une somme de $\aleph_1$ ensembles de la famille F: nous aurons évidemment $\overline{\overline{S}} \leqslant \aleph_1$. Donc, d'après la proposition **P**, l'ensemble S n'est homéomorphe à aucun ensemble linéaire. Or, il en résulte que $\overline{\overline{S}} = 2^{\aleph_0}$, puisque, comme on sait, tout ensemble plan de puissance $< 2^{\aleph_0}$ est homéomorphe d'un ensemble linéaire [1]).

Les formules $\overline{\overline{S}} \leqslant \aleph_1$ et $\overline{\overline{S}} = 2^{\aleph_0}$ donnent $2^{\aleph_0} = \aleph_1$.

L'implication **P** $\rightarrow$ **H** est ainsi établie.

L'équivalence des propositions **P** et **H** est ainsi démontrée.

[1]) C'est p. e. une conséquence facile du théorème que j'ai démontré dans *Fund. Math.* t. II, p. 89.

On the Theory of Trigonometric Series VII.

By

S. Verblunsky (Manchester).

I. Introduction.

§ 1. In §§ 2—4 of this paper, we give an extension of the general Denjoy integral (or total), and we define a class of functions which bear the same relation to the new process of integration as the resoluble functions of Denjoy bear to the process of totalisation. We call the new integral, the approximate Denjoy integral, (AD integral). The reason for introducing this integral lies in its applications to the theory of trigonometric series. If

$$\text{(I)} \qquad \varlimsup_{n\to\infty} \left| \sum_{1}^{n} c_m e^{im\theta} \right| < \infty$$

for all θ, then, we show, there is a function $h(\theta)$, such that

$$\text{(II)} \qquad c_n = \frac{1}{2\pi} \int_0^{2\pi} h(\theta)\, e^{-in\theta}\, d\theta, \qquad (n = 1, 2, \ldots)$$

where the integrals in (II) are AD integrals (Theor. V). If, in addition to (I), the series

$$\text{(III)} \qquad \sum_{1}^{\infty} c_n e^{in\theta}$$

is summable p. p. (presque partout) by Poisson's method to $H(\theta)$, then $h(\theta) = H(\theta)$ p. p. (Theor. VI). This last result is difficult to prove, but we have been able to establish it by using some theorems and methods of Khintchine [1]).

[1]) Khintchine, Fund. Math. IX (1927) 212—279.

It is perhaps not superfluous to say that an extension of the Denjoy integral is essential in order that the result expressed by (II) shall be true. It has been shown by Neder[2]) that there is a series (III) which cc verges for all θ, while

$$\text{(IV)} \qquad \sum_{1}^{\infty} \frac{c_n}{in} e^{in\theta}$$

is not uniformly convergent. Now, if (II) were true with Denjoy integrals, (IV) would be the Fourier series of a continuous (complex) function. It would thus be uniformly summable by Poisson's method, and therefore, since $c_n = o, (1)$, uniformly convergent. Any method of integration for which the result expressed by (II) is true, must therefore, be such that the indefinite integral is not necessarily continuous.

In the latter part of this paper, we give two applications of the AD integral to the question of deciding what can be said of a trigonometric series which is summable p. p. to zero, while one of the Poisson sums is infinite at an at most enumerable set (Theors. VII, VIII). In the case of Theor. VII we have thought it desirable to show by examples in what respects the theorem cannot be improved.

At the end of the paper will be found a correction to a paper which recently appeared in this periodical[3]).

II. An extension of the Denjoy integral.

§ 2. **Definition 1.** Let $f(x)$ be defined on a perfect set P with extreme points a and b. Then $f(x)$ is said to be resoluble on P if, $u_n \equiv (a_n, b_n)$ denoting the contiguous intervals of P, the function $g(x)$ is resoluble in (a, b), where

$$g(x) = f(x) \quad \text{if} \quad x \subset P,$$
$$= f(a_n) + \frac{x - a_n}{b_n - a_n}\{f(b_n) - f(a_n)\} \quad \text{if} \quad a_n < x < b_n.$$

This definition is due to Denjoy[4]).

Definition 2. Let $F(x)$ be defined in the closed interval (a, b). Then $F(x)$ will be said to be approximately resoluble in (a, b) if it possesses the following two properties:

[2]) Neder Math. Ann. 84 (1921) 117—136.
[3]) Fund. Math. XXI (1933) 168—210.
[4]) Denjoy Comptes Rendus 172 (1921) 653—655.

(I) $F(x)$ is a differential coefficient;
(II) Given any perfect set P in (a, b), there is a portion of P on which $F(x)$ is resoluble.

By (I) we mean that there is a function $\varphi(x)$ defined in (a, b) such that $\varphi'(x) = F(x)$ for $a < x < b$, while $\varphi'_+(a) = F(a)$ and $\varphi'_-(b) = F(b)$.

Theorem I. *If $F(x)$ is approximately resoluble in (a, b), then $F(x)$ has an approximate derivative p. p.*

We call ξ a point S if $F(x)$ is not resoluble in any neighbourhood of ξ (or any one-sided neighbourhood, if ξ is an end point of (a, b)). By (II) Def. 2, the set of points S is non-dense, and it is manifestly closed. Let F_1 be this non-dense closed set, and let (a_n, b_n) denote a contiguous interval of F_1. If $a_n < \alpha < \beta < b_n$, then $F(x)$ is resoluble in (α, β), and therefore has an approximate derivative p. p. in (α, β), and so p. p. in (a_n, b_n), and p. p. in CF_1.

Let P_1 be the perfect kernel of F_1. By (II) Def. 2, there is a closed set $F_2 \subset P_1$ and non-dense in P_1, such that if (a_n, b_n) be an interval contiguous to F_2, then $F(x)$ is resoluble on $P_1^{(n)}$, where $P_1^{(n)}$ denotes the portion of P_1 which lies in the open interval (a_n, b_n). Let α, β be the extreme points of $P_1^{(n)}$, and (α_ν, β_ν) its contiguous intervals. The function

$$G(x) = F(x) \qquad x \subset P_1^{(n)}$$
$$= F(\alpha_\nu) + \frac{x - \alpha_\nu}{\beta_\nu - \alpha_\nu}\{F(\beta_\nu) - F(\alpha_\nu)\} \qquad \alpha_\nu < x < \beta_\nu$$

is resoluble in (α, β) and therefore has an approximate derivative p. p. in (α, β). Since $G(x) = F(x)$ on $P^{(n)}$, at a point of $P_1^{(n)}$ at which this set has unit metric density, $F_a(x)$, the approximate derivative of $F(x)$ equals $G_a(x)$, provided the latter exists. Hence $F_a(x)$ exists p. p. in $P_1^{(n)}$, and so p. p. in $P_1 - F_2$. Since $F_1 - P_1$ is enumerable, $F_a(x)$ exists p. p. in CF_2. Proceeding in this way, we infer the desired result.

§ 3. If $g(x)$ be totalisable in (α, β), we shall denote its total, or general Denjoy integral in (α, β), by

$$D\int_{\alpha}^{\beta} g(x)\,dx.$$

Let $f(x)$ be a function defined in the interval (a, b). We say that ξ is a singularity of $f(x)$, if $f(x)$ is not totalisable in any neighbourhood of ξ (or any one-sided neighbourhood, if ξ is an end point of (a, b)). The function $f(x)$ will be said to be approximately totalisable in (a, b), if the following three conditions are satisfied, and if $F(b)-F(a)$ can be calculated by the following three operations.

Condition 1. The set of singularities of $f(x)$ in (a, b) is non-dense.

Operation 1. If (α, β) is a closed interval which is free from singularities, then

$$F(\beta)-F(\alpha)=D\int_{\alpha}^{\beta} f(t)\,dt.$$

Condition 2. If $F(\delta)-E(\gamma)$ is known for all γ, δ which satisfy $\alpha<\gamma<\delta<\beta$, then $F(x)$ is totalisable in (α, β), and the two limits

$$\lim_{h\to+0}\frac{1}{h}D\int_{\beta-h}^{\beta}\{F(x)-F(\delta)\}\,dx; \qquad \lim_{h\to+0}\frac{1}{h}D\int_{\alpha}^{\alpha+h}\{F(\gamma)-F(x)\}\,dx$$

exist.

Operation 2. If $\alpha<\gamma<\delta<\beta$, then

$$F(\beta)-F(\delta)=\lim_{h\to+0}\frac{1}{h}D\int_{\beta-h}^{\beta}\{F(x)-F(\delta)\}\,dx;$$

$$F(\gamma)-F(\alpha)=\lim_{h\to+0}\frac{1}{h}D\int_{\alpha}^{\alpha+h}\{F(\gamma)-F(x)\}\,dx.$$

Condition 3. If $F(b_n)-F(a_n)=w_n$ be known for the intervals (a_n, b_n) contiguous to a non-dense perfect set P_0 and if

$$\begin{aligned} g(x)&=f(x) && x\subset P\\ &=\frac{w_n}{b_n-a_n} && a_n<x<b_n \end{aligned}$$

then there are distinct points c, d of P, such that $g(x)$ is totalisable in (c, d).

Operation 3. If c, d have the above significance, then

$$F(d)-F(c)=D\int_{c}^{d} g(x)\,dx.$$

A function which is approximately totalisable in (a, b) will be called integrable AD in that interval. If $f(x)$ is integrable AD in (a, b), we can calculate $F(b)-F(a)$ by an enumerable set of the above operations. We write

$$F(b)-F(a)=AD\int_{a}^{b} f(x)\,dx.$$

Clearly, $F(x)-F(a)$ can be calculated in the same way for $a<x<b$.

Theorem II. *If $f(x)$ is integrable AD in (a, b), and*

$$F(x)-F(a)=AD\int_{a}^{x} f(t)\,dt, \qquad (a<x\leqslant b)$$

then $F(x)$ is approximately resoluble in (a, b).

By Condn. 2, $F(x)$ is totalisable in (a, b). Let

$$G(x)=D\int_{a}^{x} F(t)\,dt. \qquad (a\leqslant x\leqslant b)$$

Condn. 2 and Operation 2 imply,

$$G'_-(x)=F(x)\quad (a<x\leqslant b); \qquad G'_+(x)=F(x)\quad (a\leqslant x<b).$$

Hence $F(x)$ is a differential coefficient.

Let P be a perfect set in (a, b), and (a_n, b_n) its contiguous intervals. By Condn. 3, there is a portion $\tilde{\omega}$ of P whose end points are c, d, such that $g(x)$ as there defined, is totalisable in (c, d). Let

$$h(x)=D\int_{c}^{x} g(t)\,dt. \qquad (c\leqslant x\leqslant d)$$

Then $h(x)$ is resoluble in (c, d). For $x\subset\tilde{\omega}$,

$$\begin{aligned} h(x)&=D\int_{c}^{x} g(t)\,dt\\ &=F(x)-F(c) \end{aligned}$$

by Operation 3. For $a_n<x<b_n$,

$$h(x)=D\int_{c}^{a_n}+D\int_{a_n}^{x} g(t)\,dt.$$

The first term of the second member is $F(a_n)-F(c)$, by Operation 3. The second term is

$$\frac{x-a_n}{b_n-a_n}w_n=\frac{x-a_n}{b_n-a_n}\{F(b_n)-F(a_n)\}$$

by the definition of $g(t)$ and w_n. By Def. 1, the fact that $h(x)$ is resoluble in (c,d), means that $F(x)$ is resoluble on $\bar{\omega}$. Thus $F(x)$ satisfies the two conditions of Def. 2.

Theorem III. *Let $F(x)$ be approximately resoluble in (a,b). Then $F_a(x)$ is integrable AD in (a,b), and*

$$F(b)-F(a)=AD\int_a^b F_a(x)\,dx.$$

Let F_1 denote the closed set of points ξ with the property that there is no neighbourhood of ξ in which $F(x)$ is resoluble. By (II) Def. 2, F_1 is non-dense. If δ be a contiguous interval of F_1, and d be a closed interval interior to δ, then $F(x)$ is resoluble in d. Hence $F_a(x)$ is totalisable in $d=(\alpha,\beta)$, and

$$F(\beta)-F(\alpha)=D\int_\alpha^\beta F_a(x)\,dx.$$

Thus $f(x)=F_a(x)$, (defined p. p.), satisfies Condn. 1, and the operation 1 is valid.

Since $F(x)$ is a differential coefficient, it is totalisable, and it is the differential coefficient of

$$G(x)=D\int_a^x F(t)\,dt.$$

Hence,

$$F(\xi)=G'_+(\xi)=\lim_{h\to+0}\frac{1}{h}D\int_\xi^{\xi+h}F(x)\,dx,\qquad (a\leqslant\xi<b)$$

and

$$F(\eta)=G'_-(\eta)=\lim_{h\to+0}\frac{1}{h}D\int_{\eta-h}^{\eta}F(x)\,dx.\qquad (a<\eta\leqslant b)$$

Let P be a perfect set, (a_n,b_n) its contiguous intervals. Since $F(x)$ is approximately resoluble, there is a portion $\bar{\omega}$ of P on which $F(x)$ is resoluble. Let c,d be the extreme points of $\bar{\omega}$. By Def. 1, the function

$$\begin{aligned}H(x)&=F(x) && x\subset\bar{\omega}\\ &=F(a_n)+\frac{x-a_n}{b_n-a_n}\{F(b_n)-F(a_n)\} && a_n<x<b_n\end{aligned}$$

is resoluble in (c,d). Now. p. p. in $\bar{\omega}$, $H_a(x)=F_a(x)$, while for $a_n<x<b_n$,

$$H_a(x)=\frac{F(b_n)-F(a_n)}{b_n-a_n}.$$

Since $H(x)$ is resoluble in (c,d),

$$\begin{aligned}F(d)-F(c)&=H(d)-H(c)\\ &=D\int_c^d H_a(x)\,dx.\end{aligned}$$

Thus $f(x)=F_a(x)$ satisfies Condn. 3, and the operation 3 is valid. This proves the theorem.

§ 4. **Definition 3.** Let $f(x)$ be a finite function defined in the closed interval (a,b). Then $f(x)$ is said to possess the property R, if, given any perfect set P in (a,b), there is a portion of P on which $f(x)$ is continuous.

Theorem IV. *Let $F(x)$ be defined in (a,b). Suppose that*

(1) *$F(x)$ has the property R;*
(2) *$F(x)$ is a differential coefficient.*

Let E be an at most enumerable set in (a,b). Suppose further, that

(3) *At each point of CE, at least on one side, $F(x)$ has finite derivatives on a set of lower metric density $>\frac{1}{2}$.*

Then $F(x)$ is approximately resoluble.

By (3) we mean that on one side of a point $\xi\subset CE$, say the right, there is a set G, and a positive number K such that

$$\lim_{h\to+0}\frac{m\,G(\xi,\xi+h)}{h}>\frac{1}{2},$$

and such that

$$\left|\frac{F(\xi+h)-F(\xi)}{h}\right|<K$$

for $\xi+h\subset G$.

Lemma 1. Any perfect set $\tilde\omega_1$ on which $F(x)$ is continuous, contains a portion $\tilde\omega$ on which the variation of $F(x)$ is defined.

If for every portion $\tilde\omega$ of $\tilde\omega_1$,

$$\sum_{\tilde\omega}|F(b_n)-F(a_n)|=\infty,$$

where (a_n, b_n) are the intervals contiguous to $\tilde\omega_1$, and the summation is taken over the intervals contiguous to $\tilde\omega$, then by an argument of Denjoy [5]), there would be a residual R of $\tilde\omega_1$, at each point of which the following condition is satisfied. If $\xi\subset R$, and K is any positive number, the set $G=G(K,\xi)$ for which

$$\left|\frac{F(\xi+h)-F(\xi)}{h}\right|<K \qquad (\xi+h\subset G)$$

satisfies the conditions

$$\varliminf_{h\to+0}\frac{m\,G(\xi,\xi+h)}{h}\leqslant\frac{1}{2};\qquad \varliminf_{h\to+0}\frac{m\,G(\xi-h,\xi)}{h}\leqslant\frac{1}{2}.$$

Since R is un enumerable, this contradicts the hypothesis (3) of the theorem. Hence there is a portion $\tilde\omega$ of $\tilde\omega_1$ such that

$$\sum_{\tilde\omega}|F(b_n)-F(a_n)|<\infty.$$

Lemma 2. Let p be a perfect set on which $F(x)$ is continuous. Then $F(x)$ is resoluble on p.

Let α,β be the extreme points of p, and (α_n,β_n) its contiguous intervals. We define the function $G(x)$ by

$$\begin{aligned} G(x)&=F(x) && x\subset p\\ &=F(\alpha_n)+\frac{x-\alpha_n}{\beta_n-\alpha_n}\{F(\beta_n)-F(\alpha_n)\}. && \alpha_n<x<\beta_n\end{aligned}$$

[5]) Denjoy Ann. de l'Ecole Normale 33 (1916) 127—222 (207). In this paper, Denjoy is concerned with continuous functions. But the argument referred to only requires the function to be continuous on $\tilde\omega_1$.

Then $G(x)$ is continuous in (α,β). If $G(x)$ is not resoluble in (α,β), there is a perfect set q of measure zero such that, either (α) the variation of $G(x)$ is not defined on any portion of q; or else (β) the variation of $G(x)$ is defined on q and is not zero and of the same sign, on every portion of q. Now the event (α) cannot occur. For if it did, then since $G(x)$ is linear in each interval of Cp, we would have $q\subset p$. Then $F(x)$ is continuous on q. By lemma 1, q contains a portion $\tilde\omega$ on which the variation of $F(x)$ is defined. Since $F(x)=G(x)$ on $\tilde\omega$, the variation of $G(x)$ on $\tilde\omega$ is defined; a contradiction.

If now the event (β) occurs, then supposing as we may, that the variation of $G(x)$ on each portion of q is positive, there exists a set of points γ, everywhere dense in q, such that

(I) $G(x)$ has on q the unique derivative $+\infty$ at γ.

(II) For every point $\delta\subset q$ in a neighbourhood of γ, the set of values taken by $G(x)$ on $q\,d$ is of measure $>\frac{5}{6}d$, where d is the interval whose end points are γ,δ.

This follows by an argument of Denjoy [6]). Since $F(x)=G(x)$ on q, we can say,

(I)′ $F(x)$ has on q the unique derivative $+\infty$ at γ.

(II)′ For every point $\delta\subset q$ in a neighbourhood of γ, the set of values taken by $F(x)$ on $q\,d$ is of measure $>\frac{5}{6}d$.

Now (I)′ (II)′ imply that there is a residual R of q with the following property. If $\xi\subset R$, and K is any finite number, the set $G=G(K,\xi)$ for which

$$\left|\frac{F(\xi+h)-F(\xi)}{h}\right|<K \qquad (\xi+h\subset G)$$

satisfies

$$\lim_{h\to+0}\frac{m\,G(\xi,\xi+h)}{h}=0,\qquad \lim_{h\to+0}\frac{m\,G(\xi-h,\xi)}{h}=0.$$

This follows by an argument of Denjoy [7]). Since R is unenumerable, this contradicts the hypothesis of the theorem. Thus (β) cannot occur, and $G(x)$ is resoluble in (α,β).

[6]) Denjoy loc. cit. 204.

[7]) Denjoy loc. cit. 204—206. The argument only requires $F(x)$ to be continuous on q.

We can now complete the proof of the main theorem. Let P be any perfect set. Since $F(x)$ has the property R, there is a portion $\tilde{\omega}$ of P on which $F(x)$ is continuous. By lemma 2, $F(x)$ is resoluble on $\tilde{\omega}$. Since $F(x)$ is a differential coefficient by hypothesis, it is approximately resoluble.

III. Applications to Trigonometric Series.

§ 5. We write

$$A_n(x) = a_n \cos nx + b_n \sin nx, \quad B_n(x) = b_n \cos nx - a_n \sin nx.$$

Lemma 3. If for every x in $(0, 2\pi)$ which does not belong to an enumerable set E,

$$\overline{\lim_{n\to\infty}} \left| \sum_1^n A_m(x) \right| < \infty,$$

and if

$$G(x) = \sum_1^\infty A_n(x)/n$$

converges for all x, then $G(x)$ has the property R.

By hypothesis, there is a finite function $M(x) > 0$, defined in CE, such that for $x \subset CE$,

$$\left| \sum_1^m A_n(x) \right| < M(x)$$

for all m. Then for such x,

$$\left| \sum_\nu^{\nu'} A_n(x)/n \right| \leqslant \frac{1}{\nu} \left| \sum_\nu^{\nu''} A_n(x) \right| \qquad (\nu \leqslant \nu'' \leqslant \nu')$$

$$\leqslant \frac{M(x)}{\nu}.$$

Let E_m denote the set of x in $(0, 2\pi)$ at which $M(x) \leqslant m$. Then $CE = \Sigma E_m$. For any ν and $\nu' \geqslant \nu$, we have

$$\left| \sum_\nu^{\nu'} A_n(x)/n \right| \leqslant \frac{m}{\nu}. \qquad (x \subset E_m)$$

By continuity,

(1) $$\left| \sum_\nu^{\nu'} A_n(x)/n \right| \leqslant \frac{m}{\nu} \qquad (x \subset F_m)$$

where $F_m = E_m + E'_m$ is closed.

Let E be enumerated as $x_1, x_2, \ldots$ Then $(0, 2\pi)$ is the sum of the closed sets $F_1, x_1, F_2, x_2, \ldots$ Given any perfect set P in $(0, 2\pi)$, one of these closed sets contains a portion $\tilde{\omega}$ of P. The closed set in question must be an F, say F_m. By (1), there is uniform convergence on $\tilde{\omega}$. Hence $G(x)$ is continuous on $\tilde{\omega}$.

Let $f(x)$ be defined in a neighbourhood of the point ξ. Let θ be number which satisfies $0 < \theta < 1$. By the symbol

$$-\infty < D_\theta f(\xi) < \infty$$

we mean that there is a set E such that

$$\lim_{h\to 0} \frac{m E(\xi, \xi + h)}{h} \geqslant \theta,$$

and a number k such that for $x \subset E$,

$$\left| \frac{f(x) - f(\xi)}{x - \xi} \right| \leqslant k.$$

Lemma 4. Let $0 < \theta < 1$. If

$$\overline{\lim_{n\to\infty}} \left| \sum_1^n A_m(\xi) \right| < \infty, \qquad \overline{\lim_{n\to\infty}} \left| \sum_1^n B_m(\xi) \right| < \infty,$$

then

(2) $$F(x) = -\sum \frac{B_n(x)}{n}, \qquad G(x) = \sum \frac{A_n(x)}{n}$$

satisfy

(3) $$-\infty < D_\theta F(\xi) < \infty, \qquad -\infty < D_\theta G(\xi) < \infty.$$

We may without loss of generality suppose that $\xi = 0$. Then there is a $C > 0$ such that $|a_n| < C$, $|b_n| < C$ for $n = 1, 2, \ldots$ Hence the series in (2) converge p. p. as well as at ξ. In virtue of the symmetry of the enunciation, it is sufficient to prove the first relation in (3). For almost all h, we have

$$\frac{F(2h) - F(0)}{2h} = \sum_1^\infty a_n \frac{\sin 2nh}{2nh} - \sum_1^\infty b_n \frac{\sin^2 nh}{nh}.$$

Write

$$\varphi_1(h) = \sum_1^\infty a_n \frac{\sin nh}{nh}, \qquad \varphi_2(h) = \sum_1^\infty b_n \frac{\sin^2 nh}{nh}.$$

It is sufficient to prove that there is a $K>0$ and a set E such that

$$\lim_{h\to+0}\frac{mE(0,h)}{h}\geqslant\theta,\quad |\varphi_1(h)|\leqslant K\quad\text{for}\quad h\subset E,$$

and that $\varphi_2(h)$ has a similar property.

By hypothesis, there is an $A>0$ such that

$$\left|\sum_1^n a_m\right|<A,\qquad \left|\sum_1^n b_m\right|<A$$

for all n. Write

$$s_n=\sum_1^n a_m.$$

Then

$$\begin{aligned}\varphi_1(h)&=\sum_1^\infty s_n\left(\frac{\sin nh}{nh}-\frac{\sin(n+1)h}{(n+1)h}\right)\\&=\sum_1^{[h^{-1}]}+\sum_{[h^{-1}]+1}^\infty\\&=\tau_1(h)+\tau_2(h).\end{aligned}$$

In $(0,1)$, $\sin x/x$ diminishes. Hence

$$\begin{aligned}|\tau_1(h)|&<A\sum_1^{[h^{-1}]}\left(\frac{\sin nh}{nh}-\frac{\sin(n+1)h}{(n+1)h}\right)\\&<A,\end{aligned}$$

if $h<1$. We have

$$\frac{\sin nh}{nh}-\frac{\sin(n+1)h}{(n+1)h}=\frac{1-\cos h}{h}\cdot\frac{\sin nh}{n+1}-\frac{\sin h}{h}\cdot\frac{\cos nh}{n+1}+\frac{\sin nh}{nh(n+1)},$$

so that

$$\begin{aligned}\tau_2(h)&=\frac{1-\cos h}{h}\sum_{[h^{-1}]+1}^\infty s_n\frac{\sin nh}{n+1}-\frac{\sin h}{h}\sum s_n\frac{\cos nh}{n+1}+\sum s_n\frac{\sin nh}{nh(n+1)}\\&=\frac{1-\cos h}{h}J_1(h)-\frac{\sin h}{h}J_2(h)+J_3(h).\end{aligned}$$

We have

$$\begin{aligned}|J_3(h)|&<\frac{A}{h}\sum_{[h^{-1}]+1}^\infty\frac{1}{n(n+1)}\\&<A.\end{aligned}$$

Let $0<\epsilon<1$, $0<\eta<1$. Let E_1 be the set of points h in $(\epsilon\eta,\eta)$ at which $|J_1(h)|>B$, where B is a positive number. For $\epsilon\eta\leqslant h\leqslant\eta$, we have

$$\begin{aligned}|J_1(h)-J_1(\eta)|&<\sum_{[\eta^{-1}]}^{[(\epsilon\eta)^{-1}]}\frac{|s_n|}{n+1}\\&<2A\log\frac{1}{\epsilon},\end{aligned}$$

and

$$\begin{aligned}B^2mE_1&<\int_{\epsilon\eta}^{\eta}J_1^2(h)\,dh<2\int_{\epsilon\eta}^{\eta}[J_1(h)-J_1(\eta)]^2\,dh+2\int_0^{\pi}J_1^2(\eta)\,dh\\&<8\eta A^2\left(\log\frac{1}{\epsilon}\right)^2+2\pi\sum_{[\eta^{-1}]}\frac{s_n^2}{(n+1)^2}\end{aligned}$$

$$<8\eta A^2\left(\log\frac{1}{\epsilon}\right)^2+2\pi A^2\eta. \tag{4}$$

We choose $\epsilon=e^{-A}$ where $A>1$ and $e^{-A}<A^{-2}$. Then the expression in (4) does not exceed $16\eta A^4$. If now we choose $B=4A^3$, we infer that the set in $(0,\eta)$ at which $|J_1(h)|>4A^3$ is of measure less than

$$\epsilon\eta+mE_1<\frac{2\eta}{A^2}.$$

The same evaluation applies to $J_2(h)$. Hence the set in $(0,\eta)$ at which $|\varphi_1(h)|>10A^3$ is of measure less than $\frac{4\eta}{A^2}$. We can choose A so that

$$1-\frac{4}{A^2}>\theta.$$

Let $K=10A^3$. Let $E(\eta)$ denote the set of h in $(0,\eta)$ at which $|\varphi_1(h)|\leqslant K$. The measure of this set exceeds $\eta\theta$. Let the rational numbers in the open interval $(0,\eta)$ be enumerated as $\eta_1,\eta_2,\dots$

Let $E = \Sigma E(\eta_n)$. Then for $h \subset E$, $|\varphi_1(h)| \leqslant K$, and for any positive $h < \eta$, if η_ν is a sequence of rational numbers tending to h,

$$\frac{m\,E(0,h)}{h} = \lim \frac{m\,E(0,\eta_\nu)}{\eta_\nu} \geqslant \theta.$$

Consider now $\varphi_2(h)$. Write $\sigma_n = \sum_1^n b_m$. Then

$$\begin{aligned}\varphi_2(h) &= \sum_1^\infty \sigma_n \left(\frac{\sin^2 nh}{nh} - \frac{\sin^2 (n+1)h}{(n+1)h}\right)\\ &= \sum_1^{[h^{-1}]} + \sum_{[h^{-1}]+1}^\infty\\ &= T_1(h) + T_2(h).\end{aligned}$$

In $(0,1)$, $\sin^2 x/x$ is increasing. Then

$$\begin{aligned}|T_1(h)| &\leqslant A \sum_1^{[h^{-1}]} \left(\frac{\sin^2(n+1)h}{(n+1)h} - \frac{\sin^2 nh}{nh}\right)\\ &< A\end{aligned}$$

if $h < 1$. We have

$$\begin{aligned}&\frac{\sin^2 nh}{nh} - \frac{\sin^2(n+1)h}{(n+1)h} =\\ &= -\frac{\sin 2h}{2h}\cdot\frac{\sin 2nh}{n+1} - \frac{\sin^2 h}{h}\cdot\frac{\cos 2nh}{n+1} + \frac{\sin^2 nh}{nh(n+1)},\end{aligned}$$

so that

$$\begin{aligned}T_2(h) = &-\frac{\sin 2h}{2h}\sum \sigma_n \frac{\sin 2nh}{n+1} - \frac{\sin^2 h}{h}\sum \sigma_n \frac{\cos 2nh}{n+1} +\\ &+ \sum \sigma_n \frac{\sin^2 nh}{nh(n+1)}.\end{aligned}$$

The three sums are then treated precisely as $J_1(h)$, $J_2(h)$, $J_3(h)$ respectively.

§ 6. ***Lemma 5.*** *Let* $A_n(\xi) = O(1)$, $B_n(\xi) = O(1)$. *If*

$$-\sum B_n(\xi)/n, \qquad \sum A_n(\xi)/n$$

converge to $F(\xi)$, $G(\xi)$ *respectively, and if*

$$H(x) = -\sum A_n(x)/n^2, \qquad K(x) = -\sum B_n(x)/n^2,$$

then

$$H'(\xi) = F(\xi), \qquad K'(\xi) = G(\xi). \tag{5}$$

It is no loss of generality to suppose that $\xi = 0$. We may further suppose that

$$\sum b_n/n = 0, \qquad \sum a_n/n = 0. \tag{6}$$

In virtue of the symmetry of the enunciation, we need only prove the first of the relations (5). We have

$$\begin{aligned}\frac{H(2h) - H(0)}{2h} &= -\sum \frac{b_n}{n}\,\frac{\sin 2nh}{2nh} + \sum \frac{a_n}{n}\cdot\frac{\sin^2 nh}{nh}\\ &= \varphi_1(h) + \varphi_2(h).\end{aligned}$$

That $\varphi_1(h) \to 0$ in virtue of $b_n = O(1)$ and the first equation in (6) is a known theorem of Hardy and Littlewood [8].

That $\varphi_2(h) \to 0$, may be established by similar arguments. We write

$$\begin{aligned}\varphi_2(h) &= \sum_1^{[h^{-1}]} + \sum_{[h^{-1}]+1}^\infty\\ &= r_1(h) + r_2(h).\end{aligned}$$

Put

$$s_n = \sum_1^n a_m/m.$$

Then

$$r_1(h) = \sum_1^{N-1} + \sum_N^{[h^{-1}]}$$

where N is an integer such that $|S_n| < \epsilon$ for $n \geqslant N-1$. Then

$$\left|\sum_1^{N-1}\right| < \epsilon \quad \text{for} \quad 0 < h \leqslant h(\epsilon),$$

[8]) Hardy and Littlewood Proc. Lond. Math. Soc. 22 (1924) XVIII.

and

$$\sum_{N}^{[h^{-1}]} = \sum_{N}^{[h^{-1}]} (s_n - s_{n-1}) \frac{\sin^2 nh}{nh}$$

$$= -s_{N-1} \frac{\sin^2 (N-1) h}{(N-1)} + \sum_{N}^{[h^{-1}]-1} s_n \left(\frac{\sin^2 nh}{nh} - \frac{\sin^2 (n+1) h}{(n+1) h} \right) +$$
$$+ s_{[h^{-1}]} \frac{\sin^2 [h^{-1}] h}{[h^{-1}] h}.$$

The first and third terms do not exceed ϵ in absolute value. Since $\sin^2 x/x$ increases in $(0, 1)$, the second term does not exceed ϵ in absolute value. Hence

$$|r_1(h)| < 4\epsilon \quad \text{for} \quad 0 < h \leqslant h(\epsilon).$$

We have

$$r_2(h) = \sum_{[h^{-1}]+1}^{\infty} \frac{a_n}{n^2} \frac{1 - \cos 2nh}{2h}$$
$$= \frac{1}{2} \sum \frac{a_n}{n^2 h} - \sum \frac{a_n}{n} \frac{\cos 2nh}{2nh}.$$

Now

$$\left| \sum_{[h^{-1}]+1}^{\infty} \frac{a_n}{n^2 h} \right| \leqslant \frac{1}{h(1 + [h^{-1}])} \operatorname*{Max}_{\nu > h^{-1}} \left| \sum_{[h^{-1}]+1}^{\nu} a_n/n \right|$$
$$< 2\epsilon.$$

Finally, supposing as we may that $|a_n| < 1$, let $k = [\epsilon^{-\frac{1}{2}}]$. Then

$$\left| \sum_{k[h^{-1}]+1}^{\infty} \frac{a_n}{n} \cdot \frac{\cos 2nh}{2nh} \right| < \frac{1}{h(1 + k[h^{-1}])}$$
$$< k^{-1}$$
$$< 2\epsilon^{\frac{1}{2}}.$$

And

$$\sum_{[h^{-1}]+1}^{k[h^{-1}]} \frac{a_n}{n} \cdot \frac{\cos 2nh}{2nh} \tag{7}$$

can be expressed as the sum of groups of consecutive terms, such that in each group, $\frac{\cos 2nh}{2nh}$ is of one sign, and varies in one sense

as n increases. To each group, we apply Abel's lemma. The absolute value of each group, does not exceed ϵ. The number of groups is less than 2 plus the number of intervals of the form $\left(\frac{r\pi}{4}, (r+1)\frac{\pi}{4}\right)$, r an integer, which are contained in $(1, k)$. The number of groups is therefore less than $2k$ for sufficiently large k. Hence the absolute value of (7) does not exceed $2\epsilon^{\frac{1}{2}}$. This proves the lemma.

§ 7. ***Theorem V.*** *Let*

$$\varlimsup_{n \to \infty} \left| \sum_{1}^{n} c_m e^{imx} \right| < \infty \qquad (c_n = a_n - ib_n)$$

for all x, except those belonging to an enumerable set E. Let

$$\sum_{1}^{\infty} \frac{c_m e^{imx}}{im} = F(x) - i\,G(x)$$

converge for all x. Then $F_a(x)$, $G_a(x)$, the approximate derivatives of $F(x)$, $G(x)$, exist p. p., and

$$a_n = \frac{1}{\pi} AD \int_{-\pi}^{\pi} F_a(x) \cos nx \, dx = -\frac{1}{\pi} AD \int_{-\pi}^{\pi} G_a(x) \sin nx \, dx,$$

$$b_n = \frac{1}{\pi} AD \int_{-\pi}^{\pi} F_a(x) \sin nx \, dx = \frac{1}{\pi} AD \int_{-\pi}^{\pi} G_a(x) \cos nx \, dx.$$

We have

$$F(x) = -\sum B_n(x)/n, \qquad G(x) = \sum A_n(x)/n.$$

By lemma 3, $F(x)$ and $G(x)$ have the property R. By lemma 5, $F(x)$ and $G(x)$ are differential coefficients. By lemma 4, taking $\theta > \frac{1}{2}$, $F(x)$ and $G(x)$ have at each point of CE, (on both sides), finite derivatives on a set of lower metric density $> \frac{1}{2}$. Hence by Theor. IV, $F(x)$ and $G(x)$ are approximately resoluble. By Theor. III, $F_a(x)$, $G_a(x)$ exist p. p., and

$$0 = F(\pi) - F(-\pi) = AD \int_{-\pi}^{\pi} F_a(x) \, dx.$$

$$0 = G(\pi) - G(-\pi) = AD \int_{-\pi}^{\pi} G_a(x) \, dx.$$

Let μ be a positive integer. Consider the expressions

$$e^{-i\mu x}\sum_{1}^{N} c_m e^{imx} = c_\mu + \sum_{r=1}^{\mu-1} c_{\mu-r}\, e^{-irx} + \sum_{r=1}^{N-\mu} c_{\mu+r}\, e^{irx},$$

where $N>\mu$. Write $d_r = c_{\mu+r}$ $(r=1,2,\ldots)$. Then

$$\varlimsup_{n\to\infty}\left|\sum_{1}^{n} d_m e^{imx}\right| < \infty$$

for all x of CE. We have

$$\{F(x)-i\,G(x)\}\, e^{-i\mu x} = \frac{c_\mu}{i\mu} + \sum_{r=1}^{\mu-1}\frac{c_{\mu-r}}{i(\mu-r)}\, e^{-irx} + \sum_{r=1}^{\infty}\frac{c_{\mu-r}}{i(\mu-r)}\, e^{irx}.$$

Denoting by K a suitable constant, we have

$$K+\int_{-\pi}^{x}\left[\{F(t)-i\,G(t)\}\, e^{-i\mu t} - \frac{c_\mu}{i\mu}\right] dt = -\sum_{r=1}^{\mu-1}\frac{c_{\mu-r}}{i(\mu-r)}\cdot\frac{e^{-irx}}{ir} + \\ + \sum_{r=1}^{\infty}\frac{c_{\mu+r}}{i(\mu+r)}\cdot\frac{e^{irx}}{ir}.$$

Hence,

$$K i\mu + i\mu\int_{-\pi}^{x}\left[\{F(t)-i\,G(t)\}\, e^{-i\mu t} - \frac{c_\mu}{i\mu}\right] dt =$$

$$= -\sum_{r=1}^{\mu-1} c_{\mu-r}\left(\frac{1}{ir}+\frac{1}{i(\mu-r)}\right) e^{-irx} + \sum_{r=1}^{\infty} c_{\mu+r}\left(\frac{1}{ir}-\frac{1}{i(\mu+r)}\right) e^{irx},$$

and so

$$\{F(x)-i\,G(x)\}\, e^{-i\mu x} + K i\mu + i\mu\int_{-\pi}^{x}\left[\{F(t)-i\,G(t)\}\, e^{-i\mu t} - \frac{c_\mu}{i\mu}\right] dt =$$

$$= \sum_{1}^{\infty}\frac{d_m}{im}\, e^{imx} - \sum_{1}^{\mu-1}\frac{c_{\mu-r}}{ir}\, e^{-irx} + \frac{c_\mu}{i\mu}.$$

Writing

$$H(x) = -\frac{c_\mu}{i\mu} + \sum_{1}^{\mu-1}\frac{c_{\mu-r}}{ir}\, e^{-irx},$$

we have that

$$\sum_{1}^{\infty}\frac{d_m}{im}\, e^{imx}$$

converges for all x to

$$F^{(\mu)}(x) - i\,G^{(\mu)}(x) = \{F(x)-i\,G(x)\}\, e^{-i\mu x} + K i\mu + \\ + i\mu\int_{-\pi}^{x}\left[\{F(t)-i\,G(t)\}\, e^{-i\mu t} - \frac{c_\mu}{i\mu}\right] dt + H(x).$$

By what has been proved above, it follows that

$$0 = AD\int_{-\pi}^{\pi} F_a^{(\mu)}(x)\, dx = AD\int_{-\pi}^{\pi} G_a^{(\mu)}(x)\, dx.$$

Now

$$F_a^{(\mu)}(x) - i\,G_a^{(\mu)}(x) = \{F_a(x)-i\,G_a(x)\}\, e^{-i\mu x} - c_\mu + H_a(x)$$

for almost all x; and by the definition of $H(x)$,

$$\int_{-\pi}^{\pi} H_a(x)\, dx = 0.$$

Hence

$$a_\mu = \frac{1}{2\pi} AD\int_{-\pi}^{\pi}\{F_a(x)\cos\mu x - G_a(x)\sin\mu x\}\, dx, \tag{8}$$

$$b_\mu = \frac{1}{2\pi} AD\int_{-\pi}^{\pi}\{F_a(x)\sin\mu x + G_a(x)\cos\mu x\}\, dx. \tag{9}$$

We now consider the expression

$$e^{i\mu x}\sum^{N} c_m e^{imx} = \sum_{1}^{N} c_m e^{i(m+\mu)x}.$$

Write

$$\begin{aligned} e_m &= 0 & m &= 1, 2, \ldots \mu \\ &= c_{m-\mu} & m &= \mu+1, \mu+2, \ldots \end{aligned}$$

Then

$$\varlimsup_{n\to\infty}\left|\sum_{1}^{n} e_m e^{imx}\right| < \infty$$

for $x \subset CE$. We have

$$\{F(x)-i\,G(x)\}\, e^{i\mu x} = \sum_{r=\mu+1}^{\infty}\frac{e_r}{i(r-\mu)}\, e^{irx}.$$

Denoting by C a suitable constant, we have

$$C+\int_{-\pi}^{x}\{F(t)-i\,G(t)\}\,e^{i\mu t}\,dt=\sum_{r=\mu+1}^{\infty}\frac{e_r}{i(r-\mu)}\cdot\frac{e^{irx}}{ir}.$$

Hence

$$C i\mu+i\mu\int_{-\pi}^{x}\{F(t)-i\,G(t)\}\,e^{i\mu t}\,dt=\sum_{r=\mu+1}^{\infty}e_r\left(\frac{1}{i(r-\mu)}-\frac{1}{ir}\right)e^{irx},$$

and

$$\{F(x)-i\,G(x)\}\,e^{i\mu x}-C i\mu-i\mu\int_{-\pi}^{x}\{F(t)-i\,G(t)\}\,e^{i\mu t}\,dt=\sum_{r=\mu+1}^{\infty}\frac{e_r}{ir}e^{irx},$$

$$=\sum_{r=1}^{\infty}\frac{e_r}{ir}e^{irx},$$

Writing

$$F^{(-\mu)}(x)-i\,G^{(-\mu)}(x)=$$

$$=\{F(x)-i\,G(x)\}\,e^{i\mu x}-C i\mu-i\mu\int_{-\pi}^{x}\{F(t)-i\,G(t)\}\,e^{i\mu t}\,dt,$$

we have by what has been proved above,

$$0=AD\int_{-\pi}^{\pi}F_a^{(-\mu)}(x)\,dx=AD\int_{-\pi}^{\pi}G_a^{(-\mu)}(x)\,dx.$$

But

$$F_a^{(-\mu)}(x)-i\,G_a^{(-\mu)}(x)=\{F_a(x)-i\,G_a(x)\}\,e^{i\mu x}$$

for almost all x. Hence

$$0=AD\int_{-\pi}^{\pi}\{F_a(x)\cos\mu x+G_a(x)\sin\mu x\}\,dx,$$

$$0=AD\int_{-\pi}^{\pi}\{G_a(x)\cos\mu x-F_a(x)\sin\mu x\}\,dx,$$

These equations together with (8) and (9), give the desired result.

§ 7. ***Theorem VI.*** *If, in addition to the hypotheses of Theor. V.*

$$\lim_{r\to 1}\sum_{1}^{\infty}c_m r^m e^{imx}=f(x)-i\,g(x).$$

Exists p. p., then $F_a(x)=f(x)$, $G_a(x)=g(x)$ *p. p.*

We require a series of lemmas.

Lemma 6. *If*

$$\overline{\lim_{n\to\infty}}\left|\sum_{1}^{n}a_m\right|<\infty,$$

and

(C, 1) $$\sum_{1}^{\infty}a_m=S,$$

then

$$\lim_{h\to 0}\sum_{1}^{\infty}a_n\left(\frac{\sin nh}{nh}\right)^2=s.$$

We may suppose that $s=0$. Write

$$s_n=\sum_{1}^{n}a_m,\qquad \sigma_n=\sum_{1}^{n}s_m.$$

We may suppose that

(10) $$|s_n|<1, \qquad (n=1,2,\ldots)$$

and we have

(11) $$\sigma_n=o(n).$$

Write

$$\psi(h)=\left(\frac{\sin h}{h}\right)^2.$$

Then

(12) $$\begin{cases}\psi'(h)=2\dfrac{\sin h}{h}\left(\dfrac{\cos h}{h}-\dfrac{\sin h}{h^2}\right);\\[2mm] \psi''(h)=\dfrac{2\cos 2h}{h^2}-\dfrac{4\sin 2h}{h^3}+\dfrac{6\sin 2h}{h^4}.\end{cases}$$

We use the notation

$$\Delta\psi(nh)=\psi(nh)-\psi(n+1)h;$$
$$\Delta^2\psi(nh)=\psi(nh)-2\psi(n+1)h+\psi(n+2)h.$$

We have

$$\sum_{1}^{\infty}a_n\,\psi(nh)=\sum_{1}^{\infty}s_n\,\Delta\psi(nh).$$

Let $k>1$ be a positive integer. For $nh>k$, $\Delta\psi(nh)=-h\psi'(n+\theta)h$, $0<\theta<1$, so that by (12),

$$|\Delta\psi(nh)|<\frac{4}{n^2h}. \qquad (nh>k) \tag{13}$$

Hence

$$\left|\sum_{n>k/h} s_n\,\Delta\psi(nh)\right|<\frac{4}{h}\left[\frac{k}{h}\right]^{-1}$$
$$<8/k.$$

Further,

$$\sum_{1}^{[k/h]+1} s_n\,\Delta\psi(nh)=\sum_{1}^{[k/h]}\sigma_n\,\Delta^2\psi(nh)+[s_n\,\Delta\psi(nh)]_{n=[k/h]+1}.$$

By (10) and (13), the last term tends to zero as $h\to 0$. Let $N=N(\epsilon)$ be an integer such that

$$|\sigma_n|<\epsilon n \qquad (n\geqslant N) \tag{14}$$

We have to consider

$$\sum_{N}^{[k/h]}\sigma_n\,\Delta^2\psi(nh)=\sum_{N}^{[h^{-1}]}+\sum_{[h^{-1}]+1}^{[k/h]}$$
$$=J_1+J_2.$$

We see from (12) that for $h<1$, $|\psi''(h)|<C$, an absolute constant. Since $\Delta^2\psi(nh)=h^2\psi''(n+2\theta)h$, with $0<\theta<1$, we have

$$|\Delta^2\psi(nh)|<Ch^2. \qquad (nh<1)$$

Hence

$$|J_1|<C\epsilon h^2\sum_{N}^{[h^{-1}]}n$$
$$<2C\epsilon.$$

For $h\geqslant 1$, we have by (12), $|\psi''(h)|<Ch^{-2}$. Hence

$$|\Delta^2\psi(nh)|\leqslant\frac{C}{n^2}. \qquad (nh\geqslant 1)$$

Hence

$$|J_2|<C\sum_{[h^{-1}]+1}^{[k/h]}|\sigma_n|\,n^{-2}.$$

In the range of summation, we can write $|\sigma_n|<n\eta(h)$, where $\eta(h)\to 0$ with h. Hence

$$|J_2|<C\eta(h)\log k,$$

and this tends to 0 with h. Since ϵ can be made arbitrarily small and k arbitrarily large, the lemma follows.

Lemma 7. Under the conditions of lemma 6,

$$\lim_{h\to 0}\sum_{1}^{\infty}a_n\frac{1}{nh}\int_{0}^{nh}\frac{\sin^2 t}{t}\,dt=0.$$

We may suppose that $s=0$, and that (10), (11) are satisfied. The proof is similar to that of lemma 6. Write

$$\varphi(x)=\frac{1}{x}\int_{0}^{x}\frac{\sin^2 t}{t}\,dt.$$

Then $\varphi(x)\to 0$ as $x\to\infty$, so that

$$\sum_{1}^{\infty}a_n\,\varphi(nh)=\sum_{1}^{\infty}s_n\,\Delta\varphi(nh).$$

We have

$$\varphi'(x)=\frac{\sin^2 x}{x^2}-\frac{1}{x^2}\int_{0}^{x}\frac{\sin^2 t}{t}\,dt.$$

We can choose the positive integer k so that $|\varphi'(x)|<x^{-3/2}$ for $x\geqslant k$. Then

$$|\Delta\varphi(nh)|<n^{-3/2}h^{-1/2}, \qquad (nh\geqslant k)$$

and

$$\left|\sum_{n>k/h}s_n\,\Delta\varphi(nh)\right|<2k^{-1/2}.$$

We have

$$\varphi''(x)=\frac{\sin^2 x}{x^2}-\frac{3\sin^2 x}{x^3}+\frac{2}{x^3}\int_{0}^{x}\frac{\sin^2 t}{t}\,dt.$$

For $x<1$, $|\varphi''(x)|<C$, an absolute constant. Hence the sum which corresponds to J_1 of the preceding proof, does not exceed $2C\epsilon$. For $x\geqslant 1$, $|\varphi''(x)|<Cx^{-2}$, so that the proof can be completed as in lemma 6.

Lemma 8. If

$$\overline{\lim_{n\to\infty}}\left|\sum_1^n a_m\right|<\infty$$

and

$$\lim_{r\to 1}\sum a_n r^n = s,$$

then

$$\sum a_n = s. \qquad (C, 1)$$

This is a special case of a theorem of Hardy and Littlewood [9]).

§ 8. *Lemma 9.* (Khintchine [10]). *Let* $\varphi(x)$ *be integrable* L *in the interval* $(\xi-\eta,\ \xi+\eta)$, *and let*

$$\Phi(x)=\int_{\xi-r}^{x}\varphi(t)\,dt.$$

If

$$D^2\Phi(\xi)=\lim_{h\to 0}\frac{\Phi(\xi+h)+\Phi(\xi-h)-2\Phi(\xi)}{h^2}$$

exists, then

$$\lim_{h\to 0}\frac{1}{h}\int_0^h\frac{\varphi(\xi+t)-\varphi(\xi-t)}{2t}\,dt=D^2\Phi(\xi),$$

where the integral is taken as

$$\lim_{\varepsilon\to 0}\int_\varepsilon^h.$$

Lemma 10. *Let* $\varphi(x)$ *be integrable* L *in an interval. If the mean derivative of* $\varphi(x)$, *i. e.*

$$(16)\qquad \varphi_m(x)=\lim_{h\to 0}\frac{1}{h}\int_0^h\frac{\varphi(x+t)-\varphi(x)}{t}\,dt$$

is measurable and exists p. p., and if the approximate derivative $\varphi_a(x)$ *of* $\varphi(x)$ *is measurable and exists p. p., then* $\varphi_m(x)=\varphi_a(x)$, *p. p.*

9) Hardy and Littlewood Journal Lond. Math. Soc. 6 (1931) 281—286.
10) Khintchine Fund. Math. 9 (1927) 212—279 (221).

It has been proved by Khintchine that if $\varphi(x)$ is continuous, then p. p. in the set in which $\varphi_m(x)$ exists, $\varphi_a(x)$ exists and equals $\varphi_m(x)$. In the above lemma, $\varphi(x)$ is not restricted to be continuous But in virtue of the other hypotheses, we are able to establish the result by adapting the proof of Khintchine [11]).

By a theorem of Lusin [12]), there is a continuous function $\psi(x)$ such that $\psi'(x)=\varphi_m(x)$ p. p. By subtracting this function from $\varphi(x)$, we reduce the lemma to the case in which $\varphi_m(x)=0$ p. p.

Suppose then there is a set E of positive measure such that for $x\subset E$, $\varphi_m=0$ while φ_a exists and differs from 0. Then there is a sub-set J of E, of positive measure, at which say,

$$\varphi_m(x)=0,\quad \varphi_a(x)>k>0 \qquad (x\subset J)$$

We can find a diminishing sequence $h_n\to 0$ such that in the interval (h_{n+1},h_n), the oscillation of

$$\frac{1}{h}\int_0^h\frac{\varphi(x+t)-\varphi(x)}{t}\,dt$$

for each x of J does not exceed $1/n$. If follows by the theorem of Egoroff [13]), that given a positive number $\eta<k/3$, there is a sub-set H of J, of positive measure, such that

$$(17)\qquad \left|\frac{1}{t-x}\int_x^t\frac{\varphi(u)-\varphi(x)}{u-x}\,du\right|<\eta\quad (x\subset H,\ 0<t-x<\zeta)$$

where ζ is a fixed positive number, the same for all x of H.

Let x_0 be a point of H at which H has unit metric density. The set K of points x such that

$$\frac{\varphi(x)-\varphi(x_0)}{x-x_0}>k,$$

has unit metric density at x_0. So then has the set HK. Let F denote the perfect set of points x of HK with the property that every

11) Khintchine loc. cit. 276—279, 233.
12) Lusin Comptes Rendus 152 (1911) 244.
13) Egoroff Comptes Rendus 152 (1911) 244.

neighbourhood $U(x)$ of x satisfies $mU(x)\,HK>0$. Then $mF=mHK$, $x_0\subset F$, and F has unit metric density at x_0. Hence

$$\frac{\varphi(y)-\varphi(x_0)}{y-x_0}>k. \qquad (y\subset F,\ y>x_0) \tag{18}$$

Let $\epsilon<1/3$ be a positive number. For all sufficiently small $h<\zeta$, we have

$$m\,F(x_0,\ x_0+h)>(1-\epsilon)\,h. \tag{19}$$

We may suppose that there is a sequence $x_0+h=\xi_1>\xi_2>\dots$, $\xi_n\to x_0$, of points which do not belong to F. For otherwise, (18) would hold for all $y>x_0$ and sufficiently near to it. This would contradict (17).

Consider the interval $\delta_n=(\xi_{n+1},\ \xi_n)$. Let e denote a measurable set in δ_n. Then

$$\int_e \frac{\varphi(u)-\varphi(x_0)}{u-x_0}\,du$$

is an absolutely continuous function of e. It is therefore possible to find a finite number of non-overlapping intervals $\delta_{n,1},\ \delta_{n,2},\dots,\ \delta_{n,r_n}$, such that

$$\int_{\Sigma\delta_{n,i}} \frac{\varphi(u)-\varphi(x_0)}{u-x_0}\,du>(1-\epsilon)\int_{F\delta_n}\frac{\varphi(u)-\varphi(x_0)}{u-x_0}\,du. \tag{α}$$

and, since the integral last written exceeds the fixed positive number $k\,m\,F\delta_n$, such that

(β) the end points of the $\delta_{n,i}$ are limiting points of F, and therefore points of F.

Let n_0 be chosen so that

$$\sum_1^{n_0} m\,F\delta_n>(1-2\epsilon)\,h.$$

We then have $r_{n_1}+r_{n_2}+\dots r_{n_{n_0}}=p$ intervals $\delta_{n,i}$. Let these be numbered from left to right as

$$(a_1,\ b_1)\ (a_2,\ b_2)\dots\ (a_p,\ b_p).$$

These intervals consist of n_0 groups. For each group, (α) holds. Hence by (18), (20),

$$\int_{\Sigma(a_n,b_n)} \frac{\varphi(u)-\varphi(x_0)}{u-x_0}\,du>(1-\epsilon)\,k\,(1-2\epsilon)\,h \tag{21}$$

$$>(1-3\epsilon)\,k\,h.$$

We now evaluate the integrals over $(b_0,\ a_1),\ (b_1,\ a_2),\dots\ (b_{p-1},\ a_p)$, where $b_0=x_0$. We have $\varphi(u)-\varphi(x_0)=\varphi(u)-\varphi(b_n)+\varphi(b_n)-\varphi(x_0)$. Since $b_n\subset F$, we have by (18), $\varphi(b_n)-\varphi(x_0)>0$. Hence

$$I_n=\int_{b_n}^{a_{n+1}}\frac{\varphi(u)-\varphi(x_0)}{u-x_0}\,du>\int_{b_n}^{a_{n+1}}\frac{\varphi(u)-\varphi(b_n)}{u-x_0}\,du=J_n.$$

In J_n, the integrand can be written

$$\frac{u-b_n}{u-x_0}\cdot\frac{\varphi(u)-\varphi(b_n)}{u-b_n}$$

The first factor is monotone in the interval of integration. The second factor we denote by $\psi_n(u)$. By the second mean value theorem,

$$|J_n|=\left|\frac{a_{n+1}-b_n}{a_{n+1}-x_0}\int_\xi^{a_{n+1}}\psi_n(u)\,du\right| \qquad (b_n\leqslant\xi\leqslant a_{n+1})$$

$$\leqslant\left|\int_{b_n}^{a_{n+1}}\psi_n(u)\,du\right|+\left|\int_{b_n}^{\xi}\psi_n(u)\,du\right|.$$

Since $b_n\subset F\subset H$, and $a_{n+1}-b_n<\zeta$, we have by (17), $|J_n|<2(a_{n+1}-b_n)\,\eta$, and so $I_n>-2\,(a_{n+1}-b_n)\,\eta$. Hence

$$\sum_{n=0}^{p-1} I_n>-2\eta h.$$

By (21),

$$\int_{x_0}^{b_p}\frac{\varphi(u)-\varphi(x_0)}{u-x_0}\,du>(1-3\,\epsilon)\,k\,h-2\,\eta\,h.$$

Hence by (17), $\eta(b_p-x_0)>(1-3\epsilon)\,k\,h-2\,\eta\,h$. Since $h>b_p-x_0$, this gives $3\eta>(1-3\epsilon)\,k$. For sufficiently small ϵ, this contradicts $\eta<k/3$.

§ 9. We can now prove Theor. VI. By the symmetry of the enunciation, we need only prove that $F_a(x)=f(x)$ p. p. Let ξ be a point at which

$$\overline{\lim}\left|\sum_1^n A_m(\xi)\right|<\infty, \qquad \overline{\lim}\left|\sum_1^n B_m(\xi)\right|<\infty,$$

and for which $f(\xi)$ and $g(\xi)$ are defined. By lemma (8),

$$\sum A_n(\xi)=f(\xi) \quad (C,1), \qquad \sum B_n(\xi)=g(\xi). \quad (C,1)$$

By lemma 6,

$$D^2 H(\xi)=f(\xi),$$

where $H(x)$ has the meaning of lemma 5. By lemma 9,

(22) $$\lim_{h\to 0}\frac{1}{h}\int_0^h \frac{F(\xi+t)-F(\xi-t)}{2t}=f(\xi).$$

Now

$$\frac{F(\xi+2h)+F(\xi-2h)-2F(\xi)}{2h}=2\sum\frac{B_n(\xi)}{n}\cdot\frac{\sin^2 nh}{h}$$

p. p. in h. The second number is

$$\frac{1}{h}\sum\frac{B_n(\xi)}{n}-\frac{1}{h}\sum\frac{B_n(\xi)}{n}\cos 2nh.$$

If $0<\delta<q$, then

$$\int_\delta^q \frac{F(\xi+2h)+F(\xi-2h)-2F(\xi)}{2h}\,dh=$$

$$=\int_\delta^q \frac{dh}{h}\sum\frac{B_n(\xi)}{n}-\int_\delta^q\frac{dh}{h}\sum\frac{B_n(\xi)}{n}\cos 2nh.$$

But

$$\sum\frac{B_n(\xi)}{n}\cos 2nh$$

is a Fourier Lebesgue series in h, and hence

$$\int_\delta^q\frac{dh}{h}\sum\frac{B_n(\xi)}{n}\cos 2nh=\sum\frac{B_n(\xi)}{n}\int_\delta^q\frac{\cos 2nh}{h}\,dh.$$

Thus

$$\int_\delta^q\frac{F(\xi+2h)+F(\xi-2h)-2F(\xi)}{2h}\,dh=2\sum\frac{B_n(\xi)}{n}\int_\delta^q\frac{\sin^2 nh}{h}\,dh$$

(23) $$=2\sum\frac{B_n(\xi)}{n}\int_{n\delta}^{nq}\frac{\sin^2 t}{t}\,dt.$$

By lemma 7,

(24) $$\lim_{q\to 0}\sum\frac{B_n(\xi)}{nq}\int_0^q\frac{\sin^2 t}{t}\,dt=0.$$

A fortiori,

$$\lim_{\delta\to 0}\sum\frac{B_n(\xi)}{n}\int_0^\delta\frac{\sin^2 t}{t}\,dt=0.$$

In (23), let $\delta\to 0$. Divide both sides by q and let $q\to 0$. By (24).

$$\lim_{q\to 0}\frac{1}{q}\int_0^q\frac{F(\xi+2h)+F(\xi-2h)-2F(\xi)}{2h}\,dh=0.$$

Hence by (22), $F_m(\xi)=f(\xi)$. Thus p. p., $F(x)$ has the mean derivative $f(x)$. By lemma 10, $F_a(x)=f(x)$ p. p.

The following result, which we express as a lemma, should be noticed.

Lemma 11. *If*

$$\sum_1^\infty c_m e^{im\xi}$$

converges to $f(\xi)-ig(\xi)$, *then* $F_a(\xi)=f(\xi)$, $G_a(\xi)=g(\xi)$.

This is established by modifying the proof of lemma 4. We must now show that $\varphi_1(h)\to 0$ approximately, and that $\varphi_2(h)\to 0$ approximately. We remark incidentally, that $\varphi_1(h)\to 0$ approximately, is a result due to Rajchman and Zygmund [14]). The modification of the treatment of $\varphi_1(h)$ in lemma 4 will be obvious to the reader on examining the proof of lemma 19, V [15]); and then the modification of the treatment of $\varphi_2(h)$ will be equally clear.

[14]) Rajchman and Zygmund Bull. Acad. Polonaise 1925 69—80.

[15]) Fund. Math. XXI (1933) 168—210 (190).

§ 10. Let

(25) $$\tfrac{1}{2}a_0+\sum_{1}^{\infty}A_n(x)$$

be a trigonometric series. We write

$$P(r,x)=\tfrac{1}{2}a_0+\sum A_n(x)\,r^n,$$

and

$$\underline{P}(x)=\varliminf_{r\to1}P(r,x);\quad \overline{P}(x)=\varlimsup_{r\to1}P(r,x).$$

If $\overline{P}(x)=\underline{P}(x)$, we denote the common value by $P(x)$. If $a_n=o(n)$, $b_n=o(n)$, and $P(x)=0$ p. p., while $\underline{P}(x)$ is finite except at an enumerable set E at which

(26) $$\lim_{r\to1}(1-r)\,P(r,x)=0,$$

then

$$a_0=0,\quad a_n=b_n=0. \qquad (n=1,2,\dots)$$

If the condition (26) is omitted, the conclusion no longer holds. For example,

$$\tfrac{1}{2}+\sum_{1}^{\infty}\cos nx$$

satisfies $P(x)=0$ for $x\not\equiv0$ (mod. 2π).

Consider the case in which E is reducible; i. e. the derived set of E is enumerable. We have,

Theorem VII. *Let (25) be a trigonometric series with $a_n=o(n)$, $b_n=o(n)$. Let $P(x)=0$ p. p., and let $\underline{P}(x)$ be finite except at a reducible set E. If*

(27) $$-\sum A_n(x)/n^2$$

converges for all x, then the series

$$-\sum B_n(x)/n$$

is the Fourier AD series of a function which, in each interval u_m $(m=1,2,\dots)$ contiguous to $E+E'$, is of the form $-\tfrac{1}{2}a_0x+c_m$, where c_m is a constant.

The set $E+E'$ is enumerable. Let $u_m=(\alpha_m,\beta_m)$ denote a contiguous interval. Let $F(x)$ denote the sum of (27). By lemma 15, II [16]).

(28) $$F(x)=-\tfrac{1}{4}a_0x^2+c_mx+d_m \qquad (\alpha_m<x<\beta_m)$$

where c_m and d_m are constants. Since (27) converges for all x, and its coefficients are $o(1/n)$, it is a differential coefficient. Since $E+E'$ is enumerable, given any perfect set P, there is a portion $\tilde\omega$ of P which is contained in a contiguous interval u_m. By (28), $F(x)$ is resoluble on $\tilde\omega$. Hence $F(x)$ is approximately resoluble. We have

$$F_a(x)=F'(x)=-\tfrac{1}{2}a_0x+c_m, \qquad (\alpha_m<x<\beta_m)$$

and

$$F(x)-F(0)=AD\int_0^x F'(t)\,dt,$$

the integrand being defined for t not belonging to $E+E'$. In particular,

$$0=AD\int_0^{2\pi}F'(t)\,dt.$$

Since (27) is the Fourier Lebesgue series of $F(x)$, we have

(29) $$-\frac{a_n}{n^2}=\frac{1}{\pi}\int_0^{2\pi}F(x)\cos nx\,dx,\quad -\frac{b_n}{n^2}=\frac{1}{\pi}\int_0^{2\pi}F(x)\sin nx\,dx.\ (n=1,2,\dots)$$

We must show that in the formulae (29), we can integrate by parts. Consider the first formula. If $\alpha_m<\alpha<\beta<\beta_m$, then

$$\int_\alpha^\beta F(x)\cos nx\,dx=\left[F(x)\frac{\sin nx}{n}\right]_\alpha^\beta-\frac{1}{n}\int_\alpha^\beta F'(x)\sin nx\,dx$$

(30) $$=\left[F(x)\frac{\sin nx}{n}\right]_\alpha^\beta-\frac{1}{n}AD\int_\alpha^\beta F'(x)\sin nx\,dx.$$

The first member tends to a limit as $\beta\to\beta_m$, $\alpha\to\alpha_m$ independently. Since $F(x)$ is a differential coefficient, (28) holds for $\alpha_m\leqslant x\leqslant\beta_m$. Hence, the AD integral in (30) tends to a limit as $\beta\to\beta_m$, $\alpha\to\alpha_m$ independently. By § 3, Operation 2,

(31) $$\int_{\alpha_m}^{\beta_m}F(x)\cos nx\,dx=\left[F(x)\frac{\sin nx}{n}\right]_{\alpha_m}^{\beta_m}-\frac{1}{n}AD\int_{\alpha_m}^{\beta_m}F'(x)\sin nx\,dx.$$

[16]) Proc. Lond. Math. Soc. 34 (457—491) 466.

Let (ξ, η) be an interval contiguous to the derived set of $E+E'$. If there are no points of $E+E'$ in (ξ,η), then (ξ,η) is an interval u_m. If these is a finite number of such points, then by the addition of a finite number of equations of the type (31), we obtain an equation of that type with ξ, η taking the place of α_m, β_m respectively. If there is an infinite number of such points, then they can be represented as

$$\ldots a_{-\nu} < a_{-\nu+1} < \ldots < a_0 < a_1 < \ldots < a_\nu < \ldots$$

where the sequence is infinite in one or both directions. Suppose that it is infinite on the right. Then $a_\nu \to \eta$. We have an equation of type (31) for every interval (a_r, a_{r+1}), $r = 0, 1, \ldots$, and hence for every interval (a_0, a_r), $r = 1, 2, \ldots$ Further, we have an equation of type (30) for every interval (a_r, t) where $a_r < t < a_{r+1}$; hence also, for every interval (a_0, t). Thus,

$$\int_{a_0}^{t} F(x)\cos nx\,dx = \left[F(x)\frac{\sin nx}{n}\right]_{a_0}^{t} - \frac{1}{n} AD \int_{a_0}^{t} F'(x)\sin nx\,dx.$$

The first member tends to a limit as $t \to \eta$. Since $F(t)$ is a differential coefficient, so is $F(t)\sin nt$. Hence

$$\lim_{h\to+0}\frac{1}{h}\int_{\eta-h}^{\eta}\left[F(x)\frac{\sin nx}{n}\right]_{a_0}^{t} dt = \left[F(x)\frac{\sin nx}{n}\right]_{a_0}^{\eta}.$$

By § 3, Operation 2, we have

$$\int_{a_0}^{\eta} F(x)\cos nx\,dx = \left[F(x)\frac{\sin nx}{n}\right]_{a_0}^{\eta} - \frac{1}{n} AD \int_{a_0}^{\eta} F'(x)\sin nx\,dx.$$

The interval (ξ, a_0) is treated similarly. Thus we have an equation of type (31) for every interval contiguous to the derived set of $E+E'$, and for every interval contained in such an interval. Proceeding in this way, since in the enumerable well ordered set of derivatives of $E+E'$, each is non-dense in those which precede, we obtain the equation

$$\int_{0}^{2\pi} F(x)\cos nx\,dx = \left[F(x)\frac{\sin nx}{n}\right]_{0}^{2\pi} - \frac{1}{n} AD \int_{0}^{2\pi} F'(x)\sin nx\,dx.$$

By (29)

$$-\frac{a_n}{n} = \frac{1}{\pi} AD \int_0^{2\pi} F'(x)\sin nx\,dx.$$

By a similar argument,

$$\frac{b_n}{n} = \frac{1}{\pi} AD \int_0^{2\pi} F'(x)\cos nx\,dx.$$

§§ 11. In the simplest case, when the derived set of E consists of a single point (mod. 2π), it can happen that the series (27) is not convergent at that point, while the other hypotheses of the theorem are satisfied. We shall illustrate this by an example.

Let $g(x)$ be an even periodic function such that $g(x)$ is convex in $\epsilon \leqslant x \leqslant \pi$ and $g''(x)$ is continuous in that interval, for every positive $\epsilon < \pi$, and such that $g(x) \to \infty$ as $x \to 0$, and

$$\lim_{n\to\infty} n \int_0^{\pi} g(x)\cos nx\,dx = 0. \tag{32}$$

Such a function is

$$k(x) = \sqrt{\log\frac{2\pi}{x}}, \qquad 0 < x \leqslant \pi$$

$$k(-x) = k(x).$$

We have

$$\int_0^{\pi} k(x)\cos nx\,dx = \frac{1}{2n}\int_0^{\pi}\frac{\sin nx}{x\sqrt{\log\frac{2\pi}{x}}}\,dx.$$

Now if $0 < \epsilon < \pi$,

$$\lim_{n\to\infty}\int_{\epsilon}^{\pi}\frac{\sin nx}{x\sqrt{\log\frac{2\pi}{x}}}\,dx = 0,$$

while

$$\left|\int_0^{\epsilon}\right| = \left|\frac{1}{\sqrt{\log\frac{2\pi}{\epsilon}}}\int_{\epsilon'}^{\epsilon}\frac{\sin nx}{x}\,dx\right| \qquad (0 \leqslant \epsilon' \leqslant \epsilon)$$

$$\leqslant \frac{\pi}{\sqrt{\log\frac{2\pi}{\epsilon}}}.$$

Thus $k(x)$ satisfies (32). Also $k''(x)$ is continuous and positive for $0 < x \leqslant \pi$, so that $k(x)$ is convex in that interval.

We are about to construct a series

$$(33) \qquad f(x) \sim \tfrac{1}{2} a_0 + \sum_{1}^{\infty} a_n \cos nx, \qquad a_n = o\left(\frac{1}{n}\right)$$

such that,

(i) Σa_n is divergent.

(ii) there is a sequence $\pi = \alpha_0 > \alpha_1 > \dots$, $\lim \alpha_n = 0$, such that $f(x)$ is linear in each interval $(\alpha_{n+1} \leqslant x \leqslant \alpha_n)$.

Then $\Sigma n a_n \cos n x$ is summable (P) to zero for all $x \not\equiv 0 \pmod{2\pi}$, $\not\equiv \alpha_n \pmod{2\pi}$.

If $0 < \alpha < \beta \leqslant \pi$, we denote the interval (α, β) by δ, and we write

$$M(\delta) = \operatorname*{Max}_{\alpha \leqslant x \leqslant \beta} g''(x).$$

Let $\pi = \xi_1 > \xi_2 > \dots$, $\xi_n \to 0$ be a sequence of numbers. Divide the interval $\delta_n = (\xi_{n+1}, \xi_n)$ into ν equal subintervals $\delta_{n_1}, \dots \delta_{n_\nu}$. Let $\delta_{ni} \equiv (\alpha, \beta)$. Let

$$f(x) = g(\alpha) + \frac{x-\alpha}{\beta-\alpha}\{g(x) - g(\alpha)\}. \qquad (\alpha \leqslant x \leqslant \beta)$$

Then

$$(34) \quad f(x) = \frac{g(\alpha)+g(\beta)}{2} + \left(x - \frac{\alpha+\beta}{2}\right)\frac{g(\beta)-g(\alpha)}{\beta-\alpha}. \quad (\alpha \leqslant x \leqslant \beta)$$

We denote by θ a number which satisfies $\theta^2 \leqslant 1$, and which may vary at different occurrences. If $M = M(\alpha, \beta)$, then

$$g(\beta) = g\left(\frac{\alpha+\beta}{2}\right) + \frac{\beta-\alpha}{2} g'\left(\frac{\alpha+\beta}{2}\right) + \frac{\theta(\beta-\alpha)^2}{8} M,$$

$$g(\alpha) = g\left(\frac{\alpha+\beta}{2}\right) - \frac{\beta-\alpha}{2} g'\left(\frac{\alpha+\beta}{2}\right) + \frac{\theta(\beta-\alpha)^2}{8} M.$$

From these relations we substitute in (34) for $\{g(\alpha)+g(\beta)\}/2$, and for $\{g(\beta)-g(\alpha)\}/(\beta-\alpha)$, and obtain

$$\left| f(x) - g\left(\frac{\alpha+\beta}{2}\right) - \left(x - \frac{\alpha+\beta}{2}\right) g'\left(\frac{\alpha+\beta}{2}\right) \right| <$$

$$< \frac{(\beta-\alpha)^2 M}{8} + \left| x - \frac{\alpha+\beta}{2} \right| \frac{(\beta-\alpha) M}{4}$$

$$< \frac{(\beta-\alpha)^2 M}{4}. \qquad (\alpha \leqslant x \leqslant \beta)$$

Again,

$$\left| g(x) - g\left(\frac{\alpha+\beta}{2}\right) - \left(x - \frac{\alpha+\beta}{2}\right) g'\left(\frac{\alpha+\beta}{2}\right) \right| < \left(x - \frac{\alpha+\beta}{2}\right)^2 \frac{M}{2}$$

$$< \frac{(\beta-\alpha)^2 M}{8}.$$

Hence

$$|f(x) - g(x)| < (\beta-\alpha)^2 M(\alpha, \beta). \qquad (\alpha \leqslant x \leqslant \beta)$$

Write

$$h(x) = f(x) - g(x) \geqslant 0.$$

Then

$$\operatorname*{Max}_{\delta_{ni}} h(x) < (\beta-\alpha)^2 M(\alpha, \beta)$$

$$< (\xi_n - \xi_{n+1})\, \delta_{ni} M(\delta_{ni})/\nu.$$

Now

$$\lim_{\nu\to\infty} \sum_{i=1}^{\nu} \delta_{ni} M(\delta_{ni}) = \int_{\xi_{n+1}}^{\xi_n} g''(x)\, dx.$$

Hence

$$\lim_{\nu\to\infty} \sum_{i=1}^{\nu} \operatorname*{Max}_{(\delta_{ni})} h(x) = 0.$$

We choose $\nu = \nu(n)$ so that

$$(35) \qquad \sum_{i=1}^{\nu} \operatorname*{Max}_{(\delta_{ni})} h(x) < \frac{1}{2^n}.$$

We have now expressed $0 < x \leqslant \pi$ as the sum of the intervals δ_{ni} $(n = 1, 2, \dots;\ i = 1, \dots \nu(n))$. We denote the end points, starting from π, by $\pi = \alpha_0 > \alpha_1 > \dots$ Then $\lim \alpha_n = 0$. Consider the function $h(x)$. We define $h(o) = 0$. This function is absolutely continuous in (ϵ, π) for all positive $\epsilon < \pi$. To show that it is absolutely continuous in $(0, \pi)$, we must show that it is of bounded variation in that interval. Consider the interval (α_{n+1}, α_n). In this interval, $h(x)$ is concave. Thus $h(x)$ increases from 0 to a maximum, and then diminishes to 0. Its total variation in this interval is twice its maximum. By (35), the total variation of $h(x)$ in $(0,\pi)$, does not exceed 4.

Since $h(x)$ is absolutely continuous in $(0, \pi)$, and $h(0)=h(\pi)=0$, we have

$$\lim_{n\to\infty} n\int_0^\pi h(x)\cos nx\,dx = 0.$$

By (32), $f(x)=g(x)+h(x)$ satisfies (33). Finally, $f(x)\to\infty$ as $x\to 0$. Hence the series (33) is not summable (P) at $x=0$, and is therefore not convergent at that point.

§ 12. In Theor. VII, the series $-\Sigma B_n(x)/n$ is a Fourier AD series, but not necessarily a Fourier Denjoy series; i. e. the sum of (27) is approximately resoluble, but not necessarily resoluble, even in the simplest case in which the derived set of E consists of a single point (mod. 2π). We shall illustrate this by an example. We shall construct a series (33) such that

(i) Σa_n is convergent;
(ii) there is a sequence $\pi=\alpha_0>\alpha_1>\ldots$, $\lim \alpha_n = 0$, such that $f(x)$ is linear in $\alpha_{n+1}\leqslant x\leqslant \alpha_n$;
(iii) $f(x)$ is discontinuous at the origin.

Then the series (33) will be convergent for all x. The function $f(x)$ being discontinuous at the origin, is not resoluble. Manifestly, $\Sigma n a_n \cos nx$ is summable (P) to zero for all $x\not\equiv 0$, $\not\equiv \alpha_n$ (mod. 2π).

Let $n_2>5$ be a positive integer; let

$$\varphi_2(x)=\sin 2\pi n_2\left(x-\frac{\pi}{n_2}\right) \qquad \frac{\pi}{n_2}\left(1-\frac{1}{2}\right)\leqslant x\leqslant \frac{\pi}{n_2}\left(1+\frac{1}{2}\right)$$
$$=0 \qquad \text{elsewhere in } (0,\pi).$$

There is an $n_3>2n_2$ such that

$$\left| n\int_0^\pi \varphi_2\cos nx\,dx\right| < \frac{1}{2^2}. \qquad (n\geqslant n_3)$$

Let

$$\varphi_3(x)=\sin 3\pi n_3\left(x-\frac{\pi}{n_3}\right) \qquad \frac{\pi}{n_3}\left(1-\frac{1}{3}\right)\leqslant x\leqslant \frac{\pi}{n_3}\left(1+\frac{1}{3}\right)$$
$$=0 \qquad \text{elsewhere in } (0,\pi).$$

There is an $n_4>3n_3$ such that

$$\left| n\int_0^\pi \varphi_2\cos nx\,dx\right| < \frac{1}{3^2}, \qquad \left| n\int_0^\pi \varphi_3\cos nx\,dx\right| < \frac{1}{3^2}. \quad (n\geqslant n_4)$$

Let

$$\varphi_4(x)=\sin 4\pi n_4\left(x-\frac{\pi}{n_4}\right) \qquad \frac{\pi}{n_4}\left(1-\frac{1}{4}\right)\leqslant x\leqslant \frac{\pi}{n_4}\left(1+\frac{1}{4}\right)$$
$$=0 \qquad \text{elsewhere in } (0,\pi).$$

In this way we define φ_r for $r\geqslant 2$. Since $n_{r+1}>r\,n_r$, we have

$$\frac{1}{n_{r+1}}\left(1+\frac{1}{r+1}\right)<\frac{1}{n_r}\left(1-\frac{1}{r}\right),$$

so that the intervals in which the functions φ_r differ from zero, do not overlap. We write

$$\varphi=\sum_2^\infty \varphi_\nu.$$

Then for $r>3$,

$$\left| n\int_0^\pi \varphi\cos nx\,dx\right| \leqslant \left| n\int_0^\pi \left(\sum_2^{r-1}\varphi_\nu\right)\cos nx\,dx\right| +$$
$$+\left| n\int_0^\pi \varphi_r\cos nx\,dx\right| + \left| n\int_0^\pi \left(\sum_{r+1}^\infty \varphi_\nu\right)\cos nx\,dx\right|$$
$$\leqslant I_1+I_2+I_3$$

The function $\sum\limits_{r+1}^\infty \varphi_\nu$ is 0 outside the interval

$$\left(0, \frac{\pi}{(r+1)\,n_{r+1}}\left(1+\frac{1}{r+1}\right)\right),$$

and does not exceed 1 in absolute value within that interval. Hence

$$I_3<\frac{n\pi(r+2)}{(r+1)^2\,n_{r+1}}<\frac{2\pi}{r} \quad \text{for} \quad n\leqslant n_{r+1}. \tag{36}$$

Again, by construction, for $n\geqslant n_r$,

$$\left| n\int_0^\pi \varphi_2\cos nx\,dx\right| < \frac{1}{(r-1)^2},\ldots, \left| n\int_0^\pi \varphi_{r-1}\cos nx\,dx\right| < \frac{1}{(r-1)^2} \tag{37}$$

so that

$$I_2 < \frac{1}{r-1} \quad \text{for} \quad n \geqslant n_r.$$

We write

$$u_n = n \int_0^{\pi} \varphi_r \cos nx \, dx = n \int_{\frac{\pi}{n_r}(1-\frac{1}{r})}^{\frac{\pi}{n_r}(1+\frac{1}{r})} \varphi_r \cos nx \, dx.$$

Put

$$\alpha = \frac{\pi}{n_r}\left(1-\frac{1}{r}\right), \qquad \beta = \frac{\pi}{n_r}\left(1+\frac{1}{r}\right).$$

In the interval (α, β), we have

$$\varphi_r(x) = \sin \pi r n_r \left(x - \frac{\pi}{n_r}\right).$$

Since $\varphi_r(\alpha) = \varphi_r(\beta) = 0$, we have

$$u_n = -\int_\alpha^\beta \varphi_r'(x) \sin nx \, dx.$$

Let ν denote a positive integer less than n_r. Then

$$u_n - u_{n+\nu} = \int_\alpha^\beta \varphi_r'(x) \sin nx (\cos \nu x - 1) \, dx + \int_\alpha^\beta \varphi_r'(x) \cos nx \sin \nu x \, dx.$$

In the interval (α, β) of length $2\pi/r\, n_r$, the total variation of each of the functions $\cos \nu x - 1$ and $\sin \nu x$, does not exceed

$$\frac{2\pi}{r n_r} \cdot \nu < \frac{2\pi}{r}.$$

Hence

$$|u_n - u_{n+\nu}| < \frac{2\pi}{r}\left[\operatorname*{Max}_{\alpha \leqslant \gamma < \delta \leqslant \beta} \left| \int_\gamma^\delta \varphi_r' \sin nx \, dx \right| + \operatorname*{Max}_{\alpha \leqslant \gamma < \delta < \beta} \left| \int_\gamma^\delta \varphi_r' \cos nx \, dx \right| \right].$$

Now the function $\varphi_r(x)$ varies in (α, β) in the same fashion as $\sin x$ in $(-\pi, \pi)$. In particular, we can divide (α, β) into three sub-intervals in each of which $\varphi_r'(x)$ is of a constant sign. Hence for (γ, δ) in (α, β), we can express (γ, δ) as the sum of at most

three intervals, in each of which φ_r' is of a constant sign. But in an interval (λ, μ) in which φ_r' is of a constant sign, we have

$$\left| \int_\lambda^\mu \varphi_r' \begin{matrix}\cos\\ \sin\end{matrix} nx \, dx \right| < \left| \int_\lambda^\mu \varphi_r'(x) \, dx \right|$$
$$< 2.$$

Hence

(38) $$|u_n - u_{n+\nu}| < \frac{24\pi}{r}. \qquad (1 \leqslant \nu < n_r)$$

We now observe that

(39) $$u_n = 0 \quad \text{for} \quad n \equiv 0 \pmod{n_r}.$$

For

$$u_n = n \int_{\frac{\pi}{n_r}(1-\frac{1}{r})}^{\frac{\pi}{n_r}(1+\frac{1}{r})} \sin \pi r n_r \left(x - \frac{\pi}{n_r}\right) \cos nx \, dx$$

$$= n \int_{-\frac{\pi}{r n_r}}^{\frac{\pi}{r n_r}} \sin \pi r n_r t \cos n\left(t + \frac{\pi}{n_r}\right) dt.$$

When $n = k n_r$, k an integer, we have

$$u_n = (-1)^k n \int_{-\frac{\pi}{r n_r}}^{\frac{\pi}{r n_r}} \sin \pi r n_r t \cos nt \, dt$$
$$= 0$$

since the integrand is odd. By (38) and (39), we infer that

(40) $$I_2 = |u_n| < \frac{24\pi}{r}. \qquad (n = 1, 2, \ldots)$$

By (36), (37) and (40),

$$\left| n \int_\delta^\pi \varphi \cos nx \, dx \right| < \frac{26\pi}{r} + \frac{1}{r-1}. \quad (n_r \leqslant n \leqslant n_{r+1})$$

Thus

$$\lim_{n\to\infty} n\int_0^{\pi} \varphi \cos nx\,dx = 0. \tag{41}$$

The function $\varphi(x)$ is discontinuous at the origin. We define an even periodic function $g(x)$ by

$$g(x)=\varphi(x) \text{ for } 0<x\leqslant\pi;\quad g(0)=0;\quad g(-x)=g(x).$$

The function $g(x)$ is bounded. It is also approximately continuous at the origin. For if

$$\frac{\pi}{n_{r+1}}\left(1+\frac{1}{r+1}\right)\leqslant x\leqslant\frac{\pi}{n_r}\left(1-\frac{1}{r}\right),$$

and E denote the set in which $\varphi(x)\neq 0$, then

$$\frac{mE(0,x)}{x}<\frac{n_{r+1}(r+1)}{r+2}\left[\frac{2}{(r+1)\,n_{r+1}}+\frac{2}{(r+2)\,n_{r+2}}+\dots\right]$$
$$<\frac{2}{r+1}\left[1+\frac{1}{r+1}+\frac{1}{(r+1)^2}+\dots\right]$$
$$<\frac{2}{r};$$

while if

$$\frac{\pi}{n_r}\left(1-\frac{1}{r}\right)\leqslant x\leqslant\frac{\pi}{n_r}\left(1+\frac{1}{r}\right),$$

we have

$$\frac{mE(0,x)}{x}<\frac{r\,n_r}{r-1}\left[\frac{2}{r\,n_r}+\frac{2}{(r+1)\,n_{r+1}}+\dots\right]$$
$$<\frac{2r}{(r-1)^2}.$$

Hence,

$$\lim_{h\to 0}\frac{1}{2h}\int_{-h}^{h} g(x)\,dx=0.$$

Hence the Fourier series of $g(x)$ is summable (P) to zero at $x=0$. By (41), and the definition of $g(x)$, the Fourier constants of $g(x)$ are $o\left(\frac{1}{n}\right)$. Hence the Fourier series of $g(x)$ converges for $x=0$. It manifestly converges for $x\not\equiv 0$ (mod. 2π).

Consider an interval

$$\left(\frac{\pi}{n_r}\left(1-\frac{1}{r}\right),\ \frac{\pi}{n_r}\left(1+\frac{1}{r}\right)\right)\equiv(\alpha,\beta),$$

in which $\varphi=\varphi_r$. It is convex in (α,γ), where $2\gamma\equiv\alpha+\beta$, and concave in (γ,β). By the argument of § 11, we can divide (α,γ) into $\nu-1$ equal sub-intervals, such that if

$$\alpha=\alpha_{r1}<\alpha_{r2}<\dots<\alpha_{r\nu}=\gamma$$

are the points of sub-division, and if $f(x)$ is defined in (α,γ) by

$$f(x)=\varphi(x)\quad\text{for}\quad x=\alpha_{ri}\qquad(i=1,\dots\nu)$$
$$f(x)\text{ is linear in }\alpha_{ri}\leqslant x\leqslant\alpha_{ri+1}\quad(i=1,\dots,\nu-1)$$

then the total variation of $f(x)-\varphi(x)$ in (α,γ) does not exceed $\frac{1}{2^{r+1}}$. We can choose $\nu-1$ equal to an even integer, in which case $(\alpha+\gamma)/2$ will be a point of sub-division, and then, since $\varphi(x)=\varphi_r(x)$ in (α,β), we shall have $f((\alpha+\gamma)/2)=\varphi_r((\alpha+\gamma)/2)=-1$ Clearly, we can carry out a similar sub-division of (γ,β) and a corresponding definition of $f(x)$. Then the total variation of $f(x)-\varphi(x)$ in (γ,β) will be less than $\frac{1}{2^{r+1}}$, and $f((\gamma+\beta)/2)=1$.

In the part of $(0,\pi)$ which is complementary to the intervals (α,β), we put $f(x)=0$; i. e. $f(x)=\varphi(x)$. We define the periodic function $f(x)$ in $(-\pi,0)$ by $f(-x)=f(x)$, and we take $f(0)=0$. Then $f(x)-g(x)$ is absolutely continuous and periodic Hence its Fourier constants are $o\left(\frac{1}{n}\right)$. So then are those of $f(x)$. Also, the Fourier series of $f(x)-g(x)$ converges for all x; so then does the Fourier series of $f(x)$. The function $f(x)$, like $g(x)$, is discontinuous at the origin; and it manifestly possesses the property (II).

§ 13. The following theorem is analogous to Theor. VII.

Theorem VIII. *Let* $-\Sigma A_n(x)/n^2$ *be the Fourier series of a differential coefficient. Let the series*

$$\frac{1}{2}a_0+\sum_1^{\infty}A_n(x)$$

satisfy $P(x)=0$ *p. p., and* $\underline{P}(x)$ *finite except at an enumerable set* E. *Then* $-\Sigma B_n(x)/n$ *is the Fourier AD series of a function which in each interval* u_m *contiguous to* $E+E'$ *is of the form* $-\frac{1}{2}a_0x+c_m$, *where* c_m *is a constant.*

Let $F(x)$ denote the differential coefficient of which

(42) $$-\sum A_n(x)/n^2$$

is the Fourier series. Since

$$\lim_{h\to 0}\frac{1}{2h}\int_{x-h}^{x+h}F(t)\,dt=F(x),$$

for all x, the series (42) is summable (P) to $F(x)$, for all x. If then, we can show that

(43) $$F(x)=-\tfrac{1}{4}a_0x^2+c_mx+d_m, \qquad (\alpha_m<x<\beta_m)$$

where $u_m=(\alpha_m,\beta_m)$, then the proof can be completed as in Theor. VII.

Now in $\alpha_m<x<\beta_m$, $\underline{P}(x)$ is finite. By II lemma 12 [16]), $F(x)$ is upper semi-continuous in the open interval. Further, by I Theor II [17]), $\overline{D}^2F(x)\geqslant \underline{P}(x)-\frac{1}{2}a_0$ for all x of the open interval. Since $\underline{P}(x)=0$ p. p. in the interval, there is a function $g(x)$ such that $g(\alpha_m)=0$, $g(\beta_m)>-\epsilon$, $g(x)$ non-increasing and absolutely continuous in (α_m,β_m), $g'(x)=-\infty$ at the points where $\underline{P}(x)\neq 0$. Then if

$$G(x)=\int_{\alpha_m}^{x}g(t)\,dt,$$

we have $\overline{D}^2G(x)\leqslant \underline{P}(x)$ for all x of the open interval. Hence the upper semi-continuous function $F(x)-G(x)$ satisfies

$$\overline{D}^2(F(x)-G(x))\geqslant -\tfrac{1}{2}a_0 \quad \text{for} \quad \alpha_m<x<\beta_m.$$

By II lemma 14 [18]) $F(x)-G(x)+\frac{1}{4}a_0x^2$ is convex, and therefore

[17]) Proc. Lond. Math. Soc. 34 (441—456) 445.
[18]) loc. cit. [16]) 467.

continuous in the open interval. So then is $F(x)$. If $\alpha_m<\alpha<\beta<\beta_m$, then by the remark which follows II lemma 10 [19]),

$$F(x)=\int_\alpha^x dy\int_\alpha^y(\underline{P}(t)-\tfrac{1}{2}a_0)\,dt+Ax+B \qquad (\alpha<x<\beta)$$
$$=-\tfrac{1}{4}a_0x^2+px+q.$$

We can let α tend to α_m, and β tend to β_m. Then p and q must remain unaltered, and so we have an equation of the form (43)

CORRECTION.

Fund. Math. XXI 168—210.

This paper stands in need of two corrections.

(a) In the proof of lemma 15, pp. 186—187, that part of the proof which appears on p. 187 should be deleted, and the following substituted:

By (10) and lemma 13, the function $F(x+t)-F(x-t)-2tF_a(x)$ is non-increasing in $-h(\eta)\leqslant t\leqslant h(\eta)$. At $t=0$, it has the approximate derivative zero. By a theorem of Khintchine (lemma 27, p. 210), it has a differential coefficient equal to zero; i. e.

$$\lim\frac{F(x+h)-F(x-h)}{2h}=F_a(x).$$

Hence by (9),

$$\frac{F(x+h)-F(x-h)}{2h}>F_a(x).$$

By the last equation on p. 186,

$$F_a(x+\theta h)+F_a(x-\theta h)-2F_a(x)>0,$$

which implies $\overline{D}^2F_a(x)\geqslant 0$.

(b) In the proof of lemma 15a, pp. 209—210, the last two lines on p. 209 should be deleted. On p. 210, delete lines 13, 14 and 15 from the top and substitute the following:

[19]) loc. cit. [16]) 465.

Hence by (22) and (21),

$$\frac{F(\xi+\theta h)-F(\xi-\theta h)}{2\theta h} > F_a(\xi).$$

By the equation which follows (22), we infer

$$F_a(\xi+\theta\theta' h)+F_a(\xi-\theta\theta' h)-2F_a(\xi) > 0.$$

which implies $\overline{D}^2 F_a(\xi) \geqslant 0$.

Sur la théorie de la mesure dans les espaces combinatoires et son application au calcul des probabilités I. Variables indépendantes [1].

Par

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L'analogie entre la mesure et la probabilité est connue depuis longtemps [2].

La probabilité pour qu'un point appartienne à un ensemble A d'un espace donné remplit les postulats de la mesure d'ensemble. On admet notamment la règle des probabilités totales — c.-à-d. l'additivité finie ou dénombrable de la mesure. (Dans le cas où l'espace est dénombrable le postulat de l'additivité finie est souvent plus adéquat).

La théorie de la mesure pour *un* espace constitue cependant une théorie d'une seule variable éventuelle et ne semble pas donner un

[1]) Les résultats concernant la théorie de la mesure dans les produits ont été exposés par les auteurs dans un Séminaire de M. H. Steinhaus (Mai 1932). Les théorèmes relatifs ont été présentés à la séance de la Soc. Pol. Math. Section de Lwów du 2. VII. 1932 (v. aussi la note de l'un de nous insérée dans les „Verhandl. des Int. Math. Kongr. Zürich" 1932, Band II). Les applications au calcul des probabilités qui se trouvent dans la deuxième partie de ce travail ont été présentées à la séance de la Soc. Pol. Math. Section de Lwów le 18. III. 1933.

[2]) E. Borel, *Sur les probabilités dénombrables*... Rendiconti del Circolo Mat. di Palermo, 1909, p. 247—281. — A. Łomnicki, *Nouveaux fondements du calcul des probabilités.* Fund. Math. T. IV, p. 35—71. — H. Steinhaus, *Les probabilités dénombrables et leur rapport à la théorie de la mesure.* Fund. Math. T. IV, p. 287—310. — R. v. Mises, *Grundlagen der Wahrscheinlichkeitsrechnung.* Math. Zeit. Bd. 34, p. 568—619. — P. Lévy, *Calcul des probabilités.* Note (p. 325—345) Gauthier-Villars, Paris. — Cf. aussi une étude approfondie chez A. Kolmogoroff, *Grundbegriffe der Wahrscheinlichkeitsrechnung* dans les *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Berlin 1933.