Singular moduli (4).

By

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In this paper I give some arithmetical developments of the theoretical researches on singular moduli and class-invariants due to Kronecker and Dedekind. A full account of these researches is to be found in Weber's *Algebra* [15].[1]

In the ordinary notation of elliptic functions write

$$q = e^{i\pi\tau}, \qquad [I(\tau) > 0, \ |q| < 1]$$

$$f(\tau) = q^{-\frac{1}{24}} \prod_{m=1}^{\infty} (1 + q^{2m-1}), \quad f_1(\tau) = q^{-\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^{2m-1}),$$

$$f_2(\tau) = 2^{\frac{1}{2}} q^{\frac{1}{12}} \prod_{m=1}^{\infty} (1 + q^{2m}) = 2^{\frac{1}{2}} q^{\frac{1}{12}} \prod_{m=1}^{\infty} (1 - q^{4m-2})^{-1},$$

so that

$$f^8(\tau) = f_1^8(\tau) + f_2^8(\tau),$$

$$f(\tau) f_1(\tau) f_2(\tau) = \sqrt{2};$$

and let $j(\tau)$ be Dedekind's invariant such that $f^{24}(\tau)$, $-f_1^{24}(\tau)$, $-f_2^{24}(\tau)$ are the roots of the equation

$$(x-16)^3 - x j(\tau) = 0.$$

Let n be an integer of the form $8m-1$; and let the number of genera of classes of quadratic forms of negative determinant $-n$ be N, the number of classes in each genus being k; and let $Nk = h$. In the

[1] Numbers following authors' names refer to the bibliography at the end of the paper.

sequel this Gaussian class-number is indicated by appending (G. N. k) to each value of n. If

$$a_r x^2 + b_r xy + c_r y^2 \qquad (r = 1, 2, \ldots, h)$$

is a complete set of quadratic forms for which $b_r^2 - 4 a_r c_r = -n$, and if the h values of τ (with positive imaginary parts) which satisfy the equations

$$a_r \tau^2 + b_r \tau + c_r = 0$$

are called $\tau_1, \tau_2, \ldots \tau_h$, it is known that the equation

$$\prod_{r=1}^{h} [x - j(\tau_r)] = 0$$

(when the left-hand side is multiplied out) is an Abelian equation in x of degree h with integral coefficients. Hence also the equation

$$\prod_{r=1}^{h} [(x-16)^3 - x j(\tau_r)] = 0,$$

that is to say

$$\prod_{r=1}^{h} [\{x - f^{24}(\tau_r)\}\{x + f_1^{24}(\tau_r)\}\{x + f_2^{24}(\tau_r)\}] = 0,$$

when multiplied out, has integral coefficients. Since one of the τ_r is a root of the equation $\tau^2 - \tau + 2m = 0$, and is consequently equal to $\dfrac{1 + i\sqrt{n}}{2}$, and since $f(\tau) f_2\left(\dfrac{\tau+1}{2}\right) = e^{\frac{1}{24}\pi i} \sqrt{2}$, *it follows that one of the roots of the above equation of degree* $3h$ *is* $2^{12} f^{-24}(\sqrt{-n})$.

To avoid the occurrence of superfluous square-roots, I shall subsequently write

$$f(\sqrt{-n}) / \sqrt{2} = F_n;$$

and, when no confusion can arise, I shall omit the suffix and write simply F for F_n.

It is shown by Weber [15], p. 473, that, from the equation of degree $3h$ with roots $f^{24}(\tau_r)$, $-f_1^{24}(\tau_r)$ $-f_2^{24}(\tau_r)$, it is possible to extract an equation (with integral coefficients), of degree h in the class-invariant F_n, when n is not a multiple of 3; when n is a multiple of 3, the corresponding class-invariant is F_n^3.

In this paper I give a catalogue of the 75 of these equations for which n has the values 7, 15, 23, ..., 599 respectively [2]. The paper may therefore be regarded as an extension of Greenhill [2]; in that paper the values of n of the form under consideration go only as far as 95 (87 being omitted). Whereas Greenhill's results were mainly obtained by the use of modular equations, my own methods of constructing new equations are purely arithmetical. My procedure is to compute for each value of n a complete set of values of $f^{24}(\tau_r)$, $-f_1^{24}(\tau_r)$, $-f_2^{24}(\tau_r)$, and then to go through the somewhat laborious task of making the proper selection of the twenty-fourth (or eighth) roots of these numbers in such a way as to obtain an eqation of degree h with integral coefficients, one of whose roots is F_n (or F_n^3). The details of the manner in which the selection is effected are described rather more fully by Watson [12], where the same process is carried out for another set of values of n.

For the 75 values of n discussed in this paper, N always has one of the values I, II, IV; in 28 cases N is I, in 42 cases it is II, and in the remaining 5 cases it is IV. Since the paper just cited deals with the set of equations for values of n for which N is I and h is 1, 3, 5, ..., 15, there is a certain amount of overlapping between the two papers; for the 21 values of n common to the two papers, I give here the equations satisfied by F_n only, without repeating their solutions in terms of radicals; for the 56 values of n which are not discussed by Watson [12], in addition to giving the equations, I reduce them as much as seems conveniently feasible.

When n is of the form ab, where b is a prime and a is either a prime or a power of a prime ($a \neq b$), N is II. I then give an equation satisfied by F which is of degree k and which has quadratic irrationalities in its coefficients; the equation whose coefficients are the conjugate irrationalities has either $F_{a/b}$ or $-F_{a/b}$ for one of its roots, (with exceptions when k is even and $F_{a/b}$ or $-F_{a/b}$ satisfies the same equation as F_{ab}).

When N is II, the following abbreviations will be used throughout:

$$F_{ab}\, F_{b/a} = \alpha,$$

$$\frac{F_{ab}}{F_{a/b}} + \frac{F_{a/b}}{F_{ab}} = \beta, \qquad \frac{F_{ab}^3}{F_{a/b}^3} + \frac{F_{a/b}^3}{F_{ab}^3} = \gamma,$$

$$F_{ab} + F_{a/b} = S_1, \qquad F_{ab} - F_{a/b} = D_1,$$

$$F_{ab}^3 + F_{a/b}^3 = S_3, \qquad F_{ab}^3 - F_{a/b}^3 = D_3.$$

[2]) I originally intended to go only to 399, but the number of interesting class-invariants in the fifth and sixth centuries was so great that I decided to go to the limit to which this paper extends.

Then α (or α^3) and β (or γ) satisfy equations of degree k with integral coefficients, and, since

$$2\, F_{ab}^2 = \alpha\,(\beta + \sqrt{\beta^2 - 4}),$$

it is evident that F_n is determinate when α and β have been determined.

I show how to obtain α and β (or γ) when k has no prime factors other than 2 or 3. In other cases, in addition to the equations for α and β (or γ), I give equations of degree k satisfied by S_1 (or S_3) and D_1 (or D_3); of the lattter pairs of equations, one member has all its coefficients integers, the other member has integers and quadratic surds as alternate coefficients.

The values of n for which N is IV are so few that abbreviations of the type just explained are unnecessary for them.

When h is a multiple of 3, I have usually given the results in terms of Berwick's cubic irrationalities; I quote the equations which define these irrationalities from the Table which was computed by Berwick and published by Mathews [5]. To each cubic equation given I append the value of its discriminant Δ.

After these preliminary explanations I give the catalogue of equations, arranged in order of magnitude of values of n, not in order of magnitude of values of h as was more convenient in my previous paper.

n = 7. (G. I. 1).

The result

$$F = 1$$

is due to Joubert [4].

n = 15. (G. II. 1).

The equation satisfied by F_{15} is

$$F^6 - F^3 - 1 = 0,$$

which, by adjunction of $\sqrt{5}$, reduces to

$$F^3 - \frac{1+\sqrt{5}}{2} = 0,$$

a result due to Joubert [4].

Here we have

$$\alpha = 1, \qquad \gamma = 3, \qquad D_3 = 1.$$

n = 23. (G. I. 3).

The equation satisfied by F_{23}, given by Weber [13] and Greenhill [1], is [3])

$$F^3 - * - F - 1 = 0. \qquad (\Delta = -23)$$

n = 31. (G. I. 3).

The equation satisfied by F_{31}, given by Weber [13] and Greenhill [1], is

$$F^3 - F^2 - * - 1 = 0. \qquad (\Delta = -31)$$

n = 39. (G. II. 2).

The equation satisfied by F_{39}, due to Joubert [4], is

$$F^{12} - 3F^9 - 4F^6 - 2F^3 - 1 = 0,$$

which, by adjunction of $\sqrt{13}$, reduces to

$$\left[F^3 - \frac{3+\sqrt{13}}{4}\right]^2 = \frac{23+7\sqrt{13}}{8}.$$

Here we have

$$\alpha^3 = \frac{3+\sqrt{13}}{2} = D_3, \qquad \beta = \frac{1+\sqrt{13}}{2}.$$

n = 47. (G. I. 5).

The equation satisfied by F_{47}, due to Weber [13] and Greenhill [1], is

$$F^5 - * - F^3 - 2F^2 - 2F - 1 = 0;$$

references to solutions of this equation are given by Watson [12].

n = 55. (G. II. 2).

The equation satisfied by F_{55} is

$$F^4 - 2F^3 + * + F - 1 = 0,$$

which, by adjunction of $\sqrt{5}$, reduces to

$$\left[F - \frac{1}{2}\right]^2 = \frac{3+2\sqrt{5}}{4}.$$

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[3]) I use the Weierstrassian symbol $*$ to indicate that a term is absent from an equation.

These results are given by Weber [13], and also they are attributed to Russell by Greenhill [2].

Here

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{3+\sqrt{5}}{2}, \quad D_1 = 1.$$

n = 63. (G. II. 2).

The equation satisfied by F_{63}, due to Joubert [4], is

$$F^{12} - 8F^9 + * + F^3 + 1 = 0,$$

which, by adjunction of $\sqrt{21}$, reduces to

$$\left[F^3 - \frac{4+\sqrt{21}}{2}\right]^2 = \frac{27+6\sqrt{21}}{4};$$

results equivalent to these are given by Weber [13] and [15].

Here we have

$$\alpha^3 = \frac{5+\sqrt{21}}{2}, \quad \beta = \frac{1+\sqrt{21}}{2}, \quad S_3 = 4+\sqrt{21}.$$

n = 71. (G. I. 7).

The equation satisfied by F_{71}, due to Russell [9] and Weber [13], is

$$F^7 - 2F^6 - F^5 + F^4 + F^3 + F^2 - F - 1 = 0;$$

its solution is given by Watson [12].

n = 79. (G. I. 5).

The equation satisfied by F_{79}, due to Russell [9], is

$$F^5 - 3F^4 + 2F^3 - F^2 + F - 1 = 0;$$

its solution is given by Watson [12].

n = 87. (G. II. 3).

The equation satisfied by F_{87} is

$$F^{18} - 13F^{15} - 11F^{12} - 4F^9 - 4F^6 - F^3 - 1 = 0,$$

which, by adjunction of $\sqrt{29}$, reduces to

$$F^9 - \frac{13+3\sqrt{29}}{2}F^6 + (6+\sqrt{29})\,F^3 - \frac{5+\sqrt{29}}{2} = 0.$$

Results equivalent to these are given by Hanna [3].

Here α, γ and D_3 are given in terms of Berwick's cubic irrationality

$$\theta^3 - 2\theta^2 + 3\theta - 3 = 0 \qquad (\Delta = -87)$$

by the formulae

$$\alpha = \theta^2 - \theta + 2, \ \gamma = 4\theta^2 - 3\theta + 8, \ D_3 = 4\theta^2 - 3\theta + 9.$$

Further we have

$$3F^3 = \frac{13 + 3\sqrt{29}}{2} + (11 + 2\sqrt{29})\sqrt[3]{\frac{\sqrt{29} - 5}{2}} + \frac{11 + \sqrt{29}}{2}\sqrt[3]{\frac{5 + \sqrt{29}}{2}}.$$

n = 95. (G. II. 4).

The equation satisfied by F_{95} is

$$F^8 - 2F^7 - 2F^6 + F^5 + 2F^4 - F^3 + * + F - 1 = 0;$$

this equation is given by Russell [9]; Greenhill [2], who gives its solution by radicals (with some errors in signs), attributes it to G.B. Mathews.

By adjunction of $\sqrt{5}$ we have

$$\left[F^2 - \frac{1 + \sqrt{5}}{2}F - \frac{1 + \sqrt{5}}{4}\right]^2 = (2\sqrt{5} - 1)\left[\frac{1 + \sqrt{5}}{4}\right]^2.$$

Here we have

$$\alpha = \frac{1 + \sqrt{5}}{4}\left[1 + \sqrt{2\sqrt{5} - 1}\right],$$

$$\beta = \frac{5 - \sqrt{5}}{4} + \frac{3 + \sqrt{5}}{4}\sqrt{2\sqrt{5} - 1};$$

also

$$\beta = 2\alpha^3 - 3\alpha^2 - 2\alpha - 1,$$

and

$$D_1 = -\alpha^3 + \alpha^2 + \alpha + 1 = \frac{1 + \sqrt{5}}{2}.$$

n = 103. (G. I. 5).

The equation satisfied by F_{103}, due to Russell [9], is

$$F^5 - F^4 - 3F^3 - 3F^2 - 2F - 1 = 0;$$

its solution is given by Watson [12].

n = 111. (G. II. 4).

The equation satisfied by F_{111} is

$$F^{24} - 21F^{21} - 26F^{18} + 14F^{15} + 18F^{12} - 11F^9 - 6F^6 - F^3 - 1 = 0;$$

it is given (with an error in sign) by Hanna [3]. By adjunction of $\sqrt{37}$, it is expressible in the form

$$\left[F^6 - \frac{21 + 3\sqrt{37}}{4}F^3 - \frac{31 + 5\sqrt{37}}{4}\right]^2$$
$$= \frac{\sqrt{37} - 5}{8}\left[\frac{29 + 5\sqrt{37}}{2}F^3 + \frac{43 + 7\sqrt{37}}{2}\right]^2.$$

Here we have

$$\alpha^3 = \frac{31 + 5\sqrt{37}}{4} + \frac{43 + 7\sqrt{37}}{2}\sqrt{\frac{\sqrt{37} - 5}{8}},$$

$$\beta = \frac{1}{2} + \frac{7 + \sqrt{37}}{2}\sqrt{\frac{\sqrt{37} - 5}{8}},$$

also

$$\alpha^3 = \beta^3 + \beta^2 - \beta + 1, \qquad D_3 = \beta^3 - 2\beta + 2.$$

n = 119. (G. II. 5).

The equation satisfied by F_{119} is

$$F^{10} - 4F^9 + 5F^8 - 8F^7 + 9F^6 - 7F^5 + 5F^4 - 4F^3 + 2F^2 - F + 1 = 0,$$

which, by adjunction of $\sqrt{17}$, reduces to

$$F^5 - 2F^4 - \frac{\sqrt{17} - 1}{2}F^3 - 3F^2 - \frac{\sqrt{17} - 1}{2}F - 1 = 0.$$

The equations satisfied by α, β, S_1 and D_1 are

$$\alpha^5 - 3\alpha^4 + \alpha^3 + * + \alpha - 1 = 0,$$

$$\beta^5 - 3\beta^4 - 5\beta^3 + 7\beta^2 - 12\beta - 13 = 0,$$

$$S_1^5 - 4S_1^4 + 2S_1^3 - 3S_1^2 - 6S_1 - 7 = 0,$$

$$D_1^5 - 8D_1 = (D_1^2 + 1)\sqrt{17}.$$

n = 127. (G. I. 5).

The equation satisfied by F_{127}, constructed by Hanna [3], is

$$F^5 - 3F^4 - F^3 + 2F^2 + F - 1 = 0;$$

it has been solved by Mitra [8] and Watson [12].

n = 135. (G. II. 3).

The equation satisfied by F_{135} is

$$F^{18} - 33F^{15} - 30F^{12} - 2F^9 + 3F^6 + 3F^3 - 1 = 0,$$

which, by adjunction of $\sqrt{5}$, reduces to

$$F^9 - \frac{33+15\sqrt{5}}{2}F^6 - \frac{21+9\sqrt{5}}{2}F^3 - \frac{11+5\sqrt{5}}{2} = 0.$$

Hence

$$F^3 = \frac{11+5\sqrt{5}}{2} + (4+2\sqrt{5})\sqrt[3]{\frac{3+\sqrt{5}}{2}} + (5+2\sqrt{5})\sqrt[3]{\frac{1+\sqrt{5}}{2}}.$$

In terms of Berwick's cubic irrationality

$$\theta^3 - 3\theta^2 - * - 1 = 0, \qquad (\Delta = -135)$$

we have

$$\alpha = \theta, \qquad \gamma = 4\theta^2 - \theta + 3,$$

$$D_3 = 3\theta^2 + \theta + 1,$$

$$2F_{135}{}^3 = \theta^2 + \theta + 1 + \frac{7\theta^2 + 3\theta + 1}{\sqrt{5}}.$$

n = 143. (G. II. 5).

The equation satisfied by F_{143} is

$$F^{10} - 6F^9 + 12F^8 - 13F^7 + 9F^6 - 3F^5 - 3F^4 + 6F^3 - 6F^2 + 3F - 1 = 0,$$

which, by adjunction of $\sqrt{13}$, reduces to

$$F^5 - (3+\sqrt{13})F^4 + (8+2\sqrt{13})F^3 - \frac{17+5\sqrt{13}}{2}F^2 + \frac{11+3\sqrt{13}}{2}F - \frac{3+\sqrt{13}}{2} = 0;$$

results equivalent to these are given by Hanna [3].

The equations satisfied by α, β, D_1 and S_1 are

$$\alpha^5 - 3\alpha^4 + * + \alpha^2 + \alpha - 1 = 0,$$

$$\beta^5 - \beta^4 - 9\beta^3 - 16\beta^2 - 19\beta - 11 = 0,$$

$$D_1{}^5 - 6D_1{}^4 + 15D_1{}^3 - 24D_1{}^2 + 26D_1 - 13 = 0,$$

$$S_1{}^5 + 17S_1{}^3 + 14S_1 = (2S_1{}^4 + 6S_1{}^2 + 1)\sqrt{13}.$$

n = 151. (G. I. 7).

The equation satisfied by F_{151}, constructed by Hanna [3], is

$$F^7 - 3F^6 - F^5 - 3F^4 - * - F^2 - F - 1 = 0;$$

its solution is given by Watson [12].

n = 159. (G. II. 5).

The equation satisfied by F_{159} is

$$F^{30} - 47F^{27} - 146F^{24} - 196F^{21} - 219F^{18} - 121F^{15} - 63F^{12} - 7F^9 - F^6 + F^3 - 1 = 0,$$

which, by adjunction of $\sqrt{53}$, reduces to

$$F^{15} - \frac{47+7\sqrt{53}}{2}F^{12} - \frac{49+7\sqrt{53}}{2}F^9 - \frac{49+7\sqrt{53}}{2}F^6 - \frac{23+3\sqrt{53}}{2}F^3 - \frac{7+\sqrt{53}}{2} = 0.$$

The equations satisfied by α, γ, D_3 and S_3 are

$$\alpha^5 - 4\alpha^4 - 2\alpha^3 - \alpha^2 - 2\alpha - 1 = 0,$$

$$\gamma^5 - 34\gamma^4 + 205\gamma^3 - 561\gamma^2 + 747\gamma - 459 = 0,$$

$$D_3{}^5 - 47D_3{}^4 - 55D_3{}^3 - 37D_3{}^2 - 12D_3 - 9 = 0,$$

$$S_3{}^5 - 43S_3{}^3 - 70S_3 = (7S_3{}^4 + 19S_3 + 11)\sqrt{53}.$$

n = 167. (G. I. 11).

The equation satisfied by F_{167}, constructed by Hanna [3], is

$$F^{11} - 2F^{10} - 4F^9 - 9F^8 - 7F^7 - 11F^6 - 6F^5 - 10F^4 - 4F^3 - 5F^2 - F - 1 = 0.$$

I have not solved this equation.

n = 175. (G. II. 3).

The equation satisfied by F_{175}, constructed by Weber **[13]**, is

$$F^6 - 4F^5 + * + * + * + F + 1 = 0;$$

by adjunction of $\sqrt{5}$, this reduces to

$$F^3 - (2 + \sqrt{5})F^2 + \frac{1+\sqrt{5}}{2}F - \frac{3+\sqrt{5}}{2} = 0.$$

The corresponding reduction for F_{175}^{-2} has been effected by Mitra **[6]**.

In terms of Berwick's cubic irrationality

$$\theta^3 - \theta^2 + 2\theta - 3 = 0, \qquad (\Delta = -175)$$

we have

$$\alpha = \theta^2 + 2, \qquad \beta = \theta^2 + 3,$$

$$S_1 = \theta^2 + \theta + 2,$$

$$2F_{175} = \theta^2 + \theta + 2 + \frac{\theta^2 + \theta + 4}{\sqrt{5}}.$$

n = 183. (G. II. 4).

The equation satisfied by F_{183} is

$$F^{24} - 71F^{21} - 53F^{18} + 157F^{15} - 98F^{12} + 10F^9 + 11F^6 - F^3 - 1 = 0,$$

which, by adjunction of $\sqrt{61}$, is expressible in the form

$$\left[F^6 - \frac{71+9\sqrt{61}}{4}F^3 - \frac{149+19\sqrt{61}}{4}\right]^2$$

$$= \frac{7+\sqrt{61}}{2}\left[\frac{27+3\sqrt{61}}{4}F^3 + \frac{55+7\sqrt{61}}{4}\right]^2.$$

Here we have

$$\alpha^{12} - 149\alpha^9 + 6\alpha^6 + 8\alpha^3 - 1 = 0,$$

$$\beta^4 - \beta^3 - 9\beta^2 - 3\beta + 9 = 0,$$

so that

$$\alpha^3 = \frac{149+19\sqrt{61}}{4} + \frac{55+7\sqrt{61}}{4}\sqrt{\frac{7+\sqrt{61}}{2}},$$

$$\beta = \frac{1+\sqrt{61}}{4} + \sqrt{\frac{7+\sqrt{61}}{8}};$$

also

$$\alpha^3 = 2\beta^3 + 5\beta^2 - 5, \qquad D_3 = \beta^3 + 2\beta^2 - 1.$$

n = 191. (G. I. 13).

The equation satisfied by F_{191}, constructed by Hanna **[3]**, is

$$F^{13} - 6F^{12} + 10F^{11} - 16F^{10} + 22F^9 - 19F^8 + 11F^7 - 5F^6$$
$$- F^5 + 5F^4 - 4F^3 + * + 2F - 1 = 0;$$

I have not solved this equation.

n = 199. (G. I. 9).

The equation satisfied by F_{199} is

$$F^9 - 5F^8 + 3F^7 - 3F^6 - * - * - 3F^3 - * - F - 1 = 0;$$

it is solved by Watson **[12]**.

n = 207. (G. II. 3).

The equation satisfied by F_{207} is

$$F^{18} - 102F^{15} + 151F^{12} - 103F^9 + 46F^6 - 11F^3 + 1 = 0,$$

which, by adjunction of $\sqrt{69}$, reduces to

$$F^9 - (51 + 6\sqrt{69})F^6 + (17 + 2\sqrt{69})F^3 - \frac{25+3\sqrt{69}}{2} = 0.$$

Hence

$$3F^3 = 51 + 6\sqrt{69} + (50\sqrt{3} + 18\sqrt{23})\sqrt[3]{\frac{3\sqrt{3}-\sqrt{23}}{2}}$$

$$+ (17\sqrt{3} + 6\sqrt{23})\sqrt[3]{\frac{3\sqrt{3}+\sqrt{23}}{2}}.$$

It is convenient to work with the cubic for F_{23}, namely

$$\theta^3 - * - \theta - 1 = 0;$$

it is found that

$$\alpha = \theta^2 + \theta + 1, \qquad \gamma = 36\theta^2 + 45\theta + 26,$$

$$S_3 = 24\theta^2 + 31\theta + 18,$$

$$2F_{207}^3 = 24\theta^2 + 31\theta + 18 + \frac{198\theta^2 + 255\theta + 144}{\sqrt{69}}.$$

$\mathbf{n = 215}$. (G. II, 7).

The equation satisfied by F_{215} is

$$F^{14} - 6F^{13} + 4F^{12} + 11F^{11} - 13F^{10} - 7F^{9} + 16F^{8} - 4F^{7} - 13F^{6}$$
$$+ 8F^{5} + 3F^{4} - 6F^{3} + * + 2F - 1 = 0$$

which, by adjunction of $\sqrt{5}$, reduces to

$$F^7 - (3+\sqrt{5})F^6 + \sqrt{5}\,F^5 + \frac{1-\sqrt{5}}{2}F^4 - \sqrt{5}\,F^3 - F^2$$
$$+ \frac{1+\sqrt{5}}{2}F - \frac{1+\sqrt{5}}{2} = 0.$$

The equations satisfied by α, β, D_1 and S_1 are

$$\alpha^7 - 5\alpha^6 + \alpha^5 - 6\alpha^4 + 5\alpha^3 - 3\alpha^2 + 3\alpha - 1 = 0,$$
$$\beta^7 - 5\beta^6 - 6\beta^5 + 37\beta^4 - 10\beta^3 - 76\beta^2 + 97\beta - 43 = 0,$$
$$D_1^7 - 6D_1^6 + 9D_1^5 - D_1^4 - D_1^3 - 12D_1^2 - 10D_1 - 5 = 0,$$
$$S_1^7 - 9S_1^5 - S_1^3 - 40S_1 = (2S_1^6 - 3S_1^4 + 18S_1^2 + 5)\sqrt{5}\ .$$

$\mathbf{n = 223}$. (G. I, 7).

The equation satisfield by F_{223}, constructed by Hanna [3], is

$$F^7 - 5F^6 + * + F^4 - 4F^3 - F^2 - * - 1 = 0;$$

its solution is given by Watson [12].

$\mathbf{n = 231}$. (G. IV, 3).

The equation satisfied by F_{231} is

$$F^{36} - 138F^{33} - 33F^{30} + 517F^{27} - 724F^{24} + 349F^{21} - 118F^{18}$$
$$- 32F^{15} + 114F^{12} - 45F^{9} + 29F^{6} - 4F^{3} + 1 = 0,$$

which, by adjunction of $\sqrt{3}$, $\sqrt{7}$ and $\sqrt{11}$, reduces to

$$2F^9 - (69 + 15\sqrt{21} + 12\sqrt{33} + 8\sqrt{77})F^6$$
$$+ (14 + 3\sqrt{21} + 3\sqrt{33} + 2\sqrt{77})(F^3 - 1) = 0.$$

In terms of Berwick's cubic irrationality

$$\theta^3 - 2\theta^2 + \theta - 3 = 0, \qquad (\Delta = -231)$$

we have

$$F^{12} - (21\theta^2 + 3\theta + 30)F^9 - (51\theta^2 + 9\theta + 70)F^6$$
$$+ (22\theta^2 + 4\theta + 30)F^3 + (29\theta^2 + 5\theta + 40) = 0.$$

Further we have

$$F_{231}F_{77/3} + F_{33/7}F_{21/11} = \theta^2 + \theta + 1,$$
$$(F_{231}F_{33/7})^3 + (F_{77/3}F_{21/11})^3 = 18\theta^2 + 3\theta + 25$$
$$F_{231}F_{21/11} + F_{77/3}F_{33/7} = \theta^2 + 1,$$
$$F_{231}F_{77/3}F_{33/7}F_{21/11} = \theta^2 + 1.$$

$\mathbf{n = 239}$. (G. I. 15).

The equation satisfied by F_{239}, constructed by Hanna [3], is

$$F^{15} - 6F^{14} + 2F^{13} + 8F^{12} + 4F^{11} - 27F^{10} + 13F^{9} + 15F^{8} - 4F^{7}$$
$$- 20F^{6} + 13F^{5} + 5F^{4} - 4F^{3} - 4F^{2} + 4F - 1 = 0;$$

its solution is given by Watson [12].

$\mathbf{n = 247}$. (G. II. 3).

The equation satisfied by F_{247} is

$$F^6 - 4F^5 - 7F^4 - 7F^3 - 6F^2 - 3F - 1 = 0,$$

which, by adjunction of $\sqrt{13}$, reduces to

$$F^3 - (2 + \sqrt{13})F^2 + F - \frac{3+\sqrt{13}}{2} = 0.$$

$$\left[\Delta = -19 \times \left(\frac{3+\sqrt{13}}{2}\right)^4\right].$$

In terms of Berwick's cubic irrationality

$$\theta^3 - 3\theta^2 + 4\theta - 5 = 0, \qquad (\Delta = -247)$$

we have

$$\alpha = D_1 = \theta^2 - \theta + 2, \quad \beta = \theta^2 - \theta + 4,$$
$$2F_{247} = \theta^2 - \theta + 2 + \frac{5\theta^2 - 7\theta + 14}{\sqrt{13}}.$$

n = 255. (G. IV. 3).

The equation satisfied by F_{255} is

$$F^{36} - 186\,F^{33} - 194\,F^{30} + 839\,F^{27} - 702\,F^{24} - 287\,F^{21} + 1012\,F^{18} - 912\,F^{15}$$
$$+ 513\,F^{12} - 221\,F^{9} + 66\,F^{6} - 11\,F^{3} + 1 = 0,$$

which, by adjunction of $\sqrt{5}$ and $\sqrt{17}$, reduces to

$$4\,F^{9} - (186 + 84\sqrt{5} + 46\sqrt{17} + 20\sqrt{85}\,)\,F^{6}$$
$$- (11 + 3\sqrt{5} + \sqrt{17} + \sqrt{85}\,)\,F^{3} - (36 + 18\sqrt{5} + 10\sqrt{17} + 4\sqrt{85}\,) = 0.$$

In terms of Berwick's cubic irrationality

$$\theta^3 - \theta^2 - * - 3 = 0, \qquad (\Delta = -255)$$

we have

$$F^{12} - (27\,\theta^2 + 24\,\theta + 45)\,F^{9} - (98\,\theta^2 + 84\,\theta + 157)\,F^{6}$$
$$- (31\,\theta^2 + 27\,\theta + 50)\,F^{3} + (59\,\theta^2 + 51\,\theta + 95) = 0.$$

Further we have

$$(F_{255}\,F_{85/3})^3 + (F_{51/5}\,F_{15/17})^3 = 80\,\theta^2 + 69\,\theta + 129,$$
$$F_{255}\,F_{51/5} + F_{85/3}\,F_{15/17} = \theta^2 + \theta + 2,$$
$$F_{255}\,F_{15/17} + F_{85/3}\,F_{51/5} = \theta^2 + \theta + 1,$$
$$F_{255}\,F_{85/3}\,F_{51/5}\,F_{15/17} = \theta^2 + \theta + 2.$$

n = 263. (G. I. 13).

The equation satisfied by F_{263}, constructed by Watson [**12**], is

$$F^{13} - 8\,F^{12} + 16\,F^{11} - 27\,F^{10} + 38\,F^{9} - 36\,F^{8} + 22\,F^{7} - 12\,F^{6} + 13\,F^{5}$$
$$- 19\,F^{4} + 21\,F^{3} - 15\,F^{2} + 6\,F - 1 = 0.$$

I have not solved this equation.

n = 271. (G. I. 11).

The equation satisfied by F_{271}, constructed by Watson [**12**], is

$$F^{11} - 5\,F^{10} - 6\,F^{9} - 5\,F^{8} + 3\,F^{7} + 6\,F^{6} + 3\,F^{5} - 3\,F^{4} - F^{3}$$
$$- F^{2} - * - 1 = 0.$$

I have not solved this equation.

n = 279. (G. II. 6).

The equation satisfied by F_{279} is

$$F^{36} - 248\,F^{33} - 384\,F^{30} - 508\,F^{27} - 110\,F^{24} + 264\,F^{21} + 259\,F^{18} + 3\,F^{15}$$
$$- 27\,F^{12} - 23\,F^{9} + 31\,F^{6} - 6\,F^{3} + 1 = 0,$$

which, by adjunction of $\sqrt{93}$, reduces to

$$F^{8} - (124 + 13\sqrt{93})\,F^{15} - \frac{43 + 5\sqrt{93}}{2}\,F^{12} + \frac{205 + 21\sqrt{93}}{2}\,F^{9} +$$
$$+ (20 + 2\sqrt{93})\,F^{6} - \frac{87 + 9\sqrt{93}}{2}\,F^{3} + \frac{29 + 3\sqrt{93}}{2} = 0.$$

It is simpler to work with the cubic for F_{31}, namely

$$\theta^3 - \theta^2 - * - 1 = 0, \qquad (\Delta = -31)$$

than with Berwick's cubic irrationality

$$\varphi^3 - \varphi^2 + 4\,\varphi - 3 = 0; \qquad (\Delta = -279)$$

it is found that

$$2\,\alpha^3 = 4\,\theta^{10} + \theta^2 + \theta^{10}\sqrt{9\,\theta + 3}, \qquad 2\,\beta = 3\,\theta^3 + 1 + (\theta^2 - \theta + 1)\sqrt{9\,\theta + 3},$$
$$S_3 = 6\,\theta^9 + \theta^7\sqrt{9\,\theta + 3};$$

it is also to be noticed that

$$27\,\theta^2 - 9\,\theta - 6 = \sqrt{93\,(9\,\theta + 3)}.$$

n = 287. (G. II. 7).

The equation satisfied by F_{287} is

$$F^{14} - 8\,F^{13} + 9\,F^{12} + 6\,F^{11} - 5\,F^{10} - 7\,F^{9} - 8\,F^{8} + 6\,F^{7} + 2\,F^{6}$$
$$- * - F^{4} - F^{3} + 3\,F^{2} + F + 1 = 0,$$

which, by adjunction of $\sqrt{41}$, reduces to

$$F^{7} - 4\,F^{6} - \frac{7 + 3\sqrt{41}}{2}\,F^{5} - (11 + \sqrt{41})\,F^{4} - \frac{13 + 3\sqrt{41}}{2}\,F^{3}$$

$$-\frac{13+\sqrt{41}}{2}F^2-\frac{1+\sqrt{41}}{2}F-1=0.$$

The equations satisfied by α, β, S_1 and D_1 are

$$\alpha^7-5\alpha^6-6\alpha^5-12\alpha^4-12\alpha^3-10\alpha^2-4\alpha-1=0,$$

$$\beta^7-5\beta^6-18\beta^5+20\beta^4+116\beta^3+50\beta^2-238\beta-245=0,$$

$$S_1{}^7-8S_1{}^6+4S_1{}^5-3S_1{}^4+23S_1{}^3-9S_1{}^2-6S_1-7=0,$$

$$D_1{}^7-18D_1{}^5-63D_1{}^3+24D_1=(9D_1{}^4+D_1{}^2+1)\sqrt{41}.$$

n = 295. (G. II. 4).

The equation satisfied by F_{295} is

$$F^8-8F^7+9F^6-F^5-7F^4+10F^3-7F^2+3F-1=0.$$

which, by adjunction of $\sqrt{5}$, is expressible in the form

$$\left[F^2-\frac{4+\sqrt{5}}{2}F-\frac{7+3\sqrt{5}}{4}\right]^2=\frac{11+6\sqrt{5}}{4}\left[F+\frac{1+\sqrt{5}}{2}\right]^2.$$

Here we have

$$\alpha^4-7\alpha^3-3\alpha^2-\alpha-1=0,\qquad \beta^4-6\beta^3-2\beta^2+3\beta+9=0,$$

so that

$$\alpha=\frac{7+3\sqrt{5}}{4}+\sqrt{\frac{63+29\sqrt{5}}{8}},\quad \beta=\frac{3+2\sqrt{5}}{2}+\sqrt{\frac{11+6\sqrt{5}}{4}},$$

$$45\alpha=\beta^3+6\beta^2-20\beta-12;$$

also
$$45D_1=4\beta^3-21\beta^2+10\beta+42.$$

n = 303. (G. II. 5).

The equation satisfied by F_{303} is

$$F^{30}-325F^{27}-1302F^{24}-756F^{21}-720F^{18}-447F^{15}-173F^{12}$$
$$-46F^9-36F^6-2F^3-1=0,$$

which, by adjunction of $\sqrt{101}$, reduces to

$$F^{15}-\frac{325+33\sqrt{101}}{2}F^{12}-\frac{211+21\sqrt{101}}{2}F^9-\frac{47+5\sqrt{101}}{2}F^6$$
$$-(10+\sqrt{101})F^3-(10+\sqrt{101})=0.$$

The equations satisfied by α, γ, D_3 und S_3 are

$$\alpha^5-10\alpha^4-5\alpha^3+5\alpha^2+\alpha-1=0,$$

$$\gamma^5-95\gamma^4-32\gamma^3+135\gamma^2+2025\gamma-3375=0,$$

$$D_3{}^5-325D_3{}^4-167D_3{}^3-26D_3{}^2+35D_3-23=0,$$

$$S_3{}^5-255S_3{}^3-70S_3=(33S_3{}^4+38S_3{}^2+9)\sqrt{101}.$$

n = 311. (G. I. 19).

The equation satisfied by F_{311} is

$$F^{19}-4F^{18}-16F^{17}-37F^{16}-42F^{15}-38F^{14}-4F^{13}+10F^{12}+25F^{11}$$
$$+18F^{10}+9F^9+F^8-10F^7-13F^6-14F^5-8F^4-5F^3$$
$$-2F^2-F-1=0.$$

I have not solved this equation.

n = 319. (G. II. 5).

The equation satisfied by F_{319} is

$$F^{10}-6F^9-9F^8-5F^7-F^6-2F^5-10F^4-14F^3-11F^2-5F-1=0$$

which, by adjunction of $\sqrt{29}$, reduces to

$$F^5-(3+\sqrt{29})F^4+\frac{11+\sqrt{29}}{2}F^3-\frac{1+\sqrt{29}}{2}F^2+F-\frac{5+\sqrt{29}}{2}=0.$$

The equation satisfied by α is

$$\alpha^5-6\alpha^4-3\alpha^3+\alpha^2-\alpha-1=0;$$

also
$$\beta=\alpha+2,\qquad D_1=\alpha,$$

and
$$S_1{}^5+25S_1{}^3+49S_1=(2S_1{}^4+7S_1{}^2+9)\sqrt{29}.$$

n = 327. (G. II. 6).

The equation satisfied by F_{327} is

$$F^{36}-423F^{33}-2573F^{30}-4844F^{27}-5524F^{24}-4006F^{21}-1436F^{18}$$
$$+92F^{15}+174F^{12}-58F^{9}-63F^{6}-14F^{3}-1=0,$$

which, by adjunction of $\sqrt{109}$, is expressible in the form

$$\left[F^9-\frac{423+41\sqrt{109}}{4}F^6-\frac{93+9\sqrt{109}}{2}F^3-\frac{407+39\sqrt{109}}{4}\right]^2$$
$$=\frac{\sqrt{109}-1}{2}\left[\frac{199+19\sqrt{109}}{4}F^6+(21+2\sqrt{109})F^3+\frac{94+9\sqrt{109}}{2}\right]^2.$$

In terms of Berwick's cubic irrationality

$$\theta^3-4\theta^2+3\theta-3=0, \qquad (\Delta=-327)$$

the equation reduces to

$$F^6-D_3F^3-\alpha^3=0,$$

where

$$2\alpha^3=159\theta^2-99\theta+141+\frac{1661\theta^2-1035\theta+1475}{\sqrt{109}},$$

$$2D_3=42\theta^2-27\theta+37+\frac{440\theta^2-285\theta+403}{\sqrt{109}};$$

it is also found that

$$2\beta=\theta^2-\theta+2+\frac{5\theta^2-2\theta-14}{\sqrt{109}}.$$

It is to be remarked that

$$1661\theta^2-1035\theta+1475=(46\theta^2-29\theta+41)(5\theta^2-2\theta-14),$$
$$440\theta^2-285\theta+403=(12\theta^2-7\theta+10)(5\theta^2-2\theta-14)$$
$$5\theta^2-2\theta-14=\sqrt{109}(\theta^2-\theta+4).$$

n = 335. (G. II. 9).

The equation satisfied by F_{335} is

$$F^{18}-4F^{17}-20F^{16}-55F^{15}-106F^{14}-144F^{13}-163F^{12}-174F^{11}-179F^{10}$$
$$-171F^9-144F^8-102F^7-64F^6-42F^5$$
$$-33F^4-25F^3-14F^2-5F-1=0,$$

which, by adjunction of $\sqrt{5}$, reduces to

$$F^9-(2+2\sqrt{5})F^8-(2+3\sqrt{5})F^7-\frac{3+7\sqrt{5}}{2}F^6-\frac{1+5\sqrt{5}}{2}F^5$$
$$+\frac{3-3\sqrt{5}}{2}F^4+\frac{5-3\sqrt{5}}{2}F^3+\frac{3-3\sqrt{5}}{2}F^2-\sqrt{5}F-\frac{1+\sqrt{5}}{2}=0.$$

By means of Berwick's cubic irrationality

$$\theta^3-2\theta^2+5\theta-5=0 \qquad (\Delta=-335)$$

the equation is further reducible to

$$\theta\left[2F^3-(\theta^2-\theta+4)F^2-(\theta^2-\theta+3)F-(\theta^2-\theta+4)\right]$$
$$=\left[(\theta^2-\theta+5)F^2+(\theta^2+3)F+(\theta^2-\theta+3)\right]\sqrt{5};$$

it is to be noticed that

$$\frac{\sqrt{5}}{\theta}=\sqrt{\theta^2-\theta+3}.$$

Further, the equations satisfied by α, β and D_1 are

$$\alpha^9-7\alpha^8-14\alpha^7-21\alpha^6-11\alpha^5-2\alpha^4-\alpha^3+\alpha^2-*-1=0,$$
$$\beta^9-*-29\beta^7-102\beta^6-173\beta^5-263\beta^4-376\beta^3-303\beta^2-133\beta-67=0,$$
$$D_1^9-4D_1^8-13D_1^7-30D_1^6+D_1^5+32D_1^4+95D_1^3$$
$$+3D_1^2-45D_1-155=0.$$

By adjunction of the cubic irrationality θ, these equations reduce to

$$\alpha^3-(2\theta^2-\theta+7)\alpha^2-(\theta^2-\theta+4)=0,$$
$$\beta^3-(\theta^2+2)\beta^2-(4\theta^2-\theta+16)\beta-(4\theta^2-3\theta+18)=0,$$
$$D_1^3-(\theta^2-\theta+4)D_1^2-(3\theta^2-3\theta+12)D_1-(3\theta^2-2\theta+15)=0.$$

Finally, β and D_1 are expressible as rational functions of α by the formulae

$$5\beta=(\theta^2-4\theta+4)\alpha^2+(2\theta^2-6\theta+4)\alpha-(\theta^2-\theta+8),$$
$$5D_1=(-6\theta+7)\alpha^2+(3\theta^2-7\theta+12)\alpha-(2\theta^2-\theta+4).$$

n = 343. (G. I. 7).

The equation satisfied by F_{343} is

$$F^7-7F^6-7F^5-7F^4-*-*-*-1=0;$$

its solution is given by Watson [12]. The equation is very easily obtainable from Schläfli's modular equation of order 7 which connects F_{343} with F_7.

n = 351. (G. II. 6).

The equation satisfied by F_{351} is

$$F^{36}-555F^{33}+453F^{30}+6F^{27}-282F^{24}-1143F^{21}-910F^{18}$$
$$-345F^{15}-75F^{12}+20F^{9}-33F^{6}-9F^{3}-1=0,$$

which, by adjunction of $\sqrt{13}$, is expressible in the form

$$\left[F^6-\frac{555+133\sqrt{13}}{4}F^6-\frac{447+123\sqrt{13}}{4}F^3-\frac{177+49\sqrt{13}}{2}\right]^2$$
$$=(7+2\sqrt{13})\left[\frac{153+39\sqrt{13}}{4}F^6+\frac{117+33\sqrt{13}}{4}F^3+\frac{47+13\sqrt{13}}{2}\right]^2.$$

In terms of Berwick's cubic irrationality

$$\theta^3+*+3\theta-3=0 \qquad (\Delta=-351)$$

the equation reduces to

$$F^6-D_3F^3-\alpha^3=0,$$

where

$$2\alpha^3=71\theta^2+58\theta+261+\frac{257\theta^2+210\theta+943}{\sqrt{13}},$$
$$2D_3=110\theta^2+91\theta+405+\frac{400\theta^2+327\theta+1463}{\sqrt{13}};$$

it is also found that

$$2\beta=2\theta^2+\theta+7+\frac{6\theta^2+9\theta+25}{\sqrt{13}}.$$

It is to be remarked that

$$257\theta^2+210\theta+943=(33\theta^2+27\theta+121)\sqrt{13(\theta^2+4)}$$
$$400\theta^2+327\theta+1463=(51\theta^2+42\theta+188)\sqrt{13(\theta^2+4)}$$
$$6\theta^2+9\theta+25=\sqrt{13(\theta^2+4)^3}$$
$$=(\theta^2+4)(2\theta^2+3\theta+4).$$

n = 359. (G. I. 19).

The equation satisfied by F_{359} is

$$F^{19}-14F^{18}+59F^{17}-113F^{16}+91F^{15}+19F^{14}-90F^{13}+51F^{12}$$
$$+2F^{11}-5F^{10}+9F^{9}-30F^{8}+22F^{7}+7F^{6}$$
$$-14F^{5}+3F^{4}+2F^{3}-2F^{2}+2F-1=0.$$

I have not solved this equation.

n = 367. (G. I. 9).

The equation satisfied by F_{367} is

$$F^9-9F^8+3F^7-2F^6+*+2F^4-6F^3+F^2+2F-1=0;$$

it is solved by Watson [12].

n = 375. (G. II. 5).

The equation satisfied by F_{375} is

$$F^{30}-705F^{27}-3345F^{24}-1200F^{21}-1815F^{18}-344F^{15}-330F^{12}$$
$$+35F^{9}-35F^{6}+10F^{3}-1=0,$$

which, by adjunction of $\sqrt{5}$, reduces to

$$F^{15}-\frac{705+319\sqrt{5}}{2}F^{12}-(200+89\sqrt{5})F^9-\frac{245+111\sqrt{5}}{2}F^6$$
$$-(25+11\sqrt{5})F^3-\frac{29+13\sqrt{5}}{2}=0.$$

The equations satisfied by α, γ, D_3 and S_3 are

$$\alpha^5-15\alpha^4+5\alpha^3-5\alpha^2-*-1=0,$$
$$\gamma^5-165\gamma^4+955\gamma^3-3540\gamma^2+8395\gamma-7743=0,$$
$$D_3{}^5-705D_3{}^4-180D_3{}^3-175D_3{}^2+15D_3-9=0,$$
$$S_3{}^5-620S_3{}^3-615S_3=(319S_3{}^4+385S_3{}^2+99)\sqrt{5}.$$

n = 383. (G. I. 17).

The equation satisfied by F_{383} is

$$F^{17}-6F^{16}-24F^{15}-42F^{14}-31F^{13}-23F^{12}-7F^{11}-F^{10}$$

$$-4F^{9}-11F^{8}-7F^{7}-13F^{6}-F^{5}$$

$$+\ *+F^{3}+F^{2}+F-1=0.$$

I have not solved this equation.

n = 391. (G. II. 7).

The equation satisfied by F_{391} is

$$F^{14}-8F^{13}-12F^{12}-12F^{11}+\ *+4F^{9}-3F^{8}-8F^{7}+6F^{6}$$

$$+15F^{5}+7F^{4}-5F^{3}-5F^{2}+\ *+1=0,$$

which, by adjunction of $\sqrt{17}$, reduces to

$$F^{7}-(4+\sqrt{17})\,F^{6}-\frac{11+3\sqrt{17}}{2}\,F^{5}-\frac{5+\sqrt{17}}{2}\,F^{4}+\frac{5+\sqrt{17}}{2}\,F^{3}$$

$$+\frac{5+\sqrt{17}}{2}\,F^{2}-\ *-1=0.$$

The equations satisfied by α, β, S_1 and D_1 are

$$\alpha^{7}-9\alpha^{6}+10\alpha^{5}-14\alpha^{4}+8\alpha^{3}-6\alpha^{2}+2\alpha-1=0,$$

$$\beta^{7}-9\beta^{6}-29\beta^{5}+27\beta^{4}+180\beta^{3}+216\beta^{2}-324\beta-729=0,$$

$$S_1^{7}-8S_1^{6}-21S_1^{5}-20S_1^{4}-16S_1^{3}-25S_1^{2}+9S_1-1=0,$$

$$D_1^{7}-D_1^{5}+22D_1^{3}+11D_1=(2D_1^{6}+4D_1^{4}+17D_1^{2}+1)\sqrt{17}.$$

n = 399. (G. IV. 4).

The equation satisfied by F_{399} is

$$F^{48}-896F^{45}-5260F^{42}-6165F^{39}-7523F^{36}+7376F^{33}$$

$$+11199F^{30}+1486F^{27}-360F^{24}+3022F^{21}+2490F^{18}$$

$$+641F^{15}+361F^{12}+330F^{9}+125F^{6}+19F^{3}+1=0,$$

which, by adjunction of $\sqrt{21}$, $\sqrt{57}$ and $\sqrt{133}$, reduces to

$$4F^{12}-(896+198\sqrt{21}+120\sqrt{57}+78\sqrt{133})\,F^{9}$$

$$+(341+75\sqrt{21}+45\sqrt{57}+29\sqrt{133})\,F^{6}$$

$$+(563+123\sqrt{21}+75\sqrt{57}+49\sqrt{133})\,F^{3}$$

$$+(140+30\sqrt{21}+18\sqrt{57}+12\sqrt{133})=0.$$

In terms of the quartic irrationality θ, defined as

$$\theta=\frac{3}{2}+\sqrt{\frac{23+2\sqrt{133}}{4}}$$

so that

$$\theta^{4}-6\theta^{3}+2\theta^{2}+21\theta-21=0,$$

we have

$$\frac{F_{399}\,F_{133/3}}{F_{57/7}\,F_{21/19}}+\frac{F_{57/7}\,F_{21/19}}{F_{399}\,F_{133/3}}=\theta^{2}-5,$$

$$\frac{F_{399}\,F_{57/7}}{F_{133/3}\,F_{21/19}}+\frac{F_{133/3}\,F_{21/19}}{F_{399}\,F_{57/7}}=-\theta^{3}+5\theta^{2}+4\theta-15,$$

$$\frac{F_{399}\,F_{21/19}}{F_{133/3}\,F_{57/7}}+\frac{F_{133/3}\,F_{57/7}}{F_{399}\,F_{21/19}}=\theta,$$

$$(F_{399}\,F_{133/3}\,F_{57/7}\,F_{21/19})^{3}=36\theta^{3}-40\theta^{2}-123\theta+155.$$

n = 407. (G. II. 8).

The equation satisfied by F_{407} is

$$F^{16}-10F^{15}+4F^{14}-33F^{13}+17F^{12}-19F^{11}-4F^{10}-\ *-12F^{8}$$

$$-\ *-15F^{6}+2F^{5}-9F^{4}-F^{3}-2F^{2}-F-1=0,$$

which, by the adjunction of $\sqrt{37}$, is expressible in the form

$$\left[F^{4}-\frac{5+\sqrt{37}}{2}F^{3}+\frac{4+\sqrt{37}}{2}F^{2}+\frac{3+\sqrt{37}}{4}F-\frac{7+\sqrt{37}}{4}\right]^{2}$$

$$=\frac{91+15\sqrt{37}}{8}\left[(\sqrt{37}-5)\,F^{3}-\frac{7-\sqrt{37}}{2}F^{2}-F+1\right]^{2}.$$

We also have

$$\left[\alpha^{2}-\frac{7+\sqrt{37}}{4}(\alpha+1)\right]^{2}=\frac{91+15\sqrt{37}}{8}(\alpha+1)^{2}.$$

$$\left[\beta^2 - \frac{11+3\sqrt{37}}{4}\beta - \frac{7+\sqrt{37}}{4}\right]^2 = \frac{91+15\sqrt{37}}{8}\left[\frac{\sqrt{37}-5}{2}\beta + (\sqrt{37}-4)\right]^2.$$

$$\left[D_1^2 - \frac{5+\sqrt{37}}{2}D_1 + \frac{15+3\sqrt{37}}{4}\right]^2$$

$$= \frac{91+15\sqrt{37}}{8}\left[(\sqrt{37}-5)D_1 - \frac{9-\sqrt{37}}{2}\right]^2.$$

n = 415. (G. II. 5).

The equation satisfied by F_{415} is

$$F^{10} - 10F^9 - F^8 - 7F^7 - 11F^6 + 2F^5 - 14F^4 + 2F^3$$
$$- 6F^2 - * - 1 = 0,$$

which, by adjunction of $\sqrt{5}$, reduces to

$$F^5 - (5+3\sqrt{5})F^4 + \frac{19+7\sqrt{5}}{2}F^3$$

$$- \frac{17+9\sqrt{5}}{2}F^2 - (5+2\sqrt{5})F - (2+\sqrt{5}) = 0.$$

The equations satisfied by α, β, D_1 and S_1 are

$$\alpha^5 - 13\alpha^4 + 9\alpha^3 + * + \alpha - 1 = 0,$$

$$\beta^5 - 9\beta^4 - 4\beta^3 + 69\beta^2 - 5\beta - 147 = 0,$$

$$D_1^5 - 10D_1^4 + 12D_1^3 - 27D_1^2 + 20D_1 - 5 = 0.$$

$$S_1^5 + 26S_1^3 + 14S_1 = (6S_1^4 + 15S_1^2 + 1)\sqrt{5}.$$

n = 423. (G. II. 5).

The equation satisfied by F_{423} is

$$F^{30} - 1140F^{27} + 2434F^{24} - 2432F^{21} + 939F^{18} - 140F^{15} + 446F^{12}$$
$$- 274F^9 + 41F^6 - F^3 + 1 = 0,$$

which, by adjunction of $\sqrt{141}$, reduces to

$$F^{15} - (570+48\sqrt{141})F^{12} + (1199+101\sqrt{141})F^9 - (1354+114\sqrt{141})F^6$$

$$+ \frac{1223+103\sqrt{141}}{2}F^3 - (95+8\sqrt{141}) = 0.$$

The equation satisfied by α, γ, S_3 and D_3 are

$$\alpha^5 - 8\alpha^4 - 18\alpha^3 - 15\alpha^2 - 6\alpha - 1 = 0,$$

$$\gamma^5 - 1291\gamma^4 - 23318\gamma^3 - 1\,38743\gamma^2 - 2\,25823\gamma - 1\,68155 = 0$$

$$S_3^5 - 1140S_3^4 + 1445S_3^3 - 1382S_3^2 + 370S_3 - 325 = 0,$$

$$D_3^5 + 3351D_3^3 + 3708D_3 = (96D_3^4 + 420D_3^2 + 99)\sqrt{141}.$$

n = 431. (G. I. 21).

The equation satisfied by F_{431} is

$$F^{21} - 12F^{20} + 15F^{19} - 16F^{18} + 40F^{17} - 21F^{16} - 10F^{15} - 44F^{14}$$
$$- 83F^{13} - 41F^{12} - 66F^{11} + 2F^{10} + 14F^9 + 30F^8 + 36F^7$$
$$+ 10F^6 + 4F^5 - 9F^4 - 9F^3 - 6F^2 - 3F - 1 = 0.$$

In terms of the cubic irrationality

$$\theta^3 - * - \theta - 8 = 0, \qquad (\Delta = -1724)$$

we have

$$F^7 - (2\theta+4)F^6 - \frac{5\theta^2+3\theta+16}{2}F^5 - (2\theta^2+6\theta+2)F^4$$

$$- \frac{\theta^2+5\theta+22}{2}F^3 - (2\theta^2+\theta+1)F^2 - (2\theta+1)F - 1 = 0.$$

This cubic irrationality is connected with Berwick's cubic irrationality

$$2\varphi^3 - \varphi^2 + 3\varphi - 2 = 0 \qquad (16\Delta = -431)$$

by the relation

$$\varphi = \frac{2}{\theta+1}.$$

I have not solved the septimic equation.

n = 439. (G. I. 15).

The equation satisfied by F_{439} is

$$F^{15}-13F^{14}+23F^{13}-7F^{12}-24F^{11}+20F^{10}+13F^{9}-38F^{8}+29F^{7}+F^{6}-17F^{5}+7F^{4}+9F^{3}-11F^{2}+5F-1=0;$$

its solution is given by Watson [12].

n = 447. (G. II. 7).

The equation satisfied by F_{447} is

$$F^{42}-1417F^{39}-13302F^{36}-38101F^{33}-53826F^{30}-44566F^{27}-28940F^{24}-23117F^{21}-20245F^{18}-13246F^{15}-5664F^{12}-1516F^{9}-243F^{6}-22F^{3}-1=0,$$

which, by adjunction of $\sqrt{149}$, reduces to

$$F^{21}-\frac{1417+117\sqrt{149}}{2}F^{18}-\frac{5359+439\sqrt{149}}{2}F^{15}-\frac{8409+689\sqrt{149}}{2}F^{12}-\frac{6797+557\sqrt{149}}{2}F^{9}-(1488+122\sqrt{149})F^{6}-\frac{671+55\sqrt{149}}{2}F^{3}-\frac{61+5\sqrt{149}}{2}=0.$$

The equations satisfied by α, γ, D_3 and S_3 are

$$\alpha^7-20\alpha^6-\ast-9\alpha^4-6\alpha^3-6\alpha^2-4\alpha-1=0,$$
$$\gamma^7-248\gamma^6-1368\gamma^5-1886\gamma^4+4105\gamma^3-7665\gamma^2-12321\gamma-51867=0,$$
$$D_3^7-1417D_3^6-5275D_3^5-6971D_3^4-4608D_3^3-2664D_3^2-1458D_3-405=0,$$
$$S_3^7-5443S_3^5-10850S_3^3-2310S_3=(117S_3^6+829S_3^4+566S_3^2+25)\sqrt{149}.$$

n = 455. (G. IV. 5).

The equation satisfied by F_{455} is

$$F^{20}-6F^{19}-50F^{18}-142F^{17}-200F^{16}-129F^{15}+38F^{14}+191F^{13}+246F^{12}+194F^{11}+76F^{10}-30F^{9}-73F^{8}-57F^{7}-15F^{6}+16F^{5}+26F^{4}+23F^{3}+15F^{2}+6F+1=0,$$

which, by adjunction of $\sqrt{5}$ and $\sqrt{13}$, reduces to

$$F^{5}-\frac{(3+\sqrt{13})\,(1+\sqrt{5})}{2}F^{4}-\frac{1+4\sqrt{5}+\sqrt{65}}{2}(F^{3}+F^{2})+\frac{(3+\sqrt{13})(1-\sqrt{5})}{4}F-\frac{(3+\sqrt{13})(1+\sqrt{5})}{4}=0.$$

Further, if

$$F_{455}F_{91/5}+F_{65/7}F_{35/13}=a,$$
$$F_{455}F_{65/7}+F_{91/5}F_{35/13}=b,$$
$$F_{455}F_{35/13}+F_{91/5}F_{65/7}=c,$$

then also

$$F_{455}F_{91/5}F_{65/7}F_{35/13}=b,$$

and a, b, c are the quintic irrationalities given by the equations

$$a^5-12a^4-41a^3-105a^2-119a-49=0,$$
$$b^5-12b^4-16b^3-7b^2-\ast-1=0,$$
$$c^5-11c^4-5c^3-3c^2-20c-35=0.$$

n = 463. (G. I. 7).

The equation satisfied by F_{463} is

$$F^7-11F^6-9F^5-8F^4-7F^3-7F^2-3F-1=0;$$

its solution is given by Watson [12].

n = 471. (G. II. 8).

The equation satisfied by F_{471} is

$$F^{48}-1775F^{45}-4085F^{42}+29217F^{39}-60470F^{36}+44222F^{33}-36601F^{30}+22281F^{27}-18987F^{24}+9629F^{21}-4756F^{18}+942F^{15}-382F^{12}+106F^{9}-62F^{6}-6F^{3}-1=0,$$

which, by adjunction of $\sqrt{157}$, is expressible in the form

$$[4F^{12}-(1775+141\sqrt{157})F^{9}-(12169+971\sqrt{157})F^{6}-(9357+747\sqrt{157})F^{3}-(5012+400\sqrt{157})]^2$$

$$=(2\sqrt{157}-25)\,[(7231+577\sqrt{157})\,F^9+(49705+3967\sqrt{157})\,F^6$$
$$+(38229+3051\sqrt{157})\,F^3+(20474+1634\sqrt{157})\,]^2.$$

We also have

$$[\,4\,\alpha^6-(10765+859\sqrt{157})\,\alpha^3-(5012+400\sqrt{157})\,]^2$$
$$=(2\sqrt{157}-25)\,[\,(43967+3509\sqrt{157})\,\alpha^3+(20474+1634\sqrt{157})\,]^2,$$
$$[\,4\,\beta^2-10\,\beta+(19+\sqrt{157})\,]^2$$
$$=(2\sqrt{157}-25)\,[\,(50+4\sqrt{157})\,\beta-(37+3\sqrt{157})\,]^2,$$
$$[\,4\,D_3^2-(1775+141\sqrt{157})\,D_3-(1404+112\sqrt{157})\,]^2$$
$$=(2\sqrt{157}-25)\,[\,(7231+577\sqrt{157})\,D_3+(5738+458\sqrt{157})\,]^2.$$

n = 479. (G. I. 25).

The equation satisfied by F_{479} is

$$F^{25}-10\,F^{24}-23\,F^{23}-76\,F^{22}-104\,F^{21}-126\,F^{20}-109\,F^{19}$$
$$-144\,F^{18}-205\,F^{17}-317\,F^{16}-336\,F^{15}-280\,F^{14}-138\,F^{13}$$
$$-11\,F^{12}+76\,F^{11}+82\,F^{10}+48\,F^9-6\,F^8-47\,F^7$$
$$-66\,F^6-60\,F^5-41\,F^4-22\,F^3-9\,F^2-3\,F-1=0.$$

I have not solved this equation.

n = 487. (G. I. 7).

The equation satisfied by F_{487} is

$$F^7-13\,F^6+4\,F^5-4\,F^4+7\,F^3-4\,F^2+F-1=0;$$

its solution is given by Watson [**12**].

n = 495. (G. IV. 4).

The equation satisfied by F_{495} is

$$F^{48}-2200\,F^{45}-5184\,F^{42}-2555\,F^{39}+1023\,F^{36}$$
$$+9360\,F^{33}+5623\,F^{30}-1895\,F^{27}-5751\,F^{24}$$
$$-2515\,F^{21}+2338\,F^{18}+1170\,F^{15}-282\,F^{12}$$
$$-235\,F^9+81\,F^6-5\,F^3+1=0,$$

which, by adjunction of $\sqrt{5}$ and $\sqrt{33}$, reduces to

$$4\,F^{12}-(2200+984\sqrt{5}+384\sqrt{33}+172\sqrt{165})\,F^9$$
$$+(3402+1520\sqrt{5}+590\sqrt{33}+264\sqrt{165})\,F^6$$
$$-(1495+667\sqrt{5}+261\sqrt{33}+117\sqrt{165})\,F^3$$
$$+(234+104\sqrt{5}+40\sqrt{33}+18\sqrt{165})=0.$$

We also have

$$F_{495}\,F_{99/5}+F_{55/9}\,F_{45/11}=4+2\sqrt{5}+\frac{3+\sqrt{5}}{2}\sqrt{\frac{3+9\sqrt{5}}{2}},$$

$$(F_{495}\,F_{55/9})^3+(F_{99/5}\,F_{45/11})^3=\frac{1030+461\sqrt{5}}{2}+\frac{605+271\sqrt{5}}{4}\sqrt{\frac{3+9\sqrt{5}}{2}},$$

$$F_{495}\,F_{45/11}+F_{99/5}\,F_{55/9}=\frac{13+5\sqrt{5}}{4}+\frac{2+\sqrt{5}}{2}\sqrt{\frac{3+9\sqrt{5}}{2}},$$

$$F_{495}\,F_{99/5}\,F_{55/9}\,F_{45/11}=\frac{15+7\sqrt{5}}{4}+\frac{2+\sqrt{5}}{2}\sqrt{\frac{3+9\sqrt{5}}{2}}.$$

n = 503. (G. I. 21).

The equation satisfied by F_{503} is

$$F^{21}-18\,F^{20}+69\,F^{19}-87\,F^{18}-34\,F^{17}+171\,F^{16}-106\,F^{15}-74\,F^{14}$$
$$+92\,F^{13}+19\,F^{12}-27\,F^{11}-30\,F^{10}+23\,F^9+10\,F^8-12\,F^7$$
$$+3\,F^6-*-7\,F^4+6\,F^3+F^2-*-1=0.$$

In terms of the cubic irrationality

$$\theta^3-6\,\theta^2-\theta-2=0 \qquad (\Delta=-2012)$$

we have

$$F^7-(2\,\theta+2)\,F^6+\frac{\theta^2-\theta-2}{2}\,F^5-(\theta^2-5\,\theta-1)\,F^4+\theta\,F^3$$
$$+\frac{\theta^2-7\,\theta+2}{2}\,F^2-*-1=0.$$

This cubic irrationality is connected with Berwick's cubic irrationality

$$2\varphi^3-5\varphi^2+5\varphi-4=0 \qquad (16\,\Delta=-503)$$

by the relation

$$\varphi=\frac{2\theta}{\theta+1}.$$

I have not solved the septimic equation.

n = 511. (G. II. 7).

The equation satisfied by F_{511} is

$$F^{14}-16F^{13}+36F^{12}-56F^{11}+77F^{10}-84F^9+70F^8-37F^7$$
$$+16F^6-21F^5+35F^4-35F^3+21F^2-7F+1=0,$$

which, by adjunction of $\sqrt{73}$, reduces to

$$F^7-(8+\sqrt{73})F^6+\frac{45+5\sqrt{73}}{2}F^5-\frac{61+7\sqrt{73}}{2}F^4+(25+3\sqrt{73})F^3$$
$$-\frac{27+3\sqrt{73}}{2}F^2+\frac{7+\sqrt{73}}{2}F-1=0.$$

The equation satisfied by α is

$$\alpha^7-16\alpha^6+20\alpha^5-16\alpha^4+9\alpha^3-*-*-1=0;$$

also

$$\beta+2=S_1=\alpha,$$

$$D_1^7+70D_1^5+331D_1^3+376D_1=(2D_1^6+22D_1^4+50D_1^2+21)\sqrt{73}.$$

n = 519. (G. II. 9).

The equation satisfied by F_{519} is

$$F^{54}-2709F^{51}-16545F^{48}+6267F^{45}-49945F^{42}+1\,36972F^{39}$$
$$-1\,50834F^{36}+1\,13720F^{33}-71908F^{30}+41892F^{27}-30826F^{24}$$
$$+20926F^{21}-12763F^{18}+7806F^{15}-3683F^{12}+1178F^9$$
$$-242F^6+25F^3-1=0,$$

which, by adjunction of $\sqrt{173}$, reduces to

$$F^{27}-\frac{2709+207\sqrt{173}}{2}F^{24}+(1002+76\sqrt{173})F^{21}-\frac{951+75\sqrt{173}}{2}F^{18}$$

$$+(41+6\sqrt{173})F^{15}-\frac{23+5\sqrt{173}}{2}F^{12}+\frac{333+27\sqrt{173}}{2}F^9$$
$$-\frac{233+19\sqrt{173}}{2}F^6+(38+3\sqrt{173})F^3-\frac{13+\sqrt{173}}{2}=0.$$

By adjunction of the cubic irrationality

$$\theta^3-2\theta^2-3\theta-3=0, \qquad (\Delta=-519)$$

which is connected with Berwick's cubic irrationality

$$3\varphi^3-6\varphi^2+5\varphi-3=0 \qquad (81\,\Delta=-519)$$

by the relation $\theta\varphi=1+\theta$, the equation is further reducible to

$$[2F^3-(177\theta^2+219\theta+167)F^2+(87\theta^2+108\theta+81)F$$
$$-(86\theta^2+105\theta+80)]\sqrt{173}$$
$$=(2361\theta^2+2841\theta+2173)F^2-(1153\theta^2+1374\theta+1065)F$$
$$+(1136\theta^2+1383\theta+1058).$$

Also α, γ and D_3 are given in terms of θ by the equations

$$\alpha^3-(2\theta^2+\theta+2)\alpha^2-(\theta^2+2\theta+1)\alpha-(\theta^2+\theta+1)=0,$$
$$\gamma^3-(24\theta^2+36\theta+21)\gamma^2-(77\theta^2+126\theta+97)\gamma-(66\theta^2+99\theta+105)=0,$$
$$D_3^3-(177\theta^2+219\theta+167)D_3^2+(37\theta^2+48\theta+34)D_3$$
$$-(265\theta^2+324\theta+246)=0.$$

n = 527. (G. II. 9).

The equation satisfied by F_{527} is

$$F^{18}-12F^{17}-31F^{16}-24F^{15}+45F^{14}+15F^{13}+12F^{12}+20F^{11}-5F^{10}$$
$$-32F^9-10F^8-F^7-7F^6-4F^5$$
$$+8F^4+5F^3+2F^2+*+1=0,$$

which, by adjunction of $\sqrt{17}$, reduces to

$$F^9-(6+2\sqrt{17})F^8+\frac{1-\sqrt{17}}{2}F^7+(8+2\sqrt{17})F^6+\frac{9+\sqrt{17}}{2}F^5$$
$$-\frac{7+\sqrt{17}}{2}F^4-\frac{5+\sqrt{17}}{2}F^3-F^2-*-1=0.$$

By adjunction of Berwick's cubic irrationality

$$\theta^3 + * + 5\theta - 1 = 0, \qquad (\Delta = -527)$$

the equation is further reducible to

$$[2F^3 - (3\theta^2 + \theta + 14)F^2 - (4\theta^2 + 2\theta + 20)F - (2\theta^2 + \theta + 10)]\sqrt{17}$$
$$= (7\theta^2 - 3\theta + 46)F^2 + (18\theta^2 + 2\theta + 94)F + (8\theta^2 - \theta + 38).$$

Also α, β and S_1 are given in terms of θ by the equations

$$\alpha^3 - (2\theta^2 + \theta + 11)\alpha^2 - (2\theta^2 + 11)\alpha - (\theta^2 + 5) = 0,$$
$$\beta^3 - (3\theta^2 - \theta + 16)\beta^2 - (2\theta^2 + 2\theta + 13)\beta - (\theta^2 + 3) = 0,$$
$$S_1^3 - (3\theta^2 + \theta + 14)S_1^2 - (2\theta^2 + \theta + 11)S_1 - (3\theta^2 + 2\theta + 15) = 0.$$

n = 535. (G. II. 7).

The equation satisfied by F_{535} is

$$F^{14} - 16F^{13} + 20F^{12} + 7F^{11} - 14F^{10} + 10F^9 - 37F^8 + 34F^7 + 3F^6$$
$$- F^5 - 16F^4 + 6F^3 + 3F^2 - * - 1 = 0,$$

which, by adjunction of $\sqrt{5}$, reduces to

$$F^7 - (8 + 4\sqrt{5})F^6 + (18 + 8\sqrt{5})F^5 - \frac{25 + 11\sqrt{5}}{2}F^4 + (1 + \sqrt{5})F^3$$
$$- (2 + \sqrt{5})F^2 + (5 + 2\sqrt{5})F - (2 + \sqrt{5}) = 0.$$

The equations satisfied by α, β, D_1 and S_1 are

$$\alpha^7 - 19\alpha^6 + \alpha^5 + 11\alpha^4 - 11\alpha^3 - \alpha^2 + 4\alpha - 1 = 0,$$
$$\beta^7 - 6\beta^6 - 52\beta^5 - 96\beta^4 - 44\beta^3 - 90\beta^2 - 315\beta - 243 = 0,$$
$$D_1^7 - 16D_1^6 + 39D_1^5 - 50D_1^4 + 120D_1^3 - 189D_1^2 + 139D_1 - 71 = 0,$$
$$S_1^7 + 33S_1^5 + 16S_1^3 + 3S_1 = (8S_1^6 + 10S_1^4 + 5S_1^2 + 1)\sqrt{5}.$$

n = 543. (G. II. 6).

The equation satisfied by F_{543} is

$$F^{36} - 3325F^{33} - 20738F^{30} + 26913F^{27} - 26778F^{24} + 24014F^{21}$$

$$- 14714F^{18} + 6654F^{15} - 2895F^{12} + 929F^9 - 206F^6 + 21F^3 - 1 = 0,$$

which, by adjunction of $\sqrt{181}$, is expressible in the form

$$\left[F^9 - \frac{3325 + 247\sqrt{181}}{4}F^6 - \frac{269 + 20\sqrt{181}}{2}F^3 - \frac{861 + 64\sqrt{181}}{2}\right]^2$$
$$= \frac{19871 + 1477\sqrt{181}}{2}\left[\frac{7 + 3\sqrt{181}}{4}F^6 + \frac{21 - \sqrt{181}}{4}F^3 + \frac{11 + \sqrt{181}}{4}\right]^2.$$

In terms of Berwick's cubic irrationality

$$\theta^3 - \theta^2 + 2\theta - 5 = 0 \qquad (\Delta = -543)$$

the equation reduces to

$$F^6 - D_3F^3 - \alpha^3 = 0$$

where

$$2\alpha^3 = 3665\theta^2 + 2345\theta + 11175 + \frac{49309\theta^2 + 31549\theta + 1\,50349}{\sqrt{181}},$$

$$2D_3 = 490\theta^2 + 313\theta + 1494 + \frac{6586\theta^2 + 4213\theta + 20084}{\sqrt{181}};$$

it is also found that

$$2\beta = \theta^2 + \theta + 4 + \frac{21\theta^2 + 15\theta + 16}{\sqrt{181}}.$$

It is to be remarked that

$$49309\theta^2 + 31549\theta + 1\,50349 = (508\theta^2 + 325\theta + 1549)(21\theta^2 + 15\theta + 16),$$
$$7(6586\theta^2 + 4213\theta + 20084) = (475\theta^2 + 304\theta + 1448)(21\theta^2 + 15\theta + 16),$$
$$21\theta^2 + 15\theta + 16 = \sqrt{181(6\theta^2 + 3\theta + 31)}.$$

n = 551. (G. II. 13).

The equation satisfied by F_{551} is

$$F^{26} - 10F^{25} - 62F^{24} - 237F^{23} - 618F^{22} - 1183F^{21} - 1773F^{20}$$
$$- 2121F^{19} - 2049F^{18} - 1625F^{17} - 1011F^{16} - 416F^{15} - 25F^{14}$$
$$+ 148F^{13} + 111F^{12} - 95F^{11} - 323F^{10} - 469F^9 - 504F^8$$
$$- 437F^7 - 316F^6 - 198F^5 - 109F^4 - 49F^3 - 17F^2 - 5F - 1 = 0,$$

which, by adjunction of $\sqrt{29}$, reduces to

$$F^{13}-(5+\sqrt{29})\,F^{12}-(29+6\sqrt{29})\,F^{11}-\frac{179+33\sqrt{29}}{2}\,F^{10}$$

$$-\frac{353+65\sqrt{29}}{2}\,F^{9}-(256+48\sqrt{29})\,F^{8}-\frac{591+109\sqrt{29}}{2}\,F^{7}$$

$$-(275+51\sqrt{29})\,F^{6}-\frac{423+79\sqrt{29}}{2}\,F^{5}-\frac{277+51\sqrt{29}}{2}\,F^{4}$$

$$-\frac{155+29\sqrt{29}}{2}\,F^{3}-(37+7\sqrt{29})\,F^{2}-\frac{27+5\sqrt{29}}{2}\,F-\frac{5+\sqrt{29}}{2}=0.$$

The equations satisfied by α, β, D_1 and S_1 are

$$\alpha^{13}-13\,\alpha^{12}+\;*\;+7\,\alpha^{10}+3\,\alpha^{9}-11\,\alpha^{8}+28\,\alpha^{7}-21\,\alpha^{6}$$

$$-9\,\alpha^{5}+9\,\alpha^{4}-14\,\alpha^{3}-\alpha^{2}+3\,\alpha-1=0,$$

$$\beta^{13}-16\,\beta^{12}-57\,\beta^{11}+304\,\beta^{10}+1100\,\beta^{9}-1692\,\beta^{8}-9219\,\beta^{7}-2067\,\beta^{6}$$

$$+28820\,\beta^{5}+32129\,\beta^{4}-20387\,\beta^{3}-54511\,\beta^{2}-32129\,\beta-6859=0,$$

$$D_1^{13}-10\,D_1^{12}-49\,D_1^{11}-177\,D_1^{10}-463\,D_1^{9}-866\,D_1^{8}-1511\,D_1^{7}$$

$$-2259\,D_1^{6}-2965\,D_1^{5}-3499\,D_1^{4}-3227\,D_1^{3}-2417\,D_1^{2}-1140\,D_1-247=0,$$

$$S_1^{13}-67\,S_1^{11}-787\,S_1^{9}-3595\,S_1^{7}-3429\,S_1^{5}-667\,S_1^{3}-38\,S_1$$

$$=(2\,S_1^{12}+47\,S_1^{10}+374\,S_1^{8}+801\,S_1^{6}+339\,S_1^{4}+33\,S_1^{2}+1)\sqrt{29}.$$

n = 559. (G. II. 8).

The equation satisfied by F_{580} is

$$F^{16}-20\,F^{15}+78\,F^{14}-161\,F^{13}+196\,F^{12}-139\,F^{11}+18\,F^{10}+83\,F^{9}$$

$$-123\,F^{8}+98\,F^{7}-51\,F^{6}+14\,F^{5}-F^{4}-2\,F^{3}+\;*\;+F-1=0,$$

which, by adjunction of $\sqrt{13}$, is expressible in the form

$$\left[F^4-(5+\sqrt{13})\,F^3+\frac{1+3\sqrt{13}}{4}\,F^2+\frac{9+\sqrt{13}}{4}\,F-\frac{9+3\sqrt{13}}{4}\right]^2$$

$$=\frac{3+2\sqrt{13}}{4}\left[(\sqrt{13}+1)\,F^3-\frac{\sqrt{13}-1}{2}\,F^2-\frac{1+\sqrt{13}}{2}\,F+\frac{3+\sqrt{13}}{2}\right]^2.$$

We also have

$$\left[\alpha^2-\frac{15+3\sqrt{13}}{4}\,\alpha-\frac{9+3\sqrt{13}}{4}\right]^2=\frac{3+2\sqrt{13}}{4}\left[\frac{5+\sqrt{13}}{2}\,\alpha+\frac{3+\sqrt{13}}{2}\right]^2,$$

$$\left[\beta^2-\frac{9+3\sqrt{13}}{2}\,\beta-\frac{5-\sqrt{13}}{4}\right]^2=\frac{3+2\sqrt{13}}{4}\left[(1+\sqrt{13})\,\beta+\frac{\sqrt{13}-1}{2}\right]^2,$$

$$\left[D_1^2-(5+\sqrt{13})\,D_1+\frac{8+3\sqrt{13}}{2}\right]^2=\frac{3+2\sqrt{13}}{4}\,[(1+\sqrt{13})\,D_1-(2+\sqrt{13})]^2.$$

n = 567. (G. II. 6).

The equation satisfied by F_{567} is

$$F^{36}-4068\,F^{33}-3798\,F^{30}-2548\,F^{27}+4383\,F^{24}+7704\,F^{21}+9324\,F^{18}$$

$$+7704\,F^{15}+4383\,F^{12}+1547\,F^{9}+297\,F^{6}+27\,F^{3}+1=0,$$

which, by adjunction of $\sqrt{21}$, is expressible in the form

$$[2\,F^9-(2034+444\sqrt{21})\,F^6-(2034+444\sqrt{21})\,F^3-(2030+443\sqrt{21})]^2$$

$$=(27+6\sqrt{21})\,[(276+60\sqrt{21})\,F^6+(276+60\sqrt{21})\,F^3+(275+60\sqrt{21})]^2.$$

In terms of Berwick's cubic irrationality

$$\theta^3-3\,\theta^2-\;*\;-3=0,$$

the equation reduces to

$$F^6-S_3\,F^3+\alpha^3=0,$$

where

$$2\,\alpha^3=435\,\theta^2+121\,\theta+398+\frac{1993\,\theta^2+555\,\theta+1824}{\sqrt{21}},$$

$$2\,S_3=323\,\theta^2+92\,\theta+295+\frac{1483\,\theta^2+414\,\theta+1353}{\sqrt{21}}$$

it is also found that

$$2\,\beta=2\,\theta^2-\theta-2+\frac{4\,\theta^2+3\,\theta+6}{\sqrt{21}}.$$

It is to be remarked that

$$1993\,\theta^2+555\,\theta+1824=(34\,\theta^2+9\,\theta+31)\,(4\,\theta^2+3\,\theta+6).$$

$$1483\,\theta^2+414\,\theta+1353=(25\,\theta^2+8\,\theta+22)\,(4\,\theta^2+3\,\theta+6),$$

$$4\,\theta^2+3\,\theta+6=\sqrt{21\,(13\,\theta^2+4\,\theta+12)}.$$

n = 575. (G. II. 9).

The equation satisfied by F_{575} is

$$F^{18}-20\,F^{17}+64\,F^{16}-64\,F^{15}+\ast+\ast+25\,F^{12}-60\,F^{11}-\ast-\ast$$
$$-F^{8}-15\,F^{7}+6\,F^{6}+4\,F^{5}+\ast+F^{3}+5\,F^{2}+4\,F+1=0,$$

which, by adjunction of $\sqrt{5}$, reduces to

$$F^9-(10+2\sqrt{5})\,F^8-(8+9\sqrt{5})\,F^7-(22+4\sqrt{5})\,F^6-\frac{19+15\sqrt{5}}{2}\,F^5$$
$$-(16+7\sqrt{5})\,F^4-(18+7\sqrt{5})\,F^3-(12+6\sqrt{5})\,F^2-\frac{11+5\sqrt{5}}{2}\,F$$
$$-\frac{3+\sqrt{5}}{2}=0.$$

It is convenient to use the cubic irrationality F_{23}, namely

$$\theta^3-\ast-\theta-1=0 \qquad (\Delta=-23)$$

rather than Berwick's cubic irrationality

$$\varphi^3-\varphi^2+4\,\varphi-5=0, \qquad (\Delta=-575)$$

these irrationalities being connected by the relation

$$\varphi=\frac{\theta+3}{2\,\theta+1}.$$

The equation is then reducible to

$$2\,F^3-4\,(\theta^2+\theta+1)\,F^2-(6\,\theta^2+5\,\theta+2)\,F-(3\,\theta^2+3\,\theta+1)$$
$$=(\theta^2+\theta)\,[2\,F^2+(\theta^2+\theta)\,F+\theta]\,\sqrt{5}.$$

The equations giving α, β and S_1 in terms of θ are

$$\alpha^3-(2\,\theta^2+5\,\theta+4)\,\alpha^2+(3\,\theta+2)\,\alpha-(\theta^2+2\,\theta+1)=0,$$
$$\beta^3-(5\,\theta^2+5\,\theta+1)\,\beta^2-(10\,\theta^2+20\,\theta+13)\,\beta-(10\,\theta^2+15\,\theta+8)=0,$$
$$S_1^{\,3}-4\,(\theta^2+\theta+1)\,S_1^{\,2}-(2\,\theta^2+5\,\theta+5)\,S_1-(2\,\theta^2+2\,\theta-1)=0.$$

n = 583. (G. II. 4).

The equation satisfied by F_{583} is

$$F^8-16\,F^7-12\,F^6+11\,F^5+12\,F^4+5\,F^3-3\,F^2-4\,F-1=0,$$

which, by adjunction of $\sqrt{53}$, is expressible in the form

$$\left[F^2-\frac{8+\sqrt{53}}{2}\,(F+1)\right]^2=\frac{(3+\sqrt{53})\,(7+\sqrt{53})^2}{32}\,(F+1)^2.$$

Here we have

$$\alpha=\frac{8+\sqrt{53}}{2}+\sqrt{\frac{131+18\sqrt{53}}{4}}$$

so that

$$\alpha^4-16\,\alpha^3+4\,\alpha^2+3\,\alpha-1=0;$$

also $\beta=\alpha+2, \qquad D_1=\alpha.$

n = 591. (G. II. 11).

The equation satisfied by F_{591} is

$$F^{66}-4945\,F^{63}-20955\,F^{60}+1\,39902\,F^{57}-3\,26851\,F^{54}+1\,95297\,F^{51}$$
$$-98626\,F^{48}-1\,17162\,F^{45}-3\,08141\,F^{42}-47480\,F^{39}-2\,49560\,F^{36}$$
$$-66786\,F^{33}-1\,06033\,F^{30}-27245\,F^{27}-44701\,F^{24}-9429\,F^{21}$$
$$-12199\,F^{18}-2000\,F^{15}-1851\,F^{12}-F^{9}-61\,F^{6}+18\,F^{3}-1=0,$$

which, by adjunction of $\sqrt{197}$, reduces to

$$F^{33}-\frac{4945+353\sqrt{197}}{2}\,F^{30}+(1391+101\sqrt{197})\,F^{27}$$
$$-(2622+190\sqrt{197})\,F^{24}-\frac{5135+367\sqrt{197}}{2}\,F^{21}-\frac{6673+475\sqrt{197}}{2}\,F^{18}$$
$$-\frac{4229+301\sqrt{197}}{2}\,F^{15}-(1768+127\sqrt{197})\,F^{12}-\frac{1631+115\sqrt{197}}{2}\,F^{9}$$
$$-\frac{705+51\sqrt{197}}{2}\,F^{6}-(71+5\sqrt{197})\,F^{3}-(14+\sqrt{197})=0.$$

The equations satisfied by α, γ, D_3 and S_3 are

$$\alpha^{11}-35\,\alpha^{10}+7\,\alpha^{9}-11\,\alpha^{8}+19\,\alpha^{7}-26\,\alpha^{6}+35\,\alpha^{5}-37\,\alpha^{4}+38\,\alpha^{3}-25\,\alpha^{2}$$
$$+8\,\alpha-1=0,$$

$$\gamma^{11}-572\,\gamma^{10}-5098\,\gamma^{9}-21002\,\gamma^{8}-92050\,\gamma^{7}-6\;06869\,\gamma^{6}-28\;81655\,\gamma^{5}$$
$$-83\;61657\,\gamma^{4}-151\;88751\,\gamma^{3}-174\;08952\,\gamma^{2}-112\;73013\,\gamma$$
$$-37\;27863=0,$$

$$D_3^{11}-4945\,D_3^{10}+21218\,D_3^{9}-48077\,D_3^{8}-7341\,D_3^{7}+68745\,D_3^{6}-1\;98661\,D_3^{5}$$
$$-95096\,D_3^{4}+8092\,D_3^{3}-1\;28896\,D_3^{2}-23460\,D_3-22419=0.$$

$$S_3^{11}-15654\,S_3^{9}+601\,S_3^{7}-9373\,S_3^{5}-8196\,S_3^{3}-4806\,S_3$$
$$=(353\,S_3^{10}+949\,S_3^{8}+771\,S_3^{6}+292\,S_3^{4}+648\,S_3^{2}+81)\sqrt{197}\,.$$

I have not simplified these equations in any way.

$\mathbf{n=599.}$ (G. I. 25).

The equation satisfied by F_{599} is

$$F^{25}-16\,F^{24}-25\,F^{23}+3\,F^{22}+82\,F^{21}-92\,F^{20}-63\,F^{19}$$
$$-28\,F^{18}+66\,F^{17}+24\,F^{16}-35\,F^{15}+\;*\;+71\,F^{13}$$
$$+6\,F^{12}-58\,F^{11}-22\,F^{10}+36\,F^{9}+13\,F^{8}-32\,F^{7}$$
$$+5\,F^{6}+17\,F^{5}-6\,F^{4}-8\,F^{3}+F^{2}+2\,F-1=0.$$

I have not solved this equation.

BIBLIOGRAPHY.

1. A. G. Greenhill, Complex multiplication moduli of elliptic functions. *Proc. London Math. Soc.* (1), 19 (1889), 301-364.

2. — Table of complex multiplication moduli. *Proc. London Math. Soc.* (1), 21 (1891), 403-422.

3. M. Hanna, The modular equations. *Proc. London Math. Soc.* (2), 28 (1928), 46-52.

4. P. Joubert, Sur la théorie des fonctions elliptiques et son application à la théorie des nombres. *Comptes Rendus*, 50 (1860) 907-912.

5. G. B. Mathews, The reduction and classification of binary cubics. *Proc. London Math. Soc.* (2), 10 (1912), 128-138.

6. S. C. Mitra, Table of complex multiplication moduli of elliptic functions for some new cases. *Bulletin Calcutta Math. Soc.*, 19 (1928), 83-86.

7. — Table of complex multiplication moduli. *Bulletin Calcutta Math. Soc.*, 21 (1929), 91-92.

8. — Table of complex multiplication moduli of elliptic functions for some new cases. *Indian Phys.-Math. J.*, 2 (1931), 7—10.

9. R. Russell, On modular equations. *Proc. London Math. Soc.* (1), 21, (1891), 351-395.

10. G. N. Watson, Singular moduli (1). *Quart. J. of Math.*, 3 (1932), 81-98.

11. — Some singular moduli (2). *Quart. J. of Math.*, 3 (1932), 189-212.

12. — Singular moduli (3). *Proc. London Math. Soc.* (2), 40 (1936), 83-142.

13. H. Weber, Zur Theorie der elliptischen Funktionen. *Acta Math.*, 11 (1888), 333-390.

14. — Zur komplexen Multiplication elliptischer Funktionen. *Math. Ann.*, 33 (1889), 390-410.

15. — *Lehrbuch der Algebra*, 3 (Braunschweig, 1908).

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