ce qui prouve que la suite (3) converge uniformément sur N, contrairement à la propriété de cette suite. La condition 3° est donc vérifiée et le théorème A est vrai.

L'équivalence des théorèmes A et B est ainsi démontrée.

Le théorème A étant vrai (d'après MM. Banach et Kuratowski), si $2^{\aleph_0} = \aleph_1$, il résulte d'équivalence des théorèmes A et B, que *le théoreme B est vrai, si* $2^{\aleph_0} = \aleph_1$ (ce que j'ai démontré directement sur une autre place [1])).

[1]) *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie,* Classe III, 1928, p. 84—87.

A Characterization of Those Subsets of Metric Separable Space Which Are Homeomorphic with Subsets of the Linear Continuum [1]).

By

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The questions connected with the mapping of sets of points on the linear continuum have been the subject of much investigation. Veblen [2]) and Lennes [3]) defined arcs in the plane homeomorphic with closed linear intervals; the latter establishing the correspondence in non-metric terms. Janiszewski [4]) gave another definition of arc in the plane. Sierpiński [5]) settled this question in n-dimensional space. Papers by Zoretti [6]), Riesz [7]) and Denjoy [8]) on totally disconnected sets in the plane led Moore and Kline [9]) to the following

Theorem: *Necessary and sufficient conditions that a bounded, closed, plane point set M be a subset of an arc are that every closed connected subset of M containing more than one point be an arc and that no point of an arc t except its end-points be a limit point of $M - t$.*

[1]) This paper is substantially a thesis submitted to the University of Michigan in May 1928 in partial fullfilment of the requirements for the degree of Doctor of Philosophy.

[2]) O. Veblen: Trans. Amer. Math. Soc. vol 6 (1905) pp. 83—98.

[3]) N. J. Lennes: Amer. Jour. Math. vol 33 (1911) pp. 287—362.

[4]) S. Janiszewski: Jour. de l'Ecole Polyt. ser. 2, vol 6 (1912).

[5]) W. Sierpiński: Annali di Math. ser. 3, vol 26 (1916) pp. 131—151.

[6]) L. Zoretti: Jour. de Math. vol 1 (1905) pp. 12 ff.

[7]) F. Riesz: Comptes Rendus (Paris) vol 141 (1905) pp. 650—655.

[8]) A. Denjoy: Comptes Rendus (Paris) vol 151 (1910) pp. 138—140.

[9]) R. L. Moore and J. R. Kline: Annals of Math. vol 20 (1919) pp. 218—223.

Sierpiński[1]) obtained a theorem for totally disconnected sets in euclidean space which as Urysohn[2]) pointed out may be stated as follows:

Theorem: *A necessary and sufficient condition that a subset of a separable metric space be homeomorphic with a subset of the irrational numbers is that the dimension of M be zero.*

The principal problem with which this paper is concerned may be stated as follows: Given a separable, metric space R, what are necessary and sufficient conditions on a set M in R in order that there shall exist a subset of the linear continuum homeomorphic with M?

The writer is indebted to Professor R. L. Wilder of the University of Michigan for his valuable encouragement and criticism during the preparation of this paper.

II

By a closed, half-open and open arc we mean sets homeomorphic with the linear intervals $a \leqq x \leqq b$, $a \leqq x < b$, $a < x < b$ respectively. If R is a metric (not necessarily separable) space, the following theorems hold.

Theorem I: *$M \subset R$ is a closed arc from a to b if and only if M is a compact continuum containing a and b which is disconnected by the omission of any point $x \subset M-(a+b)$.*

Theorem II: *$M \subset R$ is a half-open arc with the end-point a if and only if:*

1. $M = \sum_{n=1}^{\infty} M_n$.
2. *M_n is a closed arc $[a, a_n]$.*
3. $M_n \subset M_{n+1} - a_{n+1}$.
4. *If $\{x_k\}$ is a sequence of distinct points of M such that $x_k \subset M_{k+1} - M_k$, then $\{x_k\}$ has no limit point in M.*

Theorem III: *$M \subset R$ is an open arc if and only if M is the sum of two half-open arcs with a common end-point a such that $M - a$ is disconnected.*

[1]) W. Sierpiński: Fund. Math. vol 2 (1921) pp. 81–89.
[2]) P. Urysohn: Fund. Math. vol 7 (1925) p. 76.

The proof of I follows from the usual argument in the case of the similar theorem for euclidean spaces. The proofs of II and III will be suggested to the reader from the conditions of the theorems. These theorems contain a complete classification of the subsets of metric space according as they are homeomorphic with connected subsets of the linear continuum containing more than one point.

III.

It is well known that any linear set is made up of a countable set of non-overlapping intervals, which may be either closed, half-open or open, and a totally disconnected set. If we define an *arc* as any set satisfying the conditions of either theorem I, II, or III, we have the solution to the problem stated in the foregoing section in the following

Theorem IV: *Necessary and sufficient conditions that a set M in a separable metric space be homeomorphic with a subset of the linear continuum are:*

A. The components of M are either points or arcs.
B. If p is a point component of M, then $\dim_p M = 0$
C. If a is an end-point of an arc $T \subset M$, then $\dim_a (M - T + a) = 0$.
D. If p is a point of an arc T other than an end-point, then p is not a limit point of $M - T$.

That these conditions are necessary is evident. For, if N is a subset of the linear continuum, it satisfies these conditions and, since the conditions are homeomorphic invariants, the set M, homeomorphic with N, will also satisfy these conditions.

In order to establish the sufficiency of our conditions, we will consider a space Q whose elements q are the components of the set M. If the element $q \subset Q$ is the component $K \subset M$, we will say that q represents K in Q. We will consider M as a relative space and the sets $S(x, \varepsilon) M$ as spheres $S(x, \varepsilon)$ relative to M. Since M is a subset of a metric separable space, there exists a countable everywhere dense set in M Hence M itself may be considered as a separable metric space.

Lemma I: *The set of arc components of M is countable.*

Proof: Let A be a subset of M containing one and only one interior point from each arc component of M. It follows from condition D that A contains no limit point of itself. Since in a separable metric space every uncountable set contains a limit point of itself, the set A, and consequently the set of arc components of M, is at most countable.

Definition: A domain of norm ε containing an arc component $T \subset M$ is the set of points $S(a, \varepsilon) + T + S(b, \varepsilon)$; $S(a, \varepsilon) + T$; or T according as T is a closed arc with the end-points a and b; a half-open arc with the end-point a; or an open arc. In any case we denote such a set by $S(T, \varepsilon)$.

Lemma 2: *For any* $S(T, \varepsilon)$, $M = A(T) + B(T)$ *such that*

1. $$T \subset A(T) \subset S(T, \varepsilon)$$
2. $$A(T)\,\overline{B(T)} + \overline{A(T)}\,B(T) = 0.$$

Proof: If T is the closed arc $[a, b]$, then, in consequence of condition C, we have for $\varepsilon > 0$,

1.1 $$\begin{cases} M - T + \alpha = A(\alpha) + B(\alpha) \\ \alpha \subset A(\alpha) \subset S(\alpha, \varepsilon) \\ A(\alpha)\,\overline{B(\alpha)} + \overline{A(\alpha)}\,B(\alpha) = 0 \end{cases}$$

where α is either a or b.

The required decomposition of M is obtained by putting

1.2 $$A(T) = A(a) + T + A(b)$$

1.3 $$B(T) = B(a)\,B(b).$$

That condition 1. is satisfied follows at once from 1.1. From condition D and the fact that T is a component of M, it follows that

1.4 $$[T - (a + b)]\,[\overline{M - T}] + \overline{T}[M - T] = 0.$$

That condition 2 is satisfied by the given decomposition follows from 1.1 and 1.4.

If T is a half-open arc with the end-point a, we put $A(T) = A(a) + T$ and $B(T) = B(a)$. The above argument applies with the statements involving b suppressed. If T is an open arc, we put $A(T) = T$ and $B(T) = M - T$. Condition 2 is now a direct consequence of condition D.

Lemma 3: *If K is a component of M and* $KA(T) \neq 0$, *then* $K \subset A(T)$.

Lemma 4: *For* $\varepsilon > 0$, *there is a countable set of point components* $p_{K,\varepsilon} \subset M$ *such that if p is any point component of M then there is a* $p_{K,\varepsilon}$ *for which*

1. $$M = A(p_{K,\varepsilon}) + B(p_{K,\varepsilon})$$
2. $$p_{K,\varepsilon} \subset A(p_{K,\varepsilon}) \subset S(p_{K,\varepsilon}, \varepsilon)$$
3. $$A(p_{K,\varepsilon})\,\overline{B(p_{K,\varepsilon})} + \overline{A(p_{K,\varepsilon})}\,B(p_{K,\varepsilon}) = 0$$
4. $$p \subset A(p_{K,\varepsilon}).$$

Proof: Let ε be any positive number and p be any point component of M. Since $\dim_p M = 0$, we have

1.1 $$M = A(p) + B(p)$$

1.2 $$p \subset A(p) \subset S(p, \varepsilon)$$

1.3 $$A(p)\,\overline{B(p)} + \overline{A(p)}\,B(p) = 0.$$

Let K denote the class of point components of M. The class of sets $[A(p)]$ is a covering of K by open sets in consequence of 1.3. Since K is a subset of a separable metric space, there exists a countable subset $\{A(p_{K,\varepsilon})\}$ of $[A(p)]$ which covers K.

The points $p_{K,\varepsilon}$ form the set required by the theorem in consequence of 1.1, 1.2, and 1.3.

Lemma 5: *If K is any component of M and* $KA(p_{K,\varepsilon}) \neq 0$, *then* $K \subset A(p_{K,\varepsilon})$.

We fix our attention now on the sets $A(T, \varepsilon)$ and $A(p_{K,\varepsilon})$ of lemmas 2 and 4 for a given ε. In consequence of lemmas 3 and 5, these sets are not only subsets of M but the components of these sets are components of M. If $q \subset Q$ represents the arc $T \subset M$, we will denote by $U(q, \varepsilon)$ the elements of Q which represent the components of M in $A(T, \varepsilon)$. If $q \subset Q$ represents the point component $p_{K,\varepsilon} \subset M$, we will denote by $U(q_{K,\varepsilon})$ the elements of Q which represent the component of M in $A(p_{K,\varepsilon})$. In consequence of lemmas 1. and 4., the class of sets U so defined are countable for each ε. For all positive rational values of ε, consider the class of sets $U_\varepsilon \subset Q$. The totality of such sets U_ε is again countable.

Definition of the neighborhoods in Q. If q is an element of Q then any set $U(q, \varepsilon)$ or $U(q_{K,\varepsilon})$ which contains q is a neighborhood of q and is denoted by $U(q)$.

Theorem V: *Q is a topological space.*

Proof: In order to establish this we show that Q and the system of neighborhoods $U(q)$ defined for Q satisfy the following conditions which define a topological [1]) space.

a) If $q \subset Q$, then there exists a neighborhood $U(q)$ containing q.

b) If $q_2 \subset U(q_1)$, then there exists a $U(q_2) \subset U(q_1)$.

c) If $q_3 \subset U(q_1)\, U(q_2)$ then there exists a $U(q_3) \subset U(q_1)\, U(q_2)$.

d) If $q_1 \neq q_2$, then there exist $U(q_1)$, $U(q_2)$ such that

$$U(q_1)\, U(q_2) = 0.$$

a) If $q \subset Q$ represents an arc component $T \subset M$, then there is a $U(q, \varepsilon) = U(q) \supset q$. If q represents a point component $p \subset M$, then for any rational ε there is a $p_{K,\varepsilon}$ such that $p \subset A(p_{K,\varepsilon})$. The corresponding $U(q_{K,\varepsilon})$ is a $U(q) \supset q$. Condition a) is therefore satisfied.

b) The condition b) is an immediate consequence of the definition of neighborhoods in Q, since the sets U are neighborhoods of any element q they contain.

c) Let K_1, K_2 be components of M which are represented by q_1, q_2 respectively and let $A(K_1)$, $A(K_2)$ be the subsets of M whose components are represented by the elements of Q in $U(q_1)$, $U(q_2)$ respectively. Let $B(K_i) \subset M - A(K_i)$. In consequence of lemmas 2. and 4. and the theorem that the complement of the product of two sets is the sum of their complements, we have

1.1 $$A(K_1)A(K_2)\,[\overline{B(K_1)+B(K_2)}] + \overline{A(K_1)A(K_2)}\,[B(K_1)+B(K_2)] = 0.$$

We have now to consider the several cases according as q_3 represents a point, a closed arc, a half-open arc or an open arc in M.

Case I: $q_3 \subset Q$ represents a point component $p \subset M$. In consequence of 1.1 there exists a sphere $S(p, \varepsilon)$ such that

2.1 $$p \subset S(p, \varepsilon) \subset A(K_1)\, A(K_2).$$

For a rational η such that $0 < \eta < \dfrac{\varepsilon}{2}$, there is a point $p_{K,\eta}$ and a set $A(p_{K,\eta})$ in consequence of lemma 4 for which

2.2 $$p \subset A(p_{K,\eta}) \subset S(p_{K,\eta}).$$

[1]) F. Hausdorff: Grundzüge der Mengenlehre. Leipzig (1914) p. 213.

Hence we have

2.3 $$p \subset A(p_{K,\eta}) \subset S(p, \varepsilon) \subset A(K_1)\, A(K_2).$$

$A(p_{K,\eta})$ is a subset of the components of M which define a neighborhood of q. Expressing 2.3 in terms of the elements of Q, we have

2.4 $$q_3 \subset U(q_3) \subset U(q_1)\, U(q_2).$$

Case II: $q_3 \subset Q$ represents a closed arc component $T = [a, b] \subset M$. From the inclusion $T \subset A(K_1) A(K_2)$ and 1.1, it follows that there exists an $\varepsilon > 0$ such that

3.1 $$T \subset S(T, \varepsilon) \subset A(K_1)\, A(K_2).$$

In consequence of lemma 2., there is a set $A(T, \varepsilon)$ contained in the domain of norm ε which occurs in 3.1. The components of $A(T, \varepsilon)$ give rise to a set $U(q, \varepsilon)$ which is a neighborhood of q_3 satisfying c). Similar arguments are effective in the cases where q_3 is the representative of a half-open or an open arc in M.

d) Case I: q_1, q_2 in Q represent point components p_1, p_2 in M. For $\varepsilon < \frac{1}{2} d(p_1, p_2)$, we have

4.1 $$S(p_1, \varepsilon)\, S(p_2, \varepsilon) = 0.$$

For a rational η such that $0 < \eta < \dfrac{\varepsilon}{2}$, there exist, by arguments similar to those of c), Case I, sets $A(p_{K_1,\eta})$, $A(p_{K_2,\eta})$ satisfying the conditions of lemma 4., containing p_1, p_2 respectively and lying in $S(p_1, \varepsilon)$, $S(p_2, \varepsilon)$. The corresponding sets of representatives in Q being denoted by $U(q_1)$, $U(q_2)$, we have

4.2 $$q_1 \subset U(q_1); \quad q_2 \subset U(q_2); \quad U(q_1)\, U(q_2) = 0$$

in consequence of 4.1 and the definition of neighborhood.

Case II: q_1, q_2 in Q represent respectively a point component p and a closed arc component T in M. Since $d(p, T) > 0$, there is an $\varepsilon > 0$ such that

5.1 $$S(p, \varepsilon)\, S(T, \varepsilon) = 0.$$

For a positive rational η less than $\dfrac{\varepsilon}{2}$, there exist sets $A(p_{K,\eta})$ and $A(T, \eta)$ defined by lemmas 2. and 4. such that

5.2 $$p \subset A(p_{K,\eta}) \subset S(p, \varepsilon)$$

5.3 $$T \subset A(T, \eta) \subset S(T, \varepsilon).$$

The subsets $U(q_{K,\eta})$, $U(q,\eta)$ corresponding to $A(p_{K,\eta})$, $A(T,\eta)$ are neighborhoods of q_1, q_2 respectively and in consequence of 5.1, 5.2, 5.3 we have

5.4 $$U(q_1)\,U(q_2)=0.$$

If $q_2\subset Q$ represents a half-open or open arc $T\subset M$, similar argument applies.

Case III: q_1, q_2 in Q represent closed arc components $T_1=[a_1,b_1]$, $T_2=[a_2,b_2]$ in M. Since $T_1\,T_2=0$, neither of the numbers $d(\alpha_i,T_j)$; $\alpha=a,b$; $i,j=1,2$, is 0. Let ε be a positive number less than half of the smallest of these four numbers. We have

6.1 $$S(T_1,\varepsilon)\,S(T_2,\varepsilon)=0.$$

Consider now the sets $A(T_1)$, $A(T_2)$ defined by lemma 2. such that

6.2 $$T\subset A(T_i)\subset S(T_i,\varepsilon)\qquad (i=1,2)$$

and the associated sets $U(q_i)\subset Q$. We have

6.3 $$q_i\subset U(q_i)$$

6.4 $$U(q_1)\,U(q_2)=0$$

in consequence of 6.1, 6.2. If $q_2\subset Q$ is a half-open arc, we need consider only the numbers $d(a_2,T_1)$, $d(a_1,T_2)$, $d(b_1,T_2)$. If q_2 is an open arc, we need consider only the last two numbers. If both arcs are half-open, we need consider only the first two numbers. If q_1 is half-open and q_2 is an open arc, we need consider only the first number and note that $U(q_2)=q$. If both arcs are open, we put $U(q_i)=q_i$. In each of these cases the argument is similar to that for the case of the two closed arcs. Condition d) is therefore satisfied.

IV.

We now turn our attention to the properties of Q as a topological space. The neighborhoods in Q are of two kinds according as they are defined in terms of lemmas 2. or 4.

From lemma 4. and the definition of neighborhood, it follows that there is a countable infinity of elements $q_{K,\varepsilon}$ in Q, where K takes on all positive integral and ε positive rational values, such that if q is any element of Q representing a point component of M, then q belongs to some $U(q_{K,\varepsilon})$. We will call the class $[U(q_{K,\varepsilon})]$ the primary neighborhoods of the elements of Q representing point components in M. The class of neighborhoods $U(q,\varepsilon)$ which is defined by lemma 2. for the representatives of the arc components of M is also countable as ε takes on rational values. We will call $U(q,\varepsilon)$ a primary neighborhood of the element q where $q\subset Q$ represents an arc in M. By reasoning essentially similar to that employed to establish property c) for the space Q we have the following

Lemma 6: *If $q\subset Q$ represents the component $K\subset M$ and $U(q)$ is a neighborhood of q, then there exists a primary neighborhood $U_0(q)\subset U(q)$.*

This lemma amounts to a sharpening of condition b) These primary neighborhoods play an essential role in setting up the correspondence between M and the required linear set.

Lemma 7: *The complement in Q of any neighborhood is an open set.*

Proof: Consider $q\subset Q$; $U(q)$; and $q_0\subset Q-U(q)$. Let K; $A(K)$; and K_0 be the subsets of M corresponding to these sets in Q. We have

1.1 $$M=A(K)+B(K)$$

1.2 $$A(K)\,\overline{B(K)}+\overline{A(K)}\,B(K)=0$$

1.3 $$K_0\subset B(K)$$

in consequence of lemmas 2., 3., 4., 5 and the definition of neighborhood in Q. From 1.2., 1.3., it follows that there exists a domain $S(K_0,\varepsilon)$ such that

1.4 $$K_0\subset S(K_0,\varepsilon)\subset B(K).$$

It is an almost immediate consequence of lemma 2., or 4., and 1.4 that there exists a $U(q_0)\subset Q-U(q)$.

Lemma 8: *If q is an element of Q and F is a closed subset of $Q-q$, then there exist open sets G_1 and G_2 such that*

1. $$q\subset G_1;\quad F\subset G_2$$

2. $$G_1G_2=0$$

Proof: Since q is not an element of the closed set F, there is a neighborhood $U(q)$ such that $U(q)F=0$. In consequence of the preceding lemma, $Q - \overline{U}(q)$ is an open set G_2. The neighborhoods in Q being open sets, we have

1.1 $$q \subset U(q) = G_1$$

1.2 $$F \subset Q \dot{-} \overline{U}(q) = G_2$$

1.3 $$G_1 G_2 = U(q)\,[Q - \overline{U}(q)] = 0.$$

The totality of primary neighborhoods in Q being countable and forming, in consequence of lemma 6., a system of neighborhoods equivalent to the original system in the sense of Hausdorff, we may say, in the terminology of Fréchet[1]), that Q is a perfectly separable topological space. A space satisfying the conditions of lemma 8 is called regular[2]). A topological space Q is metrizable if there can be defined the distance function $d(q_1, q_2)$, which is non-negative, symmetric, zero for identical values of the arguments and obeying the triangle law, such that

1. For every $U(q)$, there exists an $S(q, \varepsilon) \subset U(q)$.
2. For every $S(q, \varepsilon)$, there exists a $U(q) \subset S(q, \varepsilon)$

where the spheres $S(q, \varepsilon)$ are defined in terms of the given distance function. Tychonoff[3]) has obtained a theorem, which together with a result of Urysohn's[4]), leads to the

Theorem: *A regular perfectly separable topological space is metrisable.*

In consequence of the above theorem, lemma 8 and the perfect separability of Q, it follows that Q is a metrizable space. We will now consider an appropriate metric defined in Q.

Lemma 9: *The dimension of Q is zero.*

Proof: Let q be any point of Q and ε be any positive number. There exists a $U(q)$ such that

1.1 $$q \subset U(q) \subset S(q, \varepsilon).$$

In consequence of lemma 7., $Q - \overline{U}(q)$ is open and hence

1.2 $$Q = \overline{U}(q) + [Q - \overline{U}(q)]$$

1.3 $$U(q)\,[\overline{Q - \overline{U}(q)}] + \overline{U(q)}\,[Q - \overline{U}(q)] = 0$$

[1]) Cf. E. W. Chittenden: *The Metrization Problem.* Bull. Amer. Math. Soc. vol 33 (1927) p. 18.

[2]) P. Alexandroff u. P. Urysohn: Math. Annalen. vol 92 (1924) p. 263.

[3]) A. Tychonoff: Math. Annalen. vol 95 (1926) p. 139.

[4]) P. Urysohn: Math. Annalen. vol 94 (1925) p. 315.

since neither of two non-overlapping open sets contains a limit point of the other. But 1.1., 1.2., 1.3. imply $\dim_q Q = 0$ and since q is any point of Q, $\dim Q = 0$.

We may now consider Q as a metric separable zero-dimensional space. We mention without proof the well-known

Lemma 10: *The set of irrational numbers is homeomorphic with a subset of any nowhere dense perfect set on the linear continuum.*

Combining this lemma with the theorem of Sierpiński mentioned in section I, we have the result:

Theorem VI: *The space Q is homeomorphic with a subset of a nowhere dense perfect set on the linear interval* $[0, 1]$.

We are now concerned with the following problem: If Q_0 is a subset of a nowhere dense set on the interval $[0, 1]$, and homeomorphic with Q, it is desired to map Q_0 on Q_1 in such a way that if $q_1 \subset Q_1$ corresponds to $q_0 \subset Q_0$ which represents an arc $T \subset M$, then approach to q_1 from the right or left shall correspond to approach to the end-point a or b of T.

Lemma 11: *If E is a nowhere dense perfect set on* $[0, 1]$ *and $\{x_n\}$ is a countable subset of E, then E is homeomorphic with a linear set E_0 such that if $x_{0,n}$ is the image of x_n, there exists a closed interval I_n such that $E_0 I_n = x_{0,n}$.*

Proof: Let $I^{(1)}$ be an interval with center at x_1 and of length 1. Since E is nowhere dense, there exists in $I^{(1)} - EI^{(1)}$ a sequence of points $\{p_n^{(1)}\}$ such that

1.1 $$\lim_{n\to\infty} p_n^{(1)} = x_1$$

1.2 $$p_n^{(1)} > p_{n+1}^{(1)} > x_1.$$

There also exists in $I^{(1)} - EI^{(1)}$ a sequence of non-overlapping closed intervals $\{I_n^{(1)}\}$ such that any sequence $\{q_n^{(1)}\}$ where $q_n^{(1)}$ is interior to $I_n^{(1)}$ has the properties

1.3 $$\lim_{n\to\infty} q_n^{(1)} = x_1$$

1.4 $$q_n^{(1)} < q_{n+1}^{(1)} < x_1.$$

Denote the open interval determined by $I_n^{(1)}$ by $J_n^{(1)}$. Since any two open intervals are homeomorphic, there exists a function θ_1

such that

1.5 $$J_k^{(1)} = \theta_1\{(p_k^{(1)}, p_{k+1}^{(1)})\} \text{ is a homeomorphism}$$

1.6 $$y = \theta_1(y) \text{ for } y \text{ in the complement of } \sum_{k=1}^{\infty}(p_k^{(1)}, p_{k+1}^{(1)}).$$

Considered as a function defined on E, θ_1 is a $1-1$ bicontinuous mapping. That θ_1 is $1-1$ follows immediately from the choice of the $I_n^{(1)}$. To establish the continuity of θ_1, we consider any sequence $\{x'_n\} \subset E$ and a point $x' \subset E$ such that

1.7 $$\lim_{n\to\infty} x'_n = x'.$$

If $x' = x_1$, then at most a finite number of x'_n lie in any one $(p_k^{(1)}, p_{k+1}^{(1)})$ in consequence of 1.2. It follows immediately from 1.3, 1.5, 1.6 that

1.8 $$\lim_{n\to\infty} \theta_1(x'_n) = \theta_1(x') = x_1.$$

If $x' \neq x_1$, then either x' and all but a finite number of x'_n lie in the complement of $\overline{\sum_{k=1}^{\infty}(p_k^{(1)}, p_{k+1}^{(1)})}$ and in consequence of 1.6

1.9 $$\lim_{n\to\infty} \theta_1(x'_n) = \theta_1(x')$$

or $-x'$ and all but a finite number of x'_n lie in some $(p_k^{(1)}, p_{k+1}^{(1)})$ and in consequence of 1.5 we have again the relation 1.9. The continuity of the inverse of θ_1 is established in a similar manner with the aid of the fact that the $p_n^{(1)}$ and the end-points of the $I_n^{(1)}$ are in the complement of E and of $\theta_1(E) = E^{(1)}$. Denoting $\theta_1(x_1) = x_1$ by $x_1^{(1)}$, and $[p_1^{(1)}, x_1]$ by I_1, we have

1.10 $$E^{(1)} I_1 = x_1^{(1)}$$

in consequence of 1.5, 1.6.

Assume that the set $E^{(n-1)}$ has been defined homeomorphic with E and that there exist intervals $I_1, \ldots, I_{n-1}$ such that

2.1 $$E^{(n-1)} I_i = x_i^{(n-1)} \qquad (i = 1, 2, \ldots, n-1).$$

There exists an interval $I^{(n)}$ with center at $x_n^{(n-1)}$ and of length less than $1/2^n$ such that $I^{(n)}$ excludes the intervals I_i $(i=1,2,\ldots,n-1)$ in consequence of 2.1. Since $E^{(n-1)}$ is homeomorphic with the nowhere dense perfect set E, $E^{(n-1)}$ is a nowhere dense perfect set. Hence, there exists in $I^{(n)} - E^{(n-1)} I^{(n)}$ a sequence of points $\{p_k^{(n)}\}$ such that

2.2 $$\lim_{k\to\infty} p_k^{(n)} = x_n^{(n-1)}$$

2.3 $$p_k^{(n)} > p_{k+1}^{(n)} > x.$$

There also exists in $I^{(n)} - E^{(n-1)} I^{(n)}$ a sequence of non-overlapping closed intervals $\{I_k^{(n)}\}$ such that any sequence $\{q_k^{(n)}\}$ where $q_k^{(n)}$ is interior to $I_k^{(n)}$ has the properties

2.4 $$\lim_{k\to\infty} q_k^{(n)} = x_n^{(n-1)}$$

2.5 $$q_k^{(n)} < q_{k+1}^{(n)} < x.$$

The function θ_n is defined on $E^{(n-1)}$ in the same way relative to $x_n^{(n-1)}$, $\{p_k^{(n)}\}$, $\{I_k^{(n)}\}$ as θ_1 was defined relative to x_1, $\{p_k^{(1)}\}$, $\{I_k^{(1)}\}$. If we put $E^{(n)} = \theta_n(E^{(n-1)})$ and $[p_1^{(n)}, x_n^{(n-1)}] = I_n$, we have

2.6 $$E^{(n)} I_n = x_n^{(n)}.$$

Further, we have

2.7 $$\begin{aligned} E^{(n)} I_i &= x_i \\ x_i^{(n)} &= x_i^{(i)} \end{aligned} \qquad (i = 1, 2, \ldots, n-1).$$

For, I_i $(i < n)$ is in the complement of $I^{(n)}$ and, since θ_n is the identical transformation in the complement of $I^{(n)}$ (see 1.6) and since it maps points of $E^{(n-1)}$ within $I^{(n)}$ into other points also within $I^{(n)}$ (see 1.5), 2.7 is a consequence of 2.1

We have then the sequence of sets and mappings:

3.1 $$\begin{aligned} E^{(0)} &\equiv E \\ E^{(n)} &= \theta_n(E^{(n-1)}) \end{aligned} \qquad (n = 1, 2, \ldots, m, \ldots).$$

If we put

3.2 $$\Theta_n = \theta_n \theta_{n-1} \ldots \theta_1 \qquad (n = 1, 2, \ldots, m, \ldots)$$

we obtain a sequence of $1-1$ bicontinuous mappings of the set E. Since the length of $I^{(n)}$ is at most $1/2^n$ we have

3.3 $$d(\Theta_n(x), \Theta_{n+1}(x)) = d(x^{(n)}, x^{(n+1)}) \leqq 1/2^n$$

and hence for any positive integers n, m $(n < m)$

3.4 $$d(\Theta_n(x), \Theta_m(x)) \leqq \sum_{i=n}^{m} {}^1\!/_{2^i} < \frac{1}{2^{n-1}}$$

for any point $x \subset E$. This establishes the uniformity of the limit

3.5 $$\Theta(x) = \lim_{n\to\infty} \Theta_n(x).$$

If we put $E_0 = \Theta(E)$, it follows that E_0 is homeomorphic with E, since the uniform limit of a sequence of $1-1$ bicontinuous functions is a $1-1$ bicontinuous function. Further, for each $x_{0,n} = \Theta(x_n)$, we have

3.6 $$E_0 I_n = x_{0,n}.$$

For, from 2.7,

3.7 $$E^{(k)} I_n = x_n^{(k)} = x_n^{(n)} \qquad (k \geqq n).$$

Hence $\Theta_k(x)$ where $x \neq x_n$, lies in the complement of I_n for $k \geqq n$ and therefore $\Theta(x)$ also lies in the complement of I_n. But from 2.7, 3.5, we have

3.8 $$x_{0,n} = \Theta(x_n) = \lim_{k\to\infty} \Theta_k(x_n) = x_n^{(n)}.$$

Hence the image of every x_n in E_0 is the end-point of an interval complementary to E_0 and the theorem is proved.

Let $\{q_n\}$ be the subset of Q whose elements represent the arc components of M. In consequence of theorem V, lemmas 10 and 11, we have the following

Theorem VII: *Q is homeomorphic with a linear set E such that if x_n is the image in E of the element q_n in Q, then there exists an interval I_n such that $EI_n = x_n$.*

We recall the primary neighborhoods of the q_n. A primary neighborhood $U(q_n)$ is the set of elements of Q which represent the components of M in a set $A(T_n)$ required by lemma 2. in which

1.1 $$A(T_n) = A(a_n) + T_n + A(b_n)$$

if T_n is a closed arc $[a_n, b_n]$. In consequence of 1.1. and 1.2 in lemma 2. the components of the sets $A(a_n)$ and $A(b_n)$ except for a_n and b_n respectively are subsets of the set of components of M. We denote by $U(q_n, a_n)$, $U(q_n, b_n)$ respectively those subsets of $U(q_n)$ made up of the elements of Q representing the components of M

in $A(a_n)$, $A(b_n)$. We have in consequence of 1.1

1.2 $$U(q_n) = U(q_n, a_n) + q_n + U(q_n, b_n).$$

In a similar manner we obtain for the half-open arc T_n the corresponding sets

1.3 $$A(T_n) = A(a_n) + T_n$$

1.4 $$U(q_n) = U(q_n, a_n) + q_n.$$

For the open arcs T_n, we have $U(q_n) = q_n$ for all primary neighborhoods of q_n. In these terms we state

Theorem VIII: *Q is homeomorphic with a linear set P such that for any p_n corresponding to a q_n which represents an arc component $T_n \subset M$ and for every primary neighborhood $U(q_n)$, there is an open interval $I_n = (x_n, y_n) \supset p_n$ such that:*

1. If T_n is a closed arc $[a_n, b_n]$, the sets $P(x_n, p_n)$, $P(p_n, y_n)$ correspond respectively to subsets of $U(q_n, a_n)$, $U(q_n, b_n)$ (or in the reverse order).

2. If T_n is a half-open arc with the end-point a_n, then the set $P(x_n, p_n) = 0$ and the set $P(p_n, y_n)$ corresponds to a subset of $U(q_n, a_n)$ (or in the reverse order).

3. If T_n is an open arc, then $PI_n = p_n$.

Proof: Let $\{q_{n_i}\}$ be that subsequence of $\{q_n\} \subset Q$ which represents the sequence of closed arcs $T_{n_i} \subset M$. For each $T_{n_i} = [a_{n_i}, b_{n_i}]$, we have $d(a_{n_i}, b_{n_i}) > 0$. For a rational positive ε_{n_i} less than $\frac{1}{2} d(a_{n_i}, b_{n_i})$, consider a domain $S(T_{n_i}, \varepsilon_{n_i})$ and a set $A(T_{n_i}, \varepsilon_{n_i})$ satisfying the conditions of lemma 2. We have

1.1 $$\begin{gathered} A(T_{n_i}, \varepsilon_{n_i}) = A(a_{n_i}, \varepsilon_{n_i}) + T_{n_i} + A(b_{n_i}, \varepsilon_{n_i}) \subset S(T_{n_i}, \varepsilon_{n_i}) \\ A(a_{n_i}, \varepsilon_{n_i}) \subset S(a_{n_i}, \varepsilon_{n_i}); \quad A(b_{n_i}, \varepsilon_{n_i}) \subset S(b_{n_i}, \varepsilon_{n_i}). \end{gathered}$$

Since ε_{n_i} is less than half the distance between a_{n_i} and b_{n_i}, we have

1.2 $$A(a_{n_i}, \varepsilon_{n_i}) \overline{A(b_{n_i}, \varepsilon_{n_i})} + \overline{A(a_{n_i}, \varepsilon_{n_i})} A(b_{n_i}, \varepsilon_{n_i}) = 0.$$

Let K_1 be a component of M in $A(a_{n_i}, \varepsilon_{n_i})$. We recall that

1.3 $$\begin{gathered} M - T_{n_i} + a_{n_i} = A(a_{n_i}, \varepsilon_{n_i}) + B(a_{n_i}, \varepsilon_{n_i}) \\ A(a_{n_i}, \varepsilon_{n_i}) \overline{B(a_{n_i}, \varepsilon_{n_i})} + \overline{A(a_{n_i}; \varepsilon_{n_i})} B(a_{n_i}, \varepsilon_{n_i}) = 0. \end{gathered}$$

It follows in consequence of 1.3 that there exists an $S(K_1, \eta_1)$ such that

1.4 $$K_1 \subset S(K_1, \eta_1) \subset A(a_{n_i}, \varepsilon_{n_i}).$$

Similarly, if K_2 is a component of $A(b_{n_i}, \varepsilon_{n_i})$, there exists a set $S(K_2, \eta_2)$ such that

1.5 $$K_2 \subset S(K_2, \eta_2) \subset A(b_{n_i}, \varepsilon_{n_i}).$$

Consider now the subsets of Q; $U(q_{n_i}, \varepsilon_{n_i})$, $U(q_{n_i}, a_{n_i})$, $U(q_{n_i}, b_{n_i})$ associated with the sets $A(T_{n_i}, \varepsilon_{n_i})$, $A(a_{n_i}, \varepsilon_{n_i})$, $A(b_{n_i}, \varepsilon_{n_i})$. We have in consequence of 1.1, and the definition of neighborhood in Q

1.6 $$U(q_{n_i}, \varepsilon_{n_i}) = U(q_{n_i}, a_{n_i}) + q_{n_i} + U(q_{n_i}, b_{n_i}).$$

In consequence of 1.4, 1.5, there exists for $q \subset U(q_{n_i}, a_{n_i})$ or $\subset U(q_{n_i}, b_{n_i})$ a neighborhood $U(q)$ such that

1.7 $$q \subset U(q) \subset U(q_{n_i}, a_{n_i}) \qquad (\text{or } \subset U(q_{n_i}, b_{n_i})).$$

From this it follows that

1.8 $$U(q_{n_i}, a_{n_i})\,\overline{U(q_{n_i}, b_{n_i})} + \overline{U(q_{n_i}, a_{n_i})}\,U(q_{n_i}, b_{n_i}) = 0.$$

In consequence of theorem VII there exists a linear set E homeomorphic with Q such that for each point of the sequence $\{x_n\}$ in E corresponding to an element of the sequence of arc representatives $\{q_n\}$ in Q, there is an interval I_n such that

2.1 $$EI_n = x_n.$$

Hence there exists an interval $J_1 = (a'_1, b'_1)$ containing x_{n_1} such that

2.2 $$E(a'_1, x_{n_1}) = 0 \qquad (\text{or } E(x_{n_1}, b'_1) = 0).$$

2·3 EJ_1 is the image of a subset of a $U(q_{n_1})$ satisfying 1.9.

2.4 $$|a'_1 - b'_1| \leqq 1.$$

2.5 The I_n are exterior to J_1 for $n < n_1$.

2.6 a'_1, b'_1 are in the complement of E.

These conditions may be satisfied simultaneously in consequence of 2.1, the homeomorphy between E and Q, the 0-dimensionality

of E and the finiteness of the number of I_n for $n < n_1$. We denote by $U(x_{n_1}, a_{n_1})$, $U(x_{n_1}, b_{n_1})$ those subsets of EJ_1 which correspond to the elements of Q in $U(q_{n_1}, a_{n_1})$, $U(q_{n_1}, b_{n_1})$ respectively. Since the relation 1.9 is a homeomorphic invariant we have

2.7 $$U(x_{n_1}, a_{n_1})\,\overline{U(x_{n_1}, b_{n_1})} + \overline{U(x_{n_1}, a_{n_1})}\,U(x_{n_1}, b_{n_1}) = 0.$$

We define a function θ_1 on E such that

2.8 $\theta_1(U(x_{n_1}, a_{n_1}))$ is a reflection through x_{n_1} followed by a similarity contraction such that $\theta_1(U(x_{n_1}, a_{n_1})) \subset (a^1_1, x_{n_1})$.

2.9 $$\text{For } x \subset E - U(x_{n_1}, a_{n_1}), \quad \theta_1(x) = x.$$

In consequence of 2.2, 2.7, 2.8, and 2.9 the set $E^{(1)} = \theta_1(E)$ is homeomorphic with E and $x^{(1)}_{n_1} = \theta(x_{n_1}) = x_{n_1}$.

We denote $\theta_1(x)$ by $x^{(1)}$. Then either $x^{(1)} = x$ or both x and $x^{(1)}$ are in J_1 in consequence of 2.8. Hence any $x^{(1)}_n$ in the complement of J_1 is the end-point of an interval complementary to $E^{(1)}$. Since θ_1 is a reflection through x_{n_1} and a contraction within J_1, any x_n for $n \neq n_1$ goes over into $x^{(1)}_n$ which is the right or left end-point of an interval complementary to $E^{(1)}$ according as x_n is a left or right end-point of an interval complementary to E. The set $E^{(1)}$ has the property that for $n \neq n_1$, $x^{(1)}_n$ is the end-point of an interval $I^{(1)}_n$ complementary to $E^{(1)}$.

Assume that there has been defined a set $E^{(k-1)}$ homeomorphic with E such that for $n \neq n_i$ $(i = 1, 2, \dots, k-1)$, there exist intervals $I^{(k-1)}_n$ such that

3.1 $$E^{(k-1)} I^{(k-1)}_n = x^{(k-1)}_n.$$

There exists an interval $J_k = (a'_k, b'_k)$ containing $x^{(k-1)}_{n_k}$ such that

3.2 $$E^{(k-1)}(a'_k, x^{(k-1)}_{n_k}) = 0 \quad (\text{or } E^{(k-1)}(x^{(k-1)}_{n_k}, b'_k) = 0).$$

3.3 $E^{(k-1)} J_k$ is the image of a subset of a $U(q_{n_k})$ satisfying 1.9

3.4 $$|a'_k - b'_k| \leqslant 1/2^k.$$

3.5 The $I^{(k-1)}_n$ satisfying 3.1 are exterior to J_k for $n < n_k$.

3.6 a'_k, b'_k are in the complement of $E^{(k-1)}$.

3.7 The $x^{(k-1)}_{n_i}$ $(i = 1, 2, \dots, k-1)$ are in the exterior of J.

These conditions may be satisfied simultaneously in consequence of 3.1, the finiteness of the sets $\{I_n^{(k-1)}\}$ and $\{x_{ni}^{(k-1)}\}$ involved in 3.5, 3.7, and the homeomorphy between $E^{(k-1)}$ and E and hence between $E^{(k-1)}$ and Q.

We denote by $U(x_{n_k}^{(k-1)}, a_{n_k})$, $U(x_{n_k}^{(k-1)}, b_{n_k})$ the subsets of $E^{(k-1)}J_k$ which correspond to elements of Q in $U(q_{n_k}, a_{n_k})$, $U(q_{n_k}, b_{n_k})$ respectively. In consequence of 3.3 and 1.9 we have

3.8 $$U(x_{n_k}^{(k-1)}, a_{n_k})\,\overline{U(x_{n_k}^{(k-1)}, b_{n_k})} + \overline{U(x_{n_k}^{(k-1)}, a_{n_k})}\,U(x_{n_k}^{(k-1)}, b_{n_k}) = 0.$$

We define a function θ_k on $E^{(k-1)}$ such that

3.9 $\theta_k(U(x_{n_k}^{(k-1)}, a_{n_k}))$ is a reflection through $x_{n_k}^{(k-1)}$

followed by a similarity transformation so that $\theta_k(U(x_{n_k}^{(k-1)}, a_{n_k}))\subset \subset(a'_k, x_{n_k}^{(k-1)})$

3.10 For $x^{(k-1)}\subset E^{(k-1)} - U(x_{n_k}^{(k-1)}, a_{n_k})$, $\theta_k(x^{(k-1)}) = x^{(k-1)}$

In consequence of 3.2, 3.9, 3.10, the set $E^{(k)} = \theta_k(E^{(k-1)})$ is homeomorphic with $E^{(k-1)}$, and $\theta_k(x_{n_k}^{(k-1)}) = x_{n_k}^{(k-1)}$. We put $x^{(k)} = \theta_k(x^{(k-1)})$. By an argument similar to that used in the case of $E^{(1)}$ it follows that $E^{(k)}$ has the property that for $n \neq n_1, n_2, \dots, n_k$, $x_n^{(k)}$ is the end-point of an interval $I_n^{(k)}$ complementary to $E^{(k)}$.

We have thus defined a sequence of sets $E^{(n)}$ and a sequence of 1—1 bicontinuous functions θ_n such that

4.1 $$E^{(0)} \equiv E \qquad E^{(n)} = \theta_n(E^{(n-1)}) \qquad (n=1,2,\dots)$$

We put

4.2 $$\Theta_n = \theta_n\,\theta_{n-1}\dots\theta_1$$

and consider the sequence of 1—1 bicontinuous functions defined on E. For any $x\subset E$, we have

4.3 $$d(\Theta_n(x), \Theta_m(x)) = \sum_{k=n}^{m} d(x^{(k)}, x^{(k+1)}) \leqslant \sum_{k=n}^{m} {}^1/_{2^k} < \frac{1}{2^{n-1}}.$$

From this follows immediately the uniformity of the limit

4.4 $$\lim_{n\to\infty} \Theta_n(x) = \Theta(x).$$

The set $P = \Theta(E)$ is therefore homeomorphic with E and hence with Q.

P has the property 1. Consider any $p_{n_k}\subset P$ which corresponds to $q_{n_k}\subset Q$ representing a closed arc component $T_{n_k}\subset M$. Let $V(q_{n_k})$ be any primary neighborhood of q_{n_k}. There exists, in consequence of lemma 6. and property b) of the space Q, a primary neighborhood $W(q_{n_k})$ such that

5.1 $$W(q_{n_k})\subset V(q_{n_k})\,U(q_{n_k})$$

where $U(q_{n_k})$ is the neighborhood required by 3.3 and 1.9.

Hence we have

5.2 $$W(q_{n_k}, a_{n_k})\subset V(q_{n_k}, a_{n_k});\quad W(q_{n_k}, b_{n_k})\subset V(q_{n_k}, b_{n_k}).$$

Since Q and $E^{(k)}$ are homeomorphic there exists an open interval $J'_k\supset x_{n_k}^{(k)}$ such that $E^{(k)}J'_k$ is the image of a subset of $W(q_{n_k})$.

The interval J'_k may be chosen so as to be a subset of the interval J_k required in the construction of θ_k. We denote the two subintervals of J'_k determined by $x_{n_k}^{(k)}$ by $J_{k,a}$ and $J_{k,b}$ respectively. In consequence of 3.9, 3.10 the notation may be so chosen that $E^{(k)}J_{k,a}$, $E^{(k)}J_{k,b}$ correspond respectively to subsets of $W(q_{n_k}, a_{n_k})$, $W(q_{n_k}, b_{n_k})$. In view of 5.2, the set $E^{(k)}$ satisfies condition 1. with respect to the point $x_{n_k}^{(k)}$. In consequence of 3.7, 3.10, it follows that $x_{n_k}^{(m)} = x_{n_k}^{(k)}$ for $m \geqq k$. The order relation between any $x^{(k)}$ and $x_{n_k}^{(k)}$ is preserved by all θ_m for $m \geqq k$ since J_m excludes $x_{n_k}^{(m)}$ and θ_m is either identical or replaces $x^{(m-1)}\subset J_m$ by $x^{(m)}\subset J_m$. But from

5.3 $$y_n < y;\quad \lim_{n\to\infty} y_n = y_0$$

follows

5.4 $$y_0 \leqslant y.$$

Hence if $p\subset P$ corresponds to $x^{(k)}\subset E^{(k)}$, then the order relation between p and p_{n_k} is the same as that between $x^{(k)}$ and $x_{n_k}^{(k)}$. P and $E^{(k)}$, both being homeomorphic with Q, are homeomorphic.

Therefore there exists an interval $H_k\supset p_{n_k}$ such that PH_k is the image of a subset of $E^{(k)}J'_k$. In consequence of the above remarks on the order relative to $x_{n_k}^{(k)}$ we have the result that:

The two subintervals of H_k determined by p_{n_k} may be denoted by $H_{k,a}$ and $H_{k,b}$ in such a way that $PH_{k,a}$ corresponds to a subset of $E^{(k)}J_{k,a}$ and hence to a subset of $V(q_{n_k}, a_{n_k})$ while $PH_{k,b}$ corresponds to a subset of $E^{(k)}J_{k,b}$ and hence to a subset of $V(q_{n_k}, b_{n_k})$. This together with 5.2 establishes property 1.

P has the property 2. Let $p_n \subset P$ correspond to a $q_n \subset Q$ representing a half-open arc $T_n \subset M$. Let n_k be the first subscript greater than n in the sequence $\{q_{n_k}\}$. In consequence of 3.1, we have

6.1 $$E^{(k)} I_n^{(k)} = x_n^{(k)}$$

where $x_n^{(k)} \subset E^{(k)}$ corresponds to q_n. In consequence of 3.5, 3.9, 3.10, we have

6.2 $$E^{(m)} I_n^{(k)} = x_n^{(m)} = x_n^{(k)} \text{ for } m \geqq k.$$

From this and 4.4 it follows that

6.3 $$PI_n^{(k)} = p_n = x_n^{(k)}.$$

If $V(q_n)$ is any primary neighborhood of q_n, it follows from the homeomorphy between P and Q that there exists an open interval $H_n \supset p_n$ such that PH_n is the image of a subset of $V(q_n)$.

Property 2. is a consequence of 6.3 and the choice of H_n.

P has property 3. Since the primary neighborhoods of any $q_n \subset Q$ representing an open arc $T_n \subset M$ consist of the element q_n alone, q_n is an isolated element of Q. Hence $p_n \subset P$ corresponding to $q_n \subset Q$ is an isolated point of P. Property 3. is an immediate consequence of this.

VI.

We now replace the points p in P by intervals. For each point $p \subset P$, we consider the points of the sequence $\{p_n\}$ such that

1.1 $$p_n < p.$$

This defines for each point p a definite subsequence $\{p_{n_k}\}$ of $\{p_n\}$. We arrange these subscripts in their natural order

1.2 $$n_1 < n_2 < \ldots < n_k < \ldots$$

and denote this ordered set of subscripts by $N(p)$. To each point $p \subset P$ we associate the point

1.3 $$x = p + \sum_{n_k \subset N(p)} 1/2^{n_k}.$$

Let X be the set of such points x. The set X is in 1—1 correspondence with the set P and has the following properties:

1. *If $p < p'$ and x, x' correspond to p, p' respectively, then $x < x'$.*
2. *If $p_n < p$, then for the corresponding x_n, x, we have $x_n + 1/2^n < x$.*
3. *The open intervals $(x_n, x_n + 1/2^n)$ are free of points of X.*
4. *If $P \subset p - \{p_n\}$ and x corresponds to p, then the mapping of P on X is bicontinuous with respect to p and x.*
5. *If for any $\{p_k'\} \subset P$, $p_k' < p_n$ and $\lim_{k\to\infty} p_k' = p_n$, then for the corresponding points in X we have $\lim_{k\to\infty} x_k' = x_n$, and conversely.*
6. *If for any $\{p_k'\} \subset P$, $p_k' > p_n$ and $\lim_{k\to\infty} p_k' = p_n$, then for the corresponding points in X we have $\lim_{k\to\infty} x_k' = x_n + 1/2^n$ and conversely.*

These properties of X follow easily from the transformation as defined in 1.2, 1.3.

We consider now the arc components $T_n \subset M$. These fall into three classes:

1. T_{n_λ}: the closed arc components in M
2. T_{n_μ}: the half-open arc components in M
3. T_{n_ν}: the open arc components in M.

There is a 1—1 correspondence determined by the subscripts between the open intervals $(x_n, x_n + 1/2^n)$ associated with the set X and the sets $T_n \subset M$. We modify the set X in the following manner. We add to X all the closed intervals $t_{n_\lambda} = [x_{n_\lambda}, x_{n_\lambda} + 1/2^{n_\lambda}]$ corresponding to closed arcs T_{n_λ}. In consequence of properties 5., 6., of the set X and of condition 2. of theorem VIII, at most one of the end-points of any $t_{n_\mu} = (x_{n_\mu}, x_{n_\mu} + 1/2^{n_\mu})$ corresponding to a half-open arc $T_{n_\mu} \subset M$ is a limit point of the set X. We add to X the intervals t_{n_μ} together with one of its end-points selected as follows: if neither end-point of t_{n_μ} is a limit point of X, then the left end-point is added and the right end-point is discarded; if one of the end-points of t_{n_μ} is a limit point of X, then that one is added and the other is discarded. We add to X the open intervals $t_{n_\nu} = (x_{n_\nu}, x_{n_\nu} + 1/2^{n_\nu})$ corresponding to the open arcs $T_{n_\nu} \subset M$ and delete from X that one of its end-points which belongs to X.

The modified set we denote by X'. The intervals in X' we denote by t'_n. There is a 1—1 correspondence, determined by the correspondence between X and P, between the components of X' and the points of P. We will use the following definitions: 1. A sequence of linear intervals $\{I_n\}$ approaches a point y as a limit if, for any open interval J containing y, all but a finite number of the I_n lie in J. 2. A linear interval I_1 precedes an interval I_2 if every point of I_1 precedes every point of I_2. Under these definitions properties 1, 4, 5, 6 holding between X and P also hold between X' and P if the points $x \subset X$ are replaced by components $C \subset X'$ determined by them.

The intervals t'_{n_λ}, t'_{n_μ}, t'_{n_ν} in X' are closed, half-open, and open respectively and in consequence of theorems I, II, III they are homeomorphic with the components T_{n_λ}, T_{n_μ}, T_{n_ν} in M respectively. A 1—1 correspondence between the point components of M and those of X' may be set as follows: To each point component $x \subset M$ there corresponds a unique representative $q \subset Q$; to q corresponds a unique point $p \subset P$ in consequence of the homeomorphy between P and Q; since $p \subset P - \{p_n\}$ there corresponds to p a unique point component $x' \subset X'$. There exists, therefore, a function Θ such that $\Theta(T_n) = t'_n$ is a homeomorphy for all n and such that $\Theta(x) = x'$ is the 1—1 correspondence between the point components of M and X' defined above.

Definition of the correspondence for the end-points of a closed arc. As between the $T_{n_\lambda} = [a_{n_\lambda}, b_{n_\lambda}]$ and the corresponding $t'_{n_\lambda} = [x'_{n_\lambda}, x_{n_\lambda} + 1/2^{n_\lambda}]$, we determine whether a_{n_λ} or b_{n_λ} corresponds to x'_{n_λ} as follows: Let $U(q_{n_\lambda})$ be a primary neighborhood of q_{n_λ}, the representative of T_{n_λ}. There is an interval $I(p_{n_\lambda})$, where $p_{n_\lambda} \subset P$ corresponds to $q_{n_\lambda} \subset Q$, whose right end-point is p_{n_λ} such that $PI(p_{n_\lambda})$ corresponds to a subset of $U(q_{n_\lambda}, a_{n_\lambda})$ or $U(q_{n_\lambda}, b_{n_\lambda})$ in consequence of theorem VIII. From the property 5. holding between X' and P, there is an interval $J(x'_{n_\lambda})$ whose right end-point is x'_{n_λ} such that $X' J(x'_{n_\lambda})$, considered as a subset of the components of X', corresponds to a subset of $PI(p_{n_\lambda})$. (The end-point of $J(x'_{n_\lambda})$ other than x'_{n_λ} can always be chosen in the complement of X' since x'_{n_λ} is the left end-point of a component of X'). According as $PI(p_{n_\lambda})$ corresponds to a subset of $U(q_{n_\lambda}, a_{n_\lambda})$ or $U(q_{n_\lambda}, b_{n_\lambda})$, x'_{n_λ} is to correspond to a_{n_λ} or b_{n_λ}.

As between the T_{n_μ} and the t'_{n_μ}, the correspondence Θ is determined completely since each has but one end-point. As between the T_{n_ν} and the t'_{n_ν}, either of the two possible orders in which they may be put into correspondence may be used since neither component contains a limit point of its complement. We may now conclude the proof of the sufficiency of the conditions on the set M with the following

Theorem IX: *The function Θ under the above restrictions is a 1—1 bicontinuous mapping of the set M on the set X'.*

The details of the proof of this theorem involve the properties 1,..., 6 on the sets X' and P, theorem VIII, the definitions of neighborhoods for the space Q and the definition of the function Θ.