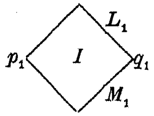
Les lignes L et M peuvent avoir, outre p et q, d'autres points en commun; mais il est facil de les remplacer par deux lignes L_1 et M_1, extraites resp. de L et M, et n'ayant que leurs extrémités p_1 et q_1 en commun, les points p_1 et q_1 étant choisis de sorte que $A+B$ coupe le plan entre eux.

Soit, notamment, sur la ligne L (orientée de p à q), q_1 le premier point de l'ensemble LMQ; un tel point existe, car selon (1): $LM(A+B)=0$, donc la frontière de Q, comme sous-ensemble de $A+B$, est disjointe de LM et il en résulte que l'ensemble LMQ est fermé. Soit, d'autre part, sur la ligne pq_1, extraite de L, p_1 le dernier point de l'ensemble LM qui précède q_1 (c'est donc bien un point qui n'appartient pas à Q, donc qui est séparé de q_1 par $A+B$).

La formule (1) entraîne

(2) $$L_1 A = 0 = M_1 B$$



qui à son tour, entraîne: $AB(L_1+M_1)=0$, ce qui veut dire que le produit AB est situé dans le complémentaire de la ligne polygonale simple fermée L_1+M_1.

Or, si l'on suppose, par impossible, que AB est un continu, on en conclut que AB est situé dans l'une des deux régions en lesquelles cette ligne coupe le plan. On peut toujours admettre que c'est la région non-bornée qui contient AB (car le cas contraire se ramène à celui-ci par inversion). Donc, en désignant par I le polygone formé par la région bornée et sa frontière $(=L_1+M_1)$, il vient: $IAB=0$, ce qui prouve que les ensembles IA et IB sont disjoints. En outre, aucun d'eux n'est une coupure de I entre p_1 et q_1, puisque selon (2), L_1 unit ces points dans $I-A$ et M_1 les unit dans $I-B$.

I étant d'après le cor. 1 uni-cohérent, il en résulte en vertu de la propriété (*), que $IA+IB$ ne coupe pas I entre p_1 et q_1; donc $A+B$ n'est pas une coupure du plan entre ces points. Mais ceci contredit la définition des points p_1 et q_1.

Ainsi, l'hypothèse, que AB est un continu, implique une contradiction.

A generalized notion of accessibility [1]).

By

G. T. Whyburn (Austin, U. S. A.).

1. Introduction.

The point P is said to be *accessible by continua* from a point set R provided that if A is any point of R, then $R+P$ contains a bounded continuum containing both A and P; P is said to be *accessible by arcs* from R if for each point A of R, $R+P$ contains a simple continuous arc AP from A to P. In this paper it is proposed to generalize the notion of an "accessible point" in these two senses to include "accessible continua" as follows: The bounded continuum K is said to be *accessible by continua* from a point set R if for each point A of R, $R+K$ contains a bounded continuum containing both A and K; K is said to be *accessible by arcs* from R provided that if A is any point of R and G denotes the collection whose elements are the continuum K together with all the points of $\bar{R}-K.\bar{R}$, then there exists a simple continuous arc AK of elements of G from A to K, which contains no point not in $R+K$.

It is obvious from the definitions that every continuum K which is accessible by arcs from a point set R is also accessible by continua from R. However, the converse is not true for all point sets R, i. e., "accessibility by continua" and "accessibility by arcs" are not equivalent for all sets R. However, it will be shown in this paper that these two notions are equivalent for all sets $[R]$ such that R is a connected open subset of some continuous curve. (Obviously,

[1]) Presented to the American Mathematical Society, April 7, 1928.

then, they are equivalent for all domains R). Hence, after establishing this fact, it is no longer necessary to distinguish between the two kinds of accessibility except when dealing with sets R which are not open subsets of any continuous curve; and in the pages that follow, the distinction will be made only in such cases. Clearly, if any single point of a bounded continuum K is accessible in either of the above senses from a point set R, then K itself is accessible, in the same sense, from R. However, a continuum K may be accessible from a set R and yet no point of K be accessible from R. (Cf. the example in § 4 below).

Definitions. The term "continuous curve" is used to designate any connected im kleinen continuum, bounded or not. The subset R of a closed set M is an open subset of M provided that $M-R$ is either vacuous or closed. A subset K of a connected set M is said to be a cutting of M, or is said to cut M, if $M-K$ is not connected; K is called an irreducible cutting[1]) of M if K cuts M but no proper subset of K cuts M; and K is called a componentwise irreducible cutting of M if K cuts M and every subset of K which cuts M contains at least one point of each component (i. e., maximal connected subset) of K. All of the theorems below concerning "accessible continua" hold true for the special case where these continua degenerate to single points. Hence, the word "continuum" as used below may be interpreted in this general sense. Unless otherwise stated, the point sets considered in this paper are assumed to lie in a Euclidean space of n dimensions. In most cases, however, the dimensionality of the space is indicated either in the titles of the sections or in the statements of the individual theorems.

Notation. The customary notation of point set theory will be employed, e. g., $K.H$ denotes the set of points common to the sets K and H, $\overline{X}=X+X'$, where X' denotes the set of all limit points of the set X, $K\subset H$ signifies that the set K is a subset of the set H, etc. If R is any point set, $F(R)$ will be used to denote the boundary of R relative to the whole space; and if R is an open subset of a continuous curve M, $F_m(R)$ is used to denote the M-boundary of R, or the boundary of R with respect to M, (i. e., the set of points $\overline{R}-R$). If K and H are point sets, $\delta(K, H)$ will be used to denote the minimum distance between K and H, i. e., the lower bound of the aggregate of numbers $[d(x, y)]$, where x and y are points of K and H respectively and $d(x, y)$ is the distance from x to y.

[1]) Cf. G. T. Whyburn, *Concerning irreducible cuttings of continua*, Fundamenta Mathematicae, vol. 13 pp. 42—57.

2. Accessibility from open subsets of a continuous curve in n dimensions.

Theorem 1. In order that the bounded subcontinuum K of a continuous curve M should be accessible by arcs from a given connected open subset R of M, it is necessary and sufficient that K should be accessible by continua from R.

Proof. The condition is necessary, because the set of all points $[Y]$ such that Y is a point of some element X of a given arc of elements of G (where G is the collection whose elements are the continuum K and the points of $\overline{R}-K.\overline{R}$), is a bounded continuum. It is also sufficient. For let A be any point of R. By hypothesis there exists a bounded continuum H such that $A+K\subset H\subset R+K$. For each integer $n>0$, let G_n be a collection of circular regions[1]) covering K and each of radius $1/n$, and let D_n be the domain obtained by adding together all the regions of G_n. There exists an integer $n_1>0$ such that $1/n_1<1/2\ \delta(A, K)$. By a theorem due to Miss Mullikin[2]), H contains a connected set Q_1 such that $Q_1.K=Q_1.F(D_{n_1})=0$, and $Q_1'.K\neq 0\neq Q_1'.F(D_{n_1})$. Let R_1 be the component of $R.D_{n_1}$ containing Q_1. There exists an integer $n_2>0$ such that $1/n_2<1/2\ \delta[K, F(D_{n_1})]$. By the Mullikin theorem, $\overline{Q}_1$ contains a connected set Q_2 such that $Q_2.K=Q_2.F(D_{n_2})=0$ and $Q_2'.K\neq 0\neq Q_2'.F(D_{n_2})$. Let R_2 be the component of $R.D_{n_2}$ containing Q_2. There exists an integer $n_3>0$ such that $1/n_3<1/2\ \delta[K, F(D_{n_2})]$, and there exist corresponding sets Q_3 and R_3, and so on. Let this process be continued indefinitely, giving a sequence $R_1, R_2, R_3, \ldots$ of connected open subsets of M, such that for each n, R_n contains R_{n+1} and $F(R_n)$ contains at least one point of K, and such that K contains the sequential limiting set of this sequence.

Now for each n, the set R_n contains a point X_n; R contains an arc[3]) AX_1; and for each n, R_n contains an arc[3]) X_nX_{n+1}. It is

[1]) A circular region is the set of all points in the space whose distance from a given point (the center) is less than a given number (the radius).

[2]) Cf. *Certain theorems relating to plane connected point sets*, Trans. Amer. Math. Soc., vol. 23 (1922), pp. 144—162.

[3]) Cf. R. L. Moore, *Concerning continuous curves in the plane* Mathematische Zeitschrift, vol. 15 (1922), pp. 254—260, Theorem 1.

easy to see that the set of points $N = K + AX_1 + \sum_1^\infty X_n X_{n+1}$ is a continuum; and if G_0 denotes the collection whose elements are the continuum K together with all the points of $N - K$, then clearly N is a continuous curve with respect to the elements of G_0. Hence N contains a simple continuous arc X of elements of G_0 from A to K. But G_0 is a subcollection of G, and N is a subset of $R + K$. Hence $R + K$ contains the arc X of elements of G from A to K, and the proof is complete.

Corollary 1. *In order that the bounded subcontinuum K of the continuous curve M should be accessible from a connected open subset R of M, it is necessary and sufficient that R should contain a sequence $R_1, R_2, R_3 \ldots$ of connected open subsets of M each containing the next and each having a limit point in K, and such that K contains the sequential limiting set of this sequence.*

Corollary 2. *In order that the bounded continuum K should be accessible by arcs from a domain D it is necessary and sufficient that K should be accessible by continua from D* [1].

Definition. The subcontinuum K of a continuous curve M is said to be *regularly accessible* [2] from a connected open subset R of M provided that K is accessible from R and for each $\varepsilon > 0$, a $\delta_\varepsilon > 0$ exists such that every point of R whose distance from K is $< \delta_\varepsilon$ lies together with K in a continuum H which is a subset of $R + K$ and every point of which is at a distance $< \varepsilon$ from some point of K.

Theorem 2. *If R is any connected open subset of a continuous curve M, then every bounded component of $F_m(R)$ is regularly accessible from R.*

Proof. Let K be any bounded component of $F_m(R)$, and let ε be any positive number. It is easily seen that a domain D_1 exists containing K and such that $F(D_1) . F_m(R) = 0$, and every point of D_1 is at a distance $< \varepsilon$ from some point of K. Now since $F_m(R) . D_1$ and $F(D_1) . R$ are mutually exclusive, closed, and bounded subsets of M, it follows by a theorem of the author's [3] that only a finite number of the components of $D_1 . R$ can have limit points in both of these sets. And since every such component must have a limit point in $F(D_1) . R$, then only a finite number can have limit points in $F_m(R) . D_1$. Hence there exists at least one component of $D_1 . R$ which has a limit point in K. Let G be the sum of all such components of $D_1 . R$. Now since K contains no limit point of $R - G$, there exists a domain D_2 within D_1 which contains K but contains no point of $R - G$. Let A be any point in $R . D_2$. I shall show that A and K lie together in a continuum in $R + K$ which is a subset of D_1.

Let R_1 be the component of G containing A. Then by practically the same argument as just given, using R_1 in place of R, it follows that R_1 contains an open subset R_2 of M which is connected, has at least one limit point in K, and every point of which is at a distance $< \varepsilon/2$ from some point of K. Likewise R_2 contains a set R_3 having a limit point in K and every point being at a distance $< \varepsilon/3$ from some point of K, and so on. Let this process be continued indefinitely, giving a sequence $R_1, R_2, R_3, \ldots$ of connected open subsets of M each containing the next, each having a limit point in K, and such that K contains the sequential limiting set of this sequence. Hence, by Corollary 1 to Theorem 1, K is accessible from R_1, and therefore A and K lie together in a continuum H which is a subset of $R_1 + K$. And since A is any point in $R . D_2$, and $R_1 \subset D_1$, it follows that K is regularly accessible from R.

Corollary. *If the M-boundary $F_m(R)$ of a connected open subset R of a continuous curve M is totally disconnected, then every point of $F_m(R)$ is regularly accessible from R.*

Theorem 3. *In order that the continuum M should be a continuous curve it is necessary and sufficient that if H is any closed subset of M and R is any component of $M - H$, then every bounded component of $\bar{R} - R$ is accessible (in either sense) from R.*

Proof. The condition is necessary by Theorem 2. It is also sufficient. For if M is not a continuous curve, then by the Moore-

[1] For the case where K consists of a single point, this result is due to Kuratowski and Knaster; cf. *Sur les continus non-bornés*, Fund. Math., vol. 5 (1924), p. 38.

[2] For the case where K is a single point, see my paper *Concerning the open subsets of a plane continuous curve*. Proc. Nat. Acad. Sc., vol. 13 (1927), pp. 650—657.

[3] *Concerning irreducible cuttings of continua*, loc. cit., Lemma 10a.

Wilder Lemma[1]), there exists a domain I, a subcontinuum W of M, and a connected subset N of M such that (1) $W \subset \bar{N}$, (2) $W \subset I$, and (3) if U is the component of $M.\bar{I}$ which contains W, then $N.U=0$. Let P be any point of W, and let C be a circular region with center P which lies, together with its boundary, wholly in I and such that there is some point of W without C. Let H denote the set of points $P+F(C).U$, and let R be the component of $M-H$ which contains N. Since $W \subset \bar{N}$ and $N.U=0$, P is a component of $\bar{R}-R$. But P is not accessible from R. For if $R+P$ contained any continuum L containing P and some point of R lying without C, L would contain a continuum K containing P and lying in I and containing a point X on $F(C)$; but then K would be a subset of U, and hence $X \subset F(C).U \subset H$, contrary to the fact that $X \subset R$ and $R.H=0$. Hence P is not accessible from R. Thus no continuum which is not a continuous curve can satisfy the condition of our theorem, and consequently the theorem is proved.

3. Accessibility by continua, in the plane.

Theorem 4. If in a plane S, R_1, R_2, and R_3 are mutually exclusive connected point sets, there does not exist, in $S-(R_1+R_2+R_3)$, three mutually exclusive bounded continua X, Y, and Z each of which is accessible by continua from R_1 and R_2 and contains at least one limit point of R_3.

Proof. Suppose, on the contrary, that three such continua X, Y, and Z do exist. It readily follows that there exist bounded continua H_1 and H_2 such that (1) $H_i (i=1,2) \subset R_i+X+Y$, (2) $H_i \supset X+Y$, and (3) $H_i-(X+Y)$ is[2]) connected. Then with the aid of a theorem proved by Janiszewski[2]) and also by Miss Mullikin[2]), it follows that H_1+H_2 cuts the plane into just two principal domains D_1 and D_2 each having boundary points in both X and Y and in both $H_1-(X+Y)$ and $H_2-(X+Y)$.

[1]) Cf. R. L. Moore, Bull. Amer. Math. Soc., vol. 29 (1923), p. 296; R. L. Wilder, Fund. Math., vol. 7. (1925), p. 371; and for the unbounded case, see an abstract of mine in the Bull. Amer. Math. Soc., vol. 34 (1928), p. 409.

[2]) Cf. Janiszewski, *Sur les coupures du plan faites par des continus*, Prace Matematyczne-Fizyczne, vol. 36 (1913), and Mullikin, loc. cit.

Now Z lies wholly in either D_1 or D_2, say in D_1. There exist points A_1 and A_2 belonging to $H_1-(X+Y)$ and $H_2-(X+Y)$ respectively and which are accessible from D_2. There exists an arc A_1OA_2 from A_1 to A_2 lying except for its end points in D_2; and since Z is accessible by continua from both R_1 and R_2, there exist continua A_1Z and A_2Z in R_1+Z and R_2+Z respectively containing A_1+Z and A_2+Z respectively. It is not difficult to show that the continuum $A_1OA_2+A_1Z+A_2Z$ separates X and Y in S. But since $Z \subset D_1$, then $R_3 \subset D_1$; and therefore $R_3 \cdot (A_1OA_2+A_1Z+A_2Z)=0$. Clearly this is impossible, since R_3+X+Y is connected. Thus the supposition that Theorem 4 is false leads to a contradiction.

Theorem 5. If in a plane S, R_1, R_2, R_3 are mutually exclusive connected point sets, and X and Y are mutually exclusive bounded subcontinua of $S-(R_1+R_2+R_3)$ each of which is accessible by continua from each of the sets R_1, R_2, and R_3, then no connected subset of $S-(R_1+R_2+R_3+X+Y)$ can contain limit points of all three of the sets R_1, R_2, and R_3.

Proof. From the hypothesis it follows that there exist three bounded continua H_1, H_2, and H_3 such that (1) $H_i (i=1,2,3) \subset R_i+X+Y$ (2) $H_i \supset X+Y$, and (3) $H_i-(X+Y)$ is connected. It readily follows with the aid of the Janiszewski-Mullikin theorem quoted above that $H_1+H_2+H_3$ cuts S into exactly three principal domains D_1, D_2, and D_3 such that $F(D_1) \subset H_1+H_2$, $F(D_2) \subset H_2+H_3$, $F(D_3) \subset H_1+H_3$, and such that the boundary of every other complementary domain of $H_1+H_2+H_3$ is a subset of some one of the continua H_1, H_2, and H_3. Now suppose, contrary to this theorem, that some connected subset N of $S-(R_1+R_2+R_3+X+Y)$ contains limit points of each of the sets R_1, R_2, and R_3. Then clearly N lies wholly in one of the domains D_1, D_2, and D_3, say in D_1. But then R_3 must contain a point of D_1; and since it contains points not in $\bar{D}_1$, e. g., all the points of $H_3-(X+Y)$, it must contain a point of $F(D_1)$, which is impossible because $F(D_1) \subset H_1+H_2 \subset R_1+R_2+X+Y$. Similarly a contradiction is obtained when we suppose N in D_2 or D_3. Thus the supposition that Theorem 5 is false leads to a contradiction.

4. Application to the cuttings of plane continuous curves.

Theorem 6. *If K is any closed and bounded componentwise irreducible cutting of a continuous curve M between*[1] *two points A and B of M, and R_c is any component of $M-K$ which contains neither A nor B, then $F_m(R_c)$ contains points of not more than two components of K.*

Proof. Suppose, on the contrary, that $F_m(R_c)$ contains a point in each of three distinct components X, Y and Z of K. But if R_a and R_b denote the components of $M-K$ containing A and B respectively, then[2] both $F_m(R_a)$ and $F_m(R_b)$ contain points in each of the sets X, Y, and Z; then since each of the sets X, Y, and Z contain components of each of the sets $F_m(R_a)$, $F_m(R_b)$, and $F_m(R_c)$, it follows by Theorem 2 that each of the continua X, Y, and Z is accessible from each of the three mutually exclusive connected subsets R_a, R_b, and R_c of $M-(X+Y+Z)$. This contradicts Theorem 4, and thus our theorem is proved.

Theorem 7. *Suppose K is a bounded irreducible cutting of a plane continuous curve M such that K has at least two components and $M-K$ has at least three components R_1, R_2, and R_3. Then K has exactly two components each of which is either a point or an indecomposable continuum.*

Proof. It follows from our hypothesis and a theorem of the author's[3] that K has just two components K_1 and K_2. Suppose, contrary to this theorem, that one of these components of K, say K_1, contains more than one point and is decomposable. Then, it is the sum of two of its proper subcontinua H and L. Let P and Q be points of $H-H.L$ and $L-H.L$ respectively. Then since[4] P is a limit point of each of the sets R_1, R_2, and R_3, but is not a limit point of $K-H$, it follows that there exist points P_1, P_2, and P_3

[1]) A subset K of a continuum M is said to be a componentwise irreducible cutting of M between two points A and B of M if K cuts M between A and B and every subset of K which cuts M between A and B contains at least one point in every component of K.

[2]) Cf. *Concerning irreducible cuttings of continua*, loc. cit., Theorem 7, a simple modification of which gives the result here used.

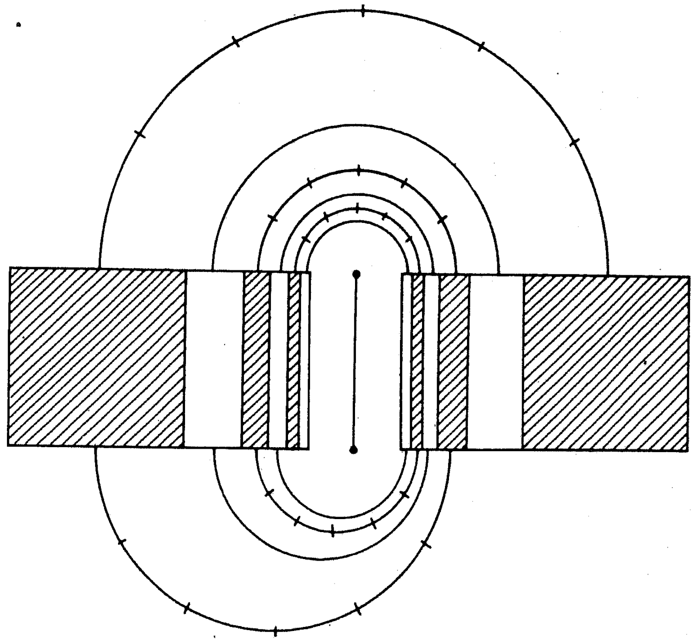
[3]) Loc. cit., Theorem 4.

[4]) Loc cit., Corollary 7a.

of H such that P_i $(i=1, 2, 3)$ is accessible from R_i. Hence H is accessible from each of the sets R_1, R_2, and R_3; by Theorem 2, K_2 also is accessible from each of these sets. But Q is a point belonging to $M-(R_1+R_2+R_3+H+K_2)$ which is a limit point of each of the sets R_1, R_2, and R_3. This contradicts Theorem 5, and thus our theorem is established.

Example. *There exists a bounded plane continuous curve M and an irreducible cutting K of M having a component I no point of which is accessible from any component of $M-K$.*

For each integer $n>0$, let A_n be the set of all points (x, y) in the plane such that $1/(n+1)<x<1/n$, $0\leqslant y\leqslant 1$; likewise let B_n be defined by the relations $-1/n<x<-1/(n+1)$, $0\leqslant y\leqslant 1$; K_n by the relations $x=1/n$, $0\leqslant y\leqslant 1$; H_n by $x=-1/n$, $0\leqslant y\leqslant 1$; Let C_n be the semicircular arc joining the points $\left(\frac{2n+1}{2n(n+1)}, 1\right)$ and



$\left(-\frac{2n+1}{2n(n+1)}, 1\right)$ and lying save for these points wholly above the line $y=1$; let D_n be the semicircular arc joining the points $\left(-\frac{2n+1}{2n(n+1)}, 0\right)$ and $\left(\frac{2(n+2)+1}{2(n+2)(n+3)}, 0\right)$ and lying otherwise wholly below the X-axis; and finally, let I be the interval (0, 1) of the Y-axis. Let

$$M=I+\sum_{n=1}^{\infty}[K_n+H_n+A_n+B_n+C_n+D_n], K=I+\sum_{n=2}^{\infty}(K_n+H_n),$$

$$R_1=K_1+H_1+\sum_{n \text{ odd and } >0}[A_n+B_n+C_n+D_n],$$

$$\text{and } R_2=\sum_{n \text{ even and } >0}[A_n+B_n+C_n+D_n].$$

Then M is a continuous curve, K is an irreducible cutting of M, R_1 and R_2 are the two components of $M-K$, $F_m(R_1)=F_m(R_2)=K$, I is a component of K, and clearly no point of I is accessible from either R_1 or R_2. Of course, by Theorem 2, I itself is accessible from both R_1 and R_2.

It is to be noted that in any example having the properties of the just constructed, $M-K$ must have just two components. For if $M-K$ has more than two components, by a theorem of the author's [1]) K itself has just two components, and obviously each of these components of K must contain a set of points dense in that component each of which is accessible from R, where R is any component of $M-K$ given in advance. Thus we have the following theorem.

Theorem 8. *Let K be any irreducible cutting of a bounded plane continuous curve M. Then either $M-K$ is the sum of two connected point sets or it is true that if R is any component whatever of $M-K$, then each component H of K contains a set of points dense in H each of which is accessible from R.*

[1]) Loc. cit., Theorem 4.

5. Accessible subcontinua of a given plane continuum. Various extensions.

This section is devoted, in the main, to extending some of the results in my paper *On certain accessible points of plane continua* [1]) about accessible points to analogous results about accessible continua. Since these extended theorems can, in general, be proved by obvious modifications of the proofs for the corresponding theorems in A. P. C., the details of proof are omitted and only the modifications which might present difficulty are indicated.

Theorem 9. (*Extension of Theorem 1 in A. P. C.*). *Let M be any continuum in the plane S, let G be any countable collection of mutually exclusive connected subsets of $S-M$, and let H be any uncountable collection of mutually exclusive bounded non-cut continua* [2]) *of M each of which is accessible by continua from at least two sets of the collection G. Then there exist continua X and Y of H, an uncountable subcollection E of H, sets R_1 and R_2 of G, and two continua L and N such that* (1) $L+N=M$, (2) $L.N=X+Y$, *and* (3) *every continuum of the collection E is a cut continuum of N, separates X and Y in N, and is accessible by continua from both R_1 and R_2.*

Corollary. (*Extension of Corollary 1a of A. P. C.*). *If H is any uncountable collection of mutually exclusive bounded non-cut* [3]) *continua of a plane continuum M each of which is accessible from at least two complementary domains of M, then there exist continua X and Y of H, an uncountable subcollection E of H, complementary domains R_1 and R_2 of M, and continua L and N such that* (1)

[1]) Mon. f. Math. u. Phys., vol. 35 (1928). This paper will be referred to hereafter as A. P. C.

[2]) A subcontinuum K of a continuum M is called a non-cut continuum or a cut continuum of M according as $M-K$ is or is not connected.

[3]) If M is bounded, we may remove the condition that the continua of H are non cut continua of M, for: *There does not exist an uncountable collection of mutually exclusive cut continua of a bounded continuum M each of which contains boundary points of at least two distinct complementary domains of M.* This theorem follows from Theorem 12 of my paper *Concerning collections of cuttings of connected point sets*, (Bull. Amer. Math. Soc., vol. 35 (1929), pp. 87—104) and an obvious extension of Theorem 10 of my paper *Concerning the cut points of continua*, Trans. Amer. Math. Soc., vol. 30 (1928), pp. 597—609.

$L+N=M$, (2) $L.N=X+Y$, *and* (3) *every continuum of the collection* E *is a cut continuum of* N, *separates* X *and* Y *in* N, *and is accessible from both* R_1 *and* R_2.

Theorem 10. (*Extension of Theorem 2 in A. P. C.*). *If* M *is any plane continuum and* K *is any collection of mutually exclusive bounded subcontinua of* M *each of which is accessible from at least two complementary domains of* M, *then all save possibly a countable number of the continua of* K *are continua of order two* [1]) *of* M, *and indeed can, for each* $\varepsilon > 0$, *be* ε*-separated in* M *by some pair of elements of* K.

In proving Theorem 10, use is made of the author's theorem that „if G is any collection of mutually exclusive bounded cut continua of any continuum M, then all save possibly a countable number of the continua of G are continua of order two of M, and can, for each $\varepsilon > 0$, be ε-separated in M by some pair of elements of G [2])", instead of the corresponding theorem about cut points used in the proof of Theorem 2 in A. P. C.

Theorem 11. (*Extension of Theorem 3 in A. P. C.*). *No plane continuum* M *contains an uncountable collection* G *of mutually exclusive subcontinua each of which contains a bounded proper subcontinuum which is accessible from at least two complementary domains of* M.

The proof of Theorem 11 parallels that of Theorem 3 in A. P. C., except that the author's theorem [3]) that „No continuum M contains an uncountable collection of mutually exclusive continua each of which contains a bounded proper subset which cuts M" is used in place of a somewhat less general theorem used in A. P. C.

[1]) A subcontinuum X of a continuum M is said to be a continuum of order n of M if for each $\varepsilon > 0$, X can be ε-separated in M by the sum of n subcontinua of M but not by $n-1$ such continua. This is an extension of the notion of a point of order n due to Menger and Urysohn (see § 6). The set X is said to be ε-separated in M by a set F provided that $M-F$ is the sum of two mutually separated sets M_x and M_0, where M_x contains X and every point of M_x+F is at a distance $<\varepsilon$ from some point of X. Cf. P. Urysohn, *Sur la ramification des lignes Cantoriennes*, Comptes Rendus, vol. 175 (1922), p. 481.

[2]) Cf. *Concerning collections of cuttings of connected point sets*, loc. cit., Theorem 12.

[3]) Loc. cit., Theorem 8.

Theorem 12. (*Extension of Theorem 4 in A. P. C.*). *If* A *and* B *are mutually exclusive non-cut continua of a bounded plane continuum* M, *then in order that* M *should be disconnected by the omission of* $A+B$ *it is necessary and sufficient that there should exist two complementary domains* R_1 *and* R_2 *of* M *such that each of the continua* A *and* B *is accessible from both* R_1 *and* R_2.

In proving Theorem 12 we assume, without loss of generality (by virtue of the principal of inversion), that each of the continua A and B lies in the unbounded complementary domain of the other. Then add to each of these continua all of its bounded complementary domains and call each of the continua thus obtained an element. Also let every point of the plane which belongs to neither of these continua be an element and let G be the collection of elements thus obtained. Then since G is an upper semi-continuous [1]) collection of bounded continua no one of which separates the plane, therefore, as established by R. L. Moore (loc. cit.), the space of elements G is topologically equivalent to the Euclidean point plane. Accordingly, Theorem 12 becomes topologically equivalent to Theorem 4 in A. P. C. and hence follows from that theorem.

It is readily seen from Theorems 12 and 9 that if R_1 and R_2 are any two complementary domains of a bounded plane continuum M, and G is any collection of mutually exclusive subcontinua of M each of which is accessible from both R_1 and R_2, then the elements of G can be cyclicly ordered in M.

Theorem 13. (*Extension of Theorem 5 in A. P. C.*). *If every component of the closed and componentwise irreducible cutting* K *of a plane continuum* M *is bounded, and* K *has more than one component, then each isolated component* [2]) *of* K *is accessible from at least two complementary domains of* M.

Corollary. (*Extension of Corollary 5a in A. P. C.*). *If the closed, bounded and componentwise irreducible cutting* K *of a plane*

[1]) A collection G of elements is said to be upper semi-continuous provided that if any element g of G contains a point of the sequential limiting set L of any sequence $g_1, g_2, g_3, \ldots$, of elements of G, then g contains all of L; cf. R. L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc., vol. 27 (1925), pp. 416—428.

[2]) The component H of K is said to be isolated if H contains no limit point of $K-H$.

continuum M has only a finite number of components, then every component of K is accessible from at least two complementary domains of M.

Theorem 14. *(Extension of Theorem 6 in A. P. C.). If K is any collection of mutually exclusive closed and bounded componentwise irreducible cuttings of a plane continuum M each having only a finite number of components, and G denotes the collection of all sets [X] such that X is a component of some element of the collection K, then all save possibly a countable number of the continua of G are continua of order two of M relative to the elements of G, i. e., all save a countable number can, for each $\varepsilon > 0$, be ε-separated in M by some pair of elements of G.*

Analogous extensions could also be stated for Theorems 8 and 9 and their respective corollaries in A. P. C.

6. Regular subcontinua of a given continuum.

Following Menger and Urysohn's definition [1]) of a *regular point* of a continuum, we will say that a bounded subcontinuum N of a continuum M is a *regular subcontinuum* of M if for each $\varepsilon > 0$, N can be ε-separated in M by a finite number of subcontinua of M; and if G is any collection of mutually exclusive subcontinua of M, we will say that N is a regular subcontinuum of M *relative to the elements of* G if for each $\varepsilon > 0$, N can be ε-separated in M by a finite number of the continua of the collection G. Similarly we will speak of subcontinua of M of order n, where n is a positive integer, and of subcontinua of M of order n relative to G.

Theorem 15. *If G is any collection of mutually exclusive bounded subcontinua of a continuum M (in n-space), then the collection E of all the continua of G which are regular subcontinua of M relative to G is upper semi-continuous.*

Proof. Suppose, on the contrary, that E contains an element e and a sequence $e_1, e_2, e_3, \dots$ of elements having a sequential limiting set L which contains at least one point P of e and at least one point Q not belonging to e. Let $2\varepsilon = \delta(Q, e)$. By hypothesis there exist a finite collection $g_1, g_2, \dots g_m$ of elements of G whose sum ε-separates e in M, and hence separates P and Q in M. But since not more than m of the continua $e_1, e_2, e_3, \dots$ can have points in common with $g_1 + g_2 + \dots + g_m$, and since both P and Q belong to L, clearly this is impossible. Therefore E is upper semi-continuous.

Since every subcollection of any upper semi-continuous collection is itself upper semi-continuous, we may state the following

Corollary. *Using the notation of Theorem 15, the collection F of all the continua of G which are continua of order n of M relative to G, where n is any positive integer given in advance, is upper semi-continuous.*

This corollary, together with Theorem 10, gives the following

Theorem 16. *If M is any plane continuum and G is any uncountable collection of mutually exclusive bounded subcontinua of M each of which is accessible from at least two complementary domains of M, then G contains an upper semi-continuous subcollection G_0 which contains all save possibly a countable number of the elements of G.*

Theorem 17. *If G is any collection of mutually exclusive closed and componentwise irreducible cuttings of a bounded plane continuum M such that for each element g of G, $M - g$ is not the sum of two connected point sets, then G is countable.*

Proof. Suppose, on the contrary, that G is uncountable. By a theorem of the author's [1]), each element of G has at most two components. Hence if E is the collection of all sets $[X]$ such that X is a component of some element of G, then, since G is uncountable, it follows by Theorem 14 that E contains an element Y which is a subcontinuum of M of order two of M relative to E. But Y is a component of some element F of G, and by hypothesis $M - F = H_1 + H_2 + H_3$, where H_1, H_2, and H_3 are mutually separated point sets. But since [2]) $H_1 + F$, $H_2 + F$, and $H_3 + F$ are continua, it is clear that if P_1, P_2, and P_3 are points of H_1, H_2, and H_3 respectively, and A denotes the set of points $P_1 + P_2 + P_3 + (F - X)$, then A and Y cannot be separated in M by any

[1]) Cf. K. Menger, *Grundzüge einer Theorie der Kurven*, Math. Ann. vol. 95 (1925), pp. 277—306; and P. Urysohn, loc. cit.

[1]) *Concerning irreducible cuttings of continua.* loc. cit., Theorem 4.

[2]) Loc. cit., Corollary 2a.

two subcontinua of M. Hence Y is not a continuum of order two of M, contrary to what we have just shown. Thus the supposition that G is not countable leads to a contradiction.

In conclusion I will point out the following interesting fact concerning regular subcontinua of a continuum. Let G be any collection of mutually exclusive subcontinua of a bounded continuum M (in n-space) each of which is a regular subcontinuum of M relative to G. Then if T denotes the point set obtained by adding together all the point sets of the collection G, it is readily seen that each component of $M-T$ is closed, and hence is a bounded continuum. And if E denotes the collection of all continua $[X]$ such that X is either an element of G or a component of $M-T$, then with the aid of a theorem of Menger's[1]) it follows that all the continua of E are regular subcontinua of M relative to E, and hence *with respect to the continua of E as elements, M is a Menger regular curve.*

[1]) K. Menger, loc. cit., Theorem 8.

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On a problem of Menger concerning regular curves[1]).

By

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In his paper *Zur allgemeinen Kurventheorie*[2]) Karl Menger raised the following question: *If M is a regular curve*[3]) *is it true that for every positive number ε the curve M is the sum of a finite number of continua of diameter less than ε such that any two have at most one point in common?*

The purpose of the present paper is to give an example which shows that the answer to Menger's question as stated is in the *negative*, but that for a regular curve M whose ramification points[4]) are not dense on any subcontinuum of M the answer is in the affirmative.

[1]) Presented to the Amer. Math. Soc., Dec. 28, 1928.

[2]) Fundamenta Mathematicae, vol. X (1927), pp. 96—115.

[3]) See Menger, *Grundzüge einer Theorie der Kurven*, Math. Ann. vol. 95 (1925), pp. 287—306. If M is a continuum and for each point P of M and each positive number ε there exists a connected open subset of M containing P and of diameter less than ε whose boundary with respect to M is finite then M is said to be a regular curve. If R is an open subset of M (i. e., no point of R is a limit point of $M-R$), then the boundary of R with respect to M is the set of points $\overline{R}\cdot(M-R)$. See R. L. Moore, *Concerning simple continuous curves*, Trans. Amer. Math. Soc., vol. 21 (1920), p. 345.

[4]) A ramification point of a continuous curve M is a point of order greater than 2. See W. Sierpiński, Comptes Rendus, vol. 160, p. 305. A point P of a regular curve M is said to be of order n if n is the smallest integer such that for every positive number ε there exists an open subset of M of diameter less than ε which contains P and whose boundary with respect to M contains at most n points. See Menger, loc. cit., and Urysohn, Comptes Rendus, vol. 175, (1922), p. 481.