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ON LOCALLY HOMOGENEOUS COMPACT PSEUDO-RIEMANNIAN MANIFOLDS

BY

MACIEJ BOCHEŃSKI, PIOTR JASTRZĘBSKI and ALEKSY TRALLE (Olsztyn)

To the memory of Professor Witold Roter

Abstract. We are interested in the problem of the existence of compact Clifford–Klein forms of pseudo-Riemannian homogeneous spaces G/H of reductive type. We give a generalization of our previous result in [Proc. Amer. Math. Soc. (2017)] with a more conceptual and simpler proof. We show that there are no solvable compact Clifford–Klein forms of pseudo-Riemannian homogeneous spaces G/H with any unimodular subgroup H contained in the semisimple part of the Levi factor of some parabolic subgroup.

1. Introduction. Let G/H be a non-compact homogeneous space of reductive type, that is, the homogeneous space formed by a semisimple Lie group G and a reductive subgroup H. This space carries a G-invariant pseudo-Riemannian metric coming from the Killing form of the Lie algebra $\mathfrak{g}$ of G. Assume that there exists a discrete subgroup Γ of G such that it acts freely on G/H and the resulting quotient $\Gamma\backslash G/H$ is compact. Then the latter locally homogeneous space is still pseudo-Riemannian. This construction opens a perspective of obtaining non-homogeneous compact relativistic space form models in mathematical physics (note that such spaces include $\mathbb{H}^{p,q} = \mathrm{SO}_0(p, q+1)/\mathrm{SO}_0(p, q)$, whose signature is (p, q) and the sectional curvature is constant and negative). The pioneering work in this direction was done by Calabi and Markus [4] and by Kulkarni [10]. The systematic study of the general problem of describing homogeneous spaces G/H with the appropriate discrete subgroups Γ was initiated by T. Kobayashi, who has made the main contributions to the whole area [8]. The problem is known as the problem of compact Clifford–Klein forms. There are obstructions to their existence, and the structure of possible Γ is not known except in some particular cases (see the survey [9]). Looking for possible structural conditions one can refer to the result proved by Benoist [1] that no nilpotent

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discrete subgroup Γ can yield a compact Clifford–Klein form. Following this line of thinking we have proved in [3] the following result. Let G be a real semisimple Lie group. Recall that $P \subsetneq G$ is a parabolic subgroup of G if the complexification P^c of P is a parabolic subgroup of the complexification G^c of G (that is, P^c contains some Borel subgroup of G^c).

Theorem 1.1. *Let G/H be a homogeneous space such that G is a connected semisimple linear Lie group of non-compact type and H is the semisimple part of the Levi factor of some parabolic subgroup of G. If $\Gamma\backslash G/H$ is a compact Clifford–Klein form, then Γ cannot be (virtually) solvable.*

The purpose of this short note is to give a simpler and more conceptual proof of a slightly more general result.

Theorem 1.2. *Let G/H be a homogeneous space such that G is a connected semisimple linear Lie group of non-compact type and H is a unimodular and connected subgroup of the semisimple part of the Levi factor of some parabolic subgroup of G. If $\Gamma\backslash G/H$ is a compact Clifford–Klein form, then Γ cannot be solvable.*

Note that this theorem covers many important examples of homogeneous spaces (see [2]). For instance, a popular testing case $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R})$ determined by the standard embedding of $\mathrm{SL}(m,\mathbb{R})$ into $\mathrm{SL}(n,\mathbb{R})$ belongs to the class described in Theorem 1.2.

Our new proof is more conceptual, because we do not use the root systems as in [3], but combine the results of Witte Morris [15] on syndetic hulls (as in [3]) with results of Morita [11] based on cohomological arguments.

2. Preliminaries. In this note we use standard facts of the theory of Lie groups and algebraic groups without further explanations, closely following [6], [12], [13], [17].

We will need the definition of a proper action. Let L be a locally compact topological group acting continuously on a locally compact Hausdorff topological space X. This action is said to be *proper* if for every compact subset $S \subset X$ the set

$$L_S := \{g \in L \mid g \cdot S \cap S \neq \emptyset\}$$

is compact. The action is called *properly discontinuous* if it is proper and the topology of L is discrete. Let G be a Lie group and H a closed subgroup of G. A discrete subgroup Γ of G is said to be a *discontinuous group* for the homogeneous space G/H if the natural Γ-action on G/H is properly discontinuous and fixed-point free. For such Γ, the quotient space $\Gamma\backslash G/H$ is a smooth manifold to be referred to as a *Clifford–Klein form*. In more detail $\Gamma\backslash G/H$ has a unique smooth manifold structure such that the quotient map $G/H \to \Gamma\backslash G/H$ is a C^∞-covering.

The proof of the main theorem also uses the following result on proper actions.

LEMMA 2.1 ([7, Lemma 1.3]). *Let G_1, G_2 be locally compact groups and let $L_1, H_1 \subset G_1$ and $L_2, H_2 \subset G_2$ be closed subgroups. Assume that $f : G_1 \to G_2$ is a continuous homomorphism such that $f(L_1) \subset L_2$, $f(H_1) \subset H_2$, $f(L_1)$ is closed in G_2 and $L_1 \cap \operatorname{Ker} f$ is compact. If the action of L_2 on G_2/H_2 is proper then the action of L_1 on G_1/H_1 is proper.*

We will need the notion of a syndetic hull [15].

DEFINITION 2.2. A *syndetic hull* of a subgroup Γ of a Lie group G is a subgroup B of G such that B is connected, B contains Γ and $\Gamma\backslash B$ is compact.

We will need the theorem below which yields conditions on the existence of some syndetic hulls.

THEOREM 2.3 ([5, Section 1.6]). *Let V be a finite-dimensional real vector space and Λ a virtually solvable subgroup of $\mathrm{GL}(V)$. Then there exists at least one closed virtually solvable subgroup $S \subset \mathrm{GL}(V)$ containing Λ such that:*

(1) *S has finitely many components and each component meets Λ;*
(2) (syndeticity) *there exists a compact set $K \subset H$ such that $S = K \cdot \Lambda$;*
(3) *S and Λ have the same Zariski closure in $\mathrm{GL}(V)$.*

Recall that a real algebraic group T is a *torus* if T is abelian and Zariski connected, and every element of T is semisimple. A torus T is $\mathbb{R}$-*split* if every element of T is diagonalizable (and therefore hyperbolic).

LEMMA 2.4 ([16]). *Let T be a torus. If T_{split} is the maximal $\mathbb{R}$-split subtorus of T, and T_{cpt} is the maximal compact subtorus of T, then*

$$T = T_{\mathrm{split}} \cdot T_{\mathrm{cpt}}$$

and $T_{\mathrm{split}} \cap T_{\mathrm{cpt}}$ is finite.

We will also need the following well known fact (see [14]).

PROPOSITION 2.5. *Let Γ be a (co-compact) lattice in a locally compact topological group L, and L_1 a normal subgroup of L. Let $\pi : L \to L/L_1$ be the natural projection. Then $\Gamma \cap L_1$ is a (co-compact) lattice in L_1 if and only if $\pi(\Gamma) \subset L/L_1$ is a (co-compact) lattice in L/L_1.*

3. Basic lemma. Let G be a real semisimple and connected Lie group with Iwasawa decomposition $G = KAN$, and $H \subset G$ a closed subgroup. The aim of this section is to prove the following.

LEMMA 3.1. *If a solvable subgroup $\Gamma \subset G$ acts properly and co-compactly on G/H, then there exists a solvable subgroup $\Gamma_0 \subset AN$ that acts properly and co-compactly on G/H.*

Proof. Since G is connected and linear, we have $G \subset \mathrm{GL}(V)$. Take the Zariski closure $L = \bar{\Gamma}$ and apply Theorem 2.3 to $\Lambda = \Gamma$. It follows that there exists a subgroup $B_1 \subset \mathrm{GL}(V)$ such that $\Gamma \subset B_1$ and $\bar{\Gamma} = \bar{B}_1 = L \subset G$ (that is, $S = B_1$ in Theorem 2.3). Since L is the Zariski closure of Γ, it is also solvable. Thus we obtain a (virtually) solvable subgroup B_1 such that $\Gamma \backslash B_1$ is compact. Consider the connected component B of B_1. Clearly, B must be solvable. Since the Lie subgroup B_1 contains a uniform lattice Γ, so does B as B_1 has finitely many connected components.

In general the inclusion $B \subset AN$ does not hold, but we may assume this in our context, because of the argument below. Note that $L = \bar{\Gamma} = \bar{B}$ is real algebraic, and hence it is a semidirect product

$$L = \bar{T} \ltimes \bar{U}$$

of a torus and a unipotent subgroup $\bar{U}$. By Lemma 2.4,

$$\bar{T} = T_{\mathrm{split}} \cdot T_{\mathrm{cpt}}.$$

Take $L_1 = T_{\mathrm{split}} \ltimes U$, a normal and co-compact subgroup of L. It follows that $B \cap L_1$ is normal in B and $B/B \cap L_1$ is a closed subgroup in the (Lie) group L/L_1. By Proposition 2.5 we see that $\Gamma_0 := \Gamma \cap (B \cap L_1)$ is a lattice in $B \cap L_1$ (and in B as L/L_1 is compact). Therefore Γ_0 acts properly and co-compactly on G/H. It suffices to show that $L_1 \subset AN$. But any solvable subgroup of G that is generated by unipotent and hyperbolic elements is conjugate to a subgroup of AN (this is basically a generalization of the fact that a collection of commuting triangularizable matrices can be simultaneously triangularized—see [6, proof of Theorem 17.6]). The proof of Lemma 3.1 is complete. ∎

4. Proof of Theorem 1.2. The conclusion follows from the existence of Γ_0 and the following result obtained by Morita.

LEMMA 4.1 ([11, Example 6.1]). *Let G be a simply connected non-unimodular Lie group and $G = S \ltimes R$ (S semisimple, R solvable) be its Levi decomposition. Take any closed unimodular subgroup H of S with finitely many connected components. Then G/H does not admit compact Clifford–Klein forms.*

Proof of Theorem 1.2. Assume that G/H has a solvable Clifford–Klein form $\Gamma \backslash G/H$. By Lemma 3.1 we may assume that $\Gamma \subset AN$. Let $P \subset G$ be a parabolic subgroup containing H. Then $P = S \ltimes R$ and $H \subset S$. Also notice that for any $g \in G$, Γ acts properly and co-compactly on G/H if and only if it acts properly and co-compactly on $G/(gHg^{-1})$. Since any parabolic subgroup contains a Borel subgroup, it follows that (possibly after conjugation of P) $AN \subset P$, and so without loss of generality we may assume that $\Gamma \subset P$. But $\Gamma \backslash P/H$ is compact as a closed subset of $\Gamma \backslash G/H$. Also, by Lemma 2.1, Γ acts properly on P/H (for $H_1 = H_2 = H$, $L_1 = L_2 = \Gamma$, $G_1 = P$, $G_2 = G$ and the inclusion $f : P \hookrightarrow G$), and therefore $\Gamma\backslash P/H$ is a compact Clifford–Klein form. This contradicts Lemma 4.1. The proof is complete.

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Maciej Bocheński, Piotr Jastrzębski, Aleksy Tralle
Department of Mathematics and Computer Science
University of Warmia and Mazury
Słoneczna 54
10-710 Olsztyn, Poland
E-mail: mabo@matman.uwm.edu.pl
piojas@matman.uwm.edu.pl
tralle@matman.uwm.edu.pl