On the convergence of lacunary trigonometric series.

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§ 1.

1. The lacunary trigonometric series is, by definition, any series of the form

$$(1)\qquad \sum_{k=1}^{\infty}(a_k \cos n_k\theta + b_k \sin n_k\theta), \qquad (n_{k+1}/n_k > q > 1)$$

where $n_1 < n_2 < \dots$ are integers and q is independent of k. The series (1) possesses many special properties. For example, it is well known, that, if a series (1), with $a_k, b_k \to 0$, is summable by Abels method in a point $\theta = \theta_0$, i. e. if

$$(2)\qquad \lim_{R\to 1}\sum_{k=1}^{\infty}(a_k \cos n_k\theta_0 + b_k \sin n_k\theta_0)R^{n_k}$$

exists and is finite, then the series (1) is convergent for $\theta = \theta_0$ to the sum equal to (2) [1]). Hence, if (1) is a Fourier-Lebesgue series

[1]) It follows immediately from Landau's well known theorem (Landau, Monatshefte für Math., **18** (1907), p. 8—28): if a) $\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots \to \infty$, b) $c_n = o\{(\lambda_n - \lambda_{n-1})/\lambda_n\}$, then the existence of

$$\lim_{\sigma\to+0}\Sigma c_n e^{-\lambda_n\sigma} = s$$

implies the convergence of Σc_n to the sum s. When $\lambda_{n+1}/\lambda_n > q > 1$, the condition b) is reduced to $c_n = o(1)$. As a matter or fact, when $\lambda_{n+1}/\lambda_n > q > 1$, a much stronger result is true, cf. Hardy and Littlewood, Proc. London Math. Soc., **25** (1926), p. 219—236.

(in particular, when the series

$$(3)\qquad \sum_{k=1}^{\infty}(a_k^2 + b_k^2)$$

converges), it converges almost everywhere [2]).

2. Recently I proved the following theorem [3]):

Theorem A. *If a series* (1) *is the Fourier-Lebesgue series of a function f, then the series* (3) *is convergent, i. e.* $f \subset L^2$.

The proof was rather complicated. From the remarks I have just made it follows that Theorem A may be considered as a corollary of the following much more general theorem.

Theorem B. *If a series* (1) *converges in a set Z of positive measure, then the series* (3) *converges.*

This theorem shows that the necessary and sufficient condition that a series (1) should be convergent almost everywhere, is that the series (3) should be convergent. It gives us a vast class of almost everywhere divergent trigonometric series with coefficients tending to 0.

3. Proof. The series (1) may be written in the form

$$(4)\qquad \sum_{k=1}^{\infty}\varrho_k \cos(n_k\theta + \theta_k), \qquad (\varrho_k \geqslant 0)$$

where θ_k is independant of θ.

From the hypothesis of the theorem it follows that there exists a set E of measure $e > 0$ and a constant C, such that

$$\left|\sum_{k=1}^{N}\varrho_k \cos(n_k\theta + \theta_k)\right| \leqslant C \qquad (N = 1, 2, \dots;\ \theta \subset E).$$

Hence

$$(5)\qquad \left|\sum_{k=M}^{N}\varrho_k \cos(n_k\theta + \theta_k)\right| \leqslant 2C = C_1, \quad (M, N = 1, 2, \dots, M \leqslant N,\ \theta \subset E).$$

[2]) Cf. also, Kolmogoroff, Fund. Math. **5** (1922), pp. 96—97.

[3]) Journal London Math. Soc., April, 1930.

We shall fix M a little later. It results from (5) that

$$\int_E \Big\{\sum_{k=M}^{N} \varrho_k \cos(n_k\theta+\theta_k)\Big\}^2 d\theta \leqslant C_1^2 e = C_2.$$

$$\sum_{k=M}^{N} \varrho_k^2 \int_E \cos^2(n_k\theta+\theta_k)\, d\theta + \tag{6}$$

$$+\sum_{\substack{k,l=M\\ k\neq l}}^{N} \varrho_k \varrho_l \int_E \cos(n_k\theta+\theta_k)\cos(n_l\theta+\theta_l)\, d\theta \leqslant C_2.$$

Let us put

$$\frac{1}{\pi}\int_E \cos m\theta\, d\theta = \xi_m, \quad \frac{1}{\pi}\int_E \sin m\,\theta\, d\theta = \eta_m, \tag{7}$$

$$\int_E \cos(n_k\theta+\theta_k)\cos(n_l\theta+\theta_l)\, d\theta = b_{k,l}. \tag{8}$$

The numbers ξ_m and η_m are Fourier coefficients of the function $\chi(\theta)$ equal to 1 for $\theta \subset E$, and to 0 for $\theta \subset CE$ (mod 2π). Supposing, for example, that $k>l$, we have

$$\frac{2}{\pi} b_{k,l} = \xi_{n_k+n_l}\cos(\theta_k+\theta_l) - \eta_{n_k+n_l}\sin(\theta_k+\theta_l) + \tag{9}$$
$$+\xi_{n_k-n_l}\cos(\theta_k-\theta_l) - \eta_{n_k-n_l}\sin(\theta_k-\theta_l).$$

$$\left(\frac{2}{\pi}b_{k,l}\right)^2 \leqslant 2(\xi_{n_k+n_l}^2+\eta_{n_k+n_l}^2+\xi_{n_k-n_l}^2+\eta_{n_k-n_l}^2) = \tag{10}$$
$$= 2(r_{n_k+n_l}^2 + r_{n_k-n_l}^2), \qquad (r_m^2=\xi_m^2+\eta_m^2),$$

From the Riemann-Lebesgue theorem it follows that, for $k\to\infty$,

$$\int_E \cos^2(n_k\theta+\theta_k)\, d\theta = \frac{1}{2}\int_E d\theta + \frac{1}{2}\int_E \cos 2(n_k\theta+\theta_k)\, d\theta \to \frac{1}{2}e. \tag{11}$$

To the second sum in (6) we apply Schwarz's inequality

$$\Big|\sum_{\substack{k,l=M\\ k\neq l}}^{N} \varrho_k\varrho_l b_{k,l}\Big| \leqslant \Big(\sum_{\substack{k,l=M\\ k\neq l}}^{N}\varrho_k^2\varrho_l^2\Big)^{\frac{1}{2}} \Big(\sum_{\substack{k,l=M\\ k\neq l}}^{N} b_{k,l}^2\Big)^{\frac{1}{2}} \leqslant \Big(\sum_{k=M}^{N}\varrho_k^2\Big)\Big(\sum_{\substack{k,l=M\\ k\neq l}}^{N} b_{k,l}^2\Big)^{\frac{1}{2}}. \tag{12}$$

I am going to prove that the series

$$\sum_{\substack{k,l=M\\ k\neq l}}^{\infty} b_{k,l}^2$$

converges. From (10) it will be sufficient to prove the convergence of the series

$$\sum_{\substack{k,l=1\\ k>l}}^{\infty} r_{n_k+n_l}^2; \quad \sum_{\substack{k,l=1\\ k>l}}^{\infty} r_{n_k-n_l}^2. \tag{13}$$

Il follows from the condition $n_{k+1}/n_k > q > 1$, that there exists a constant $\Delta=\Delta(q)$, such that every natural number m can be represented no more than Δ times in the form $n_k \pm n_l$ $(k, l=1,2,...;$ $k>l)$ [4]. Hence the sums of the series (13) cannod exceed

$$\Delta \sum_{m=1}^{\infty} r_m^2.$$

This last series, however, is convergent, ξ_m and η_m being Fourier coefficients of the bounded function. Il follows that we may find a number M, verifying two conditions

$$\Big(\sum_{\substack{k,l=M\\ k\neq l}}^{N} b_{k,l}^2\Big)^{\frac{1}{2}} < \frac{1}{8}e \qquad (N=M, M+1, M+2,...) \tag{14}$$

$$\int_E \cos^2(n_k\theta+\theta_k)\, d\theta > \frac{1}{4}e. \qquad (k\geqslant M) \tag{15}$$

Then, fixing M, we get from (6), (12), (14), (15)

$$\frac{e}{4}\sum_{k=M}^{N}\varrho_k^2 - \frac{e}{8}\sum_{k=M}^{N}\varrho_k^2 \leqslant C_2$$

[4]) Let us assume that $m=n_k+n_l$ $(k>l)$. Then $m>n_k>m/2$ and the number of n_k verifying this inequality, is less than the smallest integer x such that $q^x>2$. Similarly, if $m=n_k-n_l$, then $n_k>m$. As $n_k/n_l>q$ we have

$$n_k - n_k/q < m, \quad n_k < m\,q/(q-1),$$

and the number of n_k in the interval $(m, m\,q/(q-1))$ is also bounded.

$$\sum_{k=M}^{\infty} \varrho_k^2 \leqslant 8\, C_2/e$$

and Theorem B is proved [5]).

Evidently it is sufficient to suppose, instead of convergence, that the partial sums (or, even, a particular sequence of partial sums) of the series are finite in every point of Z.

Let

$$s_m(\theta) = \sum_{n_k \leqslant m} (a_k \cos n_k \theta + b_k \sin n_k \theta), \quad \sigma_m = \Big\{ \sum_{n_k \leqslant m} (a_k^2 + b_k^2) \Big\}^{\frac{1}{2}}$$

$$t_m(\theta) = \sum_{n_k \geqslant m} (a_k \cos n_k \theta + b_k \sin n_k \theta), \quad \tau_m = \Big\{ \sum_{n_k \geqslant m} (a_k^2 + b_k^2) \Big\}^{\frac{1}{2}}.$$

The same argument as above gives: if the series (3) diverges (resp. converges), then the inequality $s_m(\theta) = o(\sigma_m)$ (resp. $t_m(\theta) = o(\tau_m)$) may be satisfied only in a set of measure 0.

§ 2.

5. Theorem B is still true, if we suppose that the series (1), instead of being convergent, is summable (or finite) by Abel's method throughout Z [6]). The proof follows the same lines and may be left to the reader. Similarly, if the series

$$\sum_{k=1}^{\infty} (a_k^2 + b_k^2) R^{2n_k}$$

is convergent for $0 \leqslant R < R_0$ $(0 < R_0 \leqslant \infty)$, then, for $R \to R_0 - 0$, the function

$$f(R, \theta) = \sum_{k=1}^{\infty} (a_k \cos n_k \theta + b_k \sin n_k \theta) R^{n_k}$$

[5]) It may be added that, in the case of two variables the theorem and its proof remain essentially the same.

[6]) If $a_k, b_k = O(1)$, it is sufficient so suppose that

$$\lim_{R \to 1} \sum_{k=1}^{\infty} (a_k \cos n_k \theta + b_k \sin n_k \theta) R^{n_k} > -\infty \qquad (\theta \subset Z).$$

It is probable that the theorem is true in the general case.

may be

$$o \Big\{ \sum_{k=1}^{\infty} (a_k^2 + b_k^2) R^{2n_k} \Big\}^{\frac{1}{2}}$$

only in a set of θ whose measure is 0, provided that the expression $\{\}$ tends to $+\infty$.

The situation is the same for another metod of summation (sometimes called Lebesgue's method), which attributes to a trigonometric series

$$\text{(16)} \qquad \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

the sum

$$\text{(17)} \qquad s(\theta) = \lim_{h \to 0} \Big[\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \frac{\sin nh}{nh} \Big],$$

supposing that the series on the right hand converges for sufficiently small values of $|h|$. The existence of the limit (17) for $\theta = \theta_0$ is equivalent to the fact that

$$\lim_{h \to 0} \frac{F(\theta_0 + h) - F(\theta_0 - h)}{2h} = s(\theta_0),$$

where $F(\theta)$ denotes the sum of the series (16) integrated term by term. Considering, for simplicity, only the case of continuous functions we have the following theorem.

Theorem C. *If a series* (1) *is the Fourier series of a continuous functions F, which possesses a finite derivative in a set Z of positive measure, then*

$$\sum_{k=1}^{\infty} (a_k^2 + b_k^2) n_k^2 < \infty.$$

In other words, F is an indefinite integral of a function $f \subset L^2$.

The proof is the same as that of Theorem B. However, all these propositions are special cases of a more general theorem, which we are going to enunciate and to prove.

6. Let us consider an infinite table of real numbers

$$
(18)\qquad \begin{cases} \beta_{11}, & \beta_{12}, & \beta_{13}, \dots \\ \beta_{21}, & \beta_{22}, & \beta_{23}, \dots \\ \dots & \dots & \dots \\ \beta_{n1}, & \beta_{n2}, & \beta_{n3}, \dots \\ \dots & \dots & \dots \end{cases}
$$

This table defines a method of summation. We shall say that a series, whose partial sums are $s_1, s_2, \dots$, is summable $T(=T(\beta_{pq}))$ to the sum σ, if a) every series

$$\sum_{q=1}^{\infty} \beta_{pq} s_q \quad (p=1,2,\dots)$$

is convergent, and b) denoting by σ_p its sum, we have $\lim\limits_{p} \sigma_p = \sigma$. The condition a) is satisfied if, for every p, only a finite number of $\beta_{pq} \neq 0$. It is well known [7]), that the necessary and sufficient condition that a convergent series should be summable T and the two sums be equal, is that

$$
(19)\qquad
\begin{aligned}
&1^0 \quad \sum_{q=1}^{\infty} |\beta_{pq}| \leqslant M && (p=1,2,\dots)\\
&2^0 \quad \lim_{p} \sum_{q=1}^{\infty} \beta_{pq} = 1 \\
&3^0 \quad \lim_{p} \beta_{pq} = 0. && (q=1,2,\dots)
\end{aligned}
$$

In speaking about a method T, we shall always suppose, that the corresponding table verifies the three conditions (19). If the numbers β_{pq} verify only two conditions, those $(19, 2^0)$ and $(19, 3^0)$, we shall denote the method by T^*. All methods of summation used in Analysis are either T or T^*. In many important cases (the methods of Abel, Borel, etc.) the variable p tends c o n t i n u a l l y to ∞, which, of course, makes no essential difference. The methods being T^*, but not T, are, for example, those of Lebesgue and (C, γ) $(-1<\gamma<0)$.

[7]) O. Toeplitz, Prace Matematyczno-Fizyczne, **22** (1911), pp. 113—119.

Theorem D. *If a series* (1) *is summable (or finite)* T^* *in a set* Z *of positive measure, then the series* (3) *converges* [8]).

In saying that (1) is summable (finite) T^* we mean that, for every $\theta \subset Z$,

$$(20)\qquad \sum_{q=1}^{\infty} \beta_{pq}\, s_q(\theta) = \sigma_p(\theta)\ ^{9}),$$

the left hand series being convergent, and that the sequence $\{\sigma_p(\theta)\}$ is convergent (finite).

7. The proof of Theorem D is not essentially different from that of Theorem B. Let us suppose at first that in every line of the table (19) there is only a finite number of β's different from 0.

From our hypotheses it follows, that there exists a set E of positive measure $(E \subset Z)$ and number C such that $|\sigma_p(\theta)| \leqslant C$ $(p=1, 2, \dots, \ \theta \subset E)$ [10]). When the set E is fixed, we may suppose (changing, if necessary, the value of C) that n_1 is sufficiently great. Then

$$
(21)\qquad
\begin{aligned}
\sigma_p(\theta) &= \sum_{q=1}^{\infty} \beta_{pq}\, s_q(\theta) = \\
&= \sum_{k=1}^{\infty} A_{n_k}(\theta)\,(\beta_{p,n_k} + \beta_{p,n_k+1} + \dots) = \sum_{k=1}^{\infty} A_k(\theta)\, R_{n_k}(p)
\end{aligned}
$$

where

$$A_k(\theta) = a_k \cos n_k\theta + b_k \sin n_k\theta = \varrho_k \cos(n_k\theta + \theta_k),$$
$$R_m(p) = \beta_{p,m} + \beta_{p,m+1} + \dots$$

[8]) The c o n v e r s e theorem is evidently false (consider the method (C, r), $(-1<r<0)$) and is trivially true if we replace T^* by T.

[9]) $s_q(\theta)$ denotes the q'th partial sum of (1), i. e.

$$s_q(\theta) = \sum_{n_k \leqslant q} (a_k \cos n_k\theta + b_k \sin n_k\theta).$$

[10]) The functions $\sigma_p(\theta)$ $(p=1,2,\dots)$ are continuous, hence, using Egoroff's well known theorem, we get that, except in a set F of arbitrarily small measure, we have in Z, for $p > p_0 = p_0(F)$, the inequality $|\sigma_p(\theta)| \leqslant |\sigma(\theta)| + 1$. On the other hand, the function $\sigma(\theta)$ being measurable (on Z), the functions $\sigma(\theta), \sigma_1(\theta), \dots, \sigma_{p_0}(\theta)$, are less than a certain constant D, except in a set G of arbitrarily small measure $D = D(p_0, G)$). Hence in the set $E = Z - (F+G)$ we have $|\sigma_p(\theta)| \leqslant C$ $(p=1, 2, \dots;\ C = D+1)$.

Of course, the series in (21) are, in fact, finite. Then we have

$$\begin{aligned}(22)\qquad \int_E \{\sigma_p(\theta)\}^2\,d\theta = \sum_{k=1}^{\infty} \varrho_k^2 R_{n_k}^2(p) \int_E \cos^2(n_k\theta+\theta_k)\,d\theta + \\ + \sum_{\substack{k,l=1\\ k\neq l}}^{\infty} \varrho_k\varrho_l R_{n_k}(p) R_{n_l}(p) \int_E \cos(n_k\theta+\theta_k)\cos(n_l\theta+\theta_l)\,d\theta \\ \leqslant C^2\cdot m(E) = G.\end{aligned}$$

As in the proof of Theorem B, it results from (22), that there exists a constant C_2, such that

$$\sum_{k=1}^{\infty} \varrho_k^2 R_{n_k}^2(p) \leqslant C_2 \qquad (p=1,2,\ldots).$$

Taking an arbitrary integer $K>0$, and using the fact that $\lim_p R_m(p)=1$, we get by turn

$$\sum_{k=1}^{K} \varrho_k^2 R_{n_k}^2(p) \leqslant C_2,$$

$$\sum_{k=1}^{K} \varrho_k^2 \leqslant C_2,$$

$$\sum_{k=1}^{\infty} \varrho_k^2 \leqslant C_2.$$

8. We now proceed to remove the additional condition, that in every line of the table (18) only a finite number of β's is different from zero. From (20) it follows that for every $p(=1,2,\ldots)$ there exists a $Q=Q(p)$ and a set Z_p of points such that $m(Z_p)\leqslant \leqslant 2^{-p-1}m(Z)$, and that, for $\theta\subset Z-Z_p$, we have

$$\sum_{q=1}^{Q(p)} \beta_{pq}\, s_q(\theta) = \sigma_p^*(\theta) = \sigma_p(\theta)+\varepsilon_p(\theta) \qquad (|\varepsilon_p(\theta)|\leqslant 1/p).$$

Let

$$Z = Z-(Z_1+Z_2+\ldots),\quad m(Z')>\tfrac{1}{2}m(Z).$$

Then $\sigma_p^*(\theta)-\sigma_p(\theta)\to 0$ for $\theta\subset Z'$. Supposing that $Q(p)$ has also been chosen that

$$\lim_{p\to\infty}\{\beta_{p,1}+\beta_{p,2}+\ldots+\beta_{p,Q(p)}\}=1,$$

we see that, for $\theta\subset Z'$, the series (1) is summable (finite) $T^*(\beta'_{pq})$, where $\beta'_{pq}=\beta_{pq}$ for $1\leqslant q\leqslant Q(p)$ and $\beta'_{pq}=0$ otherwise. Consequently the case is reduced to that which has been previously discussed and the theorem E is completely proved.

9. Theorem D asserts, in particular, that for every method T^* there exists a trigonometric series with coefficients tending to 0, which is almost everywhere not summable T^*. It must be emphasized, however, that, a priori, we have not excluded the possibility, that the nonsummability T^* of the series may be due merely to the fact, that some of the series in (20) may be divergent almost everywhere.

In that case it would be more appropriate to say, that the method T^* cannot be applied to the series considered. It may, however, be shown very easily that, for an arbitrary method T, there exists a trigonometric series with coefficients tending to 0, for which all the series (20) are convergent $(0\leqslant\theta\leqslant 2\pi)$ and the functions $\sigma_p(\theta)$ diverge almost everywhere[11]. In fact, if the condition (19, 1^0) is fulfilled, then there exists a sequence of positive numbers $\lambda_1<\lambda_2<\ldots<\lambda_m<\ldots\to\infty$, such that all the series

$$\sum_{q=1}^{\infty} |\beta_{pq}|\,\lambda_p \qquad (p=1,2,\ldots)$$

are convergent. Now it is sufficient to put $a_k=1/\sqrt{k}$ and to take the sequence of $n_1<n_2<\ldots$ increasing so rapidly that $n_{k+1}/n_k> >q>1$, and

$$|s_m(\theta)|\leqslant \sum_{n_k\leqslant m}\varrho_k<\lambda_m.$$

10. Additional remarks. 1^0 Let a sequence of integers $n_1<n_2<\ldots$

[11]) This theorem is not new. It was proved by a different method by Prof. Mazurkiewicz (Prace Mat.-Fizyczne, **28** (1917), p. 109—118) who, besides this, proved, for an arbitrary method T, the existence of a power series, with coefficients tending to 0, non-summable T in any point of the circle $|z|=1$. The analogous problem for trigonometric series is not solved as yet.

verify the condition $n_{k+1}/n_k > q > 1$, and let $m_1, m_2, \dots$ denote the previous sequence rearranged in a quite arbitrary way. The reader may verify easily that Theorem B, as well as D, is still true for the series

$$\sum_{k=1}^{\infty} (a_k \cos m_k\theta + b_k \sin m_k\theta)\ ^{12}). \tag{23}$$

2⁰) Let $\varphi(x)$ be the function defined by the conditions $\varphi(x+1) = \varphi(x)$, $\varphi(0) = \varphi(\frac{1}{2}) = 0$, $\varphi(x) = 1$ $(0 < x < \frac{1}{2})$, $\varphi(x) = -1$ $(\frac{1}{2} < x < 1)$. Let $\varphi_k(x) = \varphi(2^k x)$ $(k = 0, 1, 2, \dots)$. Theorems B and D hold for the series of the form

$$\sum_{k=0}^{\infty} a_k \varphi_k(x)\ ^{13}).$$

The proofs remain the same; it is sufficient to remark that

$$\int_E \varphi_k^2\, dx = m(E),$$

and that the functions

$$\varphi_{k,l}(x) = \varphi_k(x)\varphi_l(x) \qquad (k = 1, 2, \dots, \ l = 0, 1, 2, \dots;\ k > l)$$

form an orthogonal system over the interval (0, 1). The same argument applies to some more general series.

§ 3.

11. We now prove a theorem, which shows that it is impossible to find an everywhere divergent series of the form (1), for which $a_k, b_k \to 0$.

Theorem E. Corresponding to every series of the form (1), *for which* $a_k, b_k \to 0$, *there exists a set* E, *everywhere dense and everywhere of the power of the continuum, in which the series* (1) *is convergent* [14]).

We divide the proof into three parts.

a) Let $F(\theta)$ denote the sum of the series obtained from (1) by formal integration, i. e.

$$F(\theta) = \sum_{k=1}^{\infty} (a_k \sin n_k\theta - b_k \cos n_k\theta)/n_k. \tag{24}$$

If $a_k, b_k \to 0$, then for every θ

$$\lim_{h\to 0} \frac{F(\theta+h) + F(\theta-h) - 2F(\theta)}{h} = 0. \tag{25}$$

Let $B_k(\theta) = a_k \sin n_k\theta - b_k \cos n_k\theta$, $(k = 1, 2, \dots)$, $N = [1/h]$ $(h > 0)$, then

$$\frac{F(\theta+2h) + F(\theta-2h) - 2F(\theta)}{4h} = -\sum_{k=1}^{\infty} B_k \frac{\sin^2 n_k h}{n_k h} =$$

$$= -\sum_{n_k \le N} - \sum_{n_k > N} = I_h + K_h.$$

$$|I_h| \le h \sum_{n_k \le N} |B_k| \cdot n_k = o(1),$$

$$|K_h| \le h^{-1} \sum_{n_k > N} |B_k|/n_k = o(1).$$

b) For every continuous function F verifying (25), there exists a set E, everywhere dense and everywhere of the power of the continuum, such that for $\theta \subset E$ the derivative $F'(\theta)$ exists and is finite [15]). In fact, let δ be an arbitrary interval in which F is defined and let θ_0 be an interior point of δ, in which F attains its extremum (supposing that such a point exists). Placing $\theta = \theta_0$ in (25) and noticing that $F(\theta_0+h) + F(\theta_0-h) - 2F(\theta_0)$ is of constant sign for sufficiently small values of $|h|$, we have

[12]) The partial sums $s_r(\theta)$ of the series (23) are defined by the equation

$$s_r(\theta) = \sum_{k=1}^{r} (a_k \cos m_k\theta + b_k \sin m_k\theta).$$

[13]) Cf. also Khintchine and Kolmogoroff (Recueil de la Soc. Math. de Moscou, 1924, p. 668—677) where a different proof (for the case of convergence) is given.

[14]) The theorem remains true even when the numbers n_k are not integers.

[15]) A. Rajchman, Prace Mat.-Fizyczne, **30** (1919), p. 19—88, esp. p. 23—24.

$$\lim_{h\to\pm 0}\frac{F(\theta_0+h)-F(\theta_0)}{h}=0,$$

i. e. $F'(\theta_0)$ exists and is equal to 0.

Let $y=l_\alpha(\theta)$ be the equation of the straight line passing through the point $(\theta_0, F(\theta_0))$ and whose angle with the θ axis is α $\left(0\leqslant|\alpha|<\frac{\pi}{2}\right)$. Let $G_\alpha(\theta)=F(\theta)-l_\alpha(\theta)$. The function $G_\alpha(\theta)$ vanishes for $\theta=\theta_0$. Without loss of generality we can suppose that $F(\theta)$ is not linear in the whole interval δ. Then there exist two numbers $\alpha_1<\alpha_2$ such that, for every α of the interval (α_1, α_2), the function $G_\alpha(\theta)$ vanishes in a point $\theta_1\neq\theta_0$ $(\theta_1=\theta_1(\alpha))$. Hence $G'_\alpha(\theta)$ exists and is equal to 0 in at least one point $\eta=\eta_\alpha$ between θ_0 and θ_1. Of course $F'(\eta_\alpha)=\alpha$. In particular, it follows that, if $\alpha_1\leqslant\alpha$, $\beta\leqslant\alpha_2$, and $\alpha\neq\beta$, then $\eta_\alpha\neq\eta_\beta$.

c) If $a_k, b_k\to 0$, then the necessary and sufficient condition that the series (1) should be convergent for $\theta=\theta_0$ with a (finite) sum s, is that

$$\lim_{h\to 0}\frac{F(\theta_0+h)-F(\theta_0-h)}{2h}=s,$$

where F is given by (24). The theorem, even in a more general form, could be deduced from well known theorems, but, for the sake of completeness, we shall give an independant proof.

Let

$$A_k=A_k(\theta_0)=a_k\cos n_k\theta_0+b_k\sin n_k\theta_0,\quad N=[1/h].$$

Then

$$\frac{F(\theta_0+h)-F(\theta_0-h)}{2h}=\sum_{k=1}^{\infty}A_k\frac{\sin n_k h}{n_k h}.$$

$$\Delta_h(\theta_0)=\frac{F(\theta_0+h)-F(\theta_0-h)}{2h}-s_N(\theta_0)=$$

$$=\sum_{n_k\leqslant N}A_k\left(\frac{\sin n_k h}{n_k h}-1\right)+\sum_{n_k>N}^{\infty}A_k\frac{\sin n_k h}{n_k h}=L_h+M_h.$$

As $\sin x/x=1+O(x^2)$, then

$$|L_h|\leqslant\sum_{n_k\leqslant N}|A_k|\cdot O(n_k^2h^2)=h^2\sum_{n\leqslant N}o(n_k^2)=o(1),$$

$$|M_h|\leqslant h^{-1}\sum_{n_k\leqslant N}|A_k|/n_k=o(1).$$

Theorem E is an immediate consequence of a) b) c).

12. From the fact that $\Delta_h(\theta_0)$ 0 with h, it follows that

$$\underline{\lim}_{n\to\infty}s_n(\theta_0)=\underline{\lim}_{h\to 0}\frac{F(\theta_0+h)-F(\theta_0-h)}{2h}$$

$$\overline{\lim}_{n\to\infty}s_n(\theta_0)=\overline{\lim}_{h\to 0}\frac{F(\theta_0+h)-F(\theta_0-h)}{2h}$$

From c) and from arguments used in b) we have the following property of series of the form (1), for which $a_k, b_k\to 0$: If $\underline{\lim}\, s_n(\theta)=\underline{l}(\theta_0)$, $\overline{\lim}\, s_n(\theta_0)=\overline{l}(\theta_0)$, then, for every $\varepsilon>0$ and for every number l such that $\underline{l}(\theta_0)<l<\overline{l}(\theta_0)$, there exists in $(\theta_0-\varepsilon, \theta_0+\varepsilon)$ a point η such that $\lim s_n(\eta)=l$. Thus, for example, the series

$$\sum_{k=1}^{\infty}\frac{\sin 10^k\theta}{\sqrt{k}},$$

although divergent almost everywhere, possesses, in an arbitrary interval, points in which it converges to every given number.

13. When $a_k, b_k=O(1)$, there exists an everywhere dense set E^* in which the Dini's numbers of the function (24) are finite [16]), but the previous argument fails to prove that E^* is of the power of the continuum.

§ 4

14. Theorem A may be generalised by proving that, if the series (1) is the Fourier series of a function $f\subset L^2$, then $f\subset L^p$, however great p may be [17]):

We state the theorem in the complex form.

[16]) It follows from the fact that in every point θ of E the expressions

$$\frac{F(\theta+h)-F(\theta-h)}{2h},\quad\frac{F(\theta+h)+F(\theta-h)-2F(\theta)}{h}$$

are bounded.

[17]) A. Zygmund, loc. cit.

Theorem F. Let

$$f(z)=\sum_{k=1}^{\infty} c_k z^{n_k}, \qquad (n_{k+1}/n_k \geqslant q > 1) \tag{26}$$

where n_1, n_2,... are positive integers and $|c_1|^2+|c_2|^2+\ldots=1$. Then there exists a constant C_p, depending only on p, such that

$$M_p(f)=M_p(f,R)=\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(Re^{i\theta})|^p\,d\theta\right\}^{\frac{1}{p}}\leqslant C_p \tag{27}$$

$$(p>2,\ 0\leqslant R<1)$$

We will give another proof of this theorem which is, perhaps, simpler and, at any rate, more elementary than that given previously.

Evidently it is sufficient to prove the theorem for all even integers, i. e. for $p=2r$ $(r=2,3,\ldots)$. From (26) we get

$$f^r=\left(\sum c_k z^{n_k}\right)^r=\sum c_l' z^{m_l}=g(z), \tag{28}$$

where the exponents m_l are of the form

$$\alpha n_{k_1}+\beta n_{k_2}+\ldots, \qquad (\alpha+\beta+\ldots=r;\ k_1>k_2>\ldots), \tag{29}$$

the coefficients α, β,... being positive integers. It is easy to prove that if q is sufficiently great ($p=2r$ fixed!), then no natural number can be represented in two different ways in the form (29). Let us suppose that it is not true, then we have the equation

$$0=a\,n_{k_1}+b\,n_{k_2}+\ldots \qquad (|a|\leqslant r,\ |b|\leqslant r,\ldots;\ k_1>k_2>\ldots)$$

where $|a|\geqslant 1$. Consequently

$$|a|\,n_{k_1}\leqslant|b|\,n_{k_2}+\ldots$$

$$n_{k_1}\leqslant r(n_{k_2}+\ldots)$$

$$1<r(q^{-1}+q^{-2}+\ldots)=r/(q-1),$$

which is certainly false if $q\geqslant r+1$. Thus, supposing that $q\geqslant r+1$, we have

$$M_{2r}^{2r}(f,R)=\frac{1}{2\pi}\int_0^{2\pi}|f(Re^{i\theta})|^{2r}\,d\theta= \tag{30}$$

$$=\frac{1}{2\pi}\int_0^{2\pi}|g(Re^{i\theta})|^2\,d\theta=\sum|c_l'|^2R^{2m_l},$$

where, if m_l is given by (29),

$$c_l=\frac{r!}{\alpha!\,b!\ldots}\,c_{k_1}^{\alpha}\,c_{k_2}^{\beta}\ldots.$$

As

$$|c_l|^2\leqslant r!\,\frac{r!}{\alpha!\,\beta!\ldots}\,|c_{k_1}|^{2\alpha}\cdot|c_{k_2}|^{2\beta}\ldots$$

we get from (30) that

$$M_{2r}^{2r}(f,R)\leqslant r!\left(\sum_{k=1}^{\infty}|c_k|^2R^{2n_k}\right)^r\leqslant r!\left(\sum_{k=1}^{\infty}|c_k|^2\right)^r=r!$$

In the general case the series (26) may be represented as a sum of s series such that for each of them the number corresponding to q is $\geqslant r+1$ [18]). Then

$$f=f_1+f_2+\ldots+f_s;$$

$$|f|^{2r}\leqslant(|f_1|+|f_2|+\ldots+|f_s|)^{2r}\leqslant s^{2r}(|f_1|^{2r}+\ldots+|f_s|^{2r})$$

$$M_{2r}^{2r}(f,R)\leqslant r!\,s^{2r} \tag{30}$$

and Theorem F is proved. Using the well known fact that $M_p(f)$ is an increasing function of p and that

$$\exp z=1+z+\frac{z^2}{2!}+\ldots,$$

it is not difficult to deduce from (31) that

$$\int_0^{2\pi}\exp|f(Re^{i\theta})|^{2-\varepsilon}\,d\theta \tag{32}$$

is bounded for $\varepsilon>0$. It must be added, however, that the previous method gives a little stronger result *viz.* that the integral (32) is bounded even for $\varepsilon=0$, if we replace there f by kf, k being an arbitrary constant. For $\varepsilon<0$ the theorem is false [19]).

[18]) For the value of s we may take the smallest integer x such, that $q^x\geqslant r+1$.

[19]) It is possible to prove that the integrability of $\exp(k|f|^2)$ is the best result, i. e. for every function $\omega(x)$ such that $\omega(x)/x^2\to\infty$ with x, there exists a function (26) such that $\exp[\omega(|fe^{i\theta})|^2)]$ is not integrable. Cf. R. E. A. C. Paley and A. Zygmund, On some series of functions, (1), Cambridge Phil. Soc. Proc., 1930.

15. From Theorem F we obtain the following proposition about Fourier series.

Theorem G. *Let (16) be the Fourier series of a function f such that*

$$\frac{1}{\pi}\int_0^{2\pi}|f|^p\,d\theta = 1, \qquad (p>1). \tag{33}$$

If $n_{k+1}/n_k > q > 1$ $(k=1,2,\ldots;\ n_1 \geqslant 1)$, then the series

$$\sum_{k=1}^{\infty}(a_{n_k}^2 + b_{n_k}^2) \tag{34}$$

is convergent and its sum less than a certain constant $D_{p,q}$ depending only on p and q.

It is W. H. Young's well known theorem, that if

$$f \subset L^p,\ g \subset L^{p'} \qquad \left(\frac{1}{p}+\frac{1}{p'}=1\right),$$

then

$$\frac{1}{\pi}\int_0^{2\pi} fg\,d\theta = \frac{a_0 a_0'}{2} + \sum_{n=1}^{\infty}(a_n a_n' + b_n b_n'), \tag{35}$$

the series on the right hand being summable $(C, 1)$ and a_n', b_n' denoting Fourier coefficients of g. Hölders inequality and the condition (33) show that the left hand integral in (35) is absolutely less that

$$\left\{\frac{1}{\pi}\int_0^{2\pi}|g|^{p'}\,d\theta\right\}^{1/p'}.$$

Let us consider the class of functions $g(\theta)$ whose Fourier developments are

$$g \sim \sum_{k=1}^{\infty}(\alpha_k \cos n_k\theta + \beta_k \sin n_k\theta),$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots$ are quite arbitrary real numbers such that

$$\sum_{k=1}^{\infty}(\alpha_k^2 + \beta_k^2) = 1. \tag{36}$$

It follows from Theorem F that the series

$$\sum_{k=1}^{\infty}(a_{n_k}\alpha_k + b_{n_k}\beta_k) \tag{37}$$

is summable $(C, 1)$. As the signs of α_k, β_k may be chosen arbitrarily, the series (37) is convergent for every sequence of α_k, β_k verifying (36) and the theorem follows at once.

Theorem G is evidently false for $p=1$, but the series (34) still converges when $\varphi(|f|)$ is integrable, where $\varphi(t) = t\exp\sqrt{\lg t}$ for $t \geqslant 1$, and $\varphi(t)=1$ for $0 \leqslant t \leqslant 1$.

Cambridge 1. V. 1930.