ *and*

$$S_x = \frac{A}{\sqrt{3}\sqrt{|A|}a_2}\eta_x^v + \frac{1}{\sqrt{3}\sqrt{|A|}}\eta_x^h, \qquad T_x = \frac{2A}{\sqrt{3}\sqrt{|A|}a_2}\eta_x^v - \frac{1}{\sqrt{3}\sqrt{|A|}}\eta_x^h.$$

Then

$$G(S_x, S_y) = -g(\eta_x, \eta_y), \quad G(S_x, T_y) = 0, \quad G(T_x, T_y) = g(\eta_x, \eta_y).$$

(6) $a_2 \neq 0,\ A \neq 0,\ a_1 = 0,\ \varepsilon = \operatorname{sgn} A = 1$ *and*

$$S_x = \frac{A}{\sqrt{A}a_2}\eta_x^v - \frac{1}{\sqrt{A}}\eta_x^h, \qquad T_x = \frac{1}{\sqrt{A}}\eta_x^h.$$

Then

$$G(S_x, S_y) = -g(\eta_x, \eta_y), \quad G(S_x, T_y) = 0, \quad G(T_x, T_y) = g(\eta_x, \eta_y).$$

If η_x are unit, so are S_x and T_x.

Moreover, only in the first case can the metric induced on the normal bundle from G be Riemannian. In all other cases the metrics are neutral.

6.2. Formulas. Let $\eta = H_\perp^{h_N} + V_\perp^{v_N} = H^{h_N} + V^{v_N}$ be a vector field normal to LM. According to Theorem 8, H and V are vector fields normal to M in TN such that $d\pi_N(\eta) = H$ and $K_N(\eta) = V$. Denote by $\widetilde{A}$ and $\widetilde{H}$ the shape operator and the second fundamental form of the immersion (4.1) of LM into TN.

PROPOSITION 11. *For the immersion* (4.1) *of LM into TN, in the above notation, we have*

$$\begin{aligned}
(6.1)\qquad & G(\widetilde{\nabla}_{\delta_a^{h_M}}\eta, \delta_b^{h_M}) = -G(\widetilde{A}_\eta(\delta_a^{h_M}), \delta_b^{h_M}) = -G(\eta, \widetilde{\nabla}_{\delta_a^{h_M}}(\delta_b^{h_M}))\\
&= -G(\eta, \widetilde{H}(\delta_a^{h_M}, \delta_b^{h_M})) + \tfrac{1}{2}a_1[R(u, K_a, H, \delta_b) + R(u, K_b, H, \delta_a)\\
&\quad + R(u, V, \delta_a, \delta_b)] + a_2 R(u, \delta_a, H, \delta_b) - \tfrac{1}{2}B[g(u, \delta_b)g(H, K_a)\\
&\quad + g(u, \delta_a)g(H, K_b)] - g(AH + a_2 V, K_{ab}) - a_1 g(a_1 V + a_2 H, \nabla_{\delta_a} K_b),
\end{aligned}$$

$$-G(\widetilde{A}_\eta(\delta_a^{h_M}), \delta_b^{v_M}) = -G(\eta, \widetilde{\nabla}_{\delta_a^{h_M}}(\delta_b^{v_M})) = \tfrac{1}{2}[a_1 R(u, \delta_b, H, \delta_a) - b_2 g(u, \delta_b) g(H, K_a)], \tag{6.2}$$

$$-G(\widetilde{A}_\eta(\delta_a^{v_M}), \delta_b^{h_M}) = -G(\eta, \widetilde{\nabla}_{\delta_a^{v_M}}(\delta_b^{h_M})) = \tfrac{1}{2}[a_1 R(u, \delta_a, H, \delta_b) - b_2 g(u, \delta_a) g(H, K_b)], \tag{6.3}$$

$$-G(\widetilde{A}_\eta(\delta_a^{v_M}), \delta_b^{v_M}) = -G(\eta, \widetilde{\nabla}_{\delta_a^{v_M}}(\delta_b^{v_M})) = 0. \tag{6.4}$$

Here $\widetilde{\nabla}$, ∇, R *denote the Levi-Civita connections on* TN *and* N *respectively and the Riemann curvature tensor of* N.

Proof. For the F-tensors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F} : TN \otimes TN \otimes TN \to TN$ we set $\mathbf{A}(x, y, z, t) = g(\mathbf{A}(x, y, z), t)$, etc., where g is the metric tensor on N. Since $g(u, H) = g(u, V) = 0$ along M, by the use of Lemma 3 and either the Gauss formula or the Weingarten formula, we obtain

$$\begin{aligned} G(\eta, \widetilde{H}(\delta_a^{h_M}, \delta_b^{h_M})) &= G(\eta, \widetilde{\nabla}_{\delta_a^{h_M}}(\delta_b^{h_M})) \\ &= A g(H, \nabla_{\delta_a}\delta_b) + A\mathbf{A}(u, \delta_a, \delta_b, H) + a_2\mathbf{B}(u, \delta_a, \delta_b, H) \\ &\quad + a_2 g(H, \nabla_{\delta_a} K_b) + A\mathbf{C}(u, \delta_a, K_b, H) + a_2\mathbf{D}(u, \delta_a, K_b, H) \\ &\quad + A\mathbf{C}(u, \delta_a, K_b, H) + a_2\mathbf{D}(u, \delta_a, K_b, H) + A\mathbf{E}(u, K_a, K_b, H) \\ &\quad + a_2\mathbf{F}(u, K_a, K_b, H) + a_2 g(V, \nabla_{\delta_a}\delta_b) + a_2\mathbf{A}(u, \delta_a, \delta_b, V) \\ &\quad + a_1\mathbf{B}(u, \delta_a, \delta_b, V) + a_1 g(V, \nabla_{\delta_a} K_b) + a_2\mathbf{C}(u, \delta_a, K_b, V) \\ &\quad + a_1\mathbf{D}(u, \delta_a, K_b, V) + a_2\mathbf{C}(u, \delta_a, K_b, V) + a_1\mathbf{D}(u, \delta_a, K_b, V) \\ &\quad + a_2\mathbf{E}(u, K_a, K_b, V) + a_1\mathbf{F}(u, K_a, K_b, V), \end{aligned} \tag{6.5}$$

$$\begin{aligned} G(\eta, \widetilde{H}(\delta_a^{h_M}, \delta_b^{v_M})) &= G(\eta, \widetilde{\nabla}_{\delta_a^{h_M}}(\delta_b^{v_M})) \\ &= a_2 g(H, \nabla_{\delta_a}\delta_b) + a_1 g(V, \nabla_{\delta_a}\delta_b) + A\mathbf{C}(u, \delta_a, \delta_b, H) + a_2\mathbf{D}(u, \delta_a, \delta_b, H) \\ &\quad + a_2\mathbf{C}(u, \delta_a, \delta_b, V) + a_1\mathbf{D}(u, \delta_a, \delta_b, V) + A\mathbf{E}(u, K_a, \delta_b, H) \\ &\quad + a_2\mathbf{F}(u, K_a, \delta_b, H) + a_2\mathbf{E}(u, K_a, \delta_b, V) + a_1\mathbf{F}(u, K_a, \delta_b, V), \end{aligned} \tag{6.6}$$

$$\begin{aligned} G(\eta, \widetilde{H}(\delta_a^{v_M}, \delta_b^{h_M})) &= G(\eta, \widetilde{\nabla}_{\delta_a^{v_M}}(\delta_b^{h_M})) \\ &= A\mathbf{C}(u, \delta_b, \delta_a, H) + a_2\mathbf{D}(u, \delta_b, \delta_a, H) + a_2\mathbf{C}(u, \delta_b, \delta_a, V) \\ &\quad + a_1\mathbf{D}(u, \delta_b, \delta_a, V) + A\mathbf{E}(u, \delta_b, K_a, H) + a_2\mathbf{F}(u, \delta_b, K_a, H) \\ &\quad + a_2\mathbf{E}(u, \delta_b, K_a, V) + a_1\mathbf{F}(u, \delta_b, K_a, V), \end{aligned} \tag{6.7}$$

$$\begin{aligned} G(\eta, \widetilde{H}(\delta_a^{v_M}, \delta_b^{v_M})) &= G(\eta, \widetilde{\nabla}_{\delta_a^{v_M}}(\delta_b^{v_M})) \\ &= A\mathbf{E}(u, \delta_a, \delta_b, H) + a_2\mathbf{F}(u, \delta_a, \delta_b, H) + a_2\mathbf{E}(u, \delta_a, \delta_b, V) + a_1\mathbf{F}(u, \delta_a, \delta_b, V). \end{aligned} \tag{6.8}$$

Then, by the use of Theorem 8, we reach the conclusion. The Gauss and Weingarten formulas both lead to the same result, but the computations with the use of the former seem to be a bit simpler. ∎

From the above formulas we immediately get our main result.

THEOREM 12. *Let M be a manifold isometrically immersed in a space N of constant curvature, and LM be the lift* (4.1) *of M to TN with non-degenerate g-natural metric G.*

If M is a totally geodesic submanifold in (N, g), then LM is totally geodesic in (TN, G).

If LM is a totally geodesic submanifold in (TN, G) and b_2 does not vanish on a dense subset of $[0, \infty)$, then M is totally geodesic in (N, g).

Proof. By (6.1)–(6.3) the first assertion is clear. To prove the second, note that the normal bundle of LM is spanned by the lifts of the vector fields normal to M, i.e. by $\eta_x^{h_N}$ and $\eta_x^{v_N}$, $x = m+1, \ldots, n$. Since $K_a = v^c h_{ca}^x \eta_x$, $K_a^{h_N}$ and $K_a^{v_N}$ are orthogonal to LM. If LM is totally geodesic, setting $\eta = K_a^{h_N} + K_a^{v_N}$ we deduce from (6.2) that $g(K_a, K_b) = 0$ for all $a, b = 1, \ldots, m$, whence, since g is Riemannian, $K_a = 0$. ∎

Recall that the submanifold M of (N, g) is said to be *pseudo-umbilical* if its second fundamental form h and the mean curvature vector $\overrightarrow{h}$ satisfy $g(h(X, Y), \overrightarrow{h}) = \lambda g(X, Y)$ for all vector fields X, Y tangent M, where λ is a function defined along M. It is clear that a totally geedesic submanifold is pseudo-umbilical.

Thus, if LM is a pseudo-umbilical submanifold in (TN, G), then its second fundamental form $\widetilde{H}$ and the mean curvature vector $\overrightarrow{\widetilde{H}}$ satisfy

$$G(\widetilde{H}(X, Y), \overrightarrow{\widetilde{H}}) = \Lambda G(X, Y)$$

for some function Λ and all vector fields tangent to LM.

THEOREM 13. *The lift LM to TN of a manifold M, $m = \dim M > 1$, isometrically immersed in a manifold (N, g), with metric induced from a non-degenerate g-natural metric G on TN, is not a pseudo-umbilical submanifold unless $\Lambda = 0$.*

Proof. Suppose $\Lambda \neq 0$. By (6.4) we have

$$G(\widetilde{H}(\delta_a^{v_M}, \delta_b^{v_M}), \overrightarrow{\widetilde{H}}) = \Lambda G(\delta_a^{v_M}, \delta_b^{v_M}) = \Lambda[a_1 g_{ab} + b_1 v_a v_b] = 0.$$

Contracting with g^{ab} we get

$$\Lambda[m a_1 + b_1 r^2] = 0,$$

while transvecting with $v^a v^b$ we obtain, for $r^2 > 0$,

$$\Lambda[a_1 + b_1 r^2] = 0,$$

whence $\Lambda(m-1)a_1 = 0$. By continuity, $a_1(0) = 0$.

Now (6.2) yields

$$\Lambda[a_2 g_{ab} + b_2 v_a v_b] = b_2 v_a g(\overrightarrow{\overrightarrow{H}}, K_b),$$

whence, by a similar argument, $a_2(t) = 0$ for all $t \in [0, \infty)$, a contradiction. This completes the proof. ∎

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