The second gives $p+j \equiv p'+j'$ or $p-p' \equiv j'-j$, and so the set of possible residues $j'-j$ gives rise to a set of integers $x$ for which $|L(x)| \leqq \nu$ from (5). As before all the $x$'s are not zero.

The third case gives $p+i=p'+j$ or $p-p'+c=j-i+c$. Hence the possible set $p-p'$ gives rise to a set of integers $x$ such that $|L(x)+c| \leqq \frac{1}{2}\mu + \frac{1}{2}\nu$ since $|j+c| \leqq \frac{1}{2}\nu$, $|i| \leqq \frac{1}{2}\mu$. We note now that the $x$'s may all be zero when $p=p'$, $i=j$.

This proves the theorem.

A sharper form of the theorem can be deduced by applying the theorem with $\mu_s$ ($s$ arbitrary) replaced by $\mu_s(1+\varepsilon)^n$, $\mu_r$ by $\mu_r/(1+\varepsilon)$, $r \neq s$, $\nu$ by $\nu/(1+\eta)$ and making $\varepsilon>0$, $\eta>0$ tend to zero in such a way that

$$(1+\varepsilon)\Pi\mu + (1+\eta)^{-n}\Pi\nu \geqq \Pi\mu + \Pi\nu$$

or

$$\varepsilon\Pi\mu \geqq (1-(1+\eta)^{-n})\Pi\nu$$

We see then that if $\Pi\mu \neq 0$, all the $\leqq$ signs in (2), (3), (4) can be replaced by $<$ signs except the one corresponding to $\mu_s$ in (2) and the corresponding $\frac{1}{2}(\mu_s+\nu_s)$ of (4).

On making the $\mu \to 0$, we see that if $\Pi\nu = \Delta$ and the inequalities $|L(x)| < \nu$ have no solutions in integers $x$ except $x=0$, then there is* for all $c$ a solution of $|L(x)+c| \leqq \frac{1}{2}\nu$, in integers $x$. There cannot be two solutions unless for at least one $s$ these make $L_s(x)+c_s = \pm\frac{1}{2}\nu_s$ respectively, for otherwise, their difference would give a solution of $|L(x)| < \nu$. This is of course known in connection with Minkowski's limiting case.

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* This is a known result. See Radò, Journal of the London Math. Soc. 10 (1935) 115—116; Perron, Ibid. 275—277.

# Universal forms $\sum a_i x_i^n$ and Waring's problem.

By

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**1. Introduction and summary.** A form $F$ is called *universal* if every positive integer is represented by $F$ with integral values $\geqq 0$ of all the variables. Write

$$3^n = 2^n q + r,\quad 0 < r < 2^n,\quad I = 2^n + q - 2.$$

THEOREM 1. *If $n>6$ and $r \leqq 2^n - q - 3$, every positive integer is a sum of $I$ $n$-th powers. Technically, $g(n) = I$.*

The inequality holds when $4 \leqq n \leqq 400$. The theorem was recently proved by the writer.[1]) For $n>8$, it is a corollary to the new theorem proved here:

THEOREM 2. *Let $d=1$ or 2 according as $q$ is odd or even. If $9 \leqq n \leqq 400$, every positive integer is a sum of $4n+2-d$ $n$-th powers and the doubles of $P = \frac{1}{2}(2^n+q-4n+d)-2$ $n$-th powers. Here $4n+$ $+2-d+2P=I$.*

Expressed otherwise, in the ideal Waring Theorem 1, we may take $2P$ of the powers equal in pairs. While Theorem 1 states that $x_1^n + \ldots + x_I^n$ is universal, Theorem 2 yields a universal form (with

[1]) Amer. Jour. Math., vol. 58 (1936), pp. 521—35. In case the inequality fails hen $g(n) = I+f$ or $I+f-1$, according as $2^n =$ or $< fq+f+q$, where $f = [(4/3)^n]$. Announced March 13 in Bull. Amer. Math. Soc., 1936, p. 341.

If the inequality fails for any $n > 400$, the dicimal $r/2^n$ begins with fifty figures 9. But 157 and 163 are the only values $\leqq 400$ of $n$ for which it begins with two figures 9 (neither with three figures 9).

coefficients 1 or 2) involving only $I-P$ variables. Except for small $n$'s, $I-P$ is just $>\frac{1}{2}I$.

We shall prove theorems yielding universal forms involving approximately $I/a$ variables, when $a$ is not too large.

Let $[u, w]$ denote a sum of $u$ $n$-th powers and $w$ products of $n$-th powers by a fixed positive integer $a$. Its *weight* is $u+aw$.

THEOREM 3. *If* $16 \leqq n \leqq 400$, $a \leqq 4n$, *every positive integer is represented by*

$$[4n+2an,\ \{(I-4n)/a\}-2-2n],$$

*where* $\{x\}$ *denotes the least integer* $\geqq x$.

Similar theorems hold when $a>4n$, $n>16$.

THEOREM 4. *Let* $a=3$. If $11 \leqq n \leqq 400$, *every positive integer is represented by* $[4n,\ \{(I-4n)/a\}-2]$. *According as* $n=9$ *or* 10, *every positive integer is represented by* [104, 148] *or* [41, 346], *each of weight* $I$.

THEOREM 5. *If* $a=4$, $11 \leqq n \leqq 400$, *every positive integer is represented by* $[4n,\ \{(I-4n)/a\}-2]$.

If $a=b+c$, every number represented by $\Sigma x_i^n + a\Sigma y_i^n$ is evidently represented by $\Sigma x_i^n + b\Sigma y_i^n + c\Sigma z_i^n$, but not conversely. Hence each of our theorems yield universal forms obtained by partitioning one or more $a$'s into arbitrary positive integral parts, which may differ for the various $a$'s.

Future investigations will relate to summands which are not $n$-th powers, but are values of one or more polynomials. With this in mind, Lemmas 1—16 are not restricted to the case of powers, but (without lengthening the proofs or making them more difficult) are given for summands chosen from any sequence $p$ of integers $p_0=0$, $p_1$, $p_2, \ldots$ of integers arranged in ascending order of magnitude. A sum of $u$ numbers of $p$ and $w$ numbers each the product of $a$ by a number of $p$ is again denoted by $[u, w]$. Beginning with Lemma 8 (which defines $q$), we assume that $p_1=1$. Now $I=p_2+q-2$. If $A \leqq R$, $B \leqq S$, we write $[A, B] \leqq [R, S]$.

We write log for base $e$, Log for base 10.

## 2. Lemmas for ascent.

LEMMA 1. *If* $z \geqq a$ *and if all integers in the interval* $J=(E, E+zp_k)$ *are represented by* $[A, B]$, *then all in* $(E, E+(z+a)p_k)$ *are represented by* $[A, B+1]$.

Since $p_0=0$, any integer in $J$ is represented also by $[A, B+1]$. Hence proof is needed only for integers $j$ satisfying

$$E+zp_k<j \leqq E+(z+a)p_k,\quad E \leqq E+(z-a)p_k<j-ap_k \leqq E+zp_k.$$

Since $j-ap_k$ lies in $J$, it is represented by $[A, B]$, whence $j$ is represented by $[A, B+1]$.

Using Lemma 1 with $z$ replaced by $z+av$ and $B$ by $B+v$, we obtain, by induction from $v$ to $v+1$:

LEMMA 2. *If* $z \geqq a$ *and if all integers in the interval* $(E, E+zp_k)$ *are represented by* $[A, B]$, *then all in* $(E, E+(z+av)p_k)$ *are represented by* $[A, B+v]$.

Take $a=1$ where $a$ occurs explicitly in Lemma 1 and its proof, but not in the symbol $[A, B]$; we get

LEMMA 3. *If* $z \geqq 1$ *and if all in* $(E, E+zp_k)$ *are represented by* $[A, B]$, *then all in* $(E, E+(z+1)p_k)$ *are represented by* $[A+1, B]$.

Using Lemma 3 with $z$ replaced by $z+w$ and $A$ by $A+w$, we obtain, by induction from $w$ to $w+1$,

LEMMA 4. *If* $z \geqq 1$ *and all integers in* $(E, E+zp_k)$ *are represented by* $[A, B]$, *then all in* $(E, E+(z+w)p_k)$ *are represented by* $[A+w, B]$.

LEMMA 5. *Let* $R_j$ *denote the least positive or zero residue modulo* $a$ *of* $G_j=[p_j/p_{j-1}]$. *Let* $Q_j=(G_j-R_j)/a$. *Let all integers in* $(E, E+ap_{m-1})$ *be represented by* $[R, S]$. *Then all in* $(E, E+p_m)$ *are represented by* $[R, S+Q_m]$.

Proof. Apply Lemma 2 with $z=a$, $k=m-1$, $v=Q_m \geqq 0$. Thus all in $(E, E+(a+aQ_m)p_{m-1})$ are represented by $[R, S+Q_m]$.

But $a \geqq 1+R_m$, $aQ_m=G_m-R_m$. Adding we get $a+aQ_m \geqq 1+G_m>p_m/p_{m-1}$. Hence our interval extends beyond $E+p_m$.

LEMMA 6. *If* $m \geqq 3$ *and if all integers in* $(E, E+ap_2)$ *are represented by* $[A, B]$, *then all in* $(E, E+p_m)$ *are represented by* $[C, D]$, $C=A+(m-3)(a-1)$, $D=B+Q_3+\ldots+Q_m$.

Proof. By Lemma 5 with $m=3$, all in $(E, E+p_3)$ are represented by $[A, B+Q_3]$. This proves Lemma 6 when $m=3$. Using induction on $m$, assume our lemma for a particular $m \geqq 3$. Then by Lemma 4 with $z=1$, $w=a-1$, $k=m$, all in $(E, E+ap_m)$ are represented by $[C+a-1, D]$. Then by Lemma 5, with $m$ replaced by $m+1$, all in $(E, E+p_{m+1})$ are represented by $[C+a-1, D+Q_{m+1}]$, which is $[C, D]$ with $m$ replaced by $m+1$.

LEMMA 7. *Let* $\{R, S\}$ *denote a sum of* $R$ *numbers* 0 *or* 1 *and* $S$ *numbers* 0 *or* $a$. *Let* $B$ *be a positive integer. Define* $b$ *as the least residue* $>0$ *of* $B$ *modulo* $a$, *whence* $1 \leqq b \leqq a$. *Each of the integers* $0, 1, \ldots, B-1$ *is represented by* $\{b-1, k\}$ *or* $\{a-1, k-1\}$, *where* $k=(B-b)/a$, *while the second form is to be suppressed if* $k=0$, *viz.*, $B \leqq a$.

For, those of our integers which are $\geqq B-b=ka$ are sums of $ka$ and $0, 1, \dots, b-1$ and hence are represented by $\{b-1, k\}$. If $k=0$, these exhaust our integers. Next, let $k \geqq 1$. The remaining integers are $ka-ta+j$ $(j=0, \dots, a-1;\ k \geqq t \geqq 1)$ and are all represented by $\{a-1, k-t\}$ and hence by $\{a-1, k-1\}$.

Henceforth, let $p_1=1$. Then the numbers of $\{R, S\}$ and their products by any $p_i$ are all represented by $[R, S]$.

LEMMA 8. *Let $p_3=qp_2+r$, $0 \leqq r < p_2$. Let $f, g, d$ be the least residues $>0$ modulo $a$ of $p_2-r$, $r$, $q$, respectively. Then every integer in the interval $(qp_2, (q+1)p_2)$ is represented by one of*

$$(1) \qquad [d+g-1, V],\ [d+a-1, V-1],\ [f, W],\ [a, W-1],$$

*where $V=(r-g+q-d)/a$, $W=p_2-r-f)/a$, while the second form (1) is to be suppressed if $r \leqq a$ (whence $g=r$), and the fourth if $p_2-r \leqq a$.*

By Lemma 7, $0, 1, \dots, r-1$ are represented by $[g-1, (r-g)/a]$ if $r \leqq a$, but by it or $[a-1, (r-g)/a-1]$ if $r>a$. To these we add $[d, (q-d)/a]$, which represents

$$(2) \qquad qp_2=dp_2+(q-d)/a \cdot ap_2,$$

and get the first two forms (1). The further integers (except the final one) in our interval are the sums of $qp_2+r=p_3$ and $0, 1, \dots, p_2-r-1$. To these we apply Lemma 7.

Finally, [2]) $(q+1)\ p_2$ is represented by $[d+1, (q-d)/a]$ in view of (2). Hence it is represented by the first form (1) if $g \geqq 2$, and by the second form (1) if $r-g \geqq a$. Let both of these inequalities fail. Then $0 \leqq r-g < a$, and the multiple $r-g$ of $a$ is zero. Thus $r=g=1$. The same is true when the second form (1) had to be suppressed. When $p_j=j^n$, it is known that $r>1$.

LEMMA 9. *Every integer in $(qp_2, (q+a)p_2)$ is represented by one of the forms obtained by adding $a-1$ to the first entries of the forms (1). As an improvement on the first two resulting forms we may take*

$$[a+g-2, V],\ [2a-2, V-1],\ [g+d-2, V+1],$$
$$[a+d-2, V],$$

*where the second and fourth forms are to be suppressed if $r \leqq a$.*

Proof. The first statement follows from Lemmas 3, 8.

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[2]) In our applications, we may exclude this end term of the interval.



For the improvement, consider the sums of $(q+z)p_2$ and $0, \dots, r-1$, where $z \leqq a-1$. First, let $z \leqq a-d-1$. By (2), each $(q+z)p_2$ is represented by $[a-1, (q-d)/a]$. When this is added to the two forms above (2), we get the first two forms in Lemma 9. Second, let $z \geqq a-d$, whence $z=a-d+w$ $(w=0, \dots, d-1)$. Then

$$(q+z)p_2=wp_2+(q-d+a)/a \cdot ap_2$$

are all represented by $[d-1, (q-d)/a+1]$. When this is added to the two forms above (2), we get the last two forms in Lemma 9.

## 3. Representation of integers $\leqq qp_2-1$.

LEMMA 10. *Let $a<p_2$. Let $c$ and $d$ be the least residues $>0$ modulo $a$ of $p_2$ and $q$, respectively. If $a<q$ define $G$ to be the greater and $L$ the smaller of $c$ and $d$ (with $G=L=c$ if $c=d$). But if $a \geqq q$, define $G=d$, $L=c$. Then every integer $\leqq qp_2-1$ is represented by one of the three forms*

$$[u_0, Q],\ [u_1, Q-1],\ [u_2, Q-2],$$

$u_0=G+L-2$, $u_1=a+G-2$, $u_2=2a-2$, $Q=(p_2+q-G-L)/a$, *with the third form suppressed when $a \geqq q$.*

The integers in question are $xp_2+y$, $0 \leqq x \leqq q-1$, $0 \leqq y \leqq p_2-1$. By Lemma 7, the $y$'s are represented by $[c-1, (p_2-c)/a]$ or $[a-1, (p_2-c)/a-1]$. The $x$'s (and hence their products by $p_2$) are represented by $[d-1, (q-d)/a]$ or $[a-1, (q-d)/a-1]$, with the latter suppressed if $q \leqq a$.

Adding each of the latter to each of the former forms, we see that all $xp_2+y$ are represented by one of

$$[c+d-2, Q],\ [a+c-2, Q-1],\ [a+d-2, Q-1],\ [2a-2, Q-2],$$

where the second and fourth are to be suppressed if $q \leqq a$. In the latter case, Lemma 10 is proved. Next, let $q>a$. Then the second and third forms are $[a+G-2, Q-1]$ and $[a+L-2, Q-1]$ in some order. The latter may by discarded since every integer represented by it is represented also by the former (with 0 as the value of $G-L$ of the single $p$'s.

The asymptotic theory (§ 7) employs a number $s$ which is $4n$ in general. For the present, let $a \leqq 1+s/2$. Define $\{z\}$ to be the least integer $\geqq z$. If $z$ is an integer $\geqq 0$, we replace each of $z$ terms $ap_x$ by $p_x+\dots+p_x$ (with $a$ summands) and see that every integer represented by $[u, w]$ is represented by $[u+az, w-z]$. When $[u, v]$ is one of the forms in Lemma 10, we desire the least integer $z_i$ for which $u_i + a z_i \geqq s$. Since $u_i \leqq 2a - 2 \leqq s$, we have $z_i \geqq 0$ and $z_i = \{(s - u_i)/a\}$.

LEMMA 11. *If* $a < p_2$ *and* $a \leqq 1 + s/2$, $z_i = \{(s - u_i)/a\}$, *then* $z_i \geqq 0$ *and every integer* $\leqq q p_2 - 1$ *is represented by one of the forms*

$$F_i = [u_i + a z_i,\ Q - i - z_i] \qquad (i = 0, 1, 2),$$

*with* $F_2$ *suppressed if* $a \geqq q$.

The weights of $F_0, F_1, F_2$ are $I, I - L, I - L - G$, respectively.

LEMMA 12. *When* $a = 2$, $s = 2t$, $t \geqq 1$, *and* $p_2$ *is even, every integer* $\leqq q p_2 - 1$ *is represented by* $F_0 = [2t + 2 - d, Q - t - 1 + d]$, $Q = \frac{1}{2}(p_2 + q - 2 - d)$.

For, if $q > 2$, $G = c = 2$, $L = d$, $u_0 = d$, $u_1 = u_2 = 2$, whence every integer represented by $F_2$ is represented by $F_1$. Also

$$z_0 = t - d + 1,\ z_1 = t - 1,\ F_1 = [2t,\ Q - t].$$

If $d = 1$ we obtain $F_1$ from $F_0$ by using 0 as one of the single powers, and if $d = 2$ by using 0 as one of the double powers.

But if $q \leqq 2$, $F_2$ is to be suppressed, while $L = 2$, $G = d$, $u_1 = u_0 = d$, $z_1 = z_0$, and $F_1$ is derived from $F_0$ by subtracting 1 from its second entry, whence we may drop also $F_1$.

To treat the general case $a \geqq 2$, write

$$s = aZ + p,\ u_i = aU_i + v_i,\ 0 \leqq p < a,\ 0 \leqq v_i < a,$$

$$d_i = 0 \text{ if } v_i \geqq p,\ d_i = 1 \text{ if } v_i < p.$$

If $U_i \geqq Z + 1$, $u_i > s$, whereas $u_i \leqq s$. Thus $U_i < Z + 1$.

$$d_i + Z - U_i - (s - u_i)/a = d_i + (v_i - p)/a \geqq 0 \text{ and } < 1.$$

Hence $z_i = d_i + Z - U_i$. Define $\max([R_i, S_i])$ to be $[\max R_i, \max S_i]$.

Case $a = 4$, $s = 4t$, $t \geqq 2$, $p_2 \equiv 0 \pmod 4$, $4 < p_2$, $4 < q$. Thus $Z = t$, $p = 0$, all $d_i = 0$, $G = c = 4$, $L = d$, $u_1 = u_2 = 6$. Since $U_1 = 1$, $v_1 = 2$, $z_1 = t - 1$, $F_1 = [4t + 2, Q - t]$. We may drop $F_2$ which is obtained from $F_1$ by reducing its second entry by 1. Next, $u_0 = 2 + d$. If $d = 1$, $u_0 = v_0 = 3$, $U_0 = 0$, $z_0 = t$, $F_0 = [4t + 3, Q - t]$ and we may drop $F_1$. But if $d \geqq 2$, $U_0 = 1$, $v_0 = d - 2$, $z_0 = t - 1$, $F_0 = [4t + d - 2, Q - t + 1]$ and we may drop $F_1$ only when $d$ has its maximum value 4.

LEMMA 13. *If* $a = 4$, $s = 4t$, *all integers* $\leqq q p_2 - 1$ *are represented by* $F_0$ *alone when* $d = 1$ *or* 4. *But if* $d = 1$ *or* 3, *some are represented by* $F_0$ *and the others by* $F_1$. *All are represented by their maximum* $[4t + 2, Q - t + 1]$, *whose weight is* $I + 4 - d = I + 2$ *or* $I + 1$.



For general $a$ and $s$, $a < p_2$, $a < q$, we separate cases as follows, find $U_i, v_i, z_i$ by inspection, and get

$$F_2 = [a(d_2 + Z + 1) - 2,\ Q - d_2 - Z - 1];$$

$$G \geqq 2,\ F_1 = [a(d_1 + Z) + G - 2,\ Q - d_1 - Z];$$

$$G = 1,\ F_1 = [a(d_1 + Z + 1) - 1,\ Q - d_1 - Z - 1];$$

$$G + L - 2 \geqq a,\ F_0 = [a(d_0 + Z - 1) + G + L - 2,\ Q - d_0 - Z + 1];$$

$$G + L - 2 < a,\ F_0 = [a(d_0 + Z) + G + L - 2,\ Q - d_0 - Z].$$

We seek $F = \max(F_0, F_1, F_2)$. Since $v_2 = a - 2$, $p < a$, we get

$$d_2 = 1 \text{ if } p = a - 1,\ d_2 = 0 \text{ if } p \leqq a - 2.$$

I. Case $G = 1$. Then $L = 1$, $G + L - 2 = 0 < a$. Hence

$$F = [aZ + A,\ Q - d_0 - Z],\ A = \max(a(d_2 + 1) - 2,\ a(d_1 + 1) - 1,\ a d_0).$$

$\mathrm{I}_1$: $m = 0$, $d_0 = 0$, $A = a - 1$. $\mathrm{I}_2$: $p = a - 1$, $d_0 = 1$, $A = 2a - 2$, $\mathrm{I}_3$: $0 < p \leqq a - 2$ (vacuous if $a = 2$), $d_0 = 1$, $A = a$.

II. Case $G \geqq 2$, $G + L - 2 < a$. Hence

$$F = [aZ - 2 + B,\ Q - Z - \min(d_0, d_1)],$$

$$B = \max(a(d_2 + 1),\ a d_1 + G,\ a d_0 + G + L).$$

$\mathrm{II}_1$: $p \leqq G - 2$, $B = \max(a, G + L)$. $\mathrm{II}_2$: $G - 2 < p < G + L - 2$, $B = a + G$.

$\mathrm{II}_3$: $p = G + L - 2 = a - 1$, $B = 2a$. $\mathrm{II}_4$: $p = G + L - 2 < a - 1$, $B = a + G$.

$\mathrm{II}_5$: $p > G + L - 2$, $p = a - 1$, $B = 2a$. $\mathrm{II}_6$: $p > G + L - 2$, $p \leqq a - 2$, $B = a + G + L$. All $d_i = 0$ for $\mathrm{II}_1$; $d_0 = 0$, $d_1 = 1$ for $\mathrm{II}_2 - \mathrm{II}_4$; $d_0 = d_1 = 1$ for $\mathrm{II}_5, \mathrm{II}_6$.

III. Case $G \geqq 2$, $G + L - 2 \geqq a$. Hence $L \geqq 2$ and

$$F = [aZ - 2 + C,\ Q - d_0 - Z + 1],$$

$$C = \max(a(d_2 + 1),\ a d_1 + G,\ a(d_0 - 1) + G + L).$$

$\mathrm{III}_1$: $G - 2 + L - a \geqq p$, all $d_i = 0$, $C = a$.

$\mathrm{III}_2$: $G - 2 < p = a - 1$, $d_0 = 1$, $C = 2a$. $\mathrm{III}_3$: $G - 2 < p \leqq a - 2$, $d_0 = 1$, $C = a + G$.

$\mathrm{III}_4$: $G - 2 + L - a < p \leqq G - 2$, $d_0 = 1$, $d_1 = d_2 = 0$, $C = G + L$.

The weights of $\mathrm{I}_1, \ldots \mathrm{III}_4$ are $I + a - 1, I + a - 2, I, I - G - L +$

max $(a, G+L)$, $I+a-L$, $I+a-1$, $I+a-L$, $I+a-G-L$, $I$, $I-G-L+2a$, same, $I+a-L$, $I$. The weight of $F$ is evidently $I$ if and only if $F=F_0$.

LEMMA 14. *If $a<p_2$, $a<q$, $2\leqq a\leqq 1+s/2$, all integers $\leqq q p_2-1$ are represented by $F$ given by $\mathrm{I}_1-\mathrm{III}_4$, whose weights range from $I$ to $I+a-1$.*

LEMMA 15. *Let $a=3$, $s\geqq 4$, $p_2$ be $>3$ and prime to 3, $q>3$. All integers $\leqq q p_2-1$ are represented by $F$ (followed by its weight), where if $G=L=1$,*

$$p=0,\ F=[3Z+2,\ Q-Z]=e,\ I+2;$$

$$p=1,\ F=[3Z+3,\ Q-Z-1],\ I;$$

$$p=2,\ F=[3Z+4,\ Q-Z-1]=h,\ I+1.$$

$G=3$, $L=1$, $p=0$, $1$, $F=e$, $I$; $p=2$, $F=[3Z+4,\ Q-Z]=f$, $I+2$. $G=3$, $L=2$, $p=0$, $F=[3Z+1,\ Q-Z+1]$, $I+1$;

$$p=1,\ F=[3Z+3,\ Q-Z]=g,\ I;\ p=2,\ F=f,\ I+1.$$

$G=L=2$, $p=0$, $F=e$, $I$; $p=1$, $F=g$, $I+1$; $p=2$, $F=f$, $I+2$. $G=2$, $L=1$, $p=0$, $F=[3Z+1,\ Q-Z]$, $I$; $p=1$, $F=g$, $I+2$; $p=2$, $F=h$, $I$.

LEMMA 16. *Let $j$ be the least integer for which $u_j>s$. Then $u_i>s$ for $i\geqq j$ and we define $z_i=0\ (i\geqq j)$. But for $i<j$, define $z_i$ and all the $F$'s as in Lemma 11. Thus $u_i+a z_i\geqq s\ (i=0,1,2)$. If $j=0$, $F=[2a-2,\ Q]$ for $a<q$, $F=[u_1,\ Q]$ for $a\geqq q$.*

**4. Formula for $t$ ascents at one step.** Henceforth take $p_x=x^n$, where $n$ is an integer $\geqq 4$.

LEMMA 17. *If $g\geqq 0$, $s\geqq g+a$, there exists a positive integer $i$ such that*[3]).

(3) $$g\leqq s-a i^n<g+n(a s^{n-1})^{1/n}.$$

This implies

LEMMA 18. *Let $g$ be an integer $\geqq 0$, $L$ an integer $\geqq a$,*

(4) $$S=\{a^{-1}(L/n)^n\}^{1/(n-1)}.$$

*If $S\geqq g+L$ and if all integers between $g$ and $g+L$ inclusive are represented by a form $F$, then every integer $s$ between $g$ and $S$ inclusive is represented by $F+a x^n$.*

[3]) Dickson, American Journal Mathematics, Vol. 49 (1927), p. 242.



For, by hypothesis, all between $g$ and $g+a$ inclusive are represented by $F$ and hence by $F+a x^n$. Henceforth, let $s\geqq g+a$. Then (3) holds for a positive integer $i$. For an $s$ in Lemma 18, $s\leqq S$, whence the final number in (3) is

$$\leqq g+n(a S^{n-1})^{1/n}=g+L,$$

by (4). By hypothesis, $s-a i^n$ (being between $g$ and $g+L$) is represented by $F$, and hence $s$ by $F+a x^n$.

Write $L_0=g+L$, $v=(1-g/L_0)/n$, $L_1=S$. Then Lemma 18 will be proved equivalent to the more symetrical

LEMMA 19. *Let $g$ be an integer $\geqq 0$, $L_0\geqq g+a$,*

(5) $$L_1=\{a^{-1}(L_0 v)^n\}^{1/(n-1)}.$$

*If $L_1\geqq L_0$ and if all integers between $g$ and $L_0$ inclusive are represented by $F$, then all between $g$ and $L_1$ inclusive are represented by $F+a x^n$.*

For, the conditions in Lemma 18 become $L\geqq a$, $L_1\geqq L_0$, and (4) with $S$ replaced by $L_1$. Eliminate $L=L_0-g$. Then $L\geqq a$ becomes $L_0\geqq g+a$, while the new (4) becomes (5).

By Lemma 19, all integers between $g$ and $L_2=\{a^{-1}(L_1 v)^n\}^{1/(n-1)}$ are represented by $F+a x^n+a y^n$ since

$$L_2/L_1=(L_1/L_0)^{n/(n-1)}\geqq 1,\ L_2\geqq L_1.$$

Likewise, all between $g$ and

(6) $$L_{t+1}=\{a^{-1}(L_t v)^n\}^{1/(n-1)}$$

are represented by $F+a(x_1^n+\ldots+x_{t+1}^n)$. Then

(7) $$\operatorname{Log} L_t=\left(\frac{n}{n-1}\right)^t(\operatorname{Log} L_0+w)-w,\ w=n\operatorname{Log} v-\operatorname{Log} a,$$

follows by induction on $t$ from (6), and is evident if $t=0$. The condition $L_1\geqq L_0$ is equivalent to $L_0 v^n\geqq a$. This proves

LEMMA 20. *Let*[4]) *$g$ be an integer $\geqq 0$,*

(8) $$v=(1-g/L_0)/n,\ L_0\geqq g+a,\ L_0 v^n\geqq a.$$

*Compute $L_t$ by (7). If all integers between $g$ and $L_0$ inclusive are represented by a form $F$, then all between $g$ and $L_t$ inclusive are represented by $F+a(x_1^n+\ldots+x_t^n)$.*

[4]) A convenient form of $(8_3)$ is that the last factor in (10) be $\geqq 0$.

When $L_0$ is sufficiently large, Lemma 20 permits us to make $t$ ascents at one step.

LEMMA 21. *Starting with $L_t$ and $A$ in place of $L_0$ and $a$, $T$ further ascents leads to an $L$ such that*

$$(9) \qquad \operatorname{Log} L_{t+T} = \left(\frac{n}{n-1}\right)^{t+T} (\operatorname{Log} L_0 + w) + \left(\frac{n}{n-1}\right)^{T} (W - w) - W,$$

*where* $W = n \operatorname{Log} V - \operatorname{Log} A$, $V = (1 - g/L_t)/n$.

We have eliminated $L_t$ between (7) and the like formula in capital letters with $L_t$ and $L_0$ replaced by $L_{t+T}$ and $L_t$ respectively.

We shall use Lemma 21 for $A = 1$, and $L_t$ so large that $1/n$ is a sufficiently close approximation to $V$, whence $W = -n \operatorname{Log} n$. But $v < 1/n$, whence $W - w > \operatorname{Log} a$. Since the last two terms of (9) are positive,

$$(10) \qquad \operatorname{Log} L_{t+T} > \left(\frac{n}{n-1}\right)^{t+T} (\operatorname{Log} L_0 + n \operatorname{Log} v - \operatorname{Log} a).$$

**5. Theory of prime numbers.** For a positive real $x$, define $\vartheta(x)$ to be the sum of the natural logarithms of the prime numbers $\leqq x$ (and $\geqq 2$). Thus $\vartheta(x) = 0$ if $0 \leqq x < 2$.

LEMMA 22. *If $n$ is any integer* $\geqq 4$,

$$(11) \qquad \vartheta(2n+1) - \vartheta(n+1) < n \log 4 - \log 3.$$

This is readily verified if $4 \leqq n \leqq 11$. Let $n \geqq 12$ and employ the binomial coefficients

$$R = \binom{2n+1}{n},\ S = \binom{2n+1}{n-1},\ T = \binom{2n+1}{n-2},\ U = \binom{2n+1}{n-3},$$
$$V = \binom{2n+1}{n-4}.$$

Then

$$S = \frac{Rn}{n+2},\ T = \frac{Rn(n-1)}{(n+2)(n+3)},\ U = \frac{Rn(n-1)(n-2)}{(n+2)(n+3)(n+4)},$$
$$V = \frac{Rn(n-1)(n-2)(n-3)}{(n+2)(n+3)(n+4)(n+5)}.$$

Thus $(1+1)^{2n+1} > 2R + 2S + 2T + 2U + 2V$ will exceed $6R$ if $S + T + U + V > 2R$. The latter is equivalent to



$$n^4 - 6n^3 - 43n^2 - 132n - 120 > 0,$$

which holds if $n \geqq 12$. Hence $2^{2n+1} > 6R$. For any prime $p$ satisfying $n+2 \leqq p \leqq 2n+1$, $p$ divides the numerator, but not the denominator, of $R = (2n+1)(2n)\ldots(n+2)/n!$ Hence

$$\Pi p < R < \frac{1}{3} 2^{2n} \qquad (n+1 < p \leqq 2n+1).$$

Taking logarithms, we get (11).

Replace $n$ by $n-1$ in Lemma 22 and use $\vartheta(2n) = \vartheta(2n-1)$.

LEMMA 23. *If $n$ is any integer* $\geqq 5$,

$$(12) \qquad \vartheta(2n) - \vartheta(n) < (n-1)\log 4 - \log 3.$$

Lemmas 22 and 23 imply

LEMMA 24. *If $n$ is any integer* $\geqq 5$,

$$\vartheta(2n+1) - \vartheta(n) < \begin{cases} n \log 4 - \log 3 & \text{if } n+1 \text{ or } 2n+1 \text{ is composite,} \\ \log(n+1) + n \log 4 - \log 3 & \text{if } n+1 \text{ is prime.} \end{cases}$$

LEMMA 25. *For any real* $y \geqq 1$,

$$D(y) = \vartheta(2y) - \vartheta(y) < cy, \quad c = 1.42.$$

First, let $y = n + f$, $0 \leqq f < \frac{1}{2}$, $n$ an integer $\geqq 5$. By Lemma 23, $D(y)$ has the value (12), which is $< cn$, since $\log 4 < c$. Also $\vartheta(2n) - \vartheta(n) < cn$ for $n = 1, 2, 3, 4$.

Second, let $y = n + \frac{1}{2} + f$, $0 \leqq f < \frac{1}{2}$, $n$ an integer $\geqq 5$. Then $D(y)$ has the value in Lemma 24. Thus $D(y) < cn < cy$ if either $n+1$ or $2n+1$ is compositive. This holds also when $n = 3$ or $4$. Next, let both $n+1$ and $2n+1$ be primes. Then will

$$D(y) < \log(n+1) + n \log 4 - \log 3 < c\left(n + \frac{1}{2}\right) \leqq cy$$

if $\log(n+1) < n(c - \log 4) + \frac{1}{2}c + \log 3 = n(.03370564) + 1.80861229$.

This holds if $n = 76$ and hence if $n \geqq 76$. There remain only the cases $n = 1, 2, 6, 18, 30, 36$. For these, we find that

$$\vartheta(2n+1) - \vartheta(n) < 1.42\left(n + \frac{1}{2}\right).$$

LEMMA 26. *For any real* $x \geqq 1$, $\vartheta(x) < 1.42\,(x-1)$.

In Lemma 25, take $x=2y$. Hence if $x \geqq 2$,

$$\vartheta(x)-\vartheta(x/2)<A\,x,\quad A=c/2.$$

Write $g(x)=\vartheta(x)-2Ax$. Then

$$g(x)<g(x/2)<g(x/4)<\ldots<g(x/2^k),$$

where $k$ is the greatest integer for which $x/2^{k-1}\geqq 2$. Then $x/2^k<2$,

$$\vartheta(x/2^k)=0,\ 2A\,(x/2^k)=A\,(x/2^{k-1})\geqq 2A,\ g(x/2^k)\leqq -2A.$$

Hence $g(x)<-2A$, $\vartheta(x)<2Ax-2A$.

LEMMA 27. *If* $G_i=[i^n/(i-1)^n]$, $m\geqq 6$, $n\geqq 9$, *then*

$$\sum_{i=4}^{m} G_i<X,\ X=G_4+5G_5+m-n-9+n\log(m-1)/4-\binom{n}{2}\frac{1}{(m-1)}.$$

Call the final term $t$. Comparing the area under the curve $y=(1+1/x)^n$ with the (smaller) sum of inscribed rectangles, we get

$$\left(\frac{6}{5}\right)^n+\ldots+\left(\frac{m}{(m-1)}\right)^n<\int_4^{m-1}\left(1+\frac{1}{x}\right)^n dx$$

$$=\left\{x+n\log x-\binom{n}{2}\frac{1}{x}-\binom{n}{3}\frac{1}{2x^2}-\binom{n}{4}\frac{1}{3x^3}-\ldots\right\}_4^{m-1}$$

$$<m-5+n\log\frac{m-1}{4}-t+\binom{n}{2}\frac{1}{4}+\binom{n}{3}\frac{1}{2\cdot 4^2}+\binom{n}{4}\frac{1}{3\cdot 4^3}+\ldots$$

But

$$\frac{1}{2\cdot 4^2}=\frac{1}{4^2}-\frac{1}{2\cdot 4^2},\quad \frac{1}{3\cdot 4^3}=\frac{1}{4^3}-\frac{2}{3\cdot 4^3},\quad \frac{1}{4\cdot 4^4}=\frac{1}{4^4}-\frac{3}{4\cdot 4^4},$$

$$z=\binom{n}{3}\frac{1}{2\cdot 4^2}+\binom{n}{4}\frac{2}{3\cdot 4^3}+\binom{n}{5}\frac{3}{4\cdot 4^4}\geqq 4,\ \text{if } n\geqq 9.$$

Next,

$$\binom{n}{2}\frac{1}{4}+\binom{n}{3}\frac{1}{4^2}+\binom{n}{4}\frac{1}{4^3}+\ldots=4\left(1+\frac{1}{4}\right)^n-4-n,$$

$$1+[(5/4)^n]>(5/4)^n,\ 4\left(1+\frac{1}{4}\right)^n<4+4G_5.$$

**6. The singular series** $S$. For any prime $p$, let $p^h$ be the highest power of $p$ dividing $n$. Write $g=h+1$ if $p>2$, $g=h+2$ if $p=2$,



(13) $$r=\frac{p^g-1}{p-1}\left(\frac{n}{p^h},\ p-1\right),\ b=(1+(n-1)^s)^{2/(s-5)}.$$

It is well known that, if $s\geqq 4n$,

$$S\geqq PQ,\quad P=\prod_{p\leqq b}p^{-rg},\quad Q=\prod_{p>b}\left(1-p^{-\frac{3}{2}}\right).$$

Since $Q>$ like product over all primes,

$$\frac{1}{Q}<\sum_{x=1}^{\infty}x^{-3/2}\leqq 1+\int_1^{\infty}x^{-3/2}\,dx=3,$$

(14) $$-\log P=\sum_{p\leqq b} r\,g\log p.$$

First, let $p>2$ and $p$ be not a divisor of $n$. Then $h=0$, $g=1$, $r=(n,\ p-1)\leqq n$. The corresponding part of (14) is $\leqq n\ \Sigma\log p<n\,\vartheta(b)<1.42\,n\,(b-1)$, by Lemma 26.

Second, let $p>2$, $p$ a divisor of $n$. Then $g=h+1$,

$$r\leqq\frac{p^{h+1}-1}{p-1}\cdot\frac{n}{p^h}=\frac{n}{p-1}\left(p-\frac{1}{p^h}\right)<\frac{np}{p-1}.$$

The factor $p^h$ of $n$ is $\leqq n$, whence $h\leqq\log n/\log p$. This fraction is $\geqq 1$ since $n\geqq p$. Hence $g=h+1$ is $\leqq$ double the fraction. Hence the corresponding part of (14) is

$$<2n\log n\sum,\qquad \sum=\sum\frac{1}{p-1},$$

summed for the prime divisors $p\geqq 3$ of $n$. The fraction is $\leqq 3/2$. Hence $\sum\leqq\frac{3}{2}m$, where $m$ is the number of distinct prime factors $p_i$ of $n$. But $n=\prod p_i^{e_i}\geqq\prod 2=2^m$. Hence the part of (14) is

$$<3\,n\,(\log n)^2/\log 2.$$

Third, let $p=2$. Then $r=2^{h+2}-1\leqq 4n-1$. As above, $h\leqq\log n/\log 2$. This fraction is $\geqq 2$ when $n\geqq 4$. The present term of (14) is

$$<4n\,(h+2)\log 2\leqq 4n\,(2\log n/\log 2)\log 2=8\,n\log n.$$

LEMMA 28. *If* $n\geqq 4$, $s\geqq 4n$, $S>b_4$, *where*

$$-\log b_4=\log 3+1.42\,n\,(b-1)+3\,n\log^2 n/\log 2+8\,n\log n.$$

**7. Results from the asymptotic theory.** In the writer's paper[5] on sums of $n$-th powers multiplied by constants $A_i, a_i$, we take $k_1 = k$, $f = \frac{1}{2}\nu$, where $\nu = 1/n$, and

$$A = 1,\ A_j = 1\ (j = 1, \ldots, s-3),\ a_k = 1;\ \text{all remaining } a_i = a. \tag{15}$$

Take $s = 4n$. Then $4n$ of the coefficients of the $n$-*th* powers are 1 and the others are $a$. Also,

$$m = 3^{n-1},\ C = a^{-1}(3/2)^{n-1},\ \chi = a,\ \varkappa = a^{\nu/(n-1)-1}. \tag{16}$$

The condition on $h$ below (67) is here

$$h^{-n} \geqq 4n - 3 + 3 \cdot 2^n + a\, 2^{n-1}.$$

Since $2^{n-1} \geqq 4n - 3$ if $n \geqq 6$, this holds if

$$h = 1/3,\ a \leqq 2q - 7,\ n \geqq 6.$$

Although the $a_i$ are now not all 1, there is no change in the proof of (120)—($128_1$). We may suppress the first term of $\log c_{15}$ in (120). We discard $A$, pp. 310—3, and employ a discussion[6] which avoids the divisor function. We obtain

LEMMA 29. *Define $r$, $b$, $C$ by (13) and (16). Write*

$$C_1 = w F,\ w = 12\left(8 n 3^{n-1}\right)^{\frac{1}{2}} 2^{(n-1)/4} n^{\frac{3}{2}(n-1)},\ F = a^{3/2} \big/ C^{\nu/2},$$

$$c = \nu (1/3)^n b_4,\ z = \nu^3/24,$$

$$R = n^2(6n - 1)/(n - 1 - 2n^2 z),\ k_0 > \operatorname{Log} R/(\operatorname{Log} n - \operatorname{Log}(n-1)),$$

$$k = 2k_0,\ \sigma = n(1-\nu)^k,\ 2J = \sigma\left(3 - \frac{1}{2}\nu\right) + z - \frac{1}{2}(\nu - \nu^2),$$

$$P^{-J} > C_1/c,\ N = (3P)^n.$$

*Use the least integer $k_0$. Then if $a \leqq 2q$, $n \geqq 9$, every integer $\geqq N$ is represented by* $[4n, 3k-2]$.

In fact, the further conditions in the analytic theory are then satisfied. First,

$$1 + \log C + \frac{1}{2}(1-\nu)\log P \leqq P^z. \tag{17}$$

Second, in $A$, (85) with $C = b_4$, and (87) with $c_8 = c_7(b_4 + \delta c_1 P^{-1})$,

$$c_1 P^{-1}/b_4 \text{ and } c_3 P^{-\frac{1}{4}}/c_8 \text{ are insignificant.} \tag{18}$$

The above $k_0$ is somewhat less than the value by Vinogradow:

$$k' = [2n \log n + n \log 6]. \tag{19}$$

**8. General method to prove universality.** Take $p_x = x^n$ and $E = 2^n q$ in Lemma 6 and the first part of Lemma 9. Thus all integers in the interval $J = (2^n q,\ 2^n q + m^n)$ are represented by $[K, D]$, where

$$K = A + (m-2)(a-1),\ D = B + \sum_{i=3}^{m} Q_i,$$

while $[A, B]$ was initially one of the forms (1), but may be replaced by one of

$$[d + a - 1,\ (r - g + q - d)/a],\ [a,\ (2^n - r - f)/a]. \tag{20}$$

Apply Lemmas 20, 21, with $g = 3^n$, $L_0 = m^n$, both in $J$. The condition $L_0 v^n \geqq a$ will later be seen to hold after restrictions are placed on $n$, $m$, $a$; for example, if $m = 2n$, $n \geqq 9$, $a \leqq 2^{n-1}$. Let $Z$ be the least integral value of $t + T$ for which the second member of (10) exceeds Log $N$, where $N$ is the constant in Lemma 29.

For $a = 2$, we desire that

$$[T + K,\ t + D] \leqq \left[4n + 2 - d,\ \frac{1}{2}(2^n + q - 4n + d) - 2\right], \tag{21}$$

where the latter is $F_0$ of Lemma 12 for $t = 2n$.

For $a > 2$ we seek a form $H$ with minimum entries such that $H \leqq F = \max(F_0, F_1, F_2)$. First, let $a \leqq 1 + 2n$. In Lemma 11, each $u_i + a z_i \geqq s = 4n$. Also $Z = [4n/a]$. For either $F_0$ below Lemma 13 we see that $F_0 \geqq H$ if

$$H = [4n,\ \{(2^n + q - 2 - 4n)/a\} - 2]. \tag{22}$$

Second, let $a > 1 + 2n$. The preceding holds by Lemma 16 if $j > 0$. But if $j = 0$, and $a < q$, $F = [2a - 2, Q]$ and $H < F$. To $H$ we add $[aY, -Y]$, where $Y$ is chosen later. We desire that

$$[T + K,\ t + D] \leqq [4n + ay,\ \{(2^n + q - 2 - 4n)/a\} - 2 - Y], \tag{23}$$

[5] Annals of Math., vol. 37 (1936), pp. 293—316, cited as $A$.

[6] Amer. Jour. Math., vol. 58 (1936), pp. 521—535, cited as $J$.

where $y = Y$ if $a > 2$. When $a = 2$, (23) is identical with (21) if

$$(24) \qquad y = 1 - d/2, \quad Y = -1 - d/2.$$

By Lemma 5, $a\,Q_j = G_j - R_j$, where $G_j = [j^n/(j-1)^n]$.

Consider the first case (20). Then (23) holds if

$$(25) \qquad T = 4n + ay - (a-1)(m-1) - d,$$

$$(26) \qquad r \leqq -at + g + d - \Sigma G_i + \Sigma R_i + 2^n - 2 - 4n - 2a - aY,$$

where the sums are for $i = 3, \ldots, m$. To the last add $0 = at + aT - aZ$ and the product of (25) by $a$. We get

$$(27) \qquad r \leqq -aZ + 2^n - \Sigma G_i + \Sigma R_i - aY + a^2(y - m + 1) + 4an$$
$$+ a(m-3) - 4n - 2 + g + d(1-a).$$

Give to $y$ the least integral value (certainly $\leqq m$) for which (25) is $\geqq 0$. Then $t = Z - T$ will be seen to define an integer $> 0$. Thus, conversely, (27) implies (26) and hence (23).

For the second case (20), (23) holds if

$$(28) \qquad T = 4n + ay - a - (m-2)(a-1),$$

$$r \geqq at + \sum_4 G_i - \sum_3 R_i + aY + 4n + 2 + 2a - f.$$

As in the first case, the latter is equivalent to $r \geqq E + 1 - f$ (which holds if $r \geqq E$), where

$$(29) \qquad E = aZ + \sum_4 G_i - \sum_3 R_i + aY + a(4 - m - 4n)$$
$$+ a^2(m - 1 - y) + 4n + 1.$$

Then (27) becomes

$$(30) \qquad r \leqq 2^n - G_3 - E + a + g - 1 - d(a-1).$$

Decrease $g$ to 1 and $-d$ to $-a$. Hence (30) holds if

$$(31) \qquad r \leqq 2^n - G_3 - E + 2a - a^2.$$

Hence every positive integer will be represented by the form in the second member of (23) if the decimal part $r/2^n$ of $(3/2)^n$ satisfies

$$(32) \qquad \frac{r}{2^n} \geqq G(n), \quad \frac{r}{2^n} \leqq 1 - \left(\frac{3}{4}\right)^n - G(n) - (a^2 - 2a)/2^n, \quad G(n) = E/2^n,$$



where $E$ is given by (29), in which $\Sigma G_i$ may be replaced by $X$ of Lemma 27.

**9. First 5 decimals in decimal part $d(n)$ of $(3/2)^n$.**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 11 | .49755 | 20 | .25673 | 29 | .03948 | 38 | .92041 |
| 12 | .74633 | 21 | .88509 | 30 | .05923 | 39 | .88062 |
| 13 | .61950 | 22 | .82764 | 31 | .58884 | 40 | .32094 |
| 14 | .92926 | 23 | .74146 | 32 | .88327 | 41 | .48141 |
| 15 | .89389 | 24 | .11219 | 33 | .82491 | 42 | .72211 |
| 16 | .84083 | 25 | .16829 | 34 | .73736 | 43 | .58317 |
| 17 | .26125 | 26 | .75244 | 35 | .60604 | 44 | .87475 |
| 18 | .89188 | 27 | .12866 | 36 | .40907 | 45 | .31213 |
| 19 | .83782 | 28 | .69299 | 37 | .61361 | 46 | .96820 |

$d(n) > .98$ when $n \leqq 400$ only for

$$d(105) = .98559, \quad d(140) = .98041, \quad d(157) = .99274,$$

$$d(163) = .99550, \quad d(360) = .98774, \quad d(361) = .98161.$$

$d(n) < .03$ only for $n = 95, 96, 153, 178-9, 250,$ 265, 274-5, 313-4, 368, 393. $d(n) < .016$ only for

$$d(95) = .01116, \quad d(153) = .01091, \quad d(313) = .01587, \quad d(393) = .01518.$$

**10. Corollaries.** The first following is from $J$, § 3:

$$-J > \frac{1}{4n}\left(1 - \frac{1}{n} - \frac{1}{4n^2}\right), \quad b - 1 < n^{2s/(s-5)},$$

$$\mathrm{Log}\, w = 1.2169081 + .3138181\, n + \left(\frac{3}{2}n - 1\right)\mathrm{Log}\, n,$$

$$\mathrm{Log}\, F = \left(\frac{3}{2} + \frac{1}{2n}\right)\log a - \left(1 - \frac{1}{n}\right)(.0880456).$$

**11. Case $a \leqq 4n$.** We may take $m = 2n$, $Y = y = m$. Since our formulas remain valid if $E$ is increased, we may increase $a$ to $4n$ and decrease each $R_i$ to zero. By (29), the new $E$ is $E_1 = \Sigma G_i + 4n(Z + 5 - 8n) + 1$.

Let $n \geqq 16$. Then $\mathrm{Log}\, w < .1325498\, n^2$, $\mathrm{Log}\, F < .0104811\, n^2$, $\log C_1$

$< .0012885\, n^4$, $-\log b_4 < .1555733\, n^4$, $-\log c < .1558838\, n^4$, $-J \leqq .2341309/n$, $\operatorname{Log} P \geqq .2915386\, n^5$, $\operatorname{Log} N \geqq .2915414\, n^6$. In (10), $L > N$ if

$$n/(n-1)^{t+T}(n \operatorname{Log} 2 - \operatorname{Log} 4\, n) > \operatorname{Log} N.$$

First, let $n = 18$. Then $t + T \geqq 259 = Z$, $X = 496$, $E_1 = 9137$, and the limits (32) are .0348549 and .9402813. But $d(n)$ lies between them if $18 \leqq n \leqq 45$.

Second, let $n = 46$. Then $Z = 878$, $X = 702358$, $E_1 = 797119$, and the limits (32) are .00000001 and .9999981. But $d(n)$ lies between them if $46 \leqq n \leqq 400$.

Third, let $n = 16$. Then $Z = 222$, $X = 309$, $E_1 = 6646$, and the limits (32) are .10141 and .82802. While $d(17)$ lies between them, $d(16)$ does not. We may dispose of $n = 16$ as follows. When $a \leqq 56$, $E_1 \leqq 6310$ and the limits are .09628 and .84755. When $a = 60$, $g = 25$, $d = 56$, $r/2^{16} < .841693$. When $a = 61$, $g = 22$, $d = 46$, $r/2^{16} < .84934$. Finally, when $a = 64, 63, 62, 59, 58, 57$, $d = 16, 26, 36, 7, 18, 29$, respectively, and $r/2^{16} < .8567$.

The final term in (23) exceeds $3k - 2$ with $k = 2k'$. This proves Theorem 3.

When $a \leqq n$, we may take $Y = y = 2n - 5$.

**12.** When $a \leqq 3$, $n \geqq 11$, we get

$$\operatorname{Log} w < .1719876\, n^2, \quad \operatorname{Log} F < .0054325\, n^2,$$

$$\log C_1 < .0033762\, n^4, \quad -J > .2267562/n,$$

$$-\log b_4 < .2741006\, n^4, \quad -\log c < .2750898\, n^4,$$

$$\operatorname{Log} P \geqq .5333317\, n^5, \quad \operatorname{Log} N \geqq .5333346\, n^6.$$

**13. Case** $a = 2$. Take $m = n + 1$. We increase $E$ to $E_1$ by decreasing each $R_i$ to 0 and get

$$E_1 = \Sigma G_i + 2Z + d + 1 - 2n.$$

For $n \geqq 11$, let $E_1$ become $E'$ when we increase $d$ to 2. The last factor in (10) is $n \operatorname{Log}(n+1) - n \operatorname{Log} n - \operatorname{Log} 2$, since $v = 1/n$ to 7 decimal places. The condition $L_0 v^n \geqq 2$ in Lemma 20 holds since $n \geqq 5$. For $n \geqq 10$ and $k = 2k'$, $3k - 2$ is less than the second entry of $F_0$ given by the second member of (21).

For $n = 11$, the least $t + T$ is $Z = 168$, $X = 57$, $E' \leqq 393$. The limits (32) are .191894 and .76587. For $n = 11, 12, 13$, $d(n)$ lies between them.

When $n = 14$, $Z = 234$, $X = 637$, $E' = 612$. The limits (32) are .0373535 and .944828. For $14 \leqq n \leqq 45$, $d(n)$ lies between them.



When $n = 46$, $Z = 1110$, $X = 702271$, $E' = 704402$. The limits (32) are .00000001 and .9999982.

LEMMA 30. *For $g^n \leqq E < 2g^n$, let all integers*[7]) *in $K = (E, E + p)$ be sums of $h$ $n$-th powers. Then all in $K$ are represented by* $[(g-1)(a-1)+1, [h/a]]$.

In various recent papers, the writer has explained his method to find $h$ by use of certain equations $L_j = r_j$, where $L_j$ is linear in $2^n, \dots, g^n$. Write $B = r_{j+1} - r_j$, $S = L_j + E$, $S = x_2 2^n + \dots + x_g g^n$, where $x_g = 0$, or 1. The weights of $S, S+1, \dots, S+B-1$ are $\leqq \Sigma x_i + B - 1 \leqq h$. Write $x_i = a y_i + B_i$, $0 \leqq B_i \leqq a - 1$, while $B_g = 0$ or 1. Then $S = \Sigma B_i i^n + a \Sigma y_i i^n$ is represented by $[\Sigma B_i, \Sigma y_i]$. By Lemma 7, $0, \dots, B - 1$ are all represented by $[a - 1, (B - b)/a]$, where $b \equiv B \pmod a$, $1 \leqq b \leqq a$. Hence $S, \dots, S + B - 1$ are all represented by $[u, v]$,

$$u = a - 1 + \Sigma B_i \leqq a - 1 + (g - 2)(a - 1) + 1,$$

$$a v = a \Sigma y_i + B - b = \Sigma x_i - \Sigma B_i + B - b \leqq h.$$

For $a = 2$, $n = 9$ or 10, we apply Lemma 30 with $E = 6^n + 2.5^n$.

**14.** When $a = 2$, $n = 10$, we get $\operatorname{Log} N \geqq .6405552\, n^6$. All integers[8]) in $K$ are sums of $149 + 3$ tenth powers. Hence by Lemma 30, all in $K$ are represented by [6, 76]. Thus all in $(E, E + 2 \cdot 2^n)$ are represented by [7, 76].

The values of $G_3, \dots, G_{11}$ are 57, 17, 9, 6, 4, 3, 3, 2, 2. The corresponding $R_i$ are 1, 1, 1, 0, 0, 1, 1, 0, 0, and the $Q_i$ are 28, 8, 4, 3, 2, 1, 1, 1, 1. Hence by Lemma 6 with $m = 11$, all $(E, L_0 = E + 11^{10})$ are represented by [15, 125]. We get

$$\operatorname{Log} v = \bar{2}.9986626, \left(\frac{10}{9}\right)^{t+T} (.1008605) \geqq \operatorname{Log} N, \quad Z = 149.$$

Here $F_0 = [41, 519]$. Take $T = 26$. Then $[15 + T, 125 + t] = [41, 248] < F_0$.

Since $k_0 = 60$, $[s, 3k - 2] \leqq F_0$. By Lemma 8, all in $(2^n q, 2^n q + 2^n)$ are represented by [2, 368]. Then by Lemma 7 with $m = 7$, all in $(2^n q, 2^n q + 7^{10})$ are represented by $[7, 4/3] < F_0$.

**15.** When $a = 2$, $n = 9$, all integers[9]) in $K$ are sums of $106 + 3$

[7]) In the applications here, we take $p = 2^n$.

[8]) Dickson, Researches on Waring's Problem, Carnegie Institution of Washington, 1935, p. 2, tablette I. We may replace 149 by 142 except for the two functions numbered 6 and 25. By examining them, we find that all in $K$ are represented by [6, 73].

[9]) Bull. Amer. Math. Soc., vol. 40 (1934), p. 489, tablette $B = C = A = 0$. Actually all in $K$ are represented by [6, 52].

ninth powers. Hence by Lemma 30, all in $K$ are represented by [6, 54]. Thus all in $(E, E+2\cdot 2^n)$ are represented by [7, 54]. The values of $G_3, \ldots, G_{10}$ are 38, 13, 7, 5, 4, 3, 2, 2. Thus the $R_i$ are 0, 1, 1, 0, 1, 0, 0 and the $Q_i$ are 19, 6, 3, 2, 2, 1, 1, 1. Hence by Lemma 6 with $m=10$, all in $(E, L_0=E+10^9)$ are represented by [14, 89]. Here $F_0=[36, 256]$. We may take $k=86$ (since $k_0=56$ and $3k-2=256$), $z=.0000105$. We get $-\log b_4 < 3.363842\,n^3$, $\log C_1 < .4890613\,n^2$, $\log C_1/c < 3.4347593\,n^3$, $-J=.0241104$, $\operatorname{Log} P \geqq 15.82892\,n^4$, $\operatorname{Log} N \geqq 15.8290\,n^5$. The final factor in (10) is .1107101. Hence $t+T \geqq 136$. Take $T=22$, $t=114$. Then $[T+14, t+89]=[36, 203] < F_0$.

In Lemma 8, $f=g=1$, $d=2$, and all in $(2^n q, 2^n q+2^n)$ are represented by [3, 141]. By Lemma 6 with $m=7$, $E=2^n q$, all $(2^n q, 2^n q+7^9)$ are represented by [8, 173] and hence by $F_0$.

This proves Theorem 2. The proof of Theorem 4 is omitted.

**16.** For $a \leqq n$, $n \geqq 11$, we get $\operatorname{Log} N \geqq .5335973\,n^6$.

**17. Case** $a=4$. We take $Y=y=0$ since (25) and (28) are then positive. Take $m=n+3$. We omit the final term in Lemma 27. In (29), we decrease each $R_i$ to 0 and get $E'=4Z+\Sigma G_i+37$.

When $n=11$, we get $Z=151$, $X<93$, $E'<734$. The limits (32) are .35840 and .59546. But $d(11)$ lies between them.

When $n=12$, $Z=171$, $X=110$, $E'=831$. The limits (32) are .20288 and .76349. But $d(12)$ and $d(13)$ lie between them.

When $n=14$, $Z=213$, $\sum_4 G_i=130$, $\sum_3 R_i=25$, and (29) gives $E=994$, The limits (32) are .060669 and .921025. For $15 \leqq n \leqq 28$, $d(n)$ lies between them. The same holds for $n \geqq 29$ and the new limits.

Let $n=14$. For $E=5^n+5\cdot 6^n+7^n$ all integers in $K=(E, E+2^n)$ are sums of 1079 + 7 14-th powers.[10] Hence by Lemma 30, all in $K$ are represented by [19, 271]. Hence all in $(E, E+a\cdot 2^n)$ are represented by [22, 271]. Then by Lemma 6, all in $(E, E+17^{14})$ are represented by [64, 370]. Since $v=1/n$ to the sixth decimal place, we may use $Z=213$ as above. Take $T=0$. Then $[T+64, 370+t]=[64, 583] \leqq [64, 4151]$, viz., (23) for $Y=y=2$. By using an interval longer than $2^n$ and the paper cited, we reach (22) and hence retain $Y=0$.

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[10]) Dickson, Monatshefte Math. Phys., vol. 43 (1936), p. 393, tablette $A=0$.

# On Waring's problem for fourth and higher powers.

By

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**1.** Introduction.

Let $k$ be a positive integer. Then $G(k)$ denotes the least number $s$ such that every sufficiently large integer is a sum of $s$ $k$-th powers (of positive integers). This notation was introduced by Hardy and Littlewood[1]) and is now generally accepted. In my paper "Proof that every large integer is a sum of seventeen biquadrates"[2]), hereafter quoted as I, I proved (simultaneously with Davenport and Heilbronn[3]) that $G(4) \leqq 17$, and conjectured that the same method could be applied to the case $k>4$ with the following result:

Let

$$m_0=\left[\frac{(k-2)\log 2+\log(k-2)-\log k}{\log k-\log(k-1)}\right] \tag{1}$$

and

$$s=2m_0+7+[2^{k-1}(k-2)(1-k^{-1})^{m_0+1}], \tag{2}$$

where $[x]$ denotes the integral part of $x$. Then

$$G(k) \leqq s. \tag{3}$$

The object of the present paper is to prove this conjecture.

When $k=4$, it follows from (1) and (2) that $s=17$, so that (3) is

[1]) Some problems of partitio numerorum I, Göttinger Nachrichten (1920), 33—54.

[2]) Proc. London Math. Soc. (2), 41 (1936), 126—142.

[3]) Ibid. 143—150.