# Memoir on the Analytical Operations and Projective Sets (I).

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## Introductory.

This memoir is intended to contribute to the socalled „descriptive theory of sets" the object of which is the investigation of different classes of sets (especially in a Euclidean space). The elementary notions in this theory are those of open and closed sets. Next to this comes the class of all the sets which can be obtained from closed sets by a finite number or a countable infinity of the operations of countable additions (i. e. additions of a finite number or a countable infinity of sets) and substractions. These sets have been introduced by Mr. E. Borel and are therefore called „Borelian sets" (*class B*).

Neither by the analytical operations then known nor by any other means tried at that time could the class of „individual" [1]) sets be extended beyond this class $B$, and Borelian sets remained the only known individual sets until Mr. Souslin introduced a new analytical operation (*operation A*) and with it a new and wider class of sets viz. that of sets $(A)$. Beside extending the class of individual sets the works of Mr. Souslin had a great methodological importance because in these works the idea of an analytical operation was for the first time distinctly introduced. A further study of the sets $A$ carried on by Souslin, Lusin, Sierpiński and

[1]) Or „effective".



others, revealed the fact that the class $A$ is a natural extension of the class $B$ because many other (not analytical) ways also lead to this class so that the $A$-sets may be defined in several different manners: 1) analytically, as a result of $A$-operation effected upon closed or more generally, upon Borelian sets; 2) as projections of the sets $G_\delta$ (or generally of Borelian sets); 3) as continuous images of $G_\delta$ (or Borelian sets); 4) as sets of values of Baire's functions in Borelian sets; 5) by means of the „crible" of Mr. Lusin [1]).

Each of these definitions gives sufficient means to develop a complete theory of $A$-sets, i. e. to answer all the questions that present themselves in the theory of any class of sets.

These questions may be subdivided into different groups: 1) those concerning the properties of sets belonging to the class in question, as e. g. their measurability, the property of Baire etc.; 2) those concerning the properties of the class itself (such are its topological invariance, the existence of a universal set or of a universal series, the power of the class etc.); 3) concerning relations among various subclasses of our class (inner classification); 4) concerning functions related to the class of sets.

Further extension of the class of sets coming under the scope of descriptive theory can be effected in two ways: 1) we may repeatedly apply the elementary operations (of countable additions and intersections and of substraction) and the operation of Mr. Souslin ($A$-operation) and thus we come to the class of all the sets which can be obtained from intervalls by a finite number or a countable infinity of the elementary operations and $A$-operations. This class has been called by Mr. Lusin *class* $(C)$. 2) We may apply besides the elementary operations, the operation of taking a projection. We come thus to the class of all the *projective sets*. Both these classes have been introduced by Mr. Lusin. They both allow an inner classification, subdividing each of them into $\aleph_1$ subclasses (having for indices the numbers of I and II class).

The theory of projective sets is very little developed. This is due to the fact that there existed no analytical expression for these sets. Almost all the results concerning projective sets were obtained by the method of continuous transformations.

In consequence the first problem which presented itself was that

[1]) See N. Lusin, *Leçons sur les ensembles analytiques*. Paris 1930.

of finding a convenient analytical expression for projective classes Another problem closely connected with that mentioned above is the problem of the relations between the class $(C)$ of Mr. Lusin (see above) and the class of projective sets.

These problems (especially the first of them) made it necessary to develop the general theory of analytical operations. The first known (not elementary) analytical operation was $(A)$-operation of Souslin. Then in 1927 Mr. Hausdorff and independently of him Mr. Kolmogoroff introduced a very wide class of analytical operations, called by Mr. Hausdorff *δs-operations.*

They are defined as follows: let $E_1, E_2, \ldots$ be arbitrary sets and $N$ a set of irrational numbers. Then

$$\Phi_N(E_1, E_2, \ldots) = \sum E_{n_1} \cdot E_{n_2} \cdot \ldots$$

where the summation is extended over all the sequences $(n_1, n_2, \ldots)$ such that

$$\frac{1|}{|n_1} + \frac{1|}{|n_2} + \ldots \in N.$$

The set $N$ has been called by us *the base of the δs-operation.*

The first part of this memoir is consecrated to the theory of $\delta s$-operations.

The importance of these operations is the consequence of their generality: every positive analytical operation [1]) on a countable infinity of sets is equivalent with a $\delta s$-function. Their theory is therefore in essence the most general theory of analytical operations on a countable infinity of sets.

In this theory the structure of the base (the set $N$, see above) is essential. This becomes plain if we consider the following theorem (theorem V of this work):

$\Phi_N(E_1, E_2, \ldots)$ can be obtained from $N, E_1, E_2, \ldots$ by the elementary operations (to which we add homeomorphic transformation and the operation of taking the product of two sets. see p. 221 below) and one projection.

In the first chapter we give the general theory of the $\delta s$-functions. § 1 is consecrated to their fundamental properties, upon which the results of the other paragraphs are based.

[1]) The definition of a positive analytical operation see p. 225.



Of the §§ 2—4 which deal with the different properties of these operations the most interesting for us at present is § 3. We prove there that the class of projections of sets belonging to a $\delta s$-class on closed (or open) sets [1]) is also a $\delta s$-class on closed (resp. open) sets. We give there also a simple analytical expression of such a class (not as a $\delta s$-function). These theorems will be applied in the second part of this work where they will supply an analytical expression for every projective class $P_\alpha$ i. e. a $\delta s$-operation $\Phi_N(E_1, E_2, \ldots)$ such that the corresponding $\delta s$-class upon closed (or open) sets is $P_\alpha$. The base $N$ of such an operation is a set of the same projective class.

Chapters II and III contain the theory of certain operations which beside being interesting in themselves, will be found usefull in the theory of projective sets, contained in the IV Chapter.

All known results appear there as a natural outcome of the general properties of $\delta s$-functions. But not only the known results can be found by this method. The analytical expression of the projective classes together with the theory of the $\delta s$-functions gives us means to find some new results especially concerning their classification.

In this domain very little was known. Two chief questions naturally presented themselves: first about the relation between the class $(C)$ of Mr. Lusin (see above, p. 215) and the class of all the projective sets, and secondly about the classification of sets which are $P_{\alpha+1}$ and $C_{\alpha+1}$ simultaneously. This last problem is still very far from being solved, though the theory of $\delta s$-functions gives us infinitely more than other methods. On the contrary the first problem (about the relations between the class $(C)$ and the projective classes) may be solved completely by the method of analytical operations. The answer is: class $(C)$ is contained in the classes $P_2$ and $C_2$ (i. e. the class of projections of $C(A)$-sets and the class of complements of these projections) simultaneously. And even more: the class $(C)$ does not even coincide with the class $P_2 \cdot C_2$ so that there exist sets which belong to $P_2$ and $C_2$ simultaneously, but not to $(C)$.

In the fifth chapter we consider the functions (of real variables) related to projective sets or more generally to the sets belonging to $\delta s$-classes. This is the natural generalisation of Baire's functions.

[1]) $\delta s$-class on a certain class $\mathfrak{H}$ of sets is the class of values of a $\delta s$-function when its „arguments" belong to $\mathfrak{H}$.

We have begun our work on questions dealt with in this memoir at the Seminary on $A$-sets led by prof. Gr. Fichtenholz in Leningrad State University in 1928—29. Some of the most important results of this work have been published in Comptes rendus Acad. Sc. [1]). They were also reported to the Congress of Mathematics of U. S. S. R in Kharkov (June 1930) [2]).

In conclusion we wish to express our deep gratitude to Prof. Gr. Fichtenholz who has continuously helped us in our work.

## PART I.

## Analytical operations.

### List of Literature [3]).


Braun, S. — *Sur la projectivité des opérations de M. Hausdorff*, C. R. de Varsovie XXIII, p. 88.

Hausdorff, F. — *Mengenlehre.* 2 Aufl. (Berlin 1927).

Kantorovitch. L. — *Sur les fonctions universelles.* Journ. de la Soc. Ph. Math. Leningr. t. II, f. 2, p. 13 (in Russian).

Kantorovitch, L. et Livenson, E. — *Sur les δs-fonctions de M. Hausdorff*, C. R. t. 191, p. 352.

Kolmogoroff, A. — *Opérations sur des ensembles*, Rec. Mathématique de Moscou, XXXV, p. 418 (in Russian).

Koźniewski, A. et Lindenbaum, A. — *Sur les opérations des additions et de multiplication dans les classes.* F. M. t. XV, p. 342.

Livenson, E. — *Sur les opérations analytiques sur des ensembles*, Travaux du Congrès des Mathématiciens de l'U. R. S. S. Kharkov, juin 1930 (in Russian).

Lusin, N. — *Leçons sur les ensembles analytiques*, Paris 1930.

Sierpiński, W. — *Sur les ensembles mesurables B*, C. R., t. 171, 5 juillet 1920. (I)

— *Sur les fonctions de M. Hausdorff*, C. R. de Varsovie XIX, p. 463. (II)

— *Sur un problème de M. Hausdorff*, F. M. t. X, p. 427. (III)

— *Sur les opérations de M. Hausdorff*, F. M. t. XV, p. 119. (IV)

— *Sur la projectivité des opérations de M. Hausdorff*, C. R. de Varsovie. t. XXIII, p. 15. (V)

— *Sur une généralisation des opérations (A).* C. R. de Varsovie, t. XXII. p. 174. (VI)

— *Sur une propriété des opérations de M. Hausdorff*, F. M. t. XVI, p. 1. (VII)


[1]) Séances 30 déc. 1929, 10 fevr., 12 mai et 2 juin 1930.

[2]) See Les travaux du Congrès: E. Livenson, *Sur les opérations analytiques sur des ensembles* and L. Kantorovitch, *Sur les ensembles projectifs* (in Russian).

[3]) Literature on analytical and projective sets will be given in the Second Part.




Sierpiński, W. — *Sur certaines opérations sur les ensembles fermés plans.* C. R. de Varsovie XXIV, p. 55. (VIII)

Szpilrajn, E. — *Un théorème sur les opérations de M. Hausdorff.* C. R. de Varsovie XXIII, p. 13.

Tarski. A. — *Sur les classes d'ensembles closes par rapport à certaines opérations élémentaires*, F. M. XVI, p. 181.

The titles of the periodicals are abridged as follows:

Fundamenta Mathematicae — F. M.

Comptes Rendus de l'Académie des Sciences à Paris — C. R.

Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie — C. R. de Varsovie.


### Notations and Preliminary Remarks.

We shall suppose that the sets with which we deal are subsets of a certain „universal" set $R$. In some parts of this work $R$ shall be supposed to be a topological or even a metric space [1]).

A. $A+B$, $\sum\limits_{\xi} A_{\xi}$, $A \subset B$, $x \in A$, have their usual meaning.

$C(E)$ denotes the complement of $E$ [2]).

$AB$, $\prod\limits_{\xi} A_{\xi}$ denotes the intersection (common part) of $A$ and $B$ (resp. of the sets $A_{\xi}$).

$\bar{A}$ denotes the „abgeschlossene Hülle" (fermeture) of $A$, i. e. the set of all the points which either belong to $A$ or are its accumulation points.

$\underset{x}{\mathcal{E}}(k)$ denotes the set of all the points $x$ which satisfy the condition $(k)$.

$=$ denotes „equal by definition".

$\varrho(x_1, x_2)$ denotes the distance of the points $x_1$ and $x_2$.

$d(E)$ denotes the diameter of set $E$.

B. If $\mathcal{H}$ is a class of sets then:

$\mathcal{H}_\sigma$ denotes the class of countable sums (i. e. sums of a finite number or a countable infinity) of sets belonging to $\mathcal{H}$.

$\mathcal{H}_\delta$ denotes the class of countable intersections of sets belonging to $\mathcal{H}$.

$\mathcal{H}_s$ denotes the class of finite sums of sets belonging to $\mathcal{H}$.

$\mathcal{H}_d$ „ „ „ „ finite intersections of sets belonging to $\mathcal{H}$.

$\mathcal{H}_\varrho$ „ „ „ „ differences „ „ „ „ „

$\mathcal{H}_c$ „ „ „ „ complements [2]) „ „ „ „ „

$\mathcal{F}$ „ „ „ „ sets closed in $R$ (if $R$ is a topological space).

$\mathcal{G}$ „ „ „ „ sets open in $R$.

[1]) As to the definition of the expressions „topological space", „metric space", etc. see Hausdorff.

[2]) Complements are taken rel. $R$.

$\mathcal{F}^{(0)}$ denotes the class of sets closed in $R^{(0)}$ where $R^{(0)}$ is a certain topological space.

$\mathcal{G}^{(0)}$ „ „ „ „ sets open in $R^{(0)}$ where $R^{(0)}$ is a certain topological space.

C. $(a, b)$ denotes sometimes the ordered pair.

$(a; b)$ denotes the open interval $\mathop{\mathcal{E}}_{x}(a < x < b)$.

$[a; b]$ denotes the closed interval $\mathop{\mathcal{E}}_{x}(a \leqslant x \leqslant b)$.

I denotes the closed interval $[0; 1]$.

$J$ denotes the set of irrational numbers of I.

$\frac{1|}{|n_1} + \frac{1|}{|n_2} + \dots$ denotes the continuous fraction $\cfrac{1}{n_1 + \cfrac{1}{n_2 + \dots}}$

$\delta_{n_1, n_2, \dots, n_k}$ denotes the set of irrational numbers $\frac{1|}{|m_1} + \frac{1|}{|m_2} + \dots$ such that:

$$m_1 = n_1; \quad m_2 = n_2; \dots; \quad m_k = n_k.$$

$\delta^1_{n_1, n_2, \dots, n_k}$ denotes the interval $\left(\frac{1|}{|n_1} + \frac{1|}{|n_2} + \dots + \frac{1|}{|n_k}; \ \frac{1|}{|n_1} + \dots + \frac{1|}{|n_k} + \frac{1|}{|1}\right)$

$(a)$ denotes the set consisting of the single element $a$.

$\Delta$ denotes the set of all the points of $I$ which can be represented (as decimal fractions) using only the signs 0 and 1. It is a perfect nowhere dense set.

$\Delta_{\nu_1^0, \nu_2^0, \dots, \nu_k^0}$ denotes the set of all points $0, \nu_1 \nu_2 \dots \epsilon \Delta$ such that $\nu_1 = \nu_1^0$; $\nu_2 = \nu_2^0; \dots; \nu_k = \nu_k^0$.

**D.** Some notions deserve a closer consideration. Such are:

1) The notion of „a system“ of things (e. g. sets) depending of a certain set of arguments.
2) of a „cortege“ and of a „sequence“.
3) of the product of two spaces.
4) of the „projection“ and the „skeleton“ of a set.

1) Suppose that we have a set $\Xi = \{\xi\}$ of things of any nature and that to every $\xi \epsilon \Xi$ corresponds another thing $E_\xi$ (for instance $\Xi$ may be the set of natural numbers and $E_\xi$ sets of points). Then we shall say that the things $E_\xi$ constitute a *system of things depending of the set $\Xi$ of arguments*, the elements of $\Xi$ we shall call arguments and we shall denote the system $\{E_\xi\}$ or $\{E_\xi\}_{\xi \epsilon \Xi}$. We shall say that two elements $E_{\xi_1}$ and $E_{\xi_2}$ of a system are equal if $E_{\xi_1} = E_{\xi_2}$ and that they coincide if $\xi_1 = \xi_2$.



A system $\{E_\xi\}_{\xi \epsilon \Xi}$ will be called a *subsystem* of another system $\{H_\xi\}_{\xi \epsilon \Theta}$ if

a) $\Xi \subset \Theta$,

b) for any $\xi \epsilon \Xi$ we have: $E_\xi = H_\xi$.

2) **Definition** $\alpha$. A *sequence* is a system of things depending of a non-finite set of natural numbers.

Unless the reverse is mentioned or follows from the context we shall always suppose that the set of arguments of a sequence is the set of *all* the natural numbers.

**Definition** $\beta$. A *cortege* is a finite system, the set of arguments for which is the set of all the natural numbers $\leqslant k$. Here $k$ is called the *rang* of the cortege. We shall denote a cortege simply $(\xi_1, \xi_2, \dots, \xi_k)$.

The subsystems of a cortege which are themselves corteges are called the *segments* of a cortege [1].

If a cortege $\mathfrak{w}' = (\xi_1, \xi_2, \dots, \xi_{k'})$ is a segment of another cortege $\mathfrak{w} = (\xi_1, \xi_2, \dots, \xi_k)$ then we shall denote sometimes $\mathfrak{w} = (\mathfrak{w}', \xi_{k'+1}, \dots, \xi_k)$.

**Definition** $\gamma$. A cortege of the *first order* is a cortege of natural numbers; a cortege of the *second order* (a *double cortege*) is a cortege of corteges of the first order and so on, so that a cortege of order $n$ is a cortege of corteges of the order $(n-1)$.

We shall denote a double cortege as follows:

$$(n_1^{(1)}, n_2^{(1)}, \dots, n_{s_1}^{(1)}; \ n_1^{(2)}, n_2^{(2)}, \dots, n_{s_2}^{(2)}; \dots; \ n_1^{(k)}, n_2^{(k)}, \dots, n_{s_k}^{(k)})$$

dividing the corteges of the first order by half columns and the elements of a cortege of the first order by comas.

Unless the reverse is mentioned or follows from the context, the word „cortege“ shall always mean a cortege of natural numbers.

3) **Definition** $\delta$. Let $E$ and $E^{(1)}$ be two different or identical sets. Then we shall call their *product* and de-

[1]) If e. g. we take the cortege $(n_1, n_2, n_3)$ then $(n_1)$ and $(n_1, n_2)$ are segments of $(n_1, n_2, n_3)$. But the subsystems such as $(n_1, n_3)$ or $(n_2)$ which are not corteges shall not be called segments of $(n_1, n_2, n_3)$.

note $E \times E^{(1)}$ the set $E^{(0)}$ of all the „ordered pairs" $(x, y)$ where

$$x \in E; \quad y \in E^{(1)}.$$

**Definition** $\varepsilon$. Let $R$ and $R^{(1)}$ be topological spaces (different or identical). Their *product* $R^{(0)} = R \times R^{(1)}$ is then also a topological space. Its points are ordered pairs $(x, y)$ where $x \in R$, $y \in R^{(1)}$ and the neighbourhoods $V^{(0)}_{(x,y)}$ of the point $(x, y)$ are the sets $V_x \times V^{(1)}_y$, where $V_x$ and $V^{(1)}_y$ are neighbourhoods of $x$ and $y$ in $R$ and $R^{(1)}$ respectively.

It may be easily verified that these neighbourhoods satisfy all the four axioms of Mr. Hausdorff [1]).

From this definition immediately follows:

1) that if a point $(x, y)$ is an accumulation point (or a condensation point) of a set of points $\{x_\xi, y_\xi\}$ then $x$ is an accumulation point (resp. a condensation point) of $\{x_\xi\}$ and $y$ is an accumulation point (a condensation point) of $\{y_\xi\}$.

2) that if $R$ and $R^{(1)}$ are metric spaces then $R^{(0)}$ is also a metric space [2]): the distance $\varrho^{(0)}(z_1, z_2)$ of two points $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ may be defined. e. g., as equal to

$$\sqrt{[\varrho(x_1, x_2)]^2 + [\varrho^{(1)}(y_1, y_2)]^2}$$

where $\varrho$ and $\varrho^{(1)}$ are the distances in $R$ and $R^{(1)}$ respectively.

3) that if $E$ and $E^{(1)}$ are open (or closed) in resp. $R$ and $R^{(1)}$ then $E^{(0)} = E \times E^{(1)}$ is open (closed) in $R^{(0)}$. In fact if $E$ and $E^{(1)}$ are closed and if $(x, y)$ is a point of accumulation of $E^{(0)}$ then, as we have seen, $x$ is a point of accumulation of $E$ and $y$ is a point of accumulation of $E^{(0)}$ so that, $E$ and $E^{(1)}$ being closed, $x \in E$ and $y \in E^{(1)}$ whence $(x, y) \in E \times E^{(1)} = E^{(0)}$ q. e. d.

If $E$ and $E^{(1)}$ are open then

$$E = \sum_{x \in E} V_x; \quad E^{(1)} = \sum_{y \in E^{(1)}} V^{(1)}_y$$

[1]) See Hausdorff, p. 228.

[2]) We do not make distinction between „les espaces métriques" and „les espaces metrisables".



where $V$ and $V^{(1)}_y$ are certain neighbourhoods of $x$ and $y$ in resp. $R$ and $R$ [1]). It follows that

$$E^{(0)} = \sum_{(x,y) \in E^{(0)}} V_x \times V^{(1)}_y$$

i. e. $E^{(0)}$ is the sum of open sets and is thus open q. e. d.

4) We shall give now the definition of the „projection" and of the „skeleton" of a set. These notions may be considered as fundamental in the second part of this work (Ch. IV—V).

**Definition** $\zeta$. Let $E^{(0)} \subset R^{(0)} = R \times R^{(1)}$; then the *projection of* $E^{(0)}$ *on* $R$ ($Pr\, E^{(0)}$ *or* $Pr_R E^{(0)}$) is the set of all the points $x \in R$, for which there exists a point $y \in R^{(1)}$ such that $(x, y) \in E^{(0)}$ (we may write

$$Pr_R E^{(0)} = \sum_{y \in R^{(1)}} \mathcal{E}_x((x, y) \in E^{(0)}).$$

**Definition** $\eta$. Let $E^{(0)} \subset R^{(0)}$ and $E^{(1)} \subset R^{(1)}$; then the *skeleton of* $E^{(0)}$ *on* $R$ *relatively* $E^{(1)}$ ($Sk\, E^{(0)}$ rel. $E^{(1)}$ or $Sk_R E^{(0)}$ rel. $E^{(1)}$) is the set of all the points $x \in R$ such that for any $y \in E^{(1)}$ we have: $(x, y) \in E^{(0)}$ (we may write

$$Sk_R E^{(0)} \text{ rel. } E^{(1)} = \prod_{y \in E^{(1)}} \mathcal{E}_x((x, y) \in E^{(0)}).$$

Evidently

(*) $$Sk\, E^{(0)} \text{ rel. } E^{(1)} = C(Pr(R \times E^{(1)} - E^{(0)})).$$

We can also define the projection and the skeleton as follows:

**Definition** $\zeta_1$. The projections of $E^{(0)}$ on $R$ and $R^{(1)}$ are the least sets $E \subset R$ and $E^{(1)} \subset R^{(1)}$ such that

$$E \times E^{(1)} \supset E^{(0)}$$

**Definition** $\eta_1$. The skeleton of $E^{(0)}$ on $R$ relatively $E^{(1)}$ is the greatest set $\check{E} \subset R$ such that

$$\check{E} \times \check{E}^{(1)} \subset E^{(0)}..$$

It is easily seen that these definitions have sense (i. e. that the least sets $E$ and $E^{(1)}$ and the greatest set $E$ exist) and that they are equivalent with the definitions $\zeta$ and $\eta$.

We shall prove now the following

**Theorem.** (1.a) *The projection of an open set is open and* (1 b) *the skeleton of a closed set relatively an arbitrary one is closed*; (2) *if $R$ is a compact (i. e. absolutely closed) space then* (2a) *the projection of a closed set is closed and* (2b) *the skeleton of an open set relatively a closed set is open.*

The relations (1a) and (2a) are almost evident and it can be easily deduced from (*) (taking into account that $E^{(1)}$ may be considered itself as a topological space) that (1a) implies (1b), and (2a) implies (2b).

## CHAPTER I.

## The General Theory of the $\delta s$-Operations of Mr. Hausdorff.

### § 1. The Definition and Nature of the $\delta s$-Operations.

**1.** The theory of sets offers many examples of operations effected upon sets.

Such are, e. g., the operations

$$\Sigma E_\xi,\quad \Pi E_\xi,\quad E_1 - E_2,\quad \Sigma \overline{E}_\xi,\quad E_1\overline{E}_2 + \overline{E}_1 E_2,$$

Some of them (such as $\Sigma E_\xi$, $\Pi E_\xi$, $E_1 - E_2$) possess the following property: in order to know whether a given point belongs or not to the result of the operation we must only know to which of the arguments it belongs. Such operations we shall call „analytical".

On the other hand such operations as:

$$\Sigma \overline{E}_\xi,\quad E_1\overline{E}_2 + \overline{E}_1 E_2$$

evidently are not analytical.

We may give now the following definitions:

**Definition 1.** A function $\Phi(\{E_\xi\})$ having sets [1]) for its arguments and for its value, is an *analytical function* if it satisfies the following condition:

[1]) Subsets of $R$.



(†) Whatever be two systems of arguments $\{E_\xi^{(1)}\}$ and $\{E_\xi^{(2)}\}$ and two points $a$ and $b$ ($a \in R$, $b \in R$), if

$$(1,1)\qquad \begin{gathered} a \in \Phi(\{E_\xi^{(1)}\}) \\ b \text{ non} \in \Phi(\{E_\xi^{(2)}\}) \end{gathered}$$

then there exists at least one $\xi$ such that either

$$(1,2)\qquad a \in E_\xi^{(1)};\quad b \text{ non} \in E_\xi^{(2)}$$

or

$$(1,2\text{bis})\qquad a \text{ non} \in E_\xi^{(1)},\quad b \in E_\xi^{(2)}.$$

**Definition 2.** A non constant [1]) function $\Phi(\{E_\xi\})$ having sets for its arguments and its value, is a *positive analytical function* if it satisfies the following condition:

(††) Whatever be two systems of arguments $\{E_\xi^{(1)}\}$ and $\{E_\xi^{(2)}\}$ and two points $a$ and $b$, (1,1) implies (1,2).

Evidently a positive analytical function is an analytical function.

**Remark.** *Every operation consisting only of additions and intersections is a positive analytical operation.*

**2.** In order to give a uniform expression to all the positive analytical operations we shall introduce the so called $\delta s$-operations of Mr. Hausdorff [2]).

**Definition 3.** Let $\mathfrak{N}$ be a set of sequences $\nu = (n_1, n_2, \ldots)$ of positive integers. Then the expression

$$(2,1)\qquad \Phi(E_1, E_2, \ldots) = \sum_{(n_1, n_2, \ldots) \in \mathfrak{N}} \prod_i E_{n_i}$$

is a $\delta s$-function of $E_1, E_2, \ldots$

*The $\delta s$-operations are positive analytical operations.*

#### Examples of $\delta s$-functions.

1) The *intersection* (common part) of a countable infinity of sets $E_1, E_2, \ldots$

$$E = E_1 \cdot E_2 \ldots = \prod_i E_i$$

[1]) The only functions satysfying (††) which reduce to a constant are the functions: $\Phi(\{E_n\}) = R$ and $\Phi(\{E_n\}) = 0$; we shall not consider them positive analytical functions.

[2]) Hausdorff, p. 89; Kolmogoroff, p. 410.

is a $\delta s$-funktion of $E_1, E_2, \ldots$ Here $\mathfrak{N}$ consists of the single sequence $(1, 2, \ldots)$

2) The *sum* of a countable infinity of sets

$$E = E_1 + E_2 + \ldots = \sum_i E_i$$

is also a $\delta s$-function.

Here $\mathfrak{N}$ consists of the sequences $(1, 1, 1, \ldots)$, $(2, 2, 2, \ldots) \ldots$

3) The *lower limit* of a countable infinity of sets $E_1, E_2, \ldots$

$$E = \sum_{k=1}^{\infty} \prod_{n=k}^{\infty} E_n$$

is a $\delta s$-function and $\mathfrak{N}$ is here the set of sequences:

$$(1, 2, 3, \ldots), \quad (2, 3, 4, \ldots), \quad (3, 4, 5, \ldots), \ldots$$

4) It may be easily seen that the *upper limit*

$$E = \prod_{k=1}^{\infty} \sum_{n=k}^{\infty} E_n$$

(though the operation is written in a different form) is a $\delta s$-function and $\mathfrak{N}$ consists in this case of all the sequences which contain an infinity of *different* integers.

3. To every sequence $\nu = (n_1, n_2, \ldots)$ of positive integers corresponds an irrationnal number

$$x = \frac{1|}{|n_1} + \frac{1|}{|n_2} + \cdots$$

and *vice versa*. We shall write $\nu \sim x$ and $x \sim \nu$.

In the same manner to every set $\mathfrak{N}$ of sequences of positive integers corresponds a set $N \sim \mathfrak{N}$ of irrational numbers viz. the set of all the numbers corresponding to the sequences belonging to $\mathfrak{N}$.

Now if $N \sim \mathfrak{N}$, we shall denote the $\delta s$-function (2,1) as follows:

$$(3,1) \qquad \sum_{(n_1, n_2, \ldots) \in \mathfrak{N}} \prod_i E_{n_i} = \Phi_N(\{E_n\}) = \underset{n}{\Phi_N}(\{E_n\}) \text{ [1]}.$$

Thus the $\delta s$-function $\Phi_N$ depends of a set $N$ of irrational numbers. *This set $N$ we shall call the base of the $\delta s$-function.*

[1]) The letter $n$ (under $\Phi$) will be omitted if no misunderstanding is possible. This letter is necessary in such cases as: $\underset{n}{\Phi_N}(\{E_n^k\})$ which means $\Phi_N(\{E_1^k, E_2^k, \ldots\})$ (the result depending of $k$) or $\underset{n}{\Phi_N}(\{E_{\varphi(n)}\})$ which means $\Phi_N(E_{\varphi(1)}, E_{\varphi(2)}, \ldots)$.



**Remark.** *We have evidently (for any class $\{N_\xi\}$ of bases)*

$$(3,2) \qquad \Phi_{\sum_\xi N_\xi}(\{E_n\}) = \sum_\xi \Phi_{N_\xi}(\{E_n\})$$

*and also*

$$(3,3) \qquad \begin{cases} \Phi_N(\{E_n \cdot P\}) = \Phi_N(\{E_n\}) \cdot P \\ \Phi_N(\{E_n + P\}) = \Phi_N(\{E_n\}) + P \\ \Phi_N(\{E_n \times P\}) = \Phi_N(\{E_n\}) \times P. \end{cases}$$

4. Let $\mathscr{H}_1, \mathscr{H}_2, \ldots$ be classes of sets and consider the class $\mathfrak{K}$ of all the sets which can be represented as

$$(4,1) \qquad \Phi_N(\{E_n\}); \quad E_n \in \mathscr{H}_n \ldots$$

This class depends of the classes $\{\mathscr{H}_n\}$ and of the set $N$.

We shall denote it

$$(4,2) \qquad \mathfrak{K} = \mathscr{H}_N(\mathscr{H}_1, \mathscr{H}_2, \ldots) = \mathscr{H}_N(\{\mathscr{H}_n\}).$$

We have thus the following

**Definition 4°.** $\mathscr{H}_N(\{\mathscr{H}_n\})$ is the class of all the sets of the form $\Phi_N(\{E_n\})$, where $E_n \in \mathscr{H}_n$ for any $n$.

The most important case is when

$$\mathscr{H}_1 = \mathscr{H}_2 = \ldots = \mathscr{H}.$$

We shall write then simply

$$(4.3) \qquad \mathfrak{K} = \mathscr{H}_N(\mathscr{H})$$

instead of $\mathscr{H}_N(\mathscr{H}, \mathscr{H}, \ldots)$ or $\mathscr{H}_N(\{\mathscr{H}\})$.

Thus $\mathscr{H}_N$ is an operation effected upon classes of sets. This operation, like $\Phi_N$, depends of a set $N$ of irrational numbers which we shall call here also *the base* of the $\mathscr{H}_N$-operation.

More generally we may define for every analytical operation $\Phi$ the corresponding operation $\mathscr{H}$ as follows:

**Definition 4.** *An operation $\mathscr{H}(\mathscr{H})$ whose argument and result are classes of sets, corresponds to the operation $\Phi(\{E_\xi\})$ effected upon systems of sets if $\mathscr{H}(\mathscr{H})$ is the class of all the sets of the form $\Phi(\{E_\xi\})$ where all the $E_\xi \in \mathscr{H}$.*

5. We shall introduce now a class $\mathfrak{D}$ of open (in $J$) sets of irrational numbers. This class will be useful many times in this work. We shall define it as follows:

**Definition 5.** $\mathfrak{D}=\{D_k\}$ is the class of all the sets $D_k$ where

(5,1) $$D_k=\delta_k+\sum_{(n_1,n_2,\dots,n_j)}\delta_{n_1,n_2,\dots,n_j,k}$$

or in other words, $D_k$ is the set of all the irrational numbers

$$x=\frac{1|}{|n_1}+\frac{1|}{|n_2}+\cdots$$

which contain among their incomplete quotients:

$$n_1,\ n_2,\dots$$

at least one equal to $k$.

6. **Definition 6.** Two analytical operations $\Psi^{(1)}$ and $\Psi^{(2)}$ are *equivalent* if the corresponding operations upon classes $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ coincide, i. e. if for any $\mathcal{H}$

$$\mathcal{H}^{(1)}(\mathcal{H})=\mathcal{H}^{(2)}(\mathcal{H}).$$

We shall need also the following definition:

**Definition 6bis.** Two analytical operations $\Psi^{(1)}$ and $\Psi^{(2)}$ are *r-equivalent in respect of* $\mathfrak{R}$, $\mathfrak{R}$ being a family of classes of sets, if for any $\mathcal{H}\in\mathfrak{R}$

$$\mathcal{H}^{(1)}(\mathcal{H})=\mathcal{H}^{(2)}(\mathcal{H}).$$

**Lemma 1.** *Every positive analytical operation $\Phi(\{E_\xi\})$ defined for any system of $\{E_\xi\}$ satisfying the condition*

(6,1) $$\Sigma E_\xi\subset R$$

*can be extended over all possible systems of $\{E_\xi\}$, i. e. there exists a positive analytical operation $\Phi_1(\{E_\xi\})$ defined for any $\{E_\xi\}$ and coinciding with $\Phi(\{E_\xi\})$ if condition* (6,1) *is fulfilled.*

Proof. We shall define $\Phi_1(\{E_\xi\})$ as follows:

$$x\in\Phi_1(\{E_\xi\})$$

if (and only if) there exists a system $\{E_\xi^{(1)}\}$ of subsets of $R$ and a point

$$a\in\Phi(\{E_\xi^{(1)}\})$$

such that for any $\xi$

$$a\in E_\xi^{(1)}\quad\text{implies}\quad x\in E_\xi.$$

One sees immediately that $\Phi_1(\{E_\xi\})$ satisfies all our conditions q. e. d.

**Remark.** In future a „positive analytical operation“ shall always mean this „extended“ operation.

**Lemma 2.** *To every (even non positive) analytical operation $\Psi(\{E_\xi\})$ defined in a certain space $R_0$, corresponds an operation $\Psi^R(\{E_\xi\})$ depending of the space $R$ and possessing the following properties:*

1) $\Psi^{R_0}(\{E_\xi\})=\Psi(\{E_\xi\})$
2) *for any $R_1$ and $R_2$ such that $R_1\subset R_2$ and for a system $\{E_\xi\}$ of sets contained in $R_2$:*

$$\Psi^{R_1}(\{R_1\,E_\xi\})=R_1\cdot\Psi^{R_2}(\{E_\xi\})$$

3) *for any $R$, $\Psi^R$ is an analytical operation.*

Proof. Take a point $x_0\in R_0$ (the case when $R_0=0$ is evident). Let $\{E_\xi\}$ be a system of sets contained in $R$. Then we shall say that a certain point $x\in R$ belongs to $\Psi^R(\{E_\xi\})$ if (and only if) the point $x_0$ belongs to $\Psi(\{E_\xi^{(1)}\})$ where

$$E_\xi^{(1)}=(x_0)\quad\text{if}\quad x\in E_\xi\quad\text{and}\quad E_\xi^{(1)}=0\quad\text{if}\quad x\text{ non}\in E_\xi.$$

The operation $\Psi^R(\{E_\xi\})$ satisfies all the conditions 1—3. 1) and 3) follow immediately from the definition of an analytical operation. To prove 2) suppose that $x\in\Psi^{R_1}(E_\xi\cdot R_1\})$. Then first of all evidently $x\in R_1$; besides $x_0\in\Psi(\{E_\xi^{(1)}\})$ where $\{E_\xi^{(1)}\}$ are the same as defined above. Hence follows from the definition of $\Psi^R(\{E_\xi\})$ that $x\in\Psi^{R_2}(\{E_\xi\})$. If on the other hand $x\in R_1\,\Psi^{R_2}(\{E_\xi\})$ then $x_0\in\Psi(\{E_\xi^{(1)}\})$; but as $x\in R_1$ we have $x\in R_1\,E_\xi$ if and only if $x\in E_\xi$.
In other words the sets $E_\xi^{(1)}$ for $R_1\cdot E_\xi$ are the same as for $E_\xi$, whence $x\in\Psi^{R_1}(\{R_1\cdot E_\xi\})$ q. e. d.

**Corollary.** *Every analytical operation $\Psi(\{E_\xi\})$ defined in the space $R$ and possessing the following property: $\Psi(\{E_\xi\})\subset\sum_\xi E_\xi$ may be extended over all the systems of sets $\{E_\xi\}$ in the same sense as in lemma 1.*

**Remark.** We shall denote sometimes this operation simply $\Psi$ instead of $\Psi^R$ (see e. g. Theorem XV).

**Theorem I.** *Every positive analytical operation $\Phi$ effected upon a sequence of sets is a $\delta s$-function of these sets. Its base $N$ is equal to $\Phi(\{D_n\})$,* (*see definition* 5)

(6,2) $$N=\Phi(\{D_n\}).$$

P r o o f. We must prove that whatever be $\{E_n\}$ we have

(6,3) $$\mathcal{E}\equiv\Phi(\{E_n\})=\Phi_N(\{E_n\})\equiv\mathcal{E}'$$

$\alpha$) $\mathcal{E}\subset\mathcal{E}'$. Let
$$a\in\Phi(\{E_n\})$$
and denote $n_1, n_2, \dots$ the sequence of *all* such numbers $n$ that $a\in E_n$ ($n_1, n_2, \dots$ must not necessarily be all different). If now
$$x=\frac{1|}{|n_1}+\frac{1|}{|n_2}+\dots$$
then $x\in D_{n_i}$ for any $i$

Consequently $a\in E_n$ implies $x\in D_n$ and therefore according to condition (††) (see definition 2)
$$x\in\Phi(\{D_n\})=N$$
whence $a\in\prod_i E_{n_i}\subset\Phi_N(\{E_n\})$ q. e. d.

$\beta$) $\mathcal{E}'\subset\mathcal{E}$. Let
$$a\in\Phi_N(\{E_n\}).$$

Then there exists such
$$x=\frac{1|}{|n_1}+\frac{1|}{|n_2}+\dots\in N=\Phi(\{D_n\})$$
that
$$a\in E_{n_1}\cdot E_{n_2}\dots$$

Hence $x\in D_n$ implies $a\in E_n$ and therefore according to (††)
$$a\in\Phi(\{E_n\}) \qquad \text{q. e. d.}$$

We have proved the relation (6,3) and consequently theorem I.

**Corollary.** *Every positive analytical operation $\Psi$ effected upon a countable system of sets $\{E_\mu\}$ is equivalent with a $\delta s$-function.*

In fact the set $\{\mu\}$ of all the indices is countable. Enumerate them in any way $\mu_1, \mu_2, \dots$ and let $H_i=E_{\mu_i}$.



Then $\Phi(\{H_i\})=\Psi(\{E_{\mu_i}\})$ is a positive analytical operation effected upon a sequence of sets. It is therefore by our theorem a $\delta s$-function. Evidently it is equivalent with $\Psi$ q. e. d.

**Theorem II.** *Every analytical operation $\Phi(\{E_n\})$ effected upon a countable infinity of sets is equivalent with a $\delta s$-function of these sets and their complements.*

P r o o f. Consider the auxiliary operation

(6,4) $$\Psi(\{H_i\})=\prod_{k=1}^{\infty}(H_{2k-1}+H_{2k})\,\Phi(\{H_{2j-1}\})+\sum_{k=1}^{\infty}H_{2k-1}\cdot H_{2k}.$$

Evidently

(6,5) $$\Psi(E_1, C(E_1), E_2, C(E_2),\dots)=\Phi(E_1, E_2,\dots)$$

If we prove that $\Psi(\{H_i\})$ is a positive analytical operation then theorem II will follow from theorem I and (6,5).

So we have only to show that if $\{H_i^{(1)}\}$ and $\{H_i^{(2)}\}$ are two systems of sets, and $a$ and $b$ two points such that

(6,6) $$a\in\Psi(\{H_i^{(1)}\})$$

(6,7) $$b \text{ non}\in\Psi(\{H_i^{(2)}\})$$

then there exists such index $i_0$ that

(6,8) $$a\in H_{i_0}^{(1)};\quad b \text{ non}\in H_{i_0}^{(2)}.$$

We shall distinguisch three cases, viz:

(6,9a) $$1)\quad a\in\sum_{k=1}^{\infty}H_{2k-1}^{(1)}H_{2k}^{(1)}$$

(6,9b) $$2)\quad a \text{ non}\in\sum_{k=1}^{\infty}H_{2k-1}^{(1)}H_{2k}^{(1)}$$

(6,9c) $$b \text{ non}\in\prod_{k=1}^{\infty}(H_{2k-1}^{(2)}+H_{2k}^{(2)})$$

(6,9b) $$3)\quad a \text{ non}\in\sum_{k=1}^{\infty}H_{2k-1}^{(1)}\cdot H_{2k}^{(1)}$$

(6,9d) $$b\in\prod_{k=1}^{\infty}(H_{2k-1}^{(2)}+H_{2k}^{(2)}).$$

*First case.* From (6,7) and (6,4) follows that

$$(6,10)\qquad b \text{ non } \epsilon \sum_{k=1}^{\infty} H_{2k-1}^{(2)} H_{2k}^{(2)}.$$

From (6,9a) and (6,10) follows easily (6,8) (for a certain $i_0$) q. e. d.

*Second case.* From (6,9b) and (6,6), (6,4) follows that

$$(6,11)\qquad a \,\epsilon \prod_{k=1}^{\infty} (H_{2k-1}^{(1)} + H_{2k}^{(1)}).$$

From (6,9c) and (6,11) follows easily (6,8) (for a certain $i_0$) q. e. d.

*Third case.* From (6,7) and (6,4) follows (6,10).

From (6,9b) and (6,6), (6,4) follows that

$$(6,12)\qquad a \,\epsilon\, \Phi(\{H_{2j-1}^{(1)}\}).$$

From (6,9d) and (6,7), (6,4) follows that

$$(6,13)\qquad b \text{ non } \epsilon\, \Phi(\{H_{2j-1}^{(2)}\}).$$

From (6,12), (6,13) and condition (†) (see the definition of analytical operations, definition 1) follows that for a certain $j_0$ either

$$(6,14a)\qquad \alpha)\quad a \,\epsilon\, H_{2j_0-1}^{(1)};\quad b \text{ non } \epsilon\, H_{2j_0-1}^{(2)}$$

or

$$(6,14b)\qquad \beta)\quad a \text{ non } \epsilon\, H_{2j_0-1}^{(1)};\quad b \,\epsilon\, H_{2j_0-1}^{(2)}.$$

In case $\alpha$) the condition (6,8) is fulfilled for $i_0 = 2j_0 - 1$.

In case $\beta$) we have

$$(6,15)\qquad a \,\epsilon\, H_{2j_0}^{(1)}$$

because by (6,6), (6,4) and (6,9b) we have for any $j$

$$(6,16)\qquad a \,\epsilon\, H_{2j-1}^{(1)} + H_{2j}^{(1)}.$$

On the other hand from (6,10) and (6,14b) follows

$$(6,17)\qquad b \text{ non } \epsilon\, H_{2j_0}^{(2)}$$

It is sufficient to approach (6,15) and (6,17) to see that in case $\beta$ (6,8) is fulfilled for $i_0 = 2j_0$.



Thus in all cases the condition (6,8) is fulfilled for a certain value of $i_0$ q. e. d.

**7.** A simple example shows than we may have

$$(7,1)\qquad \Phi_{N_1}(\{E_n\}) = \Phi_{N_2}(\{E_n\})$$

for any $\{E_n\}$ without having $N_1 = N_2$.

Let $N_1$ be the set of all the numbers $\xi_i = \frac{1}{|i} + \frac{1}{|i} + \dots$ and let further $N_2 = J$. Then evidently

$$\Phi_{N_1}(\{E_n\}) = \Phi_{N_2}(\{E_n\}) = \sum_n E_n.$$

**Definition 7.** Two sets $N_1$, and $N_2$ of irrational numbers are *equivalent as bases* if we have whatever be $\{E_n\}$

$$(7,1)\qquad \Phi_{N_1}(\{E_n\}) = \Phi_{N_2}(\{E_n\})$$

This definition is equivalent with the following

**Definition 7$_1$.** $N_1$ and $N_2$ are equivalent as bases if whatever be $\{\mathscr{H}_n\}$

$$(7,2)\qquad \mathscr{H}_{N_1}(\{\mathscr{H}_n\}) = \mathscr{H}_{N_2}(\{\mathscr{H}_n\}).$$

It is natural to ask what are the necessary and sufficient conditions of the equivalence of two sets or of two $\delta s$--operations.

We shall begin with the following

**Definition 8.** *The completed form* $\tilde{N}$ of a set $N$ of irrational numbers is the set of all the numbers

$$\xi = \frac{1}{|n_1} + \frac{1}{|n_2} + \frac{1}{|n_3} + \dots$$

for which there exists a sequence $(i_1, i_2, \dots)$ of positive integers, such that

$$(7,3)\qquad \frac{1}{|n_{i_1}} + \frac{1}{|n_{i_2}} + \dots \,\epsilon\, N$$

($i_1, i_2, \dots$ must not be necessarily all different).

We shall denote the „completed form" of a base by the sign $\sim$.

In the same manner can be defined the completed form $\widetilde{\mathfrak{N}}$ of a set $\mathfrak{N}$ of sequences of positive integers.

Evidently $N \sim \mathfrak{N}$ implies $\widetilde{N} \sim \widetilde{\mathfrak{N}}$. A set coinciding with its completed form is called *complete.*

In order to build $\widetilde{\mathfrak{N}}$ we must add to $\mathfrak{N}$ all the sequences which are formed by changing the order of elements (natural numbers) and inserting new elements into sequences belonging to $\mathfrak{N}$.

**Definition 9.** The set $\mathfrak{N}'$ (and $N' \sim \mathfrak{N}'$) of all the sequences $(n_1, n_2, \ldots)$ which: 1) belong to $\widetilde{\mathfrak{N}}$; 2) have either $n_1 < n_2 < \ldots < n_k < \ldots$ or $n_1 < n_2 < \ldots < n_k = n_{k+1} = \ldots$ shall be called the *reduced completed form* of $\mathfrak{N}$ (resp. of $N$).

The completed and the reduced completed form of $N$ are *its canonical forms.*

It is lightly seen that we have for any $N$ and $\{E_n\}$

(7,4) $$\Phi_N(\{E_n\}) = \Phi_{\widetilde{N}}(\{E_n\}) = \Phi_{N'}(\{E_n\}).$$

The set $\widetilde{N}$ admits the following analytical expression

(7,5) $$\widetilde{N} = \Phi_N(\{D_n\}).$$

In fact if $x = \frac{1|}{|m_1} + \frac{1|}{|m_2} + \ldots \epsilon \widetilde{N}$ then there exists such sequence $(i_1, i_2, \ldots)$ that $\frac{1|}{|m_{i_1}} + \frac{1|}{|m_{i_2}} + \ldots \epsilon N$.

Therefore

(7,6) $$x \epsilon D_{m_{i_1}} \cdot D_{m_{i_2}} \ldots \subset \sum_{(n_1, n_2, \ldots) \epsilon \mathfrak{N}} \prod_k D_{n_k} = \Phi_N(\{D_n\}).$$

If, on the other hand

$$x = \frac{1|}{|m_1} + \frac{1|}{|m_2} + \ldots \epsilon \Phi_N(\{D_n\})$$

then there exists a number

$$\xi = \frac{1|}{|n_1} + \frac{1|}{|n_2} + \ldots \epsilon N$$

such that $x \epsilon D_{n_1} \cdot D_{n_2} \ldots$ and therefore all the numbers $n_1, n_2 \ldots$ are found among the numbers $m_1, m_2, \ldots$ Consequently there exists a sequence $i_1, i_2, \ldots$ such that $m_{i_k} = n_k$ for any $k$ whence $x \epsilon \widetilde{N}$ q. e. d.



8. We can answer now the questions put in the preceding article.

**Theorem III.** *In order that for any sets $E_1, E_2, \ldots$ be*

$$\Phi_{N_1}(\{E_n\}) \subset \Phi_{N_2}(\{E_n\})$$

*it is necessary and sufficient that*

(8,1) $$N_1 \subset \widetilde{N}_2.$$

Proof. The sufficiency of the condition (8,1) follows immediately from (7,4) and (3,2) because we can write instead of (8,1):

$$N_2 = N_1 + (\widetilde{N}_2 - N_1).$$

The necessity of the condition (8,1) follows from (7,5) because

$$N_1 \subset \widetilde{N}_1 = \Phi_{N_1}(\{D_n\}) \subset \Phi_{N_2}(\{D_n\}) = \widetilde{N}_2.$$

**Corollary.** (First Equivalence Theorem). *Two sets of irrational numbers $N_1$ and $N_2$ are equivalent as bases if, and only if, their completed forms are identical, i. e. if*

$$\widetilde{N}_1 = \widetilde{N}_2.$$

**Theorem IV.** *In order that we have for any $\mathscr{H}$*

(8,2) $$\mathscr{H}_{N_1}(\mathscr{H}) \subset \mathscr{H}_{N_2}(\mathscr{H})$$

*the following condition is necessary and sufficient.*

(*) *There exists a function $k(n)$ the value of which is a positive integer for any natural $n$, and a base $N_1'$ equivalent to $N_1 (\widetilde{N}_1' = \widetilde{N}_1)$ such that $x \epsilon N_1$ if, and only if, there exists such*

$$y = \frac{1|}{|n_1} + \frac{1|}{|n_2} + \ldots \epsilon N_2$$

*that*

$$x = \frac{1|}{|k(n_1)} + \frac{1|}{|k(n_2)} + \ldots$$

*We may express this condition otherwise by requiring that*

(8,3) $$\widetilde{N}_1' = \Phi_{N_2}(\{D_{k(n)}\}).$$

Proof. To prove the necessity of the condition (*) or, which is the same, (8,3) suppose in (8,2) $\mathscr{H} = \mathfrak{D}$ (Def. 5). Then we shall have

$$\widetilde{N}_1 = \Phi_{N_1}(\{D_n\}) \epsilon \mathscr{H}_{N_1}(\mathfrak{D}) \subset \mathscr{H}_{N_2}(\mathfrak{D})$$

whence
$$\tilde{N}_1 = \Phi_{N_2}(\{E_n\}); \quad E_n \in \mathfrak{D}$$
i. e. $E_n = D_m$, or denoting $m = k(n)$ we come to (8,3) q. e. d.

On the other hand let condition (*) be fulfilled and let
$$E \in \mathcal{H}_{N_1}(\mathcal{H})$$
i. e.
$$E = \Phi_{N_1}(\{E_n\}) = \Phi_{N_1'}(\{E_n\}); \quad E_n \in \mathcal{H}.$$
Then we have by (*)
$$E = \underset{n}{\Phi_{N_2}}(\{E_{k(n)}\}) \in \mathcal{H}_{N_2}(\mathcal{H}).$$
Consequently $\mathcal{H}_{N_1}(\mathcal{H}) \subset \mathcal{H}_{N_2}(\mathcal{H})$ q. e. d.

If now we add the condition (**) which can be obtained by substituting $N_1$ for $N_2$ and *vice versa* in the condition (*), we shall have the following

**Corollary.** (Second Equivalence Theorem). *Two $\delta s$-functions $\Phi_{N_1}$ and $\Phi_{N_2}$ are equivalent (see Def. 6) if, and only if, their bases $N_1$ and $N_2$ satisfy* (*) *and* (**).

9. We shall show here how a $\delta s$-operation may be reduced to the operations of additions and intersections of a countable infinity of sets („countable additions and intersections") of multiplications and of one projection, all these operations being effected upon the arguments and the base of our $\delta s$-function.

Let $\{E_n\}$ be a sequence of subsets of $R$. We shall construct a set $S(\{E_n\})$ which lies in $R \times J$ and which we shall call the scheme for $\{E_n\}$.

**Definition 10.** Let $\{E_n\}$ be a sequence of sets and denote

(9,1) $$S_k = \sum_{n_1, n_2, \ldots, n_k} E_{n_k} \times \delta_{n_1, n_2, \ldots, n_k} \text{ [1]}$$

where the summation is extended over all the corteges (i. e. finite systems of natural numbers) of rang $k$.

Then *the scheme* $S(\{E_n\})$ *for* $\{E_n\}$ is defined by the formula

[1] see notations (group C and D 3).



(9,2) $$S(\{E_n\}) = \prod_k S_k = \prod_k \sum_{n_1, n_2, \ldots, n_k} E_{n_k} \times \delta_{n_1, n_2, \ldots, n_k}.$$

**Remark.** *If $E_n$ are closed in the topological space $R$ then $S(\{E_n\})$ is closed in $R \times J$.*

In fact we have only to show that $S_k$ (9,1) are then closed. Let

(9,3) $$(x, y) \in \overline{S_k}$$

and let $V_x$ be a neighbourhood of $x$ in $R$ and
$$y = \frac{1|}{|n_1^{(0)}} + \frac{1|}{|n_2^{(0)}} + \cdots$$
Then, by (9,3) there exists a point
$$(x', y') \in S_k \cdot (V_x \times \delta_{n_1^{(0)}, n_2^{(0)}, \ldots, n_k^{(0)}}) =$$
$$= \sum_{n_1, n_2, \ldots, n_k} E_{n_k} \times \delta_{n_1, n_2, \ldots, n_k} \cdot (V_x \times \delta_{n_1^{(0)}, n_2^{(0)}, \ldots, n_k^{(0)}}).$$
Consequently
$$y' \in \delta_{n_1^{(0)}, n_2^{(0)}, \ldots, n_k^{(0)}}$$
and therefore
$$x' \in E_{n_k^{(0)}}$$
We have thus proved that in every neighbourhood $V_x$ of $x$ there exists a point $x' \in E_{n_k^{(0)}}$ whence $x \in \overline{E}_{n_k^{(0)}}$ and ($E_{n_k^{(0)}}$ being closed)
$$x \in E_{n_k^{(0)}}.$$
On the other hand
$$y \in \delta_{n_1^{(0)}, n_2^{(0)}, \ldots, n_k^{(0)}}$$
and therefore
$$(x, y) \in E_{n_k^{(0)}} \times \delta_{n_1^{(0)}, n_2^{(0)}, \ldots, n_k^{(0)}} \subset \sum_{n_1, n_2, \ldots, n_k} E_{n_k} \times \delta_{n_1, n_2, \ldots, n_k} = S_k \qquad \text{q. e. d.}$$

**Theorem V.** *For any $N$ and $\{E_n\}$ we have*

(9,4) $$\mathcal{E} = \Phi_N(\{E_n\}) = Pr_R(S(\{E_n\}) \cdot (R \times N)) = \mathcal{E}'$$

Proof. *a)* $\mathcal{E} \subset \mathcal{E}'$. Let $x \in \Phi_N(\{E_n\})$.

Then there exists such

$$y = \frac{1|}{|n_1^{(0)}} + \frac{1|}{|n_2^{(0)}} + \ldots \in N$$

that $x \in E_{n_k^{(0)}}$ for any $k$.

Hence

$$(x,y) \in \prod_k (E_{n_k^{(0)}} \times \delta_{n_1^{(0)}, n_2^{(0)}, \ldots, n_k^{(0)}}) \subset \prod_k \sum_{n_1, n_2, \ldots, n_k} (E_{n_k} \times \delta_{n_1, n_2, \ldots, n_k}) = S(\{E_n\}).$$

We have also

$$(x, y) \in R \times N$$

and therefore

$$(x, y) \in S(\{E_n\}) \cdot (R \times N)$$

or

$$x \in Pr(S(\{E_n\}) \cdot (R \times N)) \quad \text{q. e. d.}$$

$\beta$) $\mathcal{E}' \subset \mathcal{E}$. Let $x \in Pr(S(\{E_n\}) \cdot (R \times N))$.

Then there exists such

$$y = \frac{1|}{|n_1^{(0)}} + \frac{1|}{|n_2^{(0)}} + \ldots \in N$$

that

$$(x, y) \in S(\{E_n\}) = \prod_k S_k.$$

We have therefore

$$(x, y) \in S_k = \sum_{n_1, n_2, \ldots, n_k} E_{n_k} \times \delta_{n_1, n_2, \ldots, n_k}$$

for any $k$. But of all the $\delta_{n_1, n_2, \ldots, n_k}$, $y$ belongs only to $\delta_{n_1^{(0)}, n_2^{(0)}, \ldots, n_k^{(0)}}$; therefore

$$(x, y) \in E_{n_k^{(0)}} \times \delta_{n_1^{(0)}, n_2^{(0)}, \ldots, n_k^{(0)}}$$

for any $k$ and

$$x \in \prod_k E_{n_k^{(0)}} \subset \Phi_N(\{E_n\}) \quad \text{q. e. d.}$$

Theorem V is now wholly demonstrated.

10. We have considered in the preceding articles the $\delta s$-functions of Mr. Hausdorff. Here we shall generalize a little the



notions of an analytical operation and of a $\delta s$-function by introducing the „quasi analytical operations" and the „$\delta s$-functions with a variable base" ($v\delta s$-functions). These new functions will be found useful in § 3 of this Chapter.

**Definition 1'.** An operation $\Phi(\{E_\xi\})$ effected upon subsets of $R$ and having for its result also a subset of $R$, is called a *quasi analytical operation* if the following condition is satisfied:

($\dagger'$) Whatever be two systems of arguments $\{E_\xi^{(1)}\}$ and $\{E_\xi^{(2)}\}$ and a point $a \in R$, if

$$\text{(10,1)} \qquad \begin{aligned} &a \in \Phi(\{E_\xi^{(1)}\}) \\ &a \text{ non} \in \Phi(\{E_\xi^{(2)}\}) \end{aligned}$$

then there exists at least one $\xi$ such that either

$$\text{(10,2)} \qquad a \in E_\xi^{(1)}; \quad a \text{ non} \in E_\xi^{(2)}$$

or

$$\text{(10,2bis)} \qquad a \text{ non} \in E_\xi^{(1)}; \quad a \in E_\xi^{(2)}$$

**Definition 2'.** An operation $\Phi(\{E_\xi\})$ effected uppon subsets of $R$ is called a *positive quasi analytical operation* if the following conditions are satisfied:

1) ($\dagger\dagger'$) Whatever be two systems of arguments $\{E_\xi^{(1)}\}$ and $\{E_\xi^{(2)}\}$ and a point $a \in R$, (10,1) implies (10,2).

2) $\Phi(\{E\}) = E$.

**Definition 3'.** Let $N(x)$ be a system of sets of irrational numbers depending of $x \in R$ and let $\{E_n\}$ denote as usual the sequence of subsets of $R$. Then $\Phi_{N(x)}^{(x)}(\{E_n\})$ is the set of all the points $x_0 \in R$ which satisfy the relation

$$\text{(10,3)} \qquad x_0 \in \Phi_{N(x_0)}(\{E_n\})$$

or, in other words,

$$x_0 \in \Phi_{N(x)}^{(x)}(\{E_n\})$$

if, and only if, there exists such

$$\xi = \frac{1|}{|n_1} + \frac{1|}{|n_2} + \ldots \in N(x_0)$$

that

$$x_0 \in \prod_i E_{n_i}.$$

The operation $\Phi^{(x)}_{N(x)}$ shall be called a $\delta s$-*function with a variable base* (*a* $v\delta s$-*function*).

**Remark.** *We can write also that*

(10,4) $$\Phi^{(x)}_{N(x)}(\{E_n\}) = \sum_{x\in R} ((x)\cdot \Phi_{N(x)}(\{E_n\})).$$

**Definition 4'.** $\mathcal{H}^{(x)}_{N(x)}(\{\mathcal{H}_n\})$ is the class of all the sets of the form $\Phi^{(x)}_{N(x)}(\{E_n\})$ where $E_n \in \mathcal{H}_n$ for any $n$.

If $\mathcal{H}_1 = \mathcal{H}_2 = \dots = \mathcal{H}$ then we shall write simply $\mathcal{H}^{(x)}_{N(x)}(\mathcal{H})$ instead of $\mathcal{H}^{(x)}_{N(x)}(\{\mathcal{H}\})$.

**Definition 6'.** Two variable bases are equivalent if we have (whatever be $\{E_n\}$)

$$\Phi^{(x)}_{N_1(x)}(\{E_n\}) = \Phi^{(x)}_{N_2(x)}(\{E_n\}).$$

**Definition 8'.** By the *completed form* of a variable base $N(x)$ we shall understand the variable base $\widetilde{N}(x)$, i. e. the variable base which for every (constant) $x$ is the completed form of the corresponding $N(x)$

Evidently

(10,5) $$\Phi^{(x)}_{N(x)}(\{E_n\}) = \Phi^{(x)}_{\widetilde{N}(x)}(\{E_n\})$$

Many properties of these $v\delta s$-functions are quite analogical to those of the ordinary $\delta s$-functions. We have namely the following theorems:

**Theorem I'.** *Every positive quasi analytical operation effected upon a countable infinity of sets is a* $v\delta s$-*function of these sets.*

Proof. From def. 2' easily follows that:

$$(x)\cdot\Phi(\{E_n\}) = \Phi(\{(x)\cdot E_n\}).$$

But this last operation is evidently a positive analytical operation defined in the space consisting of the single point $x$. By Theorem I it is then a $\delta s$-operation.

Denote its base $N(x)$. We have thus

$$\Phi(\{(x)\cdot E_n\}) = \Phi_{N(x)}(\{(x)\cdot E_n\}).$$



Hence

$$\Phi(\{E_n\}) = \Big(\sum_{x\in R}(x)\Big)\,\Phi(\{E_n\}) = \sum_{x\in R}((x)\,\Phi(\{E_n\})) = \sum_{x\in R}\Phi(\{(x)\cdot E_n\}) =$$
$$= \sum_{x\in R}\Phi_{N(x)}(\{(x)\cdot E_n\}) = \sum_{x\in R}((x)\,\Phi_{N(x)}(\{E_n\})) = \Phi^{(x)}_{N(x)}(\{E_n\}) \quad [\text{by } (10,4)]$$

q. e. d.

On the other hand every $v\delta s$-function is evidently a positive quasi analytical operation.

**Theorem II'.** *Every quasi analytical operation* $\Phi(\{E_n\})$ *effected upon a countable infinity of sets is a* $v\delta s$-*function of these sets and their complements.*

The same proof as that of Theorem II. (We have only to substitute $a$ for $b$).

**Theorem III'.** *In order that for every system* $\{E_n\}$ *we have*

(10,6) $$\Phi^{(x)}_{N_1(x)}(\{E_n\}) \subset \Phi^{(x)}_{N_2(x)}(\{E_n\})$$

*it is necessary and sufficient that for every* $x$

(10,7) $$N_1(x) \subset \widetilde{N}_2(x).$$

Proof. The sufficiency of the condition (10,7) is an immediate consequence of (10,5) and of (3,2) which is true also for $v\delta s$-functions.

To prove the necessity suppose that for a certain $x = x_0$, $N_1(x_0)$ contains a point

$$z = \frac{1|}{|n_1} + \frac{1|}{|n_2} + \dots \text{ non } \in \widetilde{N}_2(x_0);$$

if we now take $E_{n_1} = E_{n_2} = \dots = R$ and $E_n = 0$ for $n \neq n_i$ then (as may be easily seen)

$$x_0 \in \Phi^{(x)}_{N_1(x)}(\{E_n\})$$

and

$$x_0 \text{ non } \in \Phi^{(x)}_{N_2(x)}(\{E_n\})$$

which contradicts (10,6).

**Corollary.** *In order that two variable bases be equivalent it is necessary and sufficient that their completed forms were identical i. e.*

(10,8) $$N_1(x) = N_2(x) \text{ for any } x \in R.$$

**Theorem V′.** *Denote* $N^*$ *the set of all points* $(x, y)$ *such that* $x \in R$, $y \in N(x)$ *i. e.*

$$N^* = \underset{x,y}{\mathcal{E}}(x \in R,\ y \in N(x)).$$

*Then*

$$\Phi^{(x)}_{N(x)}(\{E_n\}) = Pr_R(N^* S(\{E_n\})).$$

The same proof as that of Theorem V.

## § 2. The Base of Certain $\delta s$-Operations.

11. We have seen already (Th. V) that a $\delta s$-operation can be reduced to some simple operations (sum, common part, product, projection) effected upon its arguments and its base.

It is clear therefore that the knowledge of the structure of the base is essential for the knowledge of the $\delta s$-operation. In this paragraph we shall deal chiefly with two kinds of operations, viz. 1) Compound $\delta s$-operations and 2) complementary $\delta s$-operations. The definition of these operations will be given below.

12. Consider a countable infinity of $\delta s$-operations $\Phi_{N_0}, \Phi_{N_1}, \Phi_{N_2}, \ldots$ and let $\mathfrak{L}$ be a class of sets. Denote

$$\mathcal{H}_m = \mathcal{H}_{N_m}(\mathfrak{L}) \quad (m = 1, 2, \ldots)$$

$$\mathcal{H} = \mathcal{H}_{N_0}(\{\mathcal{H}_m\}).$$

Now if there exists such $N$ that

$$\mathcal{H} = \mathcal{H}_N(\mathfrak{L})$$

then we shall call $\Phi_N$ the compound $\delta s$-function.

We have thus the following

**Definition 11.** $\Phi_N$ *is the compound function for* $\Phi_{N_0}; \Phi_{N_1}, \Phi_{N_2}, \ldots$ if for any class $\mathfrak{L}$ of sets we have

$$\mathcal{H}_N(\mathfrak{L}) = \underset{m}{\mathcal{H}_{N_0}}(\{\mathcal{H}_{N_m}(\mathfrak{L})\}). \tag{12,1}$$

**Theorem VI**[1]. 1) *For any* $N_0; N_1, N_2, \ldots$ *there exists a base* $N$ *satisfying* (12,1)

[1] The first and the second part of this theorem are not new. They can be found in Kolmogoroff (p. 420) and some particular cases in Sierpiński IV and in Hausdorff. The third part can be found in our note in C. R.



2) *This base may be supposed to coincide with the set* $\check{N}$ *of such numbers*

$$x = \frac{1|}{|q_1} + \frac{1|}{|q_2} + \ldots,$$

*that there exist numbers*

$$y_0 = \frac{1|}{|m_1} + \frac{1|}{|m_2} + \ldots \in N_0;\ y_\mu = \frac{1|}{|n_1^{(\mu)}} + \frac{1|}{|n_2^{(\mu)}} + \ldots \in N_{m_\mu} (\mu = 1, 2, \ldots) \tag{12,2}$$

*such that*

$$q_{2^{j-1}(2i-1)} = 2^{m_j - 1}(2 n_i^{(j)} - 1). \tag{12,3}$$

3) *$N$ can be obtained from $N_0; N_1, N_2, \ldots$ and $\delta_{n_1, n_2, \ldots, n_k}$ by the following operations: countable additions and intersections, multiplication and homeomorphic transformation*

Proof. 1) The first part of our theorem is an almost immediate consequence of Theorem I. In fact every natural $k$ can be represented (and in one way only) in the form

$$k = 2^{m(k) - 1}(2n(k) - 1). \tag{12,4}$$

Where $n(k)$ and $m(k)$ are positive integers. Let $\{E_n^m\}$ be a system of sets, depending of two natural indices $n$ and $m$.

Denote

$$P_k = E^{m(k)}_{n(k)}.$$

Then the operation

$$\Psi(\{P_k\}) = \underset{m}{\Phi_{N_0}}(\underset{n}{\Phi_{N_m}}(\{E_n^m\})) = \underset{m(k)}{\Phi_{N_0}}(\{\underset{n(k)}{\Phi_{N_{m(k)}}}(\{P_k\})\})$$

is a positive analytical operation (Def. 2, remark, art. 1) and is therefore a $\delta s$-function $\Phi_N$ (Theorem I, art. 6) i. e.

$$\underset{k}{\Phi_N}(\{P_k\}) = \underset{m}{\Phi_{N_0}}(\{\Phi_{N_m}(\{E_n^m\})\}). \tag{12,5}$$

It is evident that $\Phi_N$ satisfies (12,1) for any $\mathfrak{L}$.

2) It is sufficient to prove that $\widetilde{N} = \widetilde{\widetilde{N}}$ (Th. III cor). We have by (7,5) and (12,5)

$$\widetilde{N} = \underset{k}{\Phi_N}(\{D_k\}) = \underset{m}{\Phi_{N_0}}(\{\underset{n}{\Phi_{N_m}}(\{D_{2^{m-1}(2n-1)}\})\})$$

i. e. $\widetilde{N}$ is the set of all such points $x = \frac{1|}{|q_1} + \frac{1|}{|q_2} + \ldots$ that there exists such

$$y_0 = \frac{1|}{|m_1} + \frac{1|}{|m_2} + \ldots \epsilon N_0; \quad y_\mu = \frac{1|}{|n_1^{(\mu)}} + \frac{1|}{|n_2^{(\mu)}} + \ldots \epsilon N m_\mu$$

that among the numbers $q_i$ are found all the numbers of the form

$$2^{m_\nu - 1}(2 n_\lambda^{(\nu)} - 1) \quad (\nu = 1, 2, \ldots; \ \lambda = 1, 2, \ldots).$$

But the set $\tilde{\check{N}}$ may be defined in exactly the same words (see def. 8, Art. 7). Therefore $\check{N} = \tilde{\check{N}}$ and the second part of our theorem is demonstrated.

3) Denote $T_k^\nu$ the set of all such numbers

$$x = \frac{1|}{|p_1} + \frac{1|}{|p_2} + \ldots$$

that

(12,6) $$p_{2^{\nu-1}(2l-1)} = 2^{k-1}(2 q_l - 1)$$

and

$$\frac{1|}{|q_1} + \frac{1|}{|q_2} + \ldots \epsilon N_k.$$

$T_k^\nu$ is homeomorphic to $N_k \times J$. In fact let for

$$x = \frac{1|}{|p_1} + \frac{1|}{|p_2} + \ldots \epsilon T_k^\nu$$

$$\varphi(x) = (\xi, \eta) \quad \text{where}$$

$$\xi = \frac{1|}{|q_1} + \frac{1|}{|q_2} + \ldots \epsilon N_k,$$

$$\eta = \frac{1|}{|p_1} + \frac{1|}{|p_2} + \ldots + \frac{1|}{|p_{2^{\nu-1}-1}} + \frac{1|}{|p_{2^{\nu-1}+1}} + \ldots + \frac{1|}{|p_{3 \cdot 2^{\nu-1}-1}} + \frac{1|}{|p_{3 \cdot 2^{\nu-1}+1}} + \ldots$$

where $q_l$ have the same value as in (12,6).

Evidently $\varphi$ establishes a homeomorphic correspondence between $T_k^\nu$ and $N_k \times J$.

Denote now

$$T = (J \times N_0) \prod_{j=1}^{\infty} \sum_{l_1, l_2, \ldots, l_j} T_{l_j}^j \times \delta_{l_1, l_2, \ldots, l_j}$$

$\check{N}$ is homeomorphic to $T$.



For if

$$\xi = \frac{1|}{|q_1} + \frac{1|}{|q_2} + \ldots \epsilon \check{N}$$

then we have (12,2) and (12,3). Let now

$$\eta = \frac{1|}{|m_1} + \frac{1|}{|m_2} + \ldots = y_0 \epsilon N_0$$

then $(\xi, \eta) \epsilon J \times N_0$.

We have besides

$$\eta \epsilon \delta_{m_1, m_2, \ldots, m_j}$$

for any $j$, and by (12,3) and (12,6)

$$\xi \epsilon T_{m_j}^j \quad \text{for any } j.$$

Hence $(\xi, \eta) \epsilon T$.

Conversely if $(\xi, \eta) \epsilon T$ then

$$\xi = \frac{1|}{|q_1} + \frac{1|}{|q_2} + \ldots \epsilon J$$

$$\eta = \frac{1|}{|m_1} + \frac{1|}{|m_2} + \ldots \epsilon N_0$$

besides

$$(\xi, \eta) \epsilon \sum_{l_1, l_2, \ldots, l_j} T_{l_j}^j \times \delta_{l_1, l_2, \ldots, l_j}$$

for any $j$. But as $\eta$ belongs to $\delta_{l_1, l_2, \ldots, l_j}$ only if

$$l_1 = m_1, \ldots, \ l_j = m_j$$

we have $\xi \epsilon T_{m_j}^j$ for any $j$ whence (by (12,6))

$$q_{2^{j-1}(2l-1)} = 2^{m_j - 1}(2 n_l^{(j)} - 1)$$

where

$$\frac{1|}{|n_1^{(\mu)}} + \frac{1|}{|n_2^{(\mu)}} + \ldots \epsilon N m_\mu$$

hence by (12,2) and (12,3) $\xi \epsilon \check{N}$.

Thus to every point $\xi \epsilon \check{N}$ we associate a point $(\xi, \eta) \epsilon T$ and *vice versa*. It is easy to verify that the correspondence thus established is a homeomorphy q. e. d.

**13. Definition 12.** A $\delta s$-operation $\Phi_N$ is *normal* if

(13,1) $$\mathcal{H}_N(\mathcal{H}_N(\mathcal{H})) = \mathcal{H}_N(\mathcal{H})$$

for any $\mathcal{H}$.

The lefthand member of (13,1) is an $\mathcal{H}_{N'}(\mathcal{H})$ class and it is evident that for any $\mathcal{H}$

$$\mathcal{H}_N(\mathcal{H}) \subset \mathcal{H}_{N'}(\mathcal{H})\ ^{1}).$$

Consequently we have (13,1) if

$$\mathcal{H}_{N'}(\mathcal{H}) \subset \mathcal{H}_N(\mathcal{H})$$

and therefore by Theorem IV (Art. 8) and (7,5) operation $\Phi_N$ is normal if

(13,2) $$N' = \underset{n}{\Phi_{N'}}(\{D_n\}) = \underset{p}{\Phi_N}(\{D_{k(p)}\}).$$

Hence follows

**Theorem VII.** *In order that a $\delta s$-operation $\Phi_N$ be normal it is necessary and sufficient that*

$$\mathcal{H}_N(\mathcal{H}_N(\mathcal{D})) \subset \mathcal{H}_N(\mathcal{D}) \quad \text{(see def. 9).}$$

**14.** Mr. Kolmogoroff has given [2]) another definition of a normal operation viz.

**Definition 12$_1$.** Operation $\Phi_N$ is normal if for any system $\{E_n^m\}$ of sets we can find two sequences of natural numbers $m_1, m_2, \ldots$ and $n_1, n_2, \ldots$ such that

(14,1) $$\underset{m}{\Phi_N}(\{\underset{n}{\Phi_N}(\{E_n^m\})\}) = \underset{p}{\Phi_N}(\{E_{n_p}^{m_p}\}).$$

It may be easily proved that both definitions are equivalent. In fact it follows immediately from the definitions that an operation normal in Mr. Kolmogoroff's sense is also normal in the sense of the def. 12.

To prove the converse we have only to set $\mathcal{H} = \{E_n^m\}$. Then by (13,1)

[1]) This follows from the fact that $\mathcal{H} \subset \mathcal{H}_N(\mathcal{H})$ for any $\mathcal{H}$ and $N$. To verify this suppose $E \in \mathcal{H}$ and denote $E_n = E$ for any $n$. Then $E = \Phi_N(\{E_n\}) \in \mathcal{H}_N(\mathcal{H})$ q. e. d.

[2]) See Kolmogoroff, p. 417.



$$\underset{m}{\Phi_N}(\{\underset{n}{\Phi_N}(\{E_n^m\})\}) = \underset{p}{\Phi_N}(\{H_p\}); \quad H_p \in \mathcal{H}$$

i. e. $H_p = E_{n_p}^{m_p}$ and hence follows immediately (14,1).

We shall prove now that *we can chose the sequences $\{m_p\}$ and $\{n_p\}$ independently of the system $\{E_n^m\}$* (i. e. once for all such systems [1])).

Denote

(14,2) $$G_n^m = D_{2^{m-1}(2n-1)}$$

and let $\{m_p\}$ and $\{n_p\}$ be the sequences satisfying the relation

(14,3) $$\underset{m}{\Phi_N}(\{\underset{n}{\Phi_N}(\{G_n^m\})\}) = \underset{p}{\Phi_N}(\{G_{n_p}^{m_p}\}).$$

The condition (14,1) is then satisfied for $E_n^m = G_n^m$.

We shall prove now that (14,1) is satisfied for any $\{E_n^m\}$.

Take an arbitrary point $y\, (\in \Sigma E_n^m)$ and let $x (\in J)$ be such point that $x \in G_n^m$ if, and only if, $y \in E_n^m$. (Such points always exist: in fact let $\{n_i, m_i\}$ be a sequence consisting of all the pairs $n$, $m$ such that $y \in E_n^m$ and only of such pairs and denote

$$k_i = 2^{m_i - 1}(2n_i - 1)$$

then $x = \frac{1|}{|k_1} + \frac{1|}{|k_2} + \ldots$ possesses the property in question).

Now $y$ belongs to the left (right) member of (14,1) if, and only if, $x$ belongs to the left (resp. right) member of (14,3) but $x$ belongs or does not belong to both members of (14,3) simultaneously. Therefore $y$ belongs or does not belong to both members of (14,1) simultaneously q. e. d.

The following definition is analogical to the def. 6bis.

**Definition 12*bis*.** *A* $\delta s$-operation $\Phi_N$ is *r-normal in respect of a family* $\mathfrak{H}$ of classes of sets if

$$\mathcal{H}_N(\mathcal{H}_N(\mathcal{H})) = \mathcal{H}_N(\mathcal{H})$$

for any $\mathcal{H} \in \mathfrak{H}$.

**15. Definition 13.** If for a certain $N$ there exists such $N^c$ that for any $\{E_n\}$

(15,1) $$C(\Phi_N(\{E_n\})) = \Phi_{N^c}(\{C(E_n)\})$$

[1]) The question whether such sequences exist was put by Mr. Kolmogoroff.

then we shall call the operation $\Phi_{N^c}$ *complementary to* $\Phi_N$.

Evidently

$$(15,2) \qquad [\mathcal{H}_N(\mathcal{H})]_c = \mathcal{H}_{N^c}(\mathcal{H}_c).$$

To prove that every $\delta s$-operation has a complementary operation we shall introduce a new kind of analytical operations called by Mr. Hausdorff $\sigma d$-functions [1]) viz:

$$(15,3) \qquad \Psi_N(\{E_n\}) = \prod_{(n_1, n_2, \dots) \in \mathfrak{N} \sim N} \sum_i E_{n_i}.$$

It is immediately seen that $\Psi_N$ is the complementary operation to $\Phi_N$ in the sense of def 13.

We can prove now

**Theorem VIII** [2]). 1) *For every $\delta s$-operation $\Phi_N$ there exists a complementary operation $\Phi_{N^c}$.*

2) *There exists a plane point set $S \in \mathcal{G}_\delta$ such that whatever be $N$*

$$(15,4) \qquad \breve{N}^c = C(Pr((J \times N) \cdot S))\ ^{3}).$$

3) *$\breve{N}^c$ is the set of all the numbers*

$$x = \frac{1|}{|m_1} + \frac{1|}{|m_2} + \dots$$

*such that whatever be*

$$y = \frac{1|}{|n_1} + \frac{1|}{|n_2} + \dots \in N$$

*we may find such $i$ and $j$ that $n_i = m_j$.*

Proof. 1) We must prove that a $\sigma d$-operation $\Psi_N$ is a $\delta s$-operation. But this follows immediately from Theorem I (Art. 6) and from def. 2, remark (Art. 1).

2) According to the same theorem (Th. I) we have

$$(15,5) \qquad \breve{N}^c = \Psi_N(\{D_n\}) = C(\Phi_N(C(D_n)).$$

[1]) Hausdorff, p. 89.

[2]) The first and the third part of this theorem may be found in Kolmogoroff (p. 417) and Sierpiński, IV (p. 209), cf. also our note in C. R. (3°).

[3]) All the complements are taken rel. $J$ i. e. $C(P) = J - P$.



But the sets $C(D_n)$ are closed in $J$. Therefore the scheme $S$ for $\{C(D_n)\}$ (def. 10) is closed in $J \times J$ (Art. 9, remark) and therefore belongs to $\mathcal{G}_\delta$ in the plane. Now to obtain (15,4) we must only apply Theorem V.

3) It follows from (15,3) and (15,5) that

$$x = \frac{1|}{|m_1} + \frac{1|}{|m_2} + \dots \in \breve{N}^c$$

if, and only if, for any $y = \frac{1|}{|n_1} + \frac{1|}{|n_2} + \dots \in N$ we have

$$x \in \sum_i D_{n_i}$$

i. e. we can find such $i$ that $x \in D_{n_i}$.

But from the definition of $D_{n_i}$ follows that there exists such $j$ that $m_j = n_i$ q. e. d.

**16.** The theorems of this paragraph hold true to a certain extent for $v\delta s$-functions (see def. 3', Art. 10), viz:

**Theorem VI'.** *For any $N_0(x)$; $N_1(x)$, $N_2(x)$, ... there exists a variable base $N(x)$ such that*

$$\mathcal{H}^{(x)}_{N(x)}(\mathfrak{L}) = \mathcal{H}^{(x}_{N_0(x)}(\{\mathcal{H}^{(x}_{N_i(x)}(\mathfrak{L})\})$$

*for any $\mathfrak{L}$. Moreover for every (fixed) $x$, $N(x)$ is the same as defined in Theorem VI (with $N_i = N_i(x)$).*

Proof. Let $E_n^m \in \mathfrak{L}$ and denote

$$P_{2^{m-1}(2n-1)} = E_n^m \qquad \text{(see Art. 12)}$$

then for any fixed $x$

$$\Phi_{\substack{N(x)\\k}}(\{P_k\}) = \Phi_{\substack{N_0(x)\\m}}(\{\Phi_{\substack{N_m(x)\\n}}(\{E_n^m\})\}).$$

Further (see (10,4))

$$\Phi^{(x}_{N(x)}(\{P_k\}) = \sum_{x \in R} (x)\ \Phi_{N(x)}(\{P_k\}) =$$

$$= \sum_{x \in R} (x)\ \Phi_{\substack{N_0(x)\\m}}(\{\Phi_{\substack{N_m(x)\\n}}(\{E_n^m\})\}) = \sum_{x \in R} \Phi_{\substack{N_0(x)\\m}}(\{(x) \cdot \Phi_{\substack{N_m(x)\\m}}(\{E_n^m\})\}) =$$

$$=\sum_{x\in R}\underset{m}{\Phi}_{N_0(x)}\big(\{(x)\,\underset{n}{\Phi}^{(x_1)}_{N_m(x_1)}(\{E_n^m\})\}\big)=\sum_{x\in R}(x)\,\underset{m}{\Phi}_{N_0(x)}\big(\{\underset{n}{\Phi}^{(x_1)}_{N_m(x_1)}(\{E_n^m\})\}\big)=$$
$$=\underset{m}{\Phi}^{(x)}_{N_0(x)}\big(\{\underset{n}{\Phi}^{(x)}_{N_m(x)}(\{E_n^m\})\}\big).$$

Hence the theorem.

**Theorem VIII′.** *For every $v\delta s$-operation $\Phi^{(x)}_{N(x)}$ there exists a complementary operation $\Phi^{(x)}_{N^c(x)}$. Moreover for any (fixed) $x$, $\Phi_{N^c(x)}$ is complementary to $\Phi_{N(x)}$.*

The proof is analogous to that of Theorem VI′.

## § 3. The Class of Projections for an $\mathcal{H}_N(\mathcal{K})$.

**17.** It is well known that the class $\mathcal{P}$ of projections [1]) of sets belonging to a certain class $\mathcal{Q}$ offers very often a great interest.

E. g. the projections of sets of the type $\mathcal{G}_\delta$ are $(A)$-sets of Souslin [2]).

We shall study in this paragraph classes of projections of sets belonging to $\mathcal{H}_N(\mathcal{K})$-classes, $\mathcal{K}$ satisfying certain conditions. These conditions are satisfied in the cases most interesting to us viz. those of $\mathcal{K}=\mathcal{F}$ and $\mathcal{K}=\mathcal{G}$ and all our results will be applied to these classes.

**18. Definition 14.** Let classes $\mathcal{P}$ and $\mathcal{K}$ be defined in the spaces $R^{(0)}=R\times R^{(1)}$ and $R$ respectively. Then the class $\mathcal{P}$ is *projective rel.* $\mathcal{K}$ if there exists such variable base $N(y)$ $(y\in R^{(1)})$ that

(18,1) $$\mathcal{P}=\mathcal{H}^{(x,y)}_{N(y)}(\mathcal{K}^*)\ ^{3})$$

$\mathcal{K}^*$ denoting the class of all the sets of the form $K\times R^{(1)}$; $K\in\mathcal{K}$.

The following theorem will explain the name „projective".

**Theorem IX.** *If $\mathcal{P}$ is a projective class rel. $\mathcal{K}$, then we have for the class $\mathcal{P}'$ of projections of sets belonging to $\mathcal{P}$:*

[1]) We use the word „projection" in the sense of def. $\zeta$. (Notations, group $D$).

[2]) See e. g. Lusin, *Leçons sur les ensembles analytiques*, p. 141 ff.

[3]) $\Phi^{(x,y)}_{N(y)}$ and $\mathcal{H}^{(x,y)}_{N(y)}$ denote $\Phi^{(x,y)}_{N(x,y)}$ resp. $\mathcal{H}^{(x,y)}_{N(x,y)}$, where $N(x,y)=N(y)$ for any $x$.



(18,2) $$\left|\ \textit{where}\quad \begin{aligned}&\mathcal{P}'=\mathcal{H}_{N_0}(\mathcal{K})\\ &N_0=\sum_{y\in R^{(1)}}N(y).\end{aligned}\right.$$

Proof. Evidently it is sufficient to prove that

(18,3) $$\mathcal{E}\equiv Pr\,\Phi^{(x,y)}_{N(y)}(\{E_n\times R^{(1)}\})=\sum_{y\in R^{(1)}}\Phi_{N(y)}(\{E_n\})\equiv\mathcal{E}'$$

for any $\{E_n\}$ (see (3,2)).

$\alpha$) $\mathcal{E}\subset\mathcal{E}'$. Let

$$x_0\in Pr\,\Phi^{(x,y)}_{N(y)}(\{E_n\times R^{(1)}\});$$

then there exists such $y_0$ that

$$(x_0,y_0)\in\Phi^{(x,y)}_{N(y)}(\{E_n\times R^{(1)}\}).$$

Whence

$$(x_0,y_0)\in\Phi_{N(y_0)}(\{E_n\times R^{(1)}\})\quad \text{(def. 3′)}$$

or (see (3,3))

$$x_0\in\Phi_{N(y_0)}(\{E_n\})\subset\sum_{y\in R^{(1)}}\Phi_{N(y)}(\{E_n\})\quad \text{q. e. d.}$$

$\beta$) $\mathcal{E}'\subset\mathcal{E}$. Let

$$x_0\in\sum_{y\in R^{(1)}}\Phi_{N(y)}(\{E_n\}).$$

Then there exists such $y_0$ that

$$x_0\in\Phi_{N(y_0)}(\{E_n\})$$

or

$$(x_0,y_0)\in\Phi_{N(y_0)}(\{E_n\times R^{(1)}\})$$

whence

$$(x_0,y_0)\in\Phi^{(x,y)}_{N(y)}(\{E_n\times R^{(1)}\})$$

and

$$x_0\in Pr\,\Phi^{(x,y)}_{N(y)}(\{E_n\times R^{(1)}\}).\qquad \text{q. e. d.}$$

**Theorem X.** *If the class $\mathcal{P}$ is projective rel. $\mathcal{K}$ then $\mathcal{P}_c$ (see notations) is projective rel. $\mathcal{K}_c$.*

This is an immediate consequence of the definition of a projective class and of Theorem VIII′.

**Theorem XI.** *If $\mathcal{P}$ is projective rel. $\mathcal{K}$ then $\mathcal{H}_N(\mathcal{P})$ is also projective rel. $\mathcal{K}$. (We can express this by saying that the property of* *a class $\mathcal{F}$ of being projective rel. $\mathcal{H}$ is a transitive property in the sense of § 4 of this chapter).*

This is an immediate consequence of the definition of a projective class and of Theorem VI′.

**Remark.** *It follows from* (18,3) *that for any* $N$, $N^{(1)}(y)$, $\{E_n^m\}$

$$(18,4) \qquad \Pr \underset{m}{\Phi_N}(\{\underset{n}{\Phi^{(x,y)}_{N^{(1)}(y)}}(\{E_n^m \times R^{(1)}\})\}) = \sum_{y \in R^{(1)}} \underset{m}{\Phi_N}(\{\underset{n}{\Phi_{N^{(1)}(y)}}(\{E_n^m\})\}).$$

**19.** We pass now to the case of open and closed sets. Our chief aim is to prove that if $R^{1)}$ is a continuous image of $J$ then any $\mathcal{H}_N(\mathcal{G}^{(0)})$ and (with certain restrictions) any $\mathcal{H}_N(\mathcal{F}^{(0)})$ (see notations group $B$) is a projective class rel. $\mathcal{G}$ and $\mathcal{F}$ respectively. If $R^{(1)}$ is besides a compact metric space then $\mathcal{H}_N(\mathcal{F}^{(0)})$ is projective rel. $\mathcal{F}$, subject to no restrictions. It would apear that it is sufficient to treat this last case only, leaving alone the more general case of $R^{(1)}$ being a continuous image of $J$ because the case most interesting to us is when $R^{(1)} = I$. But we can not do so because the form of the operations which we obtain in the case of $R^{(1)} = \Delta$ [1]) will be of no use to us in the fourth chapter. On the other hand we cannot dispense altogether with the case of $R^{(1)} = \Delta$ because then we must put certain restrictions on the class $\mathcal{H}_N(\mathcal{F}^{(0)})$ unnecessary when $R^{(1)} = I$.

The plan of our proof is as follows: we shall prove first that the classes $\mathcal{F}^{(0)}$ and $\mathcal{G}^{(0)}$ are projective rel. resp. $\mathcal{F}$ and $\mathcal{G}$ in the cases when $R^{(1)} = \Delta$ and $R^{(1)} = J$.

It will follow that in these cases the classes $\mathcal{H}_N(\mathcal{F}^{(0)})$, $\mathcal{H}_N(\mathcal{G}^{(0)})$ are projective rel. resp. $\mathcal{F}$ and $\mathcal{G}$. In the same time (18,4) will give us the form of the operations for the classes of projections of sets belonging to $\mathcal{H}_N(\mathcal{F}^{(0)})$ and $\mathcal{H}_N(\mathcal{G}^{(0)})$ whence we shall deduce some properties of the bases of these operations. At last we shall reduce the cases when $R^{(1)}$ is a continuous image of $J$ and when $R^{(1)}$ is a compact metric space, to those mentioned above, by showing: a) that in case $R^{(1)} = \varphi J$ ($R^{(1)}$ being uncountable) the class of projections for an $\mathcal{H}_N(\mathcal{G}^{(0)})$ and (subject to certain restrictions) of an $\mathcal{H}_N(\mathcal{F}^{(0)})$ class is the same as in the case when $R^{(1)} = J$;

[1]) See notations group $C$.



b) thas in case $R^{(1)} = \varphi\Delta$ ($R^{(1)}$ being uncountable) the class of projections for an $\mathcal{H}_N(\mathcal{G}^{(0)})$ and an $\mathcal{H}_N(\mathcal{F}^{(0)})$ is the same as in the case when $R^{(1)} = \Delta$. We shall denote $R^{(\delta)} = R \times \Delta$, $R^{(j)} = R \times J$; $\mathcal{F}^{(\delta)}$ and $\mathcal{G}^{(\delta)}$ are classes of sets closed (resp. open) in $R^{(\delta)}$, and $\mathcal{F}^{(j)}$ and $\mathcal{G}^{(j)}$ are classes of sets closed (resp. open) in $R^{(j)}$.

**Theorem XII A.** *The classes $\mathcal{F}^{(\delta)}$ and $\mathcal{G}^{(\delta)}$ are projective rel. resp. $\mathcal{F}$ and $\mathcal{G}$.*

P r o o f. If this theorem is true for one of the classes $\mathcal{F}^{(\delta)}$ or $\mathcal{G}^{(\delta)}$ (e. g. for $\mathcal{G}^{(\delta)}$) then by Theorem X it is true for the other. Therefore it is sufficient to prove it for $\mathcal{G}^{(\delta)}$.

Enumerate all the corteges (finite systems) of signs 0 and 1 $(\nu_1, \nu_2, \ldots, \nu_k)$; $k = 1, 2, \ldots$; $\nu = 0, 1$ and let $\lambda(\nu_1, \nu_2, \ldots, \nu_k)$ be the number corresponding to $(\nu_1, \nu_2, \ldots, \nu_k)$.

For any $y = 0, \nu_1 \nu_2 \ldots \epsilon \Delta$ denote $N(y)$ the set of all the numbers

$$\xi_i = \frac{1}{\lambda(\nu_1, \ldots, \nu_i)} + \frac{1}{\lambda(\nu_1, \ldots, \nu_i)} + \frac{1}{\lambda(\nu_1, \ldots, \nu_i)} + \ldots$$

We shall prove that

$$\mathcal{G}^{(\delta)} = \mathcal{H}^{(x,y)}_{N(y)}(\mathcal{G}^*) \quad \text{(see def 14)}$$

and consequently $\mathcal{G}^{(\delta)}$ is projective rel. $\mathcal{G}$.

a) $\mathcal{G}^{(\delta)} \subset \mathcal{H}^{(x,y)}_{N(y)}(\mathcal{G}^*)$. Let $G^{(0)} \epsilon\, \mathcal{G}^{(\delta)}$. Denote

$$(19.1) \qquad G_{\lambda(\nu_1, \nu_2, \ldots, \nu_i)} = Sk\, G^{(0)} \text{ rel. } \Delta_{\nu_1, \nu_2, \ldots, \nu_i} \text{ [1])}$$

$$(19,2) \qquad G = \Phi^{(x,y)}_{N(y)}(\{G_\lambda \times \Delta\}).$$

We have to prove that $G^{(0)} = G$.

$\alpha$) $G^{(0)} \subset G$. Let

$$(\xi, \eta) \epsilon\, G^{(0)}; \quad \eta = 0, \nu_1 \nu_2 \ldots$$

Then (as $G^{(0)}$ is open) there exists such $V_\eta^{(\delta)}$ (a neighbourhood of $\eta$ in $\Delta$) that

$$y \epsilon V_\eta^{(\delta)} \text{ implies } (\xi, y) \epsilon\, G^{(0)}.$$

Let $k$ be great enough that

$$\Delta_{\nu_1, \nu_2, \ldots, \nu_k} \subset V_\eta^{(\delta)}$$

[1]) It is an open set (see notations group $C$ and $D$, def. $\eta$, Theor).

then by (19,1)

$$\xi \in G_{\lambda(\nu_1,\nu_2,\ldots,\nu_k)} \subset \sum_i G_{\lambda(\nu_1,\nu_2,\ldots,\nu_i)} = \underset{\lambda}{\Phi}_{N(\eta)}(\{G_\lambda\})$$

and

$$(\xi,\eta) \in \underset{\lambda}{\Phi}{}^{(x,y)}_{N(y)}(\{G_\lambda \times \Delta\}) = G \quad \text{q. e. d.}$$

$\beta$) $G \subset G^{(0)}$. Let $(\xi,\eta) \in G$; $\eta = 0, \nu_1\, \nu_2 \ldots$, then by (19,2)

$$\xi \in \underset{\lambda}{\Phi}_{N(\eta)}(\{G_\lambda\}) = \sum_i G_{\lambda(\nu_1,\nu_2,\ldots,\nu_i)}$$

then there exists such $k$ that

$$\xi \in G_{\lambda(\nu_1,\nu_2,\ldots,\nu_k)}$$

and consequently (by the definition of $G_\lambda$ (19,1)) whatever be

$$y \in \Delta_{\nu_1,\nu_2,\ldots,\nu_k}$$

$$(\xi, y) \in G^{(0)}$$

in particular $(\xi,\eta) \in G^{(0)}$ q. e. d.

The inclusion $\mathcal{G}^{(\delta)} \subset \mathcal{H}^{(x,y)}_{N(y)}(\mathcal{G}^*)$ is thus demonstrated.

b) $\mathcal{H}^{(x,y)}_{N(y)}(\mathcal{G}^*) \subset \mathcal{G}^{(\delta)}$. Let

$$G = \Phi^{(x,y)}_{N(y)}(\{G_\lambda \times \Delta\});$$

denote

$$G^{(0)} = \sum_{\nu_1,\nu_2,\ldots,\nu_k} G_{\lambda(\nu_1,\nu_2,\ldots,\nu_k)} \times \Delta_{\nu_1,\nu_2,\ldots,\nu_k}$$

$G^{(0)}$ is open because $G_\lambda$ and $\Delta_{\nu_1,\nu_2,\ldots,\nu_k}$ are open (in resp. $R$ and $\Delta$) We proceed as in (a) showing that $G = G^{(0)}$.

The theorem is now completely demonstrated.

**Remark.** *We have seen that $N(y)$ for $\mathcal{G}^{(\delta)}$ consists of all the numbers*

$$\xi_i = \frac{1|}{|\lambda(\nu_1,\nu_2,\ldots,\nu_i)} + \frac{1|}{|\lambda(\nu_1,\nu_2,\ldots,\nu_i)} + \ldots$$

*It follows from Theorem VIII′ that the variable base $N^{(1)}(y)$ for $\mathcal{F}^{(\delta)}$ may be supposed to consist of the single point*

$$\xi = \frac{1|}{|\lambda(\nu_1)} + \frac{1|}{|\lambda(\nu_1,\nu_2)} + \ldots$$



**Theorem XII B.** *$\mathcal{F}^{(J)}$ and $\mathcal{G}^{(J)}$ are projective rel. resp. $\mathcal{F}$ and $\mathcal{G}$.*

Just as in Theorem XIIA it is sufficient to prove that $\mathcal{G}^{(J)}$ is projective rel $\mathcal{G}$.

Proof. Enumerate all the corteges $(m_1, m_2, \ldots, m_k)$ of natural numbers and let $l(m_1, m_2, \ldots, m_k)$ be the number corresponding to $(m_1, m_2, \ldots, m_k)$.

For any $y = \frac{1|}{|m_1} + \frac{1|}{|m_2} + \ldots \in J$, denote by $N_1(y)$ the set of all the points

$$\xi_i' = \frac{1|}{|l(m_1,\ldots,m_i)} + \frac{1|}{|l(m_1,\ldots,m_i)} + \ldots$$

We shall prove that

$$\mathcal{G}^{(J)} = \mathcal{H}^{(x,y)}_{N_1(y)}(\mathcal{G}^*)$$

and conseqnently that $\mathcal{G}^{(J)}$ is projective rel. $\mathcal{G}$.

*a*) $\mathcal{G}^{(J)} \subset \mathcal{H}^{(x,y)}_{N_1(y)}(\mathcal{G}^*)$.

Let $G^{(0)} \in \mathcal{G}^{(J)}$. Denote

$$G'_{l(m_1,m_2,\ldots,m_k)} = Sk\, G^{(0)} \text{ rel. } \delta_{m_1,m_2,\ldots,m_k}.$$

In Theorem XIIA the corresponding sets $G_\lambda$ were open. In this case it is not always so (we can not use the theorems of introductory chapter because $J$ unlike $\Delta$ is not a compact space).

So we can not use the sets $G_l'$ themselves. Instead of them we take the greatest open sets contained in them, i. e. we denote

$$G_l = C(\overline{C(G_l')})$$

$$G = \Phi^{(x,y)}_{N_1(y)}(\{G_l \times J\}).$$

We have to prove that $G = G^{(0)}$.

*a*) $G^{(0)} \subset G$. Let

$$(\xi,\eta) \in G^{(0)}; \quad \eta = \frac{1|}{|m_1} + \frac{1|}{|m_2} + \ldots$$

Then (as $G^{(0)}$ is open) there exist such neighbourhoods $V_\xi$, $V_\eta^{(J)}$ of $\xi$ and $\eta$ in resp. $R$ and $J$ that

$$V_\xi \times V_\eta^{(J)} \subset G^{(0)}$$

Let $k$ be great enough that

$$\delta_{m_1, m_2, \ldots, m_k} \subset V_{\eta}^{(J)}$$

then

$$V_{\xi} \subset G'_{l(m_1, \ldots, m_k)}$$

whence

$$\xi \in G_{l(m_1, m_2, \ldots, m_k)} \subset \sum_i G_{l(m_1, m_2, \ldots, m_i)} = \underset{l}{\Phi}_{N_1(\eta)}(\{G_l\})$$

and

$$(\xi, \eta) \in \Phi_{N_1(y)}^{(x,y)}(\{G_l \times J\}) \quad \text{q. e. d.}$$

The rest of the proof is as in Theorem XIIA.

**Remark.** *We have seen that* $N_1(y)$ *for* $\mathcal{G}^{(J)}$ *consists of all the numbers*

$$\xi' = \frac{1}{|l(m_1, \ldots, m_i)} + \frac{1}{|l(m_1, \ldots, m_i)} + \ldots$$

*It follows from Theorem VIII' that the variable base* $N_1^{(1)}(y)$ *for* $\mathcal{F}^{(J)}$ *may be supposed to consist of the single number*

$$\xi' = \frac{1}{|l(m_1)} + \frac{1}{|l(m_1, m_2)} + \ldots$$

**Corollary of Theorems XIIA and XIIB.** *The classes of projections of sets belonging to the classes*

$$1)\ \mathcal{H}_N(\mathcal{F}^{(\delta)}); \quad 2)\ \mathcal{H}_N(\mathcal{G}^{(\delta)}); \quad 3)\ \mathcal{H}_N(\mathcal{F}^{(J)}); \quad 4)\ \mathcal{H}_N(\mathcal{G}^{(J)})$$

*are* $\delta s$*-classes*

$$1)\ \mathcal{H}_{\check{N}}(\mathcal{F}); \quad 2)\ \mathcal{H}_{\hat{N}}(\mathcal{G}); \quad 3)\ \mathcal{H}_{\check{N}'}(\mathcal{F}); \quad 4)\ \mathcal{H}_{\hat{N}'}(\mathcal{G})$$

*which can be represented by the following operations:*

$$(19,3)\left\{\begin{array}{l}
1) \sum\limits_{\nu_1, \nu_2, \ldots} \underset{n}{\Phi}_N\left(\left\{\prod\limits_k E_n^{\nu_1, \nu_2, \ldots, \nu_k}\right\}\right); \quad \nu_i = 0, 1; \quad E_n^{\nu_1, \nu_2, \ldots, \nu_k} \in \mathcal{F} \\
2) \sum\limits_{\nu_1, \nu_2, \ldots} \underset{n}{\Phi}_N\left(\left\{\sum\limits_k E_n^{\nu_1, \nu_2, \ldots, \nu_k}\right\}\right); \quad \nu_i = 0, 1; \quad E_n^{\nu_1, \nu_2, \ldots, \nu_k} \in \mathcal{G} \\
3) \sum\limits_{m_1, m_2, \ldots} \underset{n}{\Phi}_N\left(\left\{\prod\limits_k E_n^{m_1, m_2, \ldots, m_k}\right\}\right); \quad m_i \text{ are natural}; \quad E_n^{m_1, m_2, \ldots, m_k} \in \mathcal{F} \\
4) \sum\limits_{m_1, m_2, \ldots} \underset{n}{\Phi}_N\left(\left\{\sum\limits_k E_n^{m_1, m_2, \ldots, m_k}\right\}\right); \quad m_i \text{ are natural}; \quad E_n^{m_1, m_2, \ldots, m_k} \in \mathcal{G}.
\end{array}\right.$$



We have only to approach Theorems IX, X, XIIA and B and the formula (18,4) and to take into consideration that the operations $\Phi_{N^{(1)}_{(y)}}$, $\Phi_{N(y)}$, $\Phi_{N^{(1)}_{1(y)}}$, $\Phi_{N_1(y)}$ are equal resp. to:

$$\prod_k E_{\lambda(\nu_1, \nu_2, \ldots, \nu_k)}; \quad \sum_k E_{\lambda(\nu_1, \nu_2, \ldots, \nu_k)}; \quad \prod_k E_{l(m_1, m_2, \ldots, m_k)}; \quad \sum_k E_{l(m_1, m_2, \ldots, m_k)}.$$

**20. Theorem XIII.** *The sets* $\check{N}$, $\hat{N}$, $\check{N}'$, $\hat{N}'$ *of the preceding corollary can be represented in the form*

$$(20,1)\left\{\begin{array}{l}
\check{N} = \Phi_N(\{K_s\}) \\
\hat{N} = \Phi_N(\{L_s\}) \\
\check{N}' = \Phi_N(\{K'_s\}) \\
\hat{N}' = \Phi_N(\{L'_s\})
\end{array}\right.$$

*Where* $K_s$, $L_s$, $K'_s$, $L'_s$ *are certain sets of type* $\mathcal{F}_\sigma$ *in* $J$ *(independent of* $N$*).*

P r o o f. Enumerate all the systems $(s; \nu_1, \nu_2, \ldots, \nu_k)$ where

$$s = 1, 2, \ldots; \quad k = 1, 2, \ldots; \quad \nu_i = 0, 1$$

and let $\sigma(s; \nu_1, \nu_2, \ldots, \nu_k)$ be the number corresponding to

$$(s; \nu_1, \nu_2, \ldots, \nu_k).$$

Denote $K_s^r$ the set of all such numbers

$$\xi = \frac{1}{|p_1} + \frac{1}{|p_2} + \ldots$$

that there exist two sequences (depending of $\xi \in K_s^r$), $(\nu_1, \nu_2, \ldots)$ and $(n_1, n_2, n_3, \ldots)$ such that

1) $n_r = s$
2) $p_{2^{k-1}(2i-1)} = \sigma(n_i; \nu_1, \nu_2, \ldots, \nu_k)$ for $i = 1, 2, \ldots$; $k = 1, 2, \ldots$

and let

$$K_s = \sum_{r=1}^{\infty} K_s^r.$$

Evidently all the $K_s^r \in \mathcal{F}$ (in $J$) and therefore $K_s \in \mathcal{F}_\sigma$ (in $J$). Now let

$$M = \Phi_N(\{K_s\}).$$

By (7,5) and (19,3) we have only to prove that

$$M=\sum_{\nu_1,\nu_2,\dots}\underset{s}{\Phi_N}\left(\left\{\prod_k D_{\sigma(s;\nu_1,\nu_2,\dots,\nu_k)}\right\}\right).$$

Let

$$\xi=\frac{1|}{|p_1}+\frac{1|}{|p_2}+\dots\in\tilde{M}.$$

Then there exists such sequence $i_1, i_2, \dots$ that

$$\xi'\doteq\frac{1}{|p_{i_1}}+\frac{1|}{|p_{i_2}}+\dots\in M=\Phi_N(\{K_s\})$$

i. e. there exists such

$$\frac{1|}{|s_1}+\frac{1|}{|s_2}+\dots\in N$$

that

$$\xi'\in K_{s_j}=\sum_{r=1}^{\infty}K_{s_j}^r$$

for any $j$.

Or for any $j$ there exists such $r_j$ that $\xi'\in K_{s_j}^{r_j}$. We have thus shown that $\xi\in M$ if, and only if, there exist such sequences:

$(i_1, i_2, \dots)$; $(s_1, s_2, \dots)$; $(r_1, r_2, \dots)$; $(n_1, n_2, \dots)$ and $(\nu_1, \nu_2, \dots)$ that

1) $\frac{1|}{|s_1}+\frac{1|}{|s_2}+\dots\in N$
2) $n_{r_j}=s_j$ for any $j$
3) $p_{2^{k-1}(2l-1)}=\sigma(n_l;\ \nu_1, \nu_2, \dots, \nu_k)$ for any $l$ and $k$.

These conditions are equivalent with the following: there exist such sequences $(\nu_1, \nu_2, \dots)$ and $(s_1, s_2, \dots)$ that

1) $\frac{1|}{|s_1}+\frac{1|}{|s_2}+\dots\in N$
2) Among the numbers $p_i$ are found all the numbers

$$\sigma(s_j;\ \nu_1, \nu_2, \dots, \nu_k) \text{ for any } j \text{ and } k.$$

But these conditions are fulfilled if, and only if,

$$\xi\in\sum_{\nu_1,\nu_2,\dots}\underset{s}{\Phi_N}\left(\left\{\prod_k D_{\sigma(s;\nu_1,\nu_2,\dots,\nu_k)}\right\}\right)$$

i. e.

$$\xi\in\tilde{M}$$



is equivalent with

$$\xi\in\sum_{\nu_1,\nu_2,\dots}\underset{s}{\Phi_N}\left(\left\{\prod_k D_{\sigma(s;\nu_1,\nu_2,\dots,\nu_k)}\right\}\right).$$

Or

$$M=\sum_{\nu_1,\nu_2,\dots}\underset{s}{\Phi_N}\left(\left\{\prod_k D_{\sigma(s;\nu_1,\nu_2,\dots,\nu_k)}\right\}\right) \qquad \text{q. e. d.}$$

The rest of the relations (**20**,1) can be proved in the same way so that we shall give here only the definitions of $L_s^r$, $K_s'^r$, $L_s'^r$.

$L_s^r$ is the set of all the points

$$\xi=\frac{1|}{|p_1}+\frac{1|}{|p_2}+\dots$$

such that there exist three sequences

$$(\nu_1, \nu_2, \dots);\quad (n_1, n_2, \dots);\quad (k_1, k_2, \dots)$$

such that

1) $n_r=s$
2) $p_i=\sigma(n_i;\ \nu_1, \nu_2, \dots, \nu_{k_i})$.

To define $K_s'$ and $L_s'$ we must enumerate all the systems $(n;\ m_1, m_2, \dots, m_k)$ where $n=1, 2, \dots$; $k=1, 2, \dots$; $m_i=1, 2, \dots$ and proceed as in defining $K_s$ and $L_s$.

**21.** We have proved thus that the class of projections for the classes $\mathcal{H}_N(\mathcal{F}^{(0)})$ and $\mathcal{H}_N(\mathcal{G}^{(0)})$ are $\delta s$-classes over $\mathcal{F}$ resp. $\mathcal{G}$ in case $R^{(1)}=\Delta$ and $R^{(1)}=J$. We shall prove now that the cases (most important to us) when $R^{(1)}$ is a compact metric space and when $R^{(1)}$ is a continuous image of $J$ can be reduced to those mentioned above.

We shall begin with the following two lemmas.

**Lemma A.** *Every uncountable compact mertic space (more generally every continuous image of $J$) contains a set $\Delta'$ homeomorphic to $\Delta$.*

Proof. As is well known [1]) every uncountable ($A$)-set (i. e. continuous image of $J$) $R^{(1)}$ contains a perfect subset $P$.

$R^{(1)}$ is a metric space [2]). We shall define a system of open spheres $\{\Theta_{\nu_1,\nu_2,\dots,\nu_k}\}$ and $\{\Theta'_{\nu_1,\nu_2,\dots,\nu_k}\}$ $(\nu_i=0, 1)$ so that

[1]) Hausdorff, p. 180.
[2]) op. cit. p. 209.

$$(21,1)\quad\begin{cases} \text{a)} & \overline{\Theta}_{\nu_1,\nu_2,\dots,\nu_k} \subset \Theta'_{\nu_1,\nu_2,\dots,\nu_k} \\ \text{b)} & \Theta'_{\nu_1,\nu_2,\dots,\nu_k,\nu_{k+1}} \subset \Theta_{\nu_1,\nu_2,\dots,\nu_k} \\ \text{c)} & \Theta'_{\nu_1,\nu_2,\dots,\nu_k,0}\cdot\Theta'_{\nu_1,\nu_2,\dots,\nu_k,1}=0 \\ \text{d)} & P\cdot\Theta_{\nu_1,\nu_2,\dots,\nu_k}\neq 0 \\ \text{e)} & d(\Theta'_{\nu_1,\nu_2,\dots,\nu_k})\leqslant\frac{1}{k} \end{cases}$$

We can define them recurently as follows: let these sets be defined for $k=k_0$ and let them satisfy (21,1).

Take any $\Theta_{\nu_1,\nu_2,\dots,\nu_{k_0}}$. As

$$(21,1\,\text{d}))\qquad \Theta_{\nu_1,\nu_2,\dots,\nu_{k_0}}\cdot P\neq 0$$

it contains at least two points $p_0$ and $p_1$ ($P$ being perfect and $\Theta_{\nu_1,\nu_2,\dots,\nu_{k_0}}$ open).

We may evidently find two spheres $\Theta'_{\nu_1,\nu_2,\dots,\nu_{k_0},0}$ and $\Theta'_{\nu_1,\nu_2,\dots,\nu_{k_0},1}$ with centres at $p_0$ and $p_1$ and satisfying the conditions (21,1b), c) and e)).

If we now denote $\Theta_{\nu_1,\nu_2,\dots,\nu_{k_0},\nu_{k_0+1}}$ the sphere concentric with $\Theta'_{\nu_1,\nu_2,\dots,\nu_{k_0},\nu_{k_0+1}}$ with radius equal to half of the radius of the latter, then all the conditions (21,1) will be evidently satisfied for $k=k_0+1$. So we may suppose that the systems $\{\Theta_{\nu_1,\nu_2,\dots,\nu_k}\}$ and $\{\Theta'_{\nu_1,\nu_2,\dots,\nu_k}\}$ are already constructed.

Denote now $\Delta'=\prod\limits_{k=1}^{\infty}\sum\limits_{(\nu_1,\nu_2,\dots,\nu_k)}\overline{\Theta}_{\nu_1,\nu_2,\dots,\nu_k}$. As is lightly seen $\Delta'$ is homeomorphic to $\Delta$. Moreover this homeomorphy make correspond $\Delta'_{\nu_1,\nu_2,\dots,\nu_k}=\Delta'\cdot\overline{\Theta}_{\nu_1,\nu_2,\dots,\nu_k}$ to $\Delta_{\nu_1,\nu_2,\dots,\nu_k}$.

**Lemma B.** *Every uncountable $(A)$-space $R^{(1)}$ contains a set $J'$ homeomorphic to $J$.*

Proof. Consider the set $\Delta'$ of the preceding lemma.

Denote $D'$ the set of all the points of $\Delta'$ which correspond to the endpoints of the intervals contiguous to $\Delta$ (in $I$) (black intervals). Then $J'=\Delta'-D'$ is homeomorphic to $J$.

The set corresponding to $\delta_{m_1,m_2,\dots,m_i}$ (in this homeomorphy) we shall denote $\delta'_{m_1,m_2,\dots,m_i}$. It is an open set in $J'$ and thus is common part of $J'$ and a set $\vartheta_{m_1,m_2,\dots,m_i}$ open in $R^{(1)}$. We may suppose that these sets $\vartheta_{m_1,m_2,\dots,m_i}$ satisfy the conditions:

1) $\vartheta_{m_1,m_2,\dots,m_i,m_{i+1}}\subset\vartheta_{m_1,m_2,\dots,m_i}$; 2) $\vartheta_{m_1,m_2,\dots,m_i,m_{i+1}}\cdot\vartheta_{m_1,m_2,\dots,m_i,m'_{i+1}}=0$ if $m'_{i+1}\neq m_{i+1}$.



**Theorem XIV A.** *If $R^{(1)}$ is an uncountable compact metric space then*

1) *the class $\mathcal{P}$ of projections of sets belonging to a $\delta s$-class $\mathcal{H}_N(\mathcal{F}^{(0)})$ is identical with the class $\mathcal{P}'$ of projections of sets belonging to $\mathcal{H}_N(\mathcal{F}^{(\delta)})$ and similarly*

2) *the class $\mathcal{Q}$ of projections of sets belonging to $\mathcal{H}_N(\mathcal{G}^{(0)})$ is identical with the class $\mathcal{Q}'$ of projections of sets belonging to $\mathcal{H}_N(\mathcal{G}^{(\delta)})$.*

Proof. a) $\mathcal{P}\subset\mathcal{P}'$. $R^{(1)}$ is a continuous image of $\Delta$ [1]) ($R^{(1)}=\varphi\Delta$). Let $\psi$ be the „inverse function" i. e. $\psi K$, ($K\subset R^{(1)}$) is the set of all such points $x\in\Delta$ that $\varphi x\in K$; we shall write also $\psi y$ for $\psi(y)$ (if $y\in R^{(1)}$ and therefore $(y)\subset R^{(1)}$). The following well known properties of $\psi$ are needed here and further

$$(21,2)\quad\begin{cases} \text{a)} & \psi\Sigma K=\Sigma\psi K;\quad \psi\Pi K=\Pi\psi K \\ \text{b)} & \psi(R^{(1)}-K)=\Delta-\psi K \\ \text{c)} & K\in\mathcal{F}\ \text{implies}\ \psi K\in\mathcal{F} \\ \text{d)} & K\in\mathcal{G}\ (\text{rel.}\ R^{(1)})\ \text{implies}\ \psi K\in\mathcal{G}\ (\text{rel.}\ \Delta). \end{cases}$$

Denote now $\psi(x,y)=(x)\times\psi y$ for any $(x,y)\in R^{(0)}$ and let $P\in\mathcal{P}$, i. e.

$$P=Pr\,P^{(0)};\quad P^{(0)}=\Phi_N(\{F_n\});\quad F_n\in\mathcal{F}^{(0)}.$$

It is evident that

$$P=Pr\,\Phi_N(\{\psi F_n\});\quad \psi F_n\in\mathcal{F}^{(\delta)},$$

i. e.

$$\Phi_N(\{\psi F_n\})\in\mathcal{H}_N(\mathcal{F}^{(\delta)})$$

$$P\in\mathcal{P}'\qquad\text{q. e. d.}$$

The inclusion $\mathcal{Q}\subset\mathcal{Q}'$ can be proved in a similar way

$\text{b}_1$) $\mathcal{P}'\subset\mathcal{P}$. Let $\Delta'$ be a subset of $R^{(1)}$ homeomorphic to $\Delta$ ($\Delta'=\chi\Delta$, see lemma A). Denote

$$\chi(x,y)=(x,\chi y)\quad\text{for any}\quad(x,y)\in R^{(\delta)}$$

$$\chi R^{(\delta)}=R^{(\delta')}\subset R^{(0)}.$$

Evidently $R^{(\delta')}$ is closed in $R^{(0)}$ and consequently sets closed in $R^{(\delta')}$ are closed in $R^{(0)}$.

[1]) Hausdorff, p. 197.

Let now $P \epsilon \mathscr{P}'$ i. e.

$$P = Pr\, P^{(0)};\quad P_0 = \Phi_N(\{F_n\});\quad F_n \epsilon \mathscr{F}^{(\delta)}.$$

Then evidently

$$P = Pr\, P'^{(0)} \quad \text{where} \quad P'^{(0)} = \Phi_N(\{\chi F_n\});\quad \chi F_n \epsilon \mathscr{F}^{(0)}$$

or

$$P \epsilon \mathscr{P} \qquad \text{q. e. d.}$$

b₂) $\mathscr{Q}' \subset \mathscr{Q}$. Let $Q \epsilon \mathscr{Q}'$ i. e.

$$Q = Pr\, Q^{(0)};\quad Q^{(0)} = \Phi_N(\{G_n\});\quad G_n \epsilon \mathscr{G}^{(\delta)}$$

but $G_n$ as open sets can be represented in the form

$$G_n = \sum_{(x,y)\epsilon G_n} V_x^{(n;y)} \times \Gamma_{(x;y)}^{(n)} \tag{21,3}$$

where

$V_x^{(n;y)}$ is a neighbourhood of $x$ in $R$ and $\Gamma_{(x;y)}^{(n)}$ is a $\Delta_{\nu_1,\nu_2,\dots,\nu_k}$.

Denote

$$\Gamma'^{(n)}_{(x;y)} = \Theta'_{\nu_1,\nu_2,\dots,\nu_k} \quad \text{if} \quad \Gamma^{(n)}_{(x;y)} = \Delta_{\nu_1,\nu_2,\dots,\nu_k} \quad (\text{see } (21,1))$$

$$G'_n = \sum_{(x,y)\epsilon G_n} V_x^{(n;y)} \times \Gamma'^{(n)}_{(x;y)} \tag{21,4}$$

$$Q'^{(0)} = \Phi_N(\{G'_n\});\quad Q' = Pr\, Q'^{(0)}.$$

We have to prove that $Q = Q'$

$\alpha$) $Q \subset Q'$. It follows immediately from (21,1a), (21,3), (21,4) that

$$\chi G_n \subset G'_n \quad \text{for any } n.$$

Hence

$$Q = Pr\Phi_N(\{G_n\}) = Pr\Phi_N(\{\chi G_n\}) \subset Pr\Phi_N(\{G'_n\}) = Q' \quad \text{q. e. d.}$$

$\beta$) $Q' \subset Q$. Let $x \epsilon Q'$ then there exists such $y'$ that

$$(x, y') \epsilon Q'^{(0)} = \Phi_N(\{G'_n\})$$

i. e. there exists such

$$\frac{1|}{|n_1} + \frac{1|}{|n_2} + \dots \epsilon N$$

that

$$(x, y') \epsilon \prod_i G'_{n_i}$$



whence

$$x \epsilon \prod_i V_{x_i}^{(n_i;y_i)}$$

$$y' \epsilon \prod_i \Gamma'^{(n_i)}_{(x_i;y_i)}$$

But one may easily verify that

$$\prod_i \Gamma'^{(n_i)}_{(x_i;y_i)} \neq 0 \quad \text{implies} \quad \prod_i \Gamma^{(n_i)}_{(x_i;y_i)} \neq 0 \quad (\text{see } (21,1\text{d})).$$

Take $y \epsilon \prod_i \Gamma^{(n_i)}_{x_i;y}$, then evidently

$$(x, y) \epsilon \prod_i V_{x_i}^{(n_i;y_i)}\, \Gamma^{(ni)}_{(x_i;y_i)} \subset \prod_i G_{n_i} \subset \Phi_N(\{G_n\}) = Q^{(0)}$$

or

$$x \epsilon Pr\, Q^{(0)} = Q \qquad \text{q. e. d.}$$

**Theorem XIV B.** *If $R^{(1)}$ is an uncountable $(A)$-space ($R^{(1)} = J$) then*

I) *the class $\mathscr{Q}$ of projections of sets belonging to $\mathscr{H}_N(\mathscr{G}^{(0)})$ is identical with the class $\mathscr{Q}''$ of projections of sets belonging to $\mathscr{H}_N(\mathscr{G}^{(J)})$.*

II) *If moreover $R^{(1)}$ contains a set $J'$ such that*

1) *$J'$ is homeomophic to $J$ ($J' = \varphi J$)*

(21,5) 2) *$E \epsilon \mathscr{H}_N(\mathscr{F}^{(0)})$ implies $E \cdot (R \times J') \epsilon \mathscr{H}_N(\mathscr{F}^{(0)})$*

*then the class $\mathscr{P}$ of projections of sets belonging to $\mathscr{H}_N(\mathscr{F}^{(0)})$ is identical with the class $\mathscr{P}''$ of projections of sets belonging to $\mathscr{H}_N(\mathscr{F}^{(J)})$.*

Proof. The inclusions $\mathscr{P} \subset \mathscr{P}''$, $\mathscr{Q} \subset \mathscr{Q}''$ and $\mathscr{Q}'' \subset \mathscr{Q}$ are proved as in theorem XIV A. (Except that we apply lemma B instead of lemma A and substitute $\vartheta_{m_1,m_2,\dots,m_l}$ for $\Theta'_{\nu_1,\nu_2,\dots,\nu_k}$).

It remains to prove that $\mathscr{P}'' \subset \mathscr{P}$.

In fact let $P \epsilon \mathscr{P}''$, i. e.

$$P = Pr\, P^{(0)};\quad P^{(0)} = \Phi_N(\{F_n\});\quad F_n \epsilon \mathscr{F}^{(J)}$$

and denote $\varphi(x, y) = (x, \varphi y)$. Then

$$\varphi P^{(0)} = \varphi\, \Phi_N(\{F_n\}) = \Phi_N(\{\varphi F_n\}).$$

But $\varphi F_n$ are closed in $R \times J'$ and therefore

$$\varphi F_n = (R \times J') \cdot \overline{\varphi F_n}$$

or

(21,6) $\varphi P^{(0)} = \Phi_N(\{(R \times J') \cdot \overline{\varphi F_n}\}) = (R \times J') \cdot \Phi_N(\{\overline{\varphi F_n}\}).$

But we have

$$\Phi_N(\{\overline{\varphi F_n}\}) \in \mathcal{H}_N(\mathcal{F}^{(0)})$$

whence taking into account (21,5) and (21,6) we have

$$\varphi P^{(0)} \in \mathcal{H}_N(\mathcal{F}^{(0)})$$

and therefore

$$P = Pr\, \varphi P^{(0)} \in \mathcal{P} \qquad \text{q. e. d.}$$

**22.** All the main results of this § may be summarised in the following:

**Fundamental Theorem on Projections** [1]). *If $R^{(1)}$ is an uncountable compact metric space then the classes of projections (upon $R$) of sets belonging to $\mathcal{H}_N(\mathcal{F}^{(0)})$ and $\mathcal{H}_N(\mathcal{G}^{(0)})$ (for any $N$) are themselves $\delta s$-classes $\mathcal{H}_{\check{N}}(\mathcal{F})$ and $\mathcal{H}_{\hat{N}}(\mathcal{G})$ where*

$$\check{N} = \Phi_N(\{K_s\})$$

$$\hat{N} = \Phi_N(\{L_s\})$$

*$K_s$ and $L_s$ being sets (of irrational numbers) of the type $\mathcal{G}_\sigma$ (in $\mathcal{G}$). These classes are defined by the operations*

$$\sum_{\nu_1,\nu_2,\ldots} \Phi_N\left(\left\{\prod_k E_n^{\nu_1,\nu_2,\ldots,\nu_k}\right\}\right); \quad \nu_i = 0, 1; \quad E_n^{\nu_1,\nu_2,\ldots,\nu_k} \in \mathcal{F}$$

$$\sum_{\nu_1,\nu_2,\ldots} \Phi_N\left(\left\{\sum_k E_n^{\nu_1,\nu_2,\ldots,\nu_k}\right\}\right); \quad \nu_i = 0, 1; \quad E_n^{\nu_1,\nu_2,\ldots,\nu_k} \in \mathcal{G}.$$

**Remark.** An analogical theorem exists for the case when $R^{(1)}$ is a half compact $(\mathcal{F}_\sigma)$-space.

The operations in this case are:

$$\sum_m \sum_{\nu_1,\nu_2,\ldots} \Phi_N\left(\left\{\sum_k E_n^{m,\nu_1,\nu_2,\ldots,\nu_k}\right\}\right); m=1,2\ldots, n=1,2,\ldots, k=1,2,\ldots, \nu=0,1$$

and

$$\sum_m \sum_{\nu_1,\nu_2,\ldots} \Phi_N\left(\left\{\prod_k E_n^{m,\nu_1,\nu_2,\ldots,\nu_k}\right\}\right).$$

The proof is immediate.

*A similar theorem exists for the case of $R^{(1)}$ being an $(A)$-space (with certain restrictions rel. $\mathcal{H}_N(\mathcal{F}^{(0)})$ see Th. XIVB). The operations in this case are*

$$\sum_{m_1,m_2,\ldots} \Phi_N\left(\left\{\prod_k E_n^{m_1,m_2,\ldots,m_k}\right\}\right); \quad m_i = 1, 2, 3, \ldots, \quad E_n^{m_1,m_2,\ldots,m_k} \in \mathcal{F}$$

$$\sum_{m_1,m_2,\ldots} \Phi_N\left(\left\{\sum_k E_n^{m_1,m_2,\ldots,m_k}\right\}\right); \quad m_i = 1, 2, 3, \ldots, \quad E_n^{m_1,m_2,\ldots,m_k} \in \mathcal{G}.$$

## § 4. Further Properties of the Class $\mathcal{H}_N(\mathcal{K})$ [1]).

**23.** In this paragraph we shall study some properties of the class $\mathcal{H}_N(\mathcal{K})$, chiefly those properties of $\mathcal{K}$ which are unaffected by the operation $\mathcal{H}_N$. We mean by this that these properties are such that if they belong to a class $\mathcal{K}$ then they belong also to $\mathcal{H}_N(\mathcal{K})$. Such properties we shall call transitive i. e.

**Definition 15.** A property $(\mathcal{S})$ of classes of sets is called *transitive* if it is possesed by $\mathcal{H}_N(\mathcal{K})$ (with an arbitrary base) every time when it is possesed by $\mathcal{K}$.

A simple example of a transitive property is given by the proposition:

If $\mathcal{K}$ has $2^{\aleph_0}$ elements, then $\mathcal{H}_N(\mathcal{K})$ has the same power [2]).

Some of the theorems of this paragraph are true for any (even non positive) analytical operation $\Psi(\{E_i\})$ and the corresponding classes $\mathcal{H}(\mathcal{K})$ (see def. 4).

Some of the properties considered are not transitive; properties considered here are the following:

[1]) This theorem comprises and generalises all the results concerning this question viz: Our own theorem (see our note in C. R. th. 4°) generalised by Miss Braun (See Braun) and theorem of Mr. Sierpiński (Sierpiński V).

[1]) A considerable part of the theorems of this § are already known. We deemed better, however, to give the full proofs of all the theorems for the sake of systemacity. In all cases when the theorems or the proofs are not new, we give the literature.

[2]) Cp. Hausdorff, p. 181.

1) The property of having a „universal series“ (see def. 16 bis below). This is a transitive property.

2) The property of having a „universal set“ (def. 16). This is not a transitive property, but 1) always implies 2).

3) The property of having a set $E \epsilon \mathscr{K}$ such that $C(E)$ non $\epsilon\, \mathscr{K}$. This property of course is not transitive but it is closely connected with property 2).

4) The topological invariance of a class. This property is not transitive unless $\mathscr{K} = \mathscr{G}$.

**24.** Let classes $\mathscr{K}$ and $\mathscr{K}^{(0)}$ be defined in the spaces resp. $R$ and $R^{(0)} = R \times R^{(1)}$, and denote (for simlicity)

$$(24,1) \qquad \overset{a}{E} = \mathscr{E}_x((x, a) \epsilon E)$$

for any $E \subset R^{(0)}$ and $a \epsilon R^{(1)}$ and in the same manner

$$(24,2) \qquad \underset{a}{E} = \mathscr{E}_x((a, x) \epsilon E)$$

for any $E \subset R^{(0)}$ and $a \epsilon R$.

We have then the following definitions.

**Definition 16.** A set $E \subset R^{(0)}$ is called *universal* for the class $\mathscr{K}$ if whatever be $E \epsilon \mathscr{K}$ there exists such $a \epsilon R^{(1)}$ that

$$(24,3) \qquad \overset{a}{\boldsymbol{E}} = E.$$

Note that if $\boldsymbol{E}$ is a universal set for $\mathscr{K}$ then $C(\boldsymbol{E})$ is a universal set for $\mathscr{K}_c$.

**Definition 16 bis.** A sequence $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots$ of subsets of $R^{(0)}$ is called a *universal series* for $\mathscr{K}$ if whatever be sequence $E_1, E_2, \ldots$ of sets belonging to $\mathscr{K}$ there exists such $a \epsilon R^{(1)}$ that

$$(24,4) \qquad \overset{a}{\boldsymbol{E}}_\nu = E_\nu \text{ for any } \nu.$$

Note that if $\{\boldsymbol{E}_\nu\}$ is a universal series for $\mathscr{K}$ then

1) any subsequence of $\{\boldsymbol{E}_\nu\}$ is a universal series
2) all the sets $\boldsymbol{E}_\nu$ are universal sets for $\mathscr{K}$ and
3) the sequence $\{C(\boldsymbol{E}_\nu)\}$ is a universal series for $\mathscr{K}_c$.



**Definition 17.** The pair $(\mathscr{K}, \mathscr{K}^{(0)})$ possesses the property $(U)$ if there exists a set $\boldsymbol{E} \epsilon \mathscr{K}^{(0)}$ universal for $\mathscr{K}$.

**Definition 17 bis.** The pair $(\mathscr{K}, \mathscr{K}^{(0)})$ possesses the property $(U_s)$ if there exists a universal series for $\mathscr{K}$, $\{\boldsymbol{E}_\nu\}$ such that all $\boldsymbol{E}_\nu \epsilon \mathscr{K}^{(0)}$.

**Theorem XV.** *The property $(U_s)$ is transitive, i. e. if $(\mathscr{K}, \mathscr{K}^{(0)})$ possesses $(U_s)$ and if $\Psi$ is any $\delta s$-operation (or more generally any analytical operation) and $\mathscr{H}$ is the corresponding operation upon classes, then $(\mathscr{H}(\mathscr{K}), \mathscr{H}(\mathscr{K}^{(0)}))$ possesses $(U_s)$.*

Proof. Let $\{\boldsymbol{E}_\nu\}$, $(\boldsymbol{E}_\nu \epsilon \mathscr{K}^{(0)})$, be a universal series for $\mathscr{K}$ and let

$$(24,5) \qquad \boldsymbol{H}_k = \underset{n}{\Psi}(\{\boldsymbol{E}_{2^{k-1}(2n-1)}\}).$$

The sequence $\{\boldsymbol{H}_k\}$ is a universal series for $\mathscr{H}(\mathscr{K})$. (Evidently

$$\boldsymbol{H}_k \epsilon \mathscr{H}(\mathscr{K}^{(0)})).$$

In fact let $H_1, H_2, \ldots$ be any sequence of sets belonging to $\mathscr{H}(\mathscr{K})$. Then

$$H_k = \Psi(\{K_n^k\}); \quad K_n^k \epsilon \mathscr{K}$$

or denoting $E_{2^{k-1}(2n-1)} = K_n^k$ we have

$$(24,6) \qquad H_k = \underset{n}{\Psi}(\{E_{2^{k-1}(2n-1)}\}); \quad E_p \epsilon \mathscr{K}.$$

As $\{\boldsymbol{E}_\nu\}$ is a universal series for $\mathscr{R}$ we can find such $a \epsilon R^{(1)}$ that

$$E_\nu = \overset{a}{\boldsymbol{E}}_\nu \text{ for any } \nu.$$

But then (by def. 1 and (24,1), (24,5), (24,6))

$$\overset{a}{\boldsymbol{H}}_k = \underset{n}{\overset{a}{\Psi}}(\{\boldsymbol{E}_{2^{k-1}(2n-1)}\}) = \underset{n}{\Psi}(\{\overset{a}{\boldsymbol{E}}_{2^{k-1}(2n-1)}\})\,^{1)} = H_k$$

for any $k$.

Which show that $\{\boldsymbol{H}_k\}$ is a universal series for $\mathscr{H}(\mathscr{K})$ q. e. d.

**25. Theorem XVI.** *If $R$ has a countable system of neighbourhoods and $R^{(1)} = J$ then $(\mathscr{G}, \mathscr{G}^{(0)})$ possesses the property $(U_s)$.*

1) See lemma 2, art. **6**.

Proof. [1]) Enumerate all the neighbourhoods of $R$: $V_1, V_2, \dots$ and let $\boldsymbol{E}_k$ be the set of all such points $(x, y) \in R^{(0)}$ that

$$y = \frac{1|}{|q_1} + \frac{1|}{|q_2} + \dots \in J$$

(25,1) $$x \in \sum_{n=1}^{\infty} V_{q_{2^{k-1}(2n-1)}}.$$

$\boldsymbol{E}_k$ are open for if a point $(x, y) \in \boldsymbol{E}_k$ then for a certain $n$:

$$x \in V_{q_{2^{k-1}(2n-1)}}; \quad y = \frac{1|}{|q_1} + \frac{1|}{|q_2} + \dots$$

but then any point $(x', y')$ which belongs to $V_{q_{2^{k-1}(2n-1)}} \times \delta_{q_1, q_2, \dots, q_{2^{k-1}(2n-1)}}$ belongs to $\boldsymbol{E}_k$ (by (25,1)) so that we may write

(25,2) $$\boldsymbol{E}_k = \sum_{(q_1, q_2, \dots, q_{2^{k-1}(2n-1)})} V_{q_{2^{k-1}(2n-1)}} \times \delta_{q_1, q_2, \dots, q_{2^{k-1}(2n-1)}}$$

which shows that $\boldsymbol{E}_k$ is open.

We have now to prove that $\{\boldsymbol{E}_k\}$ is a universal series for $\mathcal{G}$. Let in fact $\{E_k\}$ be a sequence of open (in $R$) sets; then

$$E_k = \sum_n V_{s_n^{(k)}}$$

or denoting $q_{2^{k-1}(2n-1)} = \varrho_n^{(k)}$

(25,3) $$E_k = \sum_n V_{q_{2^{k-1}(2n-1)}}.$$

Denote now

$$y = \frac{1|}{|q_1} + \frac{1|}{|q_2} + \dots$$

We have then according to (25,2) and (25,3)

$$\overset{y}{\boldsymbol{E}}_k = E_k \text{ for any } k \qquad \text{q. e. d.}$$

**Remark I.** *The pair* $(\mathcal{F}, \mathcal{F}^{(0)})$ *being identical with* $(\mathcal{G}_c, \mathcal{G}_c^{(0)})$ (see notations) *possesses also the property* $(U_s)$.

[1]) The method here adopted is that of Mr. Sierpiński, Fund. Math. t. VII p. 198. Cp. Kolmogoroff, p. 418 and Kantorovitch, p. 16.



**Remark II.** We can replace in this theorem the condition $R^{(1)} = J$ be the following: $R^{(1)}$ *contains a set* $J'$ *homeomorphic to* $J$.

In fact let $J' = \varphi J$ and denote $\varphi(x, y) = (x, \varphi y)$. Let now

$$\boldsymbol{E}'_k = C(\varphi\, C(\boldsymbol{E}_k)).$$

Then, as one may easily verify, $\{\boldsymbol{E}'_k\}$ is a universal series for $\mathcal{G}$.

Approaching Theorems XV and XVI we have the following important

**Corollary** [1]). *If $\Psi$ is an arbitrary analytical operation, $\mathcal{K}$ the corresponding operation on classes, $R$ is a topological space which possesses a countable system of neighbourhoods, $R^{(1)}$ is a space containing a set homeomorphic to $J$, and $R^{(0)} = R \times R^{(1)}$; then the pair $(\mathcal{K}(\mathcal{G}), \mathcal{K}(\mathcal{G}^{(0)}))$ possesses the property $(U_s)$ and consequently the property $(U)$.*

**26.** A property of classes of sets, closely related to the property $(U)$ is that of having a set (belonging to $\mathcal{K}$) whose complement does not belong to $\mathcal{K}$. In order to show that under certain conditions this property is a consequence of the property $(U)$ we shall prove first the following:

**Lemma** [2]). *Let*: 1) $\boldsymbol{E}$ *be a universal set for* $\mathcal{K}(\boldsymbol{E} \subset R^{(0)})$.

2) $M \subset R^{(0)}$ *be a set such that*: a) *for any* $x \in R$, $\underset{x}{M}$ *consists of not more than one point*; b) $Pr_{R^{(1)}} M = R^{(1)}$.

Then

(26,1) $$C(Pr_R(\boldsymbol{E} \cdot M)) \text{ non } \in \mathcal{K}.$$

Proof. Suppose the contrary, i. e. that

$$C(Pr_R(\boldsymbol{E} M)) \in K.$$

Then (by def. 16 and the first condition of our lemma) there exists such $y \in R^{1)}$ that

(26,2) $$\overset{y}{\boldsymbol{E}} = C(Pr_R(\boldsymbol{E} M)).$$

But we have (by condition 2b of the lemma)

$$y \in Pr_{R^{(1)}} M.$$

[1]) This theorem in a less general form was proved by Kolmogoroff, p. 418 and Kantorovitch, p. 17.

[2]) Cp. Sierpiński (F. M. t. XIV, p. 83).

Therefore there exists such $x \in R$ that $(x, y) \in M$.

Two cases are possible *a priori*:

(26,3) 1) $(x, y) \in \boldsymbol{E}$.

Then

$$(x, y) \in \boldsymbol{E} \cdot M$$

$$x \in Pr_R(\boldsymbol{E}M)$$

$$x \text{ non } \in C(Pr_R(\boldsymbol{E}M)) = \overset{y}{\boldsymbol{E}} \qquad \text{(by (26,2))}$$

i. e.

$$(x, y) \text{ non } \in \boldsymbol{E}$$

in contradiction with (26,3).

(26,4) 2) $(x, y)$ non $\in \boldsymbol{E}$.

Then

$$(x, y) \text{ non } \in \boldsymbol{E}M$$

but $\underset{x}{M} = (y)$ (by condition 2a of the lemma); therefore

$$x \text{ non } \in Pr_R \boldsymbol{E}M$$

$$x \in C(Pr_R(\boldsymbol{E}M)) = \overset{y}{\boldsymbol{E}} \qquad \text{(by (26,2))}$$

i. e.

$$(x, y) \in \boldsymbol{E}$$

in contradiction with (26,4).

Thus both suppositions (viz. that $(x, y) \in \boldsymbol{E}$ and that $(x, y)$ non $\in \boldsymbol{E}$) lead into contradiction. Hence our original supposition (viz. that $C(Pr_R(\boldsymbol{E}M)) \in \mathscr{K}$) is false and the lemma is proved.

**Theorem XVII.** *If*: 1) *$R$ and $R^{(1)}$ are topological spaces and* $R^{(0)} = R \times R^{(1)}$

2) *$M$ is a set such that*:

a) *$\underset{x}{M}$ consists of one point which we shall denote $\varphi x$*

b) $Pr_{R^{(1)}} M = R^{(1)}$

3) *$\mathscr{K}^{(0)}$ is a class of sets in $R^{(0)}$*

4) *$\mathscr{K}$ is a class of sets in $R$ containing all the sets of the form $Pr_R(K \cdot M)$ where $K \in \mathscr{K}^{(0)}$*

5) *$(\mathscr{K}, \mathscr{K}^{(0)})$ possesses the property $(U_s)$*

6) *$N$ is a set of irrational numbers*;

*then there exists such set $L \in \mathscr{H}_N(\mathscr{K})$ that $C(L)$ non $\in \mathscr{H}_N(\mathscr{K})$.*



Proof: First of all from the definition of $M$ (2a) follows easily that for any

$$\boldsymbol{E}_1, \boldsymbol{E}_2, \dots \ (\boldsymbol{E}_i \subset R^{(0)})$$

we have:

$$Pr_R(\Pi \boldsymbol{E}_i \cdot M) = \Pi \, Pr_R \boldsymbol{E}_i M$$

whence:

$$Pr_R(\Phi_N(\{\boldsymbol{E}_n \cdot M\})) = \Phi_N(\{Pr_R(\boldsymbol{E}_n \cdot M)\}).$$

Now if $\{\boldsymbol{E}_\nu\}$ is a universal series for $\mathscr{K}$ $(\boldsymbol{E}_\nu \in \mathscr{K}^{(0)})$ then the sets

$$\boldsymbol{E} = \Phi_N(\{\boldsymbol{E}_n\})$$

and $M$ and the class $\mathscr{H}_N(\mathscr{K})$ satisfy all the conditions of our lemma and therefore

$$C(L) = C(Pr_R(\boldsymbol{E}M)) \text{ non } \in \mathscr{H}_N(\mathscr{K}).$$

We have only to show that $L \in \mathscr{H}_N(\mathscr{K})$.

In fact

$$L = Pr\, \boldsymbol{E}M = Pr\, \Phi_N(\{\boldsymbol{E}_n \cdot M\}) = \Phi_N(\{Pr(\boldsymbol{E}_n \cdot M)\}) \in \mathscr{H}_N(\mathscr{K})$$

(by cond. 4) q. e. d.

The most interesting cases are when $\mathscr{K} = \mathscr{F}$ and when $\mathscr{K} = \mathscr{G}$.

We have then the following:

**Theorem XVIIbis.** *If $R$ is a space possessing a countable system of neighbourhoods and containing a perfect set and $N$ is a set of irrationnal numbers then there exist such sets $L_1 \in \mathscr{H}_N(\mathscr{G})$ and $L_2 \in \mathscr{H}_N(\mathscr{F})$ that*

$$C(L_1) \text{ non } \in \mathscr{H}_N(\mathscr{G}); \quad C(L_2) \text{ non } \in \mathscr{H}_N(\mathscr{F}).$$

Proof. Let

$$R^{(1)} = R; \quad R^{(0)} = R \times R; \quad \mathscr{K}^{(0)} = \mathscr{G}^{(0)} \text{ or } \mathscr{F}^{(0)}, \quad \mathscr{K} = \mathscr{G} \text{ or } \mathscr{F}.$$

Then all the conditions of Theorem XVII are satisfied.

In fact: 1) $R$ and $R^{(1)}$ are topological spaces and $R^{(0)} = R \times R^{(1)}$.

2) Denoting $\varphi x = x$ and $M$ the set of points $(x, x)$ we have:

a) $\underset{x}{M} = (x) = (\varphi x)$; b) $Pr_{R^{(1)}} M = R^{(1)} = R$.

3) $\mathscr{K}^{(0)}$ is a class of sets in $R^{(0)}$.

4) $\mathscr{K}$ is a class of sets in $R$; besides $\mathscr{K}$ evidently contains all the sets of the form $Pr_R(K \cdot M)$; $K \in \mathscr{K}^{(0)}$

5) $(\mathscr{H}, \mathscr{H}^{(0)})$ possesses the property $(U_s)$ (Th. XVI and remarks I and II)

6) $N$ is a set of irrational numbers.

We have thus all the conditions of Theorem XVII satisfied and therefore its conclusion is true, i. e. there exists a set $L \in \mathscr{H}_N(\mathscr{H})$ such that $C(L)$ non $\in \mathscr{H}_N(\mathscr{H})$.

As $\mathscr{H} = \mathscr{F}$ or $\mathscr{H} = \mathscr{G}$ our theorem is now fully demonstrated.

**27.** We come now to another important property of $\mathscr{H}_N(\mathscr{G})$ viz. its topological invariance.

**Theorem XVIII.** *If $R$ is a locally compact metric space and the class $\mathscr{G}$ of sets open in $R$ is a topological invariant then $\mathscr{H}_N(\mathscr{G})$ is a topological invariant.*

Proof. We shall prove this theorem by a convenient modification of a reasoning of Mr. Sierpiński [1]).

We may suppose $N$ complete $(N = \tilde{N})$ (def 8, Art. 7).

Decompose $N$ (and $\mathfrak{N} \sim N$) into two sets $N_1 \sim \mathfrak{N}_1$ and $N_2 \sim \mathfrak{N}_2$ (see Art. 3), where $\mathfrak{N}_2$ is the set of all the sequences of $\mathfrak{N}$ which contain only a finite number of different elements and

$$\mathfrak{N}_1 = \mathfrak{N} - \mathfrak{N}_2.$$

Let now

$$P = \Phi_N(\{G_n\}); \quad G_n \in \mathscr{G}.$$

and let $Q$ be homeomorphic to $P$ $(P = \psi\, Q;\ Q = \varphi\, P)$.

Denote

$$P_1 = \Phi_{N_1}(\{G_n\}); \quad Q_1 = \varphi P_1$$
$$P_2 = \Phi_{N_2}(\{G_n\}); \quad Q_2 = \varphi P_2.$$

Evidently: 1) $P = P_1 + P_2$; $Q = Q_1 + Q_2$ (See 3,2))
2) $P_2 \in \mathscr{G}$; $Q_2 \in \mathscr{G}$.

(because $P_2$ is open by the definition of $\mathfrak{N}_2 \sim N_2$ and $Q_2$ as a set homeomorphic to $P_2$, $\mathscr{G}$ being a topological invariant).

Clearly we may suppose $G_n \supset P_2$ for every $n$ because

$$P = \Phi_N(\{G_n + P_2\}).$$

[1]) W. Sierpiński I. See also Alexandroff, Recueil Math. de Moscou v. 31, p. 310.



We shall call two corteges (finite systems of natural numbers) $(\nu_1, \nu_2, \ldots, \nu_k)$ and $(\mu_1, \mu_2, \ldots, \mu_l)$ *conjugated* if

1) $(\nu_1, \nu_2, \ldots, \nu_k, \mu_1, \mu_2, \ldots, \mu_l, \mu_l, \mu_l, \ldots) \in \mathfrak{N}$ (and therefore $\in \mathfrak{N}_2$)
2) $(\nu_1, \nu_2, \ldots, \nu_k)$ and $(\mu_1, \mu_2, \ldots, \mu_l)$ possess no common element.

Let us now define for every $n$ and for every

$$y \in \varphi(G_n \cdot P) - Q_2$$

a sphere $S_y^{(n)}$ with centre at $y$ so that denoting

$$H_n = \sum_{y \in \varphi(G_n \cdot P) - Q_2} S_y^{(n)} \tag{27,1}$$

the following conditions are fulfilled:

1° The radius of $S_y^{(n)}$ does not exceed $\frac{1}{n}$

2° a) $\overline{\psi(\overline{S_y^{(n)}} \cdot Q)}$ is compact; b) $\overline{\psi(S_y^{(n)} \cdot Q)} \subset G_n$ whence

$$H_n \cdot Q + Q_2 = \varphi(G_n \cdot P)$$

3° If

$$y \text{ non} \in \overline{H_{\nu_1} \cdot H_{\nu_2} \cdot \ldots \cdot H_{\nu_k}} \quad (\nu_i < n)$$

then

$$S_y^{(n)} \cdot \overline{H_{\nu_1} \cdot H_{\nu_2} \cdot \ldots \cdot H_{\nu_k}} = 0$$

4° If $(\nu_1, \nu_2, \ldots, \nu_k)$ and $(\mu_1, \mu_2, \ldots, \mu_l)$ are conjugated then

$$\overline{H_{\nu_1} \cdot H_{\nu_2} \cdot \ldots \cdot H_{\nu_k}} \cdot \varphi(G_{\mu_1} \cdot G_{\mu_2} \cdot \ldots \cdot G_{\mu_l} \cdot P - P_2) = 0.$$

In order not to break the integrity of our reasoning we suppose here that the spheres $S_y^{(n)}$ are already constructed (we shall prove that such spheres exist in the next Art.).

Then I say that

$$Q = \Phi_N(\{H_n + Q_2\}); \quad (H_n + Q_2 \text{ are evidently open}).$$

In fact first of all evidently

$$Q = \varphi P = \varphi\, \Phi_N(\{G_n\}) = \Phi_N(\{\varphi(G_n \cdot P)\}) =$$
$$= \Phi_N(\{H_n \cdot Q + Q_2\}) \subset \Phi_N(\{H_n + Q_2\}).$$

We have only to prove that

$$\Phi_N(\{H_n + Q_2\}) \subset Q.$$

Let

$$y \in \Phi_N(\{H_n + Q_2\})$$

then there exists such

$$\xi = \frac{1|}{|n_1} + \frac{1|}{|n_2} + \ldots \in N$$

that

$$y \in \prod_i (H_{n_i} + Q_2) = \prod_i H_{n_i} + Q_2$$

Two cases only are possible:

1) $y \in Q_2$. Then evidently $y \in Q$

2) $y \in \prod_i H_{n_i}$. Then $\xi \in N_1$. In fact suppose $\xi \in N_2$; then among the numbers $n_1, n_2, \ldots$ there is only a finite number of different, say $\nu_1, \nu_2, \ldots, \nu_k$. (So that for any $i$ there exists such $\gamma_i \leqslant k$ that $n_i = \nu_{\gamma_i}$); we may suppose besides that $\nu_1 < \nu_2 < \ldots < \nu_k$ and we have

$$y \in H_{\nu_1} \cdot H_{\nu_2} \cdot \ldots \cdot H_{\nu_k}$$

i. e. for any $i \leqslant k$ there exists such

(27,2) $$y_i \in \varphi(G_{\nu_i} \cdot P) - Q_2$$

that

(27,3) $$y \in S_{y_i}^{\nu_i}.$$

But then we come to a contradiction for it is impossible:

a) that $k = 1$, because then

$$\frac{1|}{|\nu_1} + \frac{1|}{|\nu_1} + \frac{1|}{|\nu_1} + \ldots \in N_2$$

and hence $G_{\nu_1} \subset P_2$ so that $\varphi(G_{\nu_1} \cdot P) - Q_2 = 0$ in contradiction with (27,2)

b) that

$$y_k \in \overline{H_{\nu_1} \cdot H_{\nu_2} \cdot \ldots \cdot H_{\nu_{k-1}}} \quad (k > 1)$$

because $(\nu_k)$ is conjugated with $(\nu_1, \nu_2, \ldots, \nu_{k-1})$ and (by property 4°)

$$\varphi(G_{\nu_k} \cdot P - P_2) \cdot \overline{H_{\nu_1} \cdot H_{\nu_2} \cdot \ldots \cdot H_{\nu_{k-1}}} = 0$$

c) that

$$y_k \text{ non} \in \overline{H_{\nu_1} \cdot H_{\nu_2} \cdot \ldots \cdot H_{\nu_{k-1}}} \quad (k > 1)$$



because then (by prop. 3)

$$S_{y_k}^{\nu_k} \cdot \overline{H_{\nu_1} \cdot H_{\nu_2} \cdot \ldots \cdot H_{\nu_{k-1}}} = 0$$

in contradiction with (27,3).

So the supposition $\xi \in N_2$ leads to a contradiction. Therefore $\xi \in N_1$. Evidently we may suppose $n_1 < n_2 < n_3 < \ldots$

Define [1]) as before for any $i$ such

$$y_i \in \varphi(G_{n_i} \cdot P) - Q_2$$

that

$$y \in S_{y_i}^{(n_i)}$$

Evidently $\lim y_i = y$ (for $\varrho(y, y_i) < \frac{1}{n}$ by property 1°).

We have moreover for any $i$ and for $k$ great enough $(k > k_i)$

$$y_n \in S_{y_i}^{(n_i)}$$

Denote now $x_i = \psi y_i$ and let $x$ be any limit point of $\{x_i\}$ (such point exists because $\overline{\psi(S_y^{(n)} \cdot Q)}$ is compact).

As $y_k \in S_{y_i}^{(n_i)}$ if $k > k_i$ we have $x_k \in \overline{\psi(S_{y_i}^{(n_i)} \cdot Q)}$ whence

$$x \in \psi(S_{y_i}^{(n_i)}) \subset G_{n_i} \qquad \text{(by property } 2^0\text{)}$$

for any $i$. Or

$$x \in \prod_i G_{n_i} \subset P$$

whence $\varphi x \in Q$. But $x$ being a limit point of $\{x_i\}$, $\varphi x$ must be a limit point of $\{\varphi x_i\}$ i. e. of $\{y_i\}$.

Hence $\varphi x = y$; $y \in Q$ q. e. d.

**28.** We have to show now that it is really possible to determine a system of spheres $\{S_y^n\}$ satisfying all the four conditions 1°—4° of the preceding article. We shall prove it by induction.

Suppose then that we have built the spheres $\{S_y^n\}$ for any natural $n < n_0$ $(n_0 \geqslant 1)$ and for any $y \in \varphi(G_n \cdot P) - Q_2$ so that the conditions 1°—4° (if in 4° we suppose all the $\nu_i < n_0$) are fulfilled. In the initial case $(n_0 = 1)$ our supposition is evidently true there being no natural $n < 1$.

[1]) All the rest of the proof down to the end of the article is the same as in Sierpiński I.

Let now

$$y \in \varphi(G_{n_0} \cdot P) - Q_2$$

and let $\psi y = x$; evidently $x \in G_{n_0}$; let $T_x^{n_0}$ be a sphere having $x$ for its centre and such that $\overline{T_x^{n_0}}$ is compact and

$$\overline{T_x^{n_0}} \subset G_{n_0}$$

$T_x^{n_0} \cdot P$ being open in $P$, $\varphi(T_x^{n_0} \cdot P)$ is open in $Q$; besides $\varphi(T_x^{n_0} \cdot P)$ contains $y$. Therefore we can find such sphere $S$ having $y$ for its centre, that

$$S \cdot Q \subset \varphi(T_x^{n_0} \cdot P)$$

and therefore

$$\psi(S \cdot Q) \subset T_x^{n_0} \cdot P \subset T_x^{n_0}$$

whence

$$\overline{\psi(S \cdot Q)} \subset \overline{T_x^{n_0}} \subset G_{n_0}.$$

Denote $r_{n_0}(y)$ the radius of $S$.

Take any cortege $(\nu_1, \nu_2, \ldots, \nu_k)$ such that

1) $\nu_k = n_0$; $\nu_i < n_0$ for any $i < k$
2) either $k = 1$ or

$$y \in \overline{H_{\nu_1} \cdot H_{\nu_2} \cdot \ldots \cdot H_{\nu_{k-1}}}$$

and denote $T = R$ if $k = 1$ and

(28,1) $$T = \overline{H_{\nu_1} \cdot H_{\nu_2} \cdot \ldots \cdot H_{\nu_{k-1}}} \quad \text{if} \quad k > 1.$$

Consider all the corteges $(\mu_1, \mu_2, \ldots, \mu_l)$ conjugated with $(\nu_1, \nu_2, \ldots, \nu_k)$. The sets

$$\varphi(G_{n_0} \cdot P)$$

and

$$\varphi(G_{\mu_1} \cdot G_{\mu_2} \cdot \ldots \cdot G_{\mu_l} \cdot P)$$

are open in $Q$ and consequently the sets

$$T \cdot \varphi(G_{n_0} \cdot P)$$

and

$$T\varphi(G_{\mu_1} \cdot G_{\mu_2} \cdot \ldots \cdot G_{\mu_l} \cdot P)$$

are open in $T \cdot Q$. On the other hand

(28,2) $$T\varphi(G_{n_0} \cdot P) \cdot \varphi(G_{\mu_1} \cdot G_{\mu_2} \cdot \ldots \cdot G_{\mu_l} \cdot P) \subset Q_2.$$



In fact if $k = 1$ then (28,2) follows from the definitions of $Q_2$ and of conjugated corteges. If $k > 1$ then we have by 4° (which we have supposed true for $\nu_i < n_0$) and by (28,1)

$$T\varphi(G_{n_0} \cdot G_{\mu_1} \cdot G_{\mu_2} \cdot \ldots \cdot G_{\mu_l} \cdot P - P_2) = 0$$

(because the corteges $(\nu_1, \nu_2, \ldots, \nu_{k-1})$ and $(n_0, \mu_1, \mu_2, \ldots, \mu_l)$ are conjugated).

But

$$\varphi(G_{n_0} \cdot G_{\mu_1} \cdot G_{\mu_2} \cdot \ldots \cdot G_{\mu_l} \cdot P - P_2) =$$
$$= \varphi(G_{n_0} \cdot G_{\mu_1} \cdot \ldots \cdot G_{\mu_l} \cdot P) - \varphi(P_2) = \varphi(G_{n_0} \cdot P) \cdot \varphi(G_{\mu_1} \cdot \ldots \cdot G_{\mu_l} \cdot P) - Q_2.$$

Hence (28,2).

From (28,2) follows that the sets

$$Z_1 = T \cdot \varphi(G_{n_0} \cdot P) - Q_2$$

and

$$Z_2 = T\Sigma\varphi(G_{\mu_1} \cdot G_{\mu_2} \cdot \ldots \cdot G_{\mu_l} \cdot P) - Q_2$$

(where the summation is extended over all corteges $(\mu_1, \mu_2, \ldots, \mu_l)$ conjugated with $(\nu_1, \nu_2, \ldots, \nu_k)$) are separated [1]) and therefore $\varrho(y, Z_2) > 0$ (because $y \in Z_1$). We shall denote this number $2\varrho_{\nu_1, \nu_2, \ldots, \nu_k}$.

If

$$y \text{ non } \in \overline{H_{\nu_1} \cdot H_{\nu_2} \cdot \ldots \cdot H_{\nu_{k-1}}}$$

and $\nu_k = n_0$ we shall denote

$$\varrho_{\nu_1, \nu_2, \ldots, \nu_k} = \varrho(y, \overline{H_{\nu_1} \cdot H_{\nu_2} \cdot \ldots \cdot H_{\nu_{k-1}}}).$$

We can define now for any $y \in \varphi(G_{n_0} \cdot P) - Q_2$ the sphere $S_y^{n_0}$ as the sphere having $y$ for its centre and the least of the numbers:

$$\frac{1}{n_0}, \quad r_{n_0}(y), \quad \varrho_{\nu_1, \nu_2, \ldots, \nu_k} \quad \text{for every} \quad (\nu_1, \nu_2, \ldots, \nu_k)$$

such that

$$\nu_i < n_0 \quad (i < k) \quad \nu_k = n_0$$

for its radius. We have to prove that 1°—4° are satisfied.

[1]) We can apply the following theorem: if $\Gamma_1 \subset E$ and $\Gamma_2 \subset E$ are open in $E$ then $\Gamma_1 - \Gamma_2$ and $\Gamma_2 - \Gamma_1$ are separated i. e. neither contains limit points of the other.

$1^0$ is evidently satisfied
$2^0$ is satisfied by the definition of $r_{n_0}(y)$
$3^0$ is satisfied by the definition of $\varrho_{\nu_1,\nu_2,\dots,\nu_k}$ in case

$$y \text{ non } \epsilon\, \overline{H_{\nu_1}\cdot H_{\nu_2}\cdot\ldots\cdot H_{\nu_{k-1}}}$$

$4^0$. Let $(\nu_1, \nu_2, \dots, \nu_k)$ and $(\mu_1, \mu_2, \dots, \mu_l)$ be conjugated and let $\nu_i \leqslant n_0$ $(i \leqslant k)$. If $\nu_i < n_0$ for any $i \leqslant k$ then $4^0$ is satisfied by supposition; in the case when $n_0$ is among the numbers $\nu_i$, we have evidently may suppose $\nu_i < n_0$ for any $i < k$ and $\nu_k = n_0$. We have to prove that

$$\overline{H_{\nu_1}\cdot H_{\nu_2}\cdot\ldots\cdot H_{\nu_k}}\cdot\varphi(G_{\mu_1}\cdot\ldots\cdot G_{\mu_l}\cdot P)\subset Q_2. \tag{28,3}$$

Let

$$y\,\epsilon\, T\cdot\varphi(G_{\mu_1}\cdot G_{\mu_2}\cdot\ldots\cdot G_{\mu_l}\cdot P) - Q_2 \subset Z_2$$

then

$$\varrho(y, Z_1) = \varrho > 0.$$

Now if

$$y'\,\epsilon\, H_{\nu_1}\cdot H_{\nu_2}\cdot\ldots\cdot H_{\nu_k} \quad (\nu_k = n_0)$$

then there exists such

$$y''\,\epsilon\,\varphi(G_{n_0}\cdot P) - Q_2$$

that

$$y'\,\epsilon\, H_{\nu_1}\cdot H_{\nu_2}\cdot\ldots\cdot H_{\nu_k}\cdot S^{(n_0)}_{y''} \quad (\text{if } k>1). \tag{28,4}$$

Consequently $y''\,\epsilon\, T$ (See (28,1)) (because by the definition of $S^{(n_0)}_{y}$ if $y''$ non $\epsilon\, T$ then by $3^0$ $S^{(n_0)}_{y''}\cdot T = 0$ is contradiction with (28,4)). Hence $y''\,\epsilon\, Z_1$ and consequently

$$\varrho(y, y'') = r \geqslant \varrho;\quad \varrho(y'', y') < \frac{1}{2}d(S^{(n_0)}_{y''}) \leqslant \frac{1}{2}\varrho(y'', Z_2) \leqslant \frac{r}{2}$$

(by the definition of $S^{(n_0)}_{y}$) i. e.

$$\varrho(y', y) > \frac{r}{2} \geqslant \frac{\varrho}{2}$$

for any

$$y'\,\epsilon\, H_{\nu_1}\cdot H_{\nu_2}\cdot\ldots\cdot H_{\nu_k}.$$

We see thus that the distance between $y$ and the points of $H_{\nu_1}\cdot H_{\nu_2}\cdot\ldots\cdot H_{\nu_k}$ is limited from below by $\frac{\varrho}{2}$ and therefore

$$y \text{ non } \epsilon\, \overline{H_{\nu_1}\cdot H_{\nu_2}\cdot\ldots\cdot H_{\nu_k}}.$$



But $y$ is an arbitrary point of

$$T\cdot\varphi(G_{\mu_1}\cdot G_{\mu_2}\cdot\ldots\cdot G_{\mu_l}\cdot P) - Q_2$$

whence

$$\overline{H_{\nu_1}\cdot H_{\nu_2}\cdot\ldots\cdot H_{\nu_k}}\cdot T\cdot\varphi(G_{\mu_1}\cdot G_{\mu_2}\cdot\ldots\cdot G_{\mu_l}\cdot P) - Q_2 = 0$$

or taking into account that

$$\overline{H_{\nu_1}\cdot H_{\nu_2}\cdot\ldots\cdot H_{\nu_k}}\subset T$$

we have (28,3).

**Remark.** The class $\mathscr{H}_N(\mathscr{F})$ is not always a topological invariant and that even in the case when $R = I$ (see Supplementary Papers). We have, however, the following theorem:

„If $P\,\epsilon\,\mathscr{H}_N(\mathscr{F})$ and if $\varphi$ is a homeomorphic transformation of $R$ then $\varphi P\,\epsilon\,\mathscr{H}_N(\mathscr{F})$" (ibid.).

(Continuation in vol. XIX).