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Properties of the intersection ideal $\mathcal{M} \cap \mathcal{N}$ revisited

by

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Summary. We investigate various properties of the intersection ideal $\mathcal{M} \cap \mathcal{N}$ of subsets of the reals that are related to the translations of its members. We are also concerned with cardinal coefficients associated with the ideal $\mathcal{M} \cap \mathcal{N}$.

1. Introduction. In this paper, we are interested in the ideal $\mathcal{M}\cap\mathcal{N}$ of subsets of the Cantor space 2^ω. This subject was initiated by the author [4] and [5]. In the latter article, it was proved that the class $(\mathcal{M}\cap\mathcal{N})^*$ coincides with $\mathcal{N}^*$. Throughout the paper we use rather standard terminology. We assume that $+$ is the modulo 2 coordinatewise addition in 2^ω and I, J are σ-ideals of subsets of 2^ω with $I \subseteq J$.

Definition 1.1. We say that $X \subseteq 2^\omega$ is I-*additive*, and write $X \in I^*$, if $X + A = \{x + a : x \in X,\, a \in A\} \in I$ for any $A \in I$; and we write $X \in (I, J)^*$ if $X + A \in J$ for every $A \in I$.

Definition 1.2. Let I, J be proper σ-ideals of subsets of X containing all singletons, with $I \subseteq J$. Following [1] we define

$$\operatorname{add}(I) = \min\Big\{|\mathcal{A}| : \mathcal{A} \subseteq I \text{ and } \bigcup \mathcal{A} \notin I\Big\},$$

$$\operatorname{add}(I,J) = \min\Big\{|\mathcal{A}| : \mathcal{A} \subseteq I \text{ and } \bigcup \mathcal{A} \notin J\Big\},$$

$$\operatorname{cov}(I) = \min\Big\{|\mathcal{A}| : \mathcal{A} \subseteq I \text{ and } X = \bigcup \mathcal{A}\Big\},$$

$$\operatorname{non}(I) = \min\{|A| : A \subseteq X \text{ and } A \notin I\},$$

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and

$$\operatorname{cof}(I) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq I \text{ and } \forall_{A\in I}\ \exists_{B\in\mathcal{A}}\ A \subseteq B\}.$$

We denote by $\mathcal{M}$ the σ-ideal of meager subsets of 2^ω, by $\mathcal{N}$ the σ-ideal of measure zero subsets of 2^ω, and by $\mathcal{E}$ the σ-ideal generated by the F_σ measure zero subsets of 2^ω. In [1], the authors prove that $\mathcal{E}$ is strictly contained in $\mathcal{M} \cap \mathcal{N}$.

In the definition below we introduce the following notation. Let I be a σ-ideal of subsets of 2^ω.

Definition 1.3.

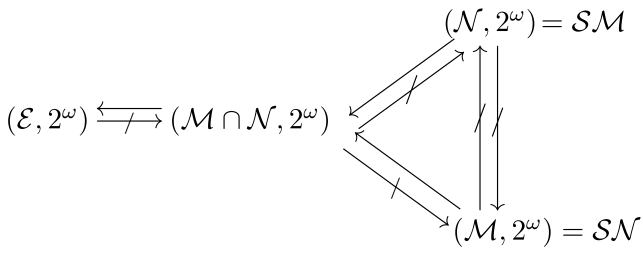
$$(I, 2^\omega) = \{X \subseteq 2^\omega : X + A \neq 2^\omega \text{ for every } A \in I\}.$$

Observation 1.4. $\mathcal{SN}$ (strongly measure zero sets) $= (\mathcal{M}, 2^\omega)$, and $\mathcal{SM}$ (strongly meager sets) $= (\mathcal{N}, 2^\omega)$.

2. Main theorems. In the first part of this section, we shall be interested in relationships between some classes of sets of the form $(I, 2^\omega)$, where I is a σ-ideal of subsets of 2^ω. This was motivated by [4, Proposition 19] where the author presents a diagram of inclusions between some families of subsets of 2^ω defined in terms of translation which are smaller than $(I, 2^\omega)$.

Assume now that $\to$ denotes inclusion and $\not\to$ means that inclusion cannot be proved in ZFC.

Theorem 2.1. *The following diagram of inclusions holds:*



Proof. We prove the non-trivial part. To see that $(\mathcal{E}, 2^\omega) \not\to (\mathcal{M}\cap\mathcal{N}, 2^\omega)$ apply Theorem 2.2 below. It is well-known that in $M[c]$, where M is a model of ZFC, and c is a Cohen real over M, $X = 2^\omega \cap M$ is an $\mathcal{SN}$ set and $X \notin \mathcal{SM}$ by Carlson's result (see [1, Lemma 3.3.12]). This proves that $(\mathcal{M}\cap\mathcal{N}, 2^\omega) \not\to (\mathcal{N}, 2^\omega)$. To show that $(\mathcal{M} \cap \mathcal{N}, 2^\omega) \not\to (\mathcal{M}, 2^\omega)$ consider $L * B_{\aleph_1}$, where L is the countable support $\aleph_2$-iteration of Laver forcing over a model of ZFC + GCH followed by the random algebra $B_{\aleph_1}$. In the resulting model, there is a Sierpiński set (thus strongly meager) $Y \in (\mathcal{M} \cap \mathcal{N}, 2^\omega)$. On the other hand, by Woodin's argument (see [1, Theorem 8.3.7]), $Y \notin (\mathcal{M}, 2^\omega)$. Crossed arrows in $(\mathcal{M}, 2^\omega) \rightleftarrows (\mathcal{N}, 2^\omega)$ follow easily from the previous considerations. ∎

THEOREM 2.2. *Assume CH. Then there exists* $X \in (\mathcal{E}, 2^\omega)$ *such that* $X \notin (\mathcal{M}\cap\mathcal{N}, 2^\omega)$.

Proof. Let $\{z_\alpha\}_{\alpha<\mathfrak{c}}$ be an enumeration of 2^ω, and let $\{F_\alpha\}_{\alpha<\mathfrak{c}}$ be a list of all Borel sets in $\mathcal{E}$. Fix a set A which belongs to $\mathcal{M}\cap\mathcal{N}\setminus\mathcal{E}$. We build $X=\{x_\alpha\}_{\alpha<\mathfrak{c}}$ and a sequence $\{r_\alpha\}_{\alpha<\mathfrak{c}}$ by induction. Suppose that we have already constructed $\{x_\alpha\}_{\alpha<\lambda<\mathfrak{c}}$ and $\{r_\alpha\}_{\alpha<\lambda<\mathfrak{c}}$. We will define x_λ and r_λ. Choose $r_\lambda \notin \bigcup_{\alpha<\lambda<\mathfrak{c}}(F_\lambda+x_\alpha)$, and let $B_\lambda = 2^\omega\setminus\bigcup_{\alpha\le\lambda}(F_\alpha+r_\alpha)$. Consider z_λ. Since the set $(z_\lambda+B_\lambda)^c = z_\lambda+B_\lambda^c$, as a member of $\mathcal{E}$, cannot cover A, we have $(z_\lambda+B_\lambda)\cap A\neq\emptyset$. Hence there are $x_\lambda\in B_\lambda$ and $a_\lambda\in A$ such that $z_\lambda = x_\lambda + a_\lambda$. Using this procedure for every $\lambda<\mathfrak{c}$ we obtain $X=\{x_\alpha\}_{\alpha<\mathfrak{c}}$. We find that $r_\alpha\notin X+F_\alpha$ for every $\alpha<\mathfrak{c}$, and $X+A=2^\omega$, thus X is as required. ■

It is easy to see that Theorem 2.2 can be proved in a more general setting. Namely, if I, J are Borel supported σ-ideals of subsets of 2^ω with $I\subsetneq J$, then the following holds.

THEOREM 2.3. *Assume CH. Then* $(J, 2^\omega)\subsetneq(I, 2^\omega)$.

Proof. Similar to the proof of Theorem 2.2. ■

A well-known theorem of Carlson (see [2]) states that $(\mathcal{E}, 2^\omega)$ forms a σ-ideal. The author of this article has not been able to answer the following question.

PROBLEM 2.4. *Is it consistent with ZFC that* $(\mathcal{M}\cap\mathcal{N}, 2^\omega)$ *is not closed under taking finite unions?*

THEOREM 2.5. *It is consistent with ZFC that every member of* $(\mathcal{M}\cap\mathcal{N}, 2^\omega)$ *is at most countable.*

Proof. There is a model of ZFC (see [3, Lemmas 5.1 and 5.2]) in which for every uncountable $X\subseteq 2^\omega$, there are $A\in\mathcal{M}$ and $B\in\mathcal{N}$ such that $X+A=2^\omega$ and $X+B=2^\omega$. Assume that $X+(A\cap B)\neq 2^\omega$. Then there is $t\in 2^\omega$ such that $X+t\subseteq(2^\omega\setminus A)\cup(2^\omega\setminus B)$. Thus there are uncountable $X', X''\subseteq X$ with $X'+t\subseteq 2^\omega\setminus A$ or $X''+t\subseteq 2^\omega\setminus B$. But this is impossible by [3, Lemmas 5.1 and 5.2]. ■

To finish this article we consider cardinal invariants related to the intersection ideal $\mathcal{M}\cap\mathcal{N}$. The idea of applying the property of two σ-ideals, defined below, was pointed out to the author by A. Krawczyk.

DEFINITION 2.6. We say that two (proper) σ-ideals I, J of subsets of a set X are *orthogonal* if there is a set $\overline{X}\in I$ such that $\overline{X}^c\in J$.

LEMMA 2.7. *Suppose that I and J are orthogonal. Then* $\mathrm{add}(I\cap J, I) = \mathrm{add}(I)$.

Proof. Clearly, $\mathrm{add}(I \cap J, I) \geq \mathrm{add}(I)$. Suppose that $\kappa = \mathrm{add}(I) < \mathrm{add}(I \cap J, I)$. Then there is $\{G_\xi\}_{\xi<\kappa} \subset I$ such that $\bigcup_{\xi<\kappa} G_\xi \notin I$. This implies that $\bigcup_{\xi<\kappa}(G_\xi \cap \overline{X}^c) \notin I$. ■

The following lemma holds for arbitrary σ-ideals.

Lemma 2.8.

$$\mathrm{cov}(I \cap J) = \max\{\mathrm{cov}(I), \mathrm{cov}(J)\},$$
$$\mathrm{non}(I \cap J) = \min\{\mathrm{non}(I), \mathrm{non}(J)\}.$$

Proof. Left to the reader. ■

Lemma 2.9. *Suppose that I and J are orthogonal σ-ideals. Then*

$$\mathrm{cof}(I \cap J) = \max\{\mathrm{cof}(I), \mathrm{cof}(J)\}.$$

Proof. Assume that $\mathrm{cof}(I) \leq \mathrm{cof}(J)$ and consider $\mathcal{F} = \{G \cap \overline{X} : G \in J\} \subseteq I \cap J$. Obviously, $\mathrm{cof}(\mathcal{F}) \leq \mathrm{cof}(I \cap J)$. Since $\mathrm{cof}(\mathcal{F}) = \mathrm{cof}(J)$, we conclude that $\mathrm{cof}(I \cap J) = \mathrm{cof}(J)$. ■

As a corollary we obtain the following.

Proposition 2.10. *We have*

$$\begin{aligned}
\mathrm{add}(\mathcal{M} \cap \mathcal{N}, \mathcal{N}) &= \mathrm{add}(\mathcal{N}),\\
\mathrm{add}(\mathcal{M} \cap \mathcal{N}, \mathcal{M}) &= \mathrm{add}(\mathcal{M}),\\
\mathrm{cov}(\mathcal{M} \cap \mathcal{N}) &= \max\{\mathrm{cov}(\mathcal{M}), \mathrm{cov}(\mathcal{N})\},\\
\mathrm{non}(\mathcal{M} \cap \mathcal{N}) &= \min\{\mathrm{non}(\mathcal{M}), \mathrm{non}(\mathcal{N})\},\\
\mathrm{cof}(\mathcal{M} \cap \mathcal{N}) &= \max\{\mathrm{cof}(\mathcal{M}), \mathrm{cof}(\mathcal{N})\}.
\end{aligned}$$

Proof. This follows from the fact that $\mathcal{M}$ and $\mathcal{N}$ are orthogonal and from the above lemmas. ■

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