C O L L O Q U I U M M A T H E M A T I C U M

VOL. 150 2017 NO. 1

ON FIBERS OF FAT ASSOCIATED BUNDLES

BY

MACIEJ BOCHEŃSKI, ANNA SZCZEPKOWSKA, ALEKSY TRALLE
and ARTUR WOIKE (Olsztyn)

Dedicated to the memory of Professor Witold Roter

Abstract. We are interested in associated fat bundles, which are an important tool in constructing Riemannian metrics of positive and non-negative curvature. We want to understand the behavior of the fatness condition under changes of structure groups of bundles. We show that if the structure group does not coincide with the holonomy group, and the G-bundle is fat, then the fiber of the bundle must be finitely covered by a sphere or a complex projective space of a particular dimension.

1. Introduction. Let $P(M,G)$ be a G-principal bundle endowed with a connection form θ and with curvature form Ω. Let $\operatorname{Ker}\theta = \mathcal{H} \subset TP$ be the corresponding horizontal distribution. Assume that the Lie algebra $\mathfrak{g}$ of G is endowed with an invariant non-degenerate bilinear form B. We say that a vector $u \in \mathfrak{g}$ is *fat*, or that the connection form θ is u-fat, if the bilinear 2-form $B(\Omega(\cdot,\cdot),u)$ is non-degenerate on $\mathcal{H}$. The role of the fatness condition in Riemannian geometry follows from its relation to the O'Neill tensor [4] of a specific fiberwise metric on the associated bundle. In more detail, consider an associated bundle

$$F \to P \times_G F \to M.$$

Endow F with a G-invariant Riemannian metric g_F, and M with a Riemannian metric g_M. Equip $P \times_G F$ with the *connection metric*, defining it to be the Riemannian metric which equals g_F on F, $g(X^*,Y^*) = g_M(X,Y)$ for horizontal lifts X^*, Y^* of $X, Y \in TM$ respectively, and declaring TF and $\mathcal{H}$ to be orthogonal with respect to g. For that metric the following holds. Let A_X denote the O'Neill tensor.

THEOREM 1.1 ([26]). *The connection metric g on $P \times_G F$ is complete and defines a Riemannian submersion $\pi : P \times_G F \to M$ with totally geodesic*

2010 *Mathematics Subject Classification*: Primary 57S30; Secondary 22F30, 22E40, 22E46.
Key words and phrases: fat bundle, connection, homogeneous space.
Received 30 October 2016.
Published online 1 September 2017.

DOI: 10.4064/cm7138s-1-2017

fibers F and with holonomy group a subgroup of G. Conversely, every Riemannian submersion over M with totally geodesic fibers arises in this fashion. Moreover, the O'Neill tensor for such submersion satisfies the equality

$$\theta(A_X Y) = -\Omega(X, Y).$$

The condition of fatness is an important tool in constructing manifolds of positive and non-negative curvature [13], [25]. There are also notions of symplectic and contact fatness, which are different but parallel to the Riemannian one. They are used as a tool of constructing symplectic and contact manifolds with prescribed properties [5], [10], [11], [19], [20], [14], [16], [21], [25], [7], [8]. In the Riemannian context, the following definition of fatness is used: a Riemannian submersion $\pi : E \to M$ with totally geodesic fibers is fat if $A_X U \neq 0$ for all horizontal vector fields X and vertical vector fields U ("all vertizontal curvatures are positive"). For the associated bundles, the characterization of fatness of any connection is known [26] and can be formulated as follows, with $\mathfrak{l}$ denoting the Lie algebra of the Lie subgroup L.

THEOREM 1.2 ([26]). *The Riemannian submersion $P \times_G F \to M$ with totally geodesic fibers G/L is fat if and only if the 2-form $B(\Omega(X, Y), u)$ is non-degenerate on the horizontal distribution for all $u \in \mathfrak{l}^\perp$.*

Keeping the above theorem in mind, from now on whenever we say that an associated bundle $G/L \to P/L \to M$ is fat, we mean that the set $\mathfrak{l}^\perp$ consists of fat vectors. Note that it was proved [26, Proposition 2.6] that every fat submersion necessarily has a homogeneous fiber. The fatness condition is well understood only in several particular cases. For example, a theorem of Derdziński and Rigas [9] shows that the only fat $\mathrm{SO}(4)/\mathrm{SO}(3)$-bundle over S^4 is the Hopf bundle $S^7 \to S^4$. Bérard-Bergery [3] classified all homogeneous fat bundles, that is, associated bundles of the form $H/L \to K/L \to K/H$, where K, H, L are compact Lie groups. There are some necessary topological conditions for fatness [12]. The conclusion is that fat bundles are scarce. In our previous paper [6] we explored the possibility of extending the results of [3] to the case of invariant connections in G-structures over compact homogeneous spaces. However, we were able to get a kind of classification of fat bundles only for the canonical connection. One can notice the following:

1. The fatness condition is preserved neither under the reduction nor under the extension of the structure group.
2. It depends on the choice of the connection.

Therefore, one can try to construct new fat bundles either by changing the connection, or by changing the structure group.

In this paper we show that the second approach is not possible with one exception: the fibers of the given G-bundle are finitely covered by spheres or complex projective spaces of some particular dimensions (Theorem 4.1).

2. Preliminaries. The basic tools in this work are invariant connections and Lie group theory. Therefore, we closely follow the terminology and notation in [17], [18], [15] and [24] without further explanations. We denote the Lie algebras of Lie groups by the corresponding Gothic letters, so the Lie algebra of the Lie group G is denoted by $\mathfrak{g}$, etc. We consider principal bundles $G \to P \to M$ and in most cases use the notation $P(M,G)$. Throughout the whole article we consider homogeneous spaces $M = K/H$ and assume that both K and H are compact Lie groups.

We recall the notion of the holonomy group following [17]. Let $P(M,G)$ denote a principal bundle with connection form θ on P. Take a point $x \in M$ and $p \in \pi^{-1}(x)$. It can be proved that for a piecewise smooth loop $\gamma : [0,1] \to M$ based at x the connection θ defines a unique horizontal lift $\tilde{\gamma} : [0,1] \to P$ for which $\tilde{\gamma}(0) = p$ and $\tilde{\gamma}(1) \in \pi^{-1}(x)$. Introduce an equivalence relation on P as follows: $p \sim q \Leftrightarrow p$ can be joined to q by a piecewise smooth horizontal curve. The *holonomy group* at a point $p \in P$ of a given connection in the principal bundle is $\mathrm{Hol}(p) = \{g \in G \mid p \sim p \cdot g\}$. It can be proved that $\mathrm{Hol}(p)$ is a Lie subgroup of G. Throughout this paper the Lie algebra of the holonomy group will be denoted by $\mathfrak{h}^*$. Denote by $P(p)$ the set of points in P which can be joined to p by a horizontal curve. Recall the classical theorem of Ambrose and Singer [1], [2], [17].

THEOREM 2.1. *Let $G \to P \to M$ be a principal bundle with M connected and paracompact. Let θ be a connection form and Ω the curvature form of a given principal bundle. Let $p \in P$. The Lie algebra of* $\mathrm{Hol}(p)$ *is equal to the subspace of $\mathfrak{g}$ spanned by all elements of the form $\Omega_v(X,Y)$, where $v \in P(p)$ and X, Y are arbitrary horizontal vectors at v.*

3. Onishchik triples

DEFINITION 3.1. We say that a triple of Lie algebras $(\mathfrak{g}, \mathfrak{l}, \mathfrak{h})$ is an *Onishchik triple*, or an *Onishchik decomposition*, if

$$\mathfrak{h} \subset \mathfrak{g}, \quad \mathfrak{l} \subset \mathfrak{g}, \quad \mathfrak{g} = \mathfrak{h} + \mathfrak{l}$$

(the inclusions are inclusions of Lie algebras, while the sum is understood as the sum of vector spaces). We will say that the Onishchik triple is *proper* if $\mathfrak{h} \neq \mathfrak{g}$ and $\mathfrak{l} \neq \mathfrak{g}$.

Onishchik classified such triples [22] (which he called *Lie algebra decompositions*). These decompositions are important and useful in so many different areas that we believe they deserve a new name. In this article we will need this classification. Therefore, we present a short summary in the case where $\mathfrak{g}$ is compact and simple. As usual, we consider the complexification $\mathfrak{g}^c$ and a Cartan subalgebra $\mathfrak{j} \subset \mathfrak{g}^c$. Having fixed them, we consider the corresponding root system Δ, and choose the set of simple roots $\alpha_1, \ldots, \alpha_m$.

It is known that any irreducible linear representation of $\mathfrak{g}$ is determined (up to equivalence) by the highest weight of the representation. This is a vector $\Lambda \in \mathfrak{j}$ which can be described by the integers

$$\Lambda_i = \frac{2\langle \Lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}, \qquad i = 1, \dots, m.$$

We write $\Lambda = (\Lambda_1, \dots, \Lambda_m)$ and denote by φ_i the irreducible representation of a simple Lie algebra $\mathfrak{g}$ with highest weight Λ which has $\Lambda_i = 1$ and $\Lambda_j = 0$, $i \neq j$. The following classification result was proved by Onishchik.

THEOREM 3.1 ([22]). *Let $\mathfrak{g}$ be a compact simple Lie algebra. All the proper Onishchik triples $(\mathfrak{g}, \mathfrak{l}, \mathfrak{h})$ and possible embeddings*

$$i' : \mathfrak{h} \to \mathfrak{g}, \qquad i'' : \mathfrak{l} \to \mathfrak{g}$$

are given in Table 1.

Table 1

$\mathfrak{g}$	$\mathfrak{h}$	i'	$\mathfrak{l}$	i''	Restrictions
A_{2n-1}	C_n	φ_1	A_{2n-2}	$\varphi_1 + N$	$n > 1$
A_{2n-1}	C_n	φ_1	$A_{2n-2} \oplus T$	$\varphi_1 + N$	$n > 1$
B_3	G_2	φ_2	B_2	$\varphi_1 + 2N$	
B_3	G_2	φ_2	$B_2 \oplus T$	$\varphi_1 + 2N$	
B_3	G_2	φ_2	D_3	$\varphi_1 + N$	
D_{n+1}	B_n	$\varphi_1 + N$	A_n	$\varphi_1 + \varphi_n$	$n > 2$
D_{n+1}	B_n	$\varphi_1 + N$	$A_n \oplus T$	$\varphi_1 + \varphi_n$	$n > 2$
D_{2n}	B_{2n-1}	$\varphi_1 + N$	C_n	$\varphi_1 + \varphi_1$	$n > 1$
D_{2n}	B_{2n-1}	$\varphi_1 + N$	$C_n \oplus T$	$\varphi_1 + \varphi_1$	$n > 1$
D_{2n}	B_{2n-1}	$\varphi_1 + N$	$C_n \oplus A_1$	$\varphi_1 + \varphi_1$	$n > 1$
D_8	B_7	$\varphi_1 + N$	B_4	φ_4	
D_4	B_3	φ_3	B_2	$\varphi_1 + 3N$	
D_4	B_3	φ_3	$B_2 \oplus T$	$\varphi_1 + 3N$	
D_4	B_3	φ_3	$B_2 \oplus A_1$	$\varphi_1 + 3N$	
D_4	B_3	φ_3	D_3	$\varphi_1 + 2N$	
D_4	B_3	φ_3	$D_3 \oplus T$	$\varphi_1 + 2N$	
D_4	B_3	φ_3	B_3	$\varphi_1 + N$	

REMARK 3.2. In Table 1, N stands for the trivial representation. The types of simple Lie algebras are denoted as usual, following [24].

DEFINITION 3.3 ([23]). We say that a triple (G, H, L) of Lie groups is an *Onishchik triple of Lie groups* if $G = HL$.

THEOREM 3.2 ([23]). *Let G be a connected Lie group and H, L be Lie subgroups, where at least one of H, L is compact. If $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$ is an Onishchik triple of Lie algebras, then (G, H, L) is an Onishchik triple of Lie groups.*

4. Holonomy and fibers of fat bundles

PROPOSITION 4.1. *Let $G \to P \to M$ be a principal fiber bundle with a connection form θ and curvature form Ω. Fix $p \in P$. Denote by $\mathfrak{h}'$ the subalgebra in $\mathfrak{g}$ generated by all $\Omega_p(X,Y)$, $X, Y \in \mathcal{H}_p$. If there exists a fat associated bundle*

$$G/L \to P \times_G (G/L) \to M$$

with respect to the given connection and $\mathfrak{g} \neq \mathfrak{h}'$, then $(\mathfrak{g}, \mathfrak{h}', \mathfrak{l})$ is a proper Onishchik triple.

Proof. Notice that the fatness condition is equivalent to all 2-forms $B(\Omega_p(X,Y), u)$ being non-degenerate for all $u \in \mathfrak{l}^{\perp}$, $p \in P$ and $X, Y \in \mathcal{H}_p$. The latter implies $(\mathfrak{h}')^{\perp} \cap \mathfrak{l}^{\perp} = \{0\}$. But $(\mathfrak{h}')^{\perp} \cap \mathfrak{l}^{\perp} = \{0\}$ iff $\mathfrak{g} = \mathfrak{h}' + \mathfrak{l}$. ∎

COROLLARY 4.2. *Let $G/L \to P \times_G (G/L) \to M$ be a fiber bundle associated with a principal bundle with a simple compact Lie group G. If it is fat with respect to some connection, then either the holonomy Lie algebra $\mathfrak{h}^*$ is contained in Table 2, or $\mathfrak{h}^* = \mathfrak{g}$.*

Proof. By Theorem 2.1, $\mathfrak{h}' \subset \mathfrak{h}^*$. Hence, if $\mathfrak{h}^* \neq \mathfrak{h}'$, the triple $(\mathfrak{g}, \mathfrak{h}^*, \mathfrak{l})$ must be a proper Onishchik triple as well. But looking at Table 1, one can see that this is not possible. Thus, either $\mathfrak{h}' = \mathfrak{h}^*$, or $\mathfrak{h}^* = \mathfrak{g}$. ∎

THEOREM 4.1. *Assume that a fiber bundle $G/L \to P/L \to M$ is associated with a principal bundle $P(M,G)$ where G is a simple and compact Lie group such that $\mathfrak{g} \neq \mathfrak{l}$. If such an associated bundle is fat with respect to a connection, then G/L is necessarily finitely covered by a homogeneous space of the form*

$$\begin{aligned} S^{8k-1} &= \mathrm{SU}(4k)/\mathrm{SU}(4k-1) \quad (k \geq 1), \\ S^5 &= \mathrm{SU}(4)/\mathrm{Sp}(2), \\ \mathbb{CP}^{2k-1} &= \mathrm{SU}(4k)/(\mathrm{SU}(4k-1) \times S^1)\ (k \geq 1), \\ S^{8k-1} &= \mathrm{SO}(8k)/\mathrm{SO}(8k-1)\ (k \geq 1), \\ S^{15} &= \mathrm{SO}(16)/\mathrm{SO}(15), \\ S^7 &= \mathrm{SO}(8)/\mathrm{SO}(7). \end{aligned}$$

The proof will follow from the propositions below.

PROPOSITION 4.3. *Assume that $\mathfrak{g}$, $\mathfrak{h}'$, $\mathfrak{l}$ and V are vector spaces such that $\mathfrak{h}', \mathfrak{l} \subset \mathfrak{g}$, $\mathfrak{g} = \mathfrak{h}' + \mathfrak{l}$, $\mathfrak{g}$ is equipped with a non-degenerate inner product $\langle \cdot, \cdot \rangle$, and the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{h}'$ and $\mathfrak{l}$ respectively is also nondegenerate. Let $R : V \times V \to \mathfrak{h}'$ be a bilinear map. If*

$$\exists_{0 \neq X \in V} \ \dim R(X, V) < \dim \mathfrak{l}^{\perp} = \dim \mathfrak{g} - \dim \mathfrak{l}$$

then

$$\exists_{u \in \mathfrak{l}^{\perp}} \forall_{Y \in V} \ \langle R(X,Y), u \rangle = 0.$$

Table 2

$\mathfrak{g}$	$\mathfrak{h}^*$	$\mathfrak{l}$	Restrictions
$\mathfrak{su}(4k)$	$\mathfrak{sp}(2k)$	$\mathfrak{su}(4k-1)$	$k \geq 1$
$\mathfrak{su}(4k)$	$\mathfrak{su}(4k-1)$	$\mathfrak{sp}(2k)$	$k \geq 1$
$\mathfrak{su}(4k)$	$\mathfrak{sp}(2k)$	$\mathfrak{su}(4k-1) \oplus \mathbb{R}$	$k \geq 1$
$\mathfrak{su}(4k)$	$\mathfrak{su}(4k-1) \oplus \mathbb{R}$	$\mathfrak{sp}(2k)$	$k \geq 1$
$\mathfrak{so}(8k)$	$\mathfrak{so}(8k-1)$	$\mathfrak{su}(4k)$	$k \geq 1$
$\mathfrak{so}(8k)$	$\mathfrak{su}(4k)$	$\mathfrak{so}(8k-1)$	$k \geq 1$
$\mathfrak{so}(8k)$	$\mathfrak{so}(8k-1)$	$\mathfrak{su}(4k) \oplus \mathbb{R}$	$k \geq 1$
$\mathfrak{so}(8k)$	$\mathfrak{su}(4k) \oplus \mathbb{R}$	$\mathfrak{so}(8k-1)$	$k \geq 1$
$\mathfrak{so}(8k)$	$\mathfrak{so}(8k-1)$	$\mathfrak{sp}(2k)$	$k \geq 1$
$\mathfrak{so}(8k)$	$\mathfrak{sp}(2k)$	$\mathfrak{so}(8k-1)$	$k \geq 1$
$\mathfrak{so}(8k)$	$\mathfrak{so}(8k-1)$	$\mathfrak{sp}(2k) \oplus \mathbb{R}$	$k \geq 1$
$\mathfrak{so}(8k)$	$\mathfrak{sp}(2k) \oplus \mathbb{R}$	$\mathfrak{so}(8k-1)$	$k \geq 1$
$\mathfrak{so}(8k)$	$\mathfrak{so}(8k-1)$	$\mathfrak{sp}(2k) \oplus \mathfrak{su}(2)$	$k \geq 1$
$\mathfrak{so}(8k)$	$\mathfrak{sp}(2k) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(8k-1)$	$k \geq 1$
$\mathfrak{so}(16)$	$\mathfrak{so}(15)$	$\mathfrak{so}(9)$	
$\mathfrak{so}(16)$	$\mathfrak{so}(9)$	$\mathfrak{so}(15)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{so}(5)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(5)$	$\mathfrak{so}(7)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{so}(5) \oplus \mathbb{R}$	
$\mathfrak{so}(8)$	$\mathfrak{so}(5) \oplus \mathbb{R}$	$\mathfrak{so}(7)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{so}(5) \oplus \mathfrak{su}(2)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(5) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(7)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{so}(6)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(6)$	$\mathfrak{so}(7)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{so}(6) \oplus \mathbb{R}$	
$\mathfrak{so}(8)$	$\mathfrak{so}(6) \oplus \mathbb{R}$	$\mathfrak{so}(7)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{so}(7)$	

Proof. Assume that $\dim R(X,V) < \dim \mathfrak{l}^{\perp}$ for some $0 \neq X \in V$. Let $\mathrm{pr}_{\mathfrak{l}}$ (respectively $\mathrm{pr}_{\mathfrak{l}^{\perp}}$) denote the orthogonal projection onto $\mathfrak{l}$ (respectively $\mathfrak{l}^{\perp}$) with respect to the orthogonal decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{l}^{\perp}$. Since $\dim R(X,V) < \dim \mathfrak{l}^{\perp}$, it follows that $\dim \mathrm{pr}_{\mathfrak{l}^{\perp}} R(X,V) < \dim \mathfrak{l}^{\perp}$. We have the orthogonal decomposition $\mathfrak{l}^{\perp} = \mathrm{pr}_{\mathfrak{l}^{\perp}} R(X,V) \oplus (\mathrm{pr}_{\mathfrak{l}^{\perp}} R(X,V))^{\perp}$ (with respect to the inner product of $\mathfrak{g}$ restricted to $\mathfrak{l}^{\perp}$) with $\dim(\mathrm{pr}_{\mathfrak{l}^{\perp}} R(X,V))^{\perp} \geq 1$. We can choose $0 \neq u \in (\mathrm{pr}_{\mathfrak{l}^{\perp}} R(X,V))^{\perp}$. Obviously $u \in \mathfrak{l}^{\perp}$. Consider the direct sum $\mathrm{pr}_{\mathfrak{l}} R(X,V) \oplus \mathrm{pr}_{\mathfrak{l}^{\perp}} R(X,V)$. Note that $R(X,V) \subset \mathrm{pr}_{\mathfrak{l}} R(X,V) \oplus \mathrm{pr}_{\mathfrak{l}^{\perp}} R(X,V)$. Take any $Y \in V$. Then $R(X,Y) = v_1 + v_2$, $v_1 \in \mathrm{pr}_{\mathfrak{l}} R(X,V) \subset \mathfrak{l}$ and $v_2 \in \mathrm{pr}_{\mathfrak{l}^{\perp}} R(X,V)$. Since $u \in \mathfrak{l}^{\perp}$ and $u \in (\mathrm{pr}_{\mathfrak{l}^{\perp}} R(X,V))^{\perp}$, we have

$$\langle R(X,Y), u\rangle = \langle v_1 + v_2, u\rangle = \langle v_1, u\rangle + \langle v_2, u\rangle = 0.$$

Then $u \in \mathfrak{l}^{\perp}$ and $\langle R(X,Y), u\rangle = 0$ for all $Y \in V$. ∎

Corollary 4.4. *If $\mathfrak{g}$, $\mathfrak{h}'$, $\mathfrak{l}$, V and R satisfy the assumptions of Proposition* 4.3 *then:*

(1) $(X,Y) \mapsto \langle R(X,Y), u\rangle$ *is degenerate for some* $u \in \mathfrak{l}^\perp$*;*
(2) *if in addition* $(\mathfrak{g}, \mathfrak{l}, \mathfrak{h}')$ *is an Onishchik triple and* R *corresponds to the curvature form then the bundle associated with a principal bundle* $P(M,G)$ *cannot be fat with respect to any connection.*

Remark 4.5. Assume that $\mathfrak{h}'$, V are vector spaces and $R : V \times V \to \mathfrak{h}'$ is a skew-symmetric bilinear mapping. Then

$$\forall_{0\neq X\in V}\,(\dim R(X,V) \le \dim V - 1 \wedge \dim R(V,X) \le \dim V - 1)\,.$$

This follows from the fact that for a skew-symmetric R the kernel of both $R(X,\cdot)$ and $R(\cdot, X)$ contains the one-dimensional subspace $\{\alpha X\}$.

Proposition 4.6. *The irreducible Onishchik triples* $(\mathfrak{g}, \mathfrak{l}, \mathfrak{h}')$ *in Table* 3 *do not correspond to fat associated bundles with respect to any connection.*

Table 3

$\mathfrak{g}$	$\mathfrak{l}$	$\mathfrak{h}'$	Restrictions
$\mathfrak{su}(4k)$	$\mathfrak{sp}(2k)$	$\mathfrak{su}(4k-1)$	$k \ge 2$
$\mathfrak{su}(4k)$	$\mathfrak{sp}(2k)$	$\mathfrak{su}(4k-1)\oplus\mathbb{R}$	$k \ge 2$
$\mathfrak{so}(8k)$	$\mathfrak{su}(4k)$	$\mathfrak{so}(8k-1)$	$k \ge 1$
$\mathfrak{so}(8k)$	$\mathfrak{su}(4k)\oplus\mathbb{R}$	$\mathfrak{so}(8k-1)$	$k \ge 1$
$\mathfrak{so}(8k)$	$\mathfrak{sp}(2k)$	$\mathfrak{so}(8k-1)$	$k \ge 1$
$\mathfrak{so}(8k)$	$\mathfrak{sp}(2k)\oplus\mathbb{R}$	$\mathfrak{so}(8k-1)$	$k \ge 1$
$\mathfrak{so}(8k)$	$\mathfrak{sp}(2k)\oplus\mathfrak{su}(2)$	$\mathfrak{so}(8k-1)$	$k \ge 1$
$\mathfrak{so}(16)$	$\mathfrak{so}(9)$	$\mathfrak{so}(15)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(5)$	$\mathfrak{so}(7)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(5)\oplus\mathbb{R}$	$\mathfrak{so}(7)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(5)\oplus\mathfrak{su}(2)$	$\mathfrak{so}(7)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(6)$	$\mathfrak{so}(7)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(6)\oplus\mathbb{R}$	$\mathfrak{so}(7)$	

Proof. Apply the previous considerations to $R(X,Y) = \Omega(X,Y)$. Note that in this case R is skew-symmetric since it corresponds to curvature. Now it is easy to check that each of these cases satisfies the assumptions of Proposition 4.3. Indeed, first note that if $\mathfrak{g} = \mathfrak{su}(4k)$ then $\mathfrak{g} \subset \mathfrak{so}(8k)$ and $\dim V = 8k$. In the remaining cases $\mathfrak{g} = \mathfrak{so}(8k)$ for some k, and then also $\dim V = 8k$. Since the dimensional assumption of Proposition 4.3 does not depend on the dimension of $\mathfrak{h}'$, it is sufficient to check it for all possible pairs of $\mathfrak{g}$ and $\mathfrak{l}$:

- $\mathfrak{g} = \mathfrak{su}(4k)$, $\mathfrak{l} = \mathfrak{sp}(2k)$, $k \ge 1$: Then $\dim V = 8k$, $\dim R(X,V) \le 8k-1$, $\dim \mathfrak{g} = 16k^2 - 1$, $\dim \mathfrak{l} = 8k^2 + 2k$, $\dim \mathfrak{l}^\perp = 8k^2 - 2k - 1$ and
$$\forall_{k\ge 2}\ \dim R(X,V) \le 8k-1 < 8k^2 - 2k - 1 = \dim \mathfrak{l}^\perp.$$

- $\mathfrak{g} = \mathfrak{so}(8k)$, $\mathfrak{l} = \mathfrak{su}(4k)$, $k \geq 1$: Then $\dim V = 8k$, $\dim R(X,V) \leq 8k-1$, $\dim \mathfrak{g} = 32k^2 - 4k$, $\dim \mathfrak{l} = 16k^2 - 1$, $\dim \mathfrak{l}^\perp = 16k^2 - 4k + 1$ and
$$\forall_{k\geq 1}\ \dim R(X,V) \leq 8k-1 < 16k^2 - 4k + 1 = \dim \mathfrak{l}^\perp.$$
- $\mathfrak{g} = \mathfrak{so}(8k)$, $\mathfrak{l} = \mathfrak{su}(4k) \oplus \mathbb{R}$, $k \geq 1$: Then $\dim V = 8k$, $\dim R(X,V) \leq 8k-1$, $\dim \mathfrak{g} = 32k^2 - 4k$, $\dim \mathfrak{l} = 16k^2$, $\dim \mathfrak{l}^\perp = 16k^2 - 4k$ and
$$\forall_{k\geq 1}\ \dim R(X,V) \leq 8k-1 < 16k^2 - 4k = \dim \mathfrak{l}^\perp.$$
- $\mathfrak{g} = \mathfrak{so}(8k)$, $\mathfrak{l} = \mathfrak{sp}(2k)$, $k \geq 1$: Then $\dim V = 8k$, $\dim R(X,V) \leq 8k-1$, $\dim \mathfrak{g} = 32k^2 - 4k$, $\dim \mathfrak{l} = 8k^2 + 2k$, $\dim \mathfrak{l}^\perp = 24k^2 - 6k$ and
$$\forall_{k\geq 1}\ \dim R(X,V) \leq 8k-1 < 24k^2 - 6k = \dim \mathfrak{l}^\perp.$$
- $\mathfrak{g} = \mathfrak{so}(8k)$, $\mathfrak{l} = \mathfrak{sp}(2k) \oplus \mathbb{R}$, $k \geq 1$: Then $\dim V = 8k$, $\dim R(X,V) \leq 8k-1$, $\dim \mathfrak{g} = 32k^2 - 4k$, $\dim \mathfrak{l} = 8k^2 + 2k + 1$, $\dim \mathfrak{l}^\perp = 24k^2 - 6k - 1$ and
$$\forall_{k\geq 1}\ \dim R(X,V) \leq 8k-1 < 24k^2 - 6k - 1 = \dim \mathfrak{l}^\perp.$$
- $\mathfrak{g} = \mathfrak{so}(8k)$, $\mathfrak{l} = \mathfrak{sp}(2k) \oplus \mathfrak{su}(2)$, $k \geq 1$: Then $\dim V = 8k$, $\dim R(X,V) \leq 8k-1$, $\dim \mathfrak{g} = 32k^2 - 4k$, $\dim \mathfrak{l} = 8k^2 + 2k + 3$, $\dim \mathfrak{l}^\perp = 24k^2 - 6k - 3$ and
$$\forall_{k\geq 1}\ \dim R(X,V) \leq 8k-1 < 24k^2 - 6k - 3 = \dim \mathfrak{l}^\perp.$$
- $\mathfrak{g} = \mathfrak{so}(16)$, $\mathfrak{l} = \mathfrak{so}(9)$: Then $\dim V = 16$, $\dim R(X,V) \leq 15$, $\dim \mathfrak{g} = 120$, $\dim \mathfrak{l} = 36$, $\dim \mathfrak{l}^\perp = 84$ and
$$\dim R(X,V) \leq 15 < 84 = \dim \mathfrak{l}^\perp.$$
- $\mathfrak{g} = \mathfrak{so}(8)$, $\mathfrak{l} = \mathfrak{so}(5)$: Then $\dim V = 8$, $\dim R(X,V) \leq 7$, $\dim \mathfrak{g} = 28$, $\dim \mathfrak{l} = 10$, $\dim \mathfrak{l}^\perp = 18$ and
$$\dim R(X,V) \leq 7 < 18 = \dim \mathfrak{l}^\perp.$$
- $\mathfrak{g} = \mathfrak{so}(8)$, $\mathfrak{l} = \mathfrak{so}(5) \oplus \mathbb{R}$: Then $\dim V = 8$, $\dim R(X,V) \leq 7$, $\dim \mathfrak{g} = 28$, $\dim \mathfrak{l} = 11$, $\dim \mathfrak{l}^\perp = 17$ and
$$\dim R(X,V) \leq 7 < 17 = \dim \mathfrak{l}^\perp.$$
- $\mathfrak{g} = \mathfrak{so}(8)$, $\mathfrak{l} = \mathfrak{so}(5) \oplus \mathfrak{su}(2)$: Then $\dim V = 8$, $\dim R(X,V) \leq 7$, $\dim \mathfrak{g} = 28$, $\dim \mathfrak{l} = 13$, $\dim \mathfrak{l}^\perp = 15$ and
$$\dim R(X,V) \leq 7 < 15 = \dim \mathfrak{l}^\perp.$$
- $\mathfrak{g} = \mathfrak{so}(8)$, $\mathfrak{l} = \mathfrak{so}(6)$: Then $\dim V = 8$, $\dim R(X,V) \leq 7$, $\dim \mathfrak{g} = 28$, $\dim \mathfrak{l} = 15$, $\dim \mathfrak{l}^\perp = 13$ and
$$\dim R(X,V) \leq 7 < 13 = \dim \mathfrak{l}^\perp.$$

- $\mathfrak{g} = \mathfrak{so}(8)$, $\mathfrak{l} = \mathfrak{so}(6) \oplus \mathbb{R}$: Then $\dim V = 8$, $\dim R(X,V) \leq 7$, $\dim \mathfrak{g} = 28$, $\dim \mathfrak{l} = 16$, $\dim \mathfrak{l}^{\perp} = 12$ and

$$\dim R(X,V) \leq 7 < 12 = \dim \mathfrak{l}^{\perp}.$$

Finally, the result follows from Corollary 4.4. ■

Only the irreducible Onishchik triples $(\mathfrak{g}, \mathfrak{l}, \mathfrak{h}')$ in Table 4 with a simple $\mathfrak{g}$ may correspond to a fat associated bundle.

Table 4

$\mathfrak{g}$	$\mathfrak{l}$	$\mathfrak{h}'$	Restrictions
$\mathfrak{su}(4)$	$\mathfrak{sp}(2)$	$\mathfrak{su}(3)$	
$\mathfrak{su}(4)$	$\mathfrak{sp}(2)$	$\mathfrak{su}(3) \oplus \mathbb{R}$	
$\mathfrak{su}(4k)$	$\mathfrak{su}(4k-1)$	$\mathfrak{sp}(2k)$	$k \geq 1$
$\mathfrak{su}(4k)$	$\mathfrak{su}(4k-1) \oplus \mathbb{R}$	$\mathfrak{sp}(2k)$	$k \geq 1$
$\mathfrak{so}(8k)$	$\mathfrak{so}(8k-1)$	$\mathfrak{su}(4k)$	$k \geq 1$
$\mathfrak{so}(8k)$	$\mathfrak{so}(8k-1)$	$\mathfrak{su}(4k) \oplus \mathbb{R}$	$k \geq 1$
$\mathfrak{so}(8k)$	$\mathfrak{so}(8k-1)$	$\mathfrak{sp}(2k)$	$k \geq 1$
$\mathfrak{so}(8k)$	$\mathfrak{so}(8k-1)$	$\mathfrak{sp}(2k) \oplus \mathbb{R}$	$n \geq 1$
$\mathfrak{so}(8k)$	$\mathfrak{so}(8k-1)$	$\mathfrak{sp}(2k) \oplus \mathfrak{su}(2)$	$k \geq 1$
$\mathfrak{so}(16)$	$\mathfrak{so}(15)$	$\mathfrak{so}(9)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{so}(5)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{so}(5) \oplus \mathbb{R}$	
$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{so}(5) \oplus \mathfrak{su}(2)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{so}(6)$	
$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{so}(6) \oplus \mathbb{R}$	
$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{so}(7)$	

Assume that the bundle $G/L \to P/L \to M$ is a fat bundle associated with the principal bundle $P(M,G)$ and assume that G is simple and $\mathfrak{h}' \neq \mathfrak{g}$. Looking at the table of all possible Onishchik triples we see that the fiber G/L is necessarily covered by $\mathbb{CP}^{2k-1}$ $(k \geq 1)$ or one of the following spheres:

$$S^5,\ S^7,\ S^{15},\ S^{8k-1}\ (k \geq 1),$$

which completes the proof. Note that the fiber corresponds to some pair $(\mathfrak{g}, \mathfrak{l})$ from Table 4. Choose compact Lie group G corresponding to $\mathfrak{g}$ and a compact Lie subgroups $L \subset G$ and $H' \subset G$ corresponding to $\mathfrak{l}$ and $\mathfrak{h}'$. By [23, Proposition 7, p. 89] (see also Theorem 3.2), (G, H', L) is Onishchik triple. Hence, by [23, Corollary 2 on p. 228], G must be $\mathrm{SO}(m)$ or $\mathrm{SU}(l)$ with $m = 8k, k \geq 1, m = 8, 16$ and $l = 4k, k \geq 1$ (as in Table 4), while L must have the connected component L_0 equal to $\mathrm{SO}(m-1)$ or $\mathrm{S}(\mathrm{U}(l-1) \times \mathrm{U}(1))$. Thus, G/L is either a sphere or a projective space (in the simply connected case), or is finitely covered by these spaces.

REFERENCES

[1] I. Agricola, *The Srní lectures on non-integrable geometries with torsion*, Arch. Math. (Brno) 42 (2006), Suppl., 5–84.

[2] W. Ambrose and I. M. Singer, *A theorem on holonomy*, Trans. Amer. Math. Soc. 75 (1953), 428–453.

[3] L. Bérard-Bergery, *Sur certaines fibrations d'espaces homogènes riemanniens*, Compos. Math. 30 (1975), 43–61.

[4] A. Besse, *Einstein Manifolds*, Springer, Berlin, 1987.

[5] I. Biswas, M. Fernández, V. Muñoz and A. Tralle, *Formality of Sasakian manifolds*, J. Topology 9 (2016), 161–180.

[6] M. Bocheński, A. Szczepkowska, A. Tralle and A. Woike, *A classification of fat associated bundles over compact homogeneous spaces*, Differential Geom. Appl. 49 (2016), 131–141.

[7] M. Bocheński, A. Szczepkowska, A. Tralle and A. Woike, *On symplectically fat twistor bundles*, Ann. Global Anal. Geom. 48 (2015), 181–194.

[8] M. Bocheński, A. Szczepkowska, A. Tralle and A. Woike, *New constructions of symplectically fat twistor bundles*, J. Fixed Point Theory Appl. 17 (2015), 65–81.

[9] A. Derdziński and A. Rigas, *Unflat connections in 3-sphere bundles over* S^4, Trans. Amer. Math. Soc. 265 (1981), 153–165.

[10] J. Fine and D. Panov, *Hyperbolic geometry and non-Kaehler manifolds with trivial canonical bundle*, Geom. Topol. 14 (2010), 1723–1763.

[11] J. Fine and D. Panov, *Circle invariant fat bundles and symplectic Fano 6-manifolds*, J. London Math. Soc. 91 (2015), 709–730.

[12] L. Florit and W. Ziller, *Topological obstructions to fatness*, Geom. Topol. 15 (2011), 891–925.

[13] K. Grove and W. Ziller, *Curvature and symmetry of Milnor spheres*, Ann. of Math. 152 (2000), 331–367.

[14] B. Hajduk and A. Tralle, *On simply connected K-contact non-Sasakian manifolds*, J. Fixed Point Theory Appl. 16 (2014), 229–241.

[15] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Amer. Math. Soc., Providence, RI, 2001.

[16] J. Kędra, A. Tralle and A. Woike, *On nondegenerate coupling forms,* J. Geom. Phys. 61 (2011), 462–475.

[17] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. 1, Interscience, New York, 1961.

[18] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. 2, Interscience, New York, 1969.

[19] E. Lerman, *How fat is a fat bundle?*, Lett. Math. Phys. 15 (1988), 335–339.

[20] E. Lerman, *Contact fiber bundles*, J. Geom. Phys. 49 (2004), 52–66.

[21] V. Muñoz and A. Tralle, *Simply connected K-contact and Sasakian manifolds of dimension 7*, Math. Z. 281 (2015), 457–470.

[22] A. L. Onishchik, *Inclusion relations between transitive compact transformation groups*, Trudy Moskov. Mat. Obshch. 11 (1962), 199–242 (in Russian).

[23] A. L. Onishchik, *Topology of Transitive Transformation Groups*, Johann Ambrosius Barth, Leipzig, 1994.

[24] A. Onishchik and E. Vinberg, *Lie Groups and Lie Algebras III*, Springer, Berlin, 2004.

[25] A. Weinstein, *Fat bundles and symplectic manifolds*, Adv. Math. 37 (1980), 239–250.

[26] W. Ziller, *Fatness revisited*, preprint,
http://www.math.upenn.edu/~wziller/papers/Fat-09.pdf.

Maciej Bocheński, Anna Szczepkowska, Aleksy Tralle, Artur Woike
Department of Mathematics and Computer Science
University of Warmia and Mazury
Słoneczna 54
10-710 Olsztyn, Poland
E-mail: mabo@matman.uwm.edu.pl
anna.szczepkowska@matman.uwm.edu.pl
tralle@matman.uwm.edu.pl
awoike@matman.uwm.edu.pl