Proof. In the first place, whether M lies in a plane or not $F(C_i)$ must be connected, since by virtue of Theorem 2 we have

$$F(C_i)=F(\sum C_j)=\lim C_j, \tag{1}$$

and the last set clearly is connected since each C_j is connected.

Therefore if $F(C_i)$ contains more than one point, it must contain more than two points and, hence, by Theorem 5, G would have to be of order $\leqq 2$. whereas G is infinite. Therefore $F(C_i)=p$ where p is a point of K.

The second part of the theorem follows immediately from (1).

Theorem 7. *If M is a two dimensional sphere and $T(M)=M$ is a homeomorphism such that one rotation group under T is of order >1, then K is a simple closed curve*[1]).

Proof. Take a rotation group of order >1 and call two of its elements C_1 and C_2. Then by Theorem 4, $F(C_1)$ is contained in a simple closed curve J and, since C_1 is open in M and therefore $F(C_1)$ separates M, it follows that $F(C_1)=J=F(C_2)$. Now we show that $C_1+C_2+J=M$. By the Jordan Curve Theorem, $M-J=D_1+D_2$, where D_1 and D_2 are connected regions. Now C_1 is contained either in D_1 or in D_2, say in D_1; then clearly $C_1=D_1$ since D_1 is connected and contains no point of $F(C_1)$. Similarly $C_2=D_2$. Whence $C_1+C_2+J=M$ and therefore $J=K$.

Corollary. *Under conditions of Theorem 7, there exists only one rotation group under T and this group is of order 2.*

[1]) Since this theorem was obtained the author has learned that a similar conclusion under less general conditions has been proven independently by W. Dancer and extended by R. L. Wilder. The same method used by Wilder to extend Dancer's result may be used to extend this result to higher dimensional Euclidean spheres. See abstracts by Dancer and Wilder in the Bulletin of the Amer. Math. Soc., Vol 41, pp. 342 and 484, respectively and Fund. Math. 27.

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Mean values of trigonometrical polynomials.

By

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CHAPTER I.

1. The object of this chapter is to establish a number of inequalities between various mean values of trigonometrical polynomials.

Let $x_0<x_1<x_2<\ldots<x_{2n}$ be a system of $2n+1$ points equally distributed (mod 2π) over the interval $(0, 2\pi)$, *i. e.*

$$x_\nu=x_0+\nu\frac{2\pi}{2n+1} \qquad (\nu=0, 1, \ldots, 2n).$$

Let

$$S(x)=\frac{1}{2}a_0+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx) \tag{1}$$

be an arbitrary trigonometrical polynomial, real or complex, of order not exceeding n. It is well known that

$$\frac{1}{2\pi}\int_0^{2\pi}|S(x)|^2\,dx=\frac{1}{2n+1}\sum_{\nu=0}^{2n}|S(x_\nu)|^2. \tag{2}$$

In this chapter we extend this relation to the case of exponents other than 2. It is plain that the sign of equality in (1) shall have to be replaced by a sign of inequality.

Without loss of generality, we may suppose that $x_0=0$, for otherwise it is sufficient to consider the polynomial $S(x+x_0)$ instead of $S(x)$.

Given an integer $\mu>0$, let a function $\varphi_\mu(x)$ be defined by the following conditions

$$\varphi_\mu(t)=j\frac{2\pi}{\mu} \qquad \textit{for} \qquad j\frac{2\pi}{\mu}\leqslant t<(j+1)\frac{2\pi}{\mu},$$

where $j=0, \pm 1, \pm 2, \ldots$ The function $\varphi_\mu(t)$ is a step function, with jumps $2\pi/\mu$ at the points $2\pi j/\mu$. The relation (2), with $x_0=0$, may be written

$$\int_0^{2\pi}|S(x)|^2\,dx=\int_0^{2\pi}|S(x)|^2\,d\varphi_{2n+1}(x), \tag{3}$$

and this form will be more convenient to us than (2).

In the sequel, S will constantly denote an arbitrary trigonometrical polynomial (1) of order not exceeding n. By $\bar{S}$ we shall mean the conjugate polynomial, *i. e.*

$$\bar{S}(x)=\sum_{k=1}^{n}(a_k\sin kx-b_k\cos kx). \tag{4}$$

Theorems 1 and 2 which follow are not new[1]), but we have included them for the sake of completeness.

Theorem 1. *There is an absolute constant A such that*

$$\Big\{\int_0^{2\pi}|S|^p\,d\varphi_{2n+1}\Big\}^{1/p}\leqslant A\Big\{\int_0^{2\pi}|S|^p\,dx\Big\}^{1/p} \qquad (1\leqslant p\leqslant\infty). \tag{5}$$

Theorem 2. *There is a constant B_p depending only on p and such that*

$$\Big\{\int_0^{2\pi}|S|^p\,dx\Big\}\leqslant B_p\Big\{\int_0^{2\pi}|S|^p\,d\varphi_{2n+1}\Big\}^{1/p} \qquad (1<p<\infty). \tag{6}$$

It will be shown on examples that for $p=1$ and $p=\infty$ the inequality (6) is false.

Theorems 1 and 2 have analogues for the conjugate polynomials.

Theorem 3. *There is a constant $\bar{A}_p$ depending only on p and such that*

$$\Big\{\int_0^{2\pi}|\bar{S}|^p\,d\varphi_{2n+1}\Big\}^{1/p}\leqslant \bar{A}_p\Big\{\int_0^{2\pi}|S|^p\,dx\Big\}^{1/p} \qquad (1<p<\infty). \tag{7}$$

[1]) See Marcinkiewicz [2].

Similarly,

$$\Big\{\int_0^{2\pi}|\bar{S}|^p\,dx\Big\}^{1/p}\leqslant \bar{B}_p\Big\{\int_0^{2\pi}|S|^p\,d\varphi_{2n+1}\Big\}^{1/p} \qquad (1<p<\infty) \tag{8}$$

with $\bar{B}_p$ depending on p only.

The following three theorems may be considered as limiting cases of Theorems 2 and 3.

Theorem 4. *If*

$$\operatorname*{Max}_{\nu}\left|S\left(\frac{2\pi\nu}{2n+1}\right)\right|\leqslant 1, \tag{9}$$

then there is an absolute constant λ_0 such that, for every $0\leqslant\lambda<\lambda_0$,

$$\int_0^{2\pi}\exp\lambda|S|\,dx\leqslant\mu_\lambda \tag{10}$$

$$\int_0^{2\pi}\exp\lambda|\bar{S}|\,dx\leqslant\mu_\lambda, \tag{11}$$

where μ_λ depends on λ only.

Theorem 5. *There exist two absolute constants A and B such that*

$$\int_0^{2\pi}|S|\,dx\leqslant A\int_0^{2\pi}|S|\log^+|S|\,d\varphi_{2n+1}+B, \tag{12}$$

$$\int_0^{2\pi}|\bar{S}|\,dx\leqslant A\int_0^{2\pi}|S|\log^+|S|\,d\varphi_{2n+1}+B. \tag{13}$$

Theorem 6. *For every $0<\mu<1$,*

$$\int_0^{2\pi}|S(x)|^\mu\,dx\leqslant K_\mu\Big\{\int_0^{2\pi}|S(x)|\,d\varphi_{2n+1}\Big\}^\mu \tag{14}$$

$$\int_0^{2\pi}|\bar{S}(x)|^\mu\,dx\leqslant K_\mu\Big\{\int_0^{2\pi}|S(x)|\,d\varphi_{2n+1}\Big\}^\mu, \tag{15}$$

where K_μ depends on μ only. Moreover

$$K_\mu\leqslant\frac{K}{1-\mu},$$

K denoting an absolute constant.

We have already mentioned that Theorem 2 is false for $p=1$ and $p=\infty$. Theorems 4, 5, 6 are substitute theorems for those cases. Another substitute theorem is the following

Theorem 7. *For every integer* $k>0$,

$$\int_0^{2\pi}|S|\,dx\leqslant\left(1+\frac{2n}{k}\right)\int_0^{2\pi}|S|\,d\varphi_{2n+k} \tag{16}$$

$$\operatorname*{Max}_x|S(x)|\leqslant\left(1+\frac{2n}{k}\right)\operatorname*{Max}_\nu\left|S\left(\frac{2\pi\nu}{2n+k}\right)\right|. \tag{17}$$

In particular, if $\mu>(1+\varepsilon)2n$ *is an integer, then*

$$\int_0^{2\pi}|S|\,dx\leqslant\left(1+\frac{1}{\varepsilon}\right)\int_0^{2\pi}|S|\,d\varphi_\mu \tag{18}$$

$$\operatorname*{Max}_x|S(x)|\leqslant\left(1+\frac{1}{\varepsilon}\right)\operatorname*{Max}_\nu\left|S\left(\frac{2\pi\nu}{\mu}\right)\right|. \tag{19}$$

In the following paragraphs we shall give proofs of the theorems enunciated as well as of some other results.

2. ***Lemma*** α. *For every* $p\geqslant1$,

$$\left\{\int_0^{2\pi}|S'(x)|^p dx\right\}^{1/p}\leqslant n\left\{\int_0^{2\pi}|S(x)|^p dx\right\}^{1/p}. \tag{20}$$

This inequality is known [1]). It represents a generalization of Bernstein's well-known theorem, to which it reduces for $p=\infty$.

It is sufficient to prove Theorem 1 for p finite and greater than 1, for thence we obtain the remaining cases by a simple passage to the limit.

We observe that

$$\int_0^{2\pi}|S|^p\,d\varphi_{2n+1}=\int_0^{2\pi}|S|^p\,d(\varphi_{2n+1}-x)+\int_0^{2\pi}|S|^p\,dx.$$

[1]) See Zygmund [1]. A weaker inequality, with the coefficient n on the right replaced by $2n$, will be found in Zygmund [2], 155. This weaker result is sufficient for our purposes.

The function $\varphi_{2n+1}-x$ does not exceed $2\pi/(2n+1)$ in absolute value, and vanishes at the ends of the interval $(0,2\pi)$. Hence, integrating by parts and taking account of (20) we obtain

$$\begin{aligned}\int_0^{2\pi}|S|^p\,d\varphi_{2n+1}&\leqslant\frac{2\pi p}{2n+1}\int_0^{2\pi}|S|^{p-1}\,|S'|\,dx+\int_0^{2\pi}|S|^p\,dx\\&\leqslant\frac{\pi p}{n}\left(\int_0^{2\pi}|S|^p\,dx\right)^{1/p'}\left(\int_0^{2\pi}|S'|^p\,dx\right)^{1/p}+\int_0^{2\pi}|S|^p\,dx\\&\leqslant(\pi p+1)\int_0^{2\pi}|S|^p\,dx.\end{aligned} \tag{21}$$

This proves (5), with $A=\operatorname{Max}(\pi p+1)^{1/p}$ [1]).

3. The best value of the constant A is unknown but it certainly exceeds 1. For let

$$S(x)=\cos nx,\qquad \alpha=\pi/(2n+1).$$

Then

$$\left|S\left(\frac{2\pi\nu}{2n+1}\right)\right|=\left|\cos\frac{\pi\nu}{2n+1}\right|,$$

$$\begin{aligned}\int_0^{2\pi}|S|\,d\varphi_{2n+1}&=2\alpha\sum_{\nu=-n}^{n}|\cos\nu\alpha|=4\alpha\left(\frac{1}{2}+\sum_{\nu=1}^{n}|\cos\nu\alpha|\right)\\&=4\alpha\left(\frac{1}{2}+\sum_{\nu=1}^{n}\cos\nu\alpha\right)=4\alpha\,D_n(\alpha),\end{aligned}$$

where

$$D_n(u)=\tfrac{1}{2}+\cos u+\ldots+\cos nu=\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{1}{2}u} \tag{22}$$

denotes Dirichlet's kernel. It is now sufficient to observe that

$$4\alpha\,D_n(\alpha)=2\alpha=2\alpha/\sin\tfrac{1}{2}\alpha>4=\int_0^{2\pi}|S|\,dx.$$

In the inequality (5) we may replace A by 1 in the cases $p=2$ and $p=\infty$, and it may be asked whether the same is true for the

[1]) The inequality (5) holds if $d\varphi_{2n+1}$ is replaced by $d\varphi_\mu$, where $\mu\geqslant\varepsilon n$. The constant A will then depend on ε.

values of p belonging to the interval $(2, \infty)$. But this is not so, as can easily be seen on the example

$$S(x) = \sin nx, \qquad p=3 \text{ [1])}.$$

4. We now pass on to the proof of Theorem 2. We shall require the first part of the following proposition.

Lemma β. *Let $f(x)$ be a function of the class L^p where $1<p<\infty$, and of period 2π. Let $s_n[f]=s_n(x, f)$ be the n-th partial sum of the Fourier series of f, and $\bar{s}_n(x, f)$ the polynomial conjugate to s_n. Then*

$$\left\{\int_0^{2\pi} |s_n(x, f)|^p \, dx\right\}^{1/p} \leqslant R_p \left\{\int_0^{2\pi} |f|^p \, dx\right\}^{1/p} \tag{23}$$

$$\left\{\int_0^{2\pi} |\bar{s}_n(x, f)|^p \, dx\right\}^{1/p} \leqslant \bar{R}_p \left\{\int_0^{2\pi} |f|^p \, dx\right\}^{1/p}, \tag{24}$$

where R_p and $\bar{R}_p$ are constants depending on p only [2]).

Moreover we shall use the remark that, for any trigonometrical polynomial $T(x)$ of order not exceeding m, and for any $\mu>m$, we have

$$\int_0^{2\pi} T(x)\, dx = \int_0^{2\pi} T(x)\, d\varphi_\mu(x). \tag{25}$$

Let $p'=p/(p-1)$. There is a function $g(x)$ such that

$$\left(\int_0^{2\pi} |S|^p \, dx\right)^{1/p} = \int_0^{2\pi} S\, g\, dx, \qquad \int_0^{2\pi} |g|^{p'} dx = 1. \tag{26}$$

Hence

$$\begin{aligned} \left(\int_0^{2\pi} |S|^p dx\right)^{1/p} &= \int_0^{2\pi} S\, g\, dx = \int_0^{2\pi} S\, s_n[g]\, dx \\ &= \int_0^{2\pi} S\, s_n[g]\, d\varphi_{2n+1} \leqslant \left(\int_0^{2\pi} |S|^p d\varphi_{2n+1}\right)^{1/p} \left(\int_0^{2\pi} |s_n[g]|^{p'} d\varphi_{2n+1}\right)^{1/p'} \\ &\leqslant \left(\int_0^{2\pi} |S|^p d\varphi_{2n+1}\right)^{1/p} \cdot A \left(\int_0^{2\pi} |s_n[g]|^{p'} dx\right)^{1/p'} \\ &\leqslant A R_{p'} \left(\int_0^{2\pi} |S|^p d\varphi_{2n+1}\right)^{1/p} \left(\int_0^{2\pi} |g|^{p'} dx\right)^{1/p'} \\ &= A R_{p'} \left(\int_0^{2\pi} |S|^p d\varphi_{2n+1}\right)^{1/p}. \end{aligned} \tag{27}$$

[1]) When $p=2k$ $(k=1, 2, \ldots)$, we can obtain for A a slightly better value than before, *viz.* $A \leqslant k^{1/2k}$. For the proof it is sufficient to apply the equation analogous to (2) to the polynomial S^k, which is of the order nk.

[2]) M. Riesz [1], [2]; Zygmund [2], 153.

This proves (6). The best value of B_p satisfies the inequality

$$B_p \leqslant A R_{p'}. \tag{28}$$

It is known that

$$R_p = R_{p'} \qquad (1<p<\infty) \text{ [1])} \tag{29}$$

and it may be proved that

$$Kp \leqslant R_p \leqslant Lp \qquad \text{for} \quad p \geqslant 2, \text{ [2])}, \tag{30}$$

where K and L are two positive absolute constants.

5. The following result completes Theorem 2.

Theorem 8. *The least value of the constant B_p in Theorem 2 satisfies the relations*

$$R_p \leqslant B_p \leqslant A R_p, \tag{31}$$

where A denotes the constant of Theorem 1 and R_p the least constant in the inequality (23).

The second inequality (31) follows from (28) and (29), and we may restrict ourselves to the first inequality (31).

We shall require the following lemma, which is a special case of a known result.

Lemma γ. *Let $f(t)$ be an L-integrable function of period 2π, and let*

$$g_u(t) = f(u+t).$$

Given an arbitrary function $h(x)$ of period 2π, let $I_n[x, h]$ denote the trigonometrical polynomial, of order not exceeding n, which at the points $2\pi\nu/(2n+1)$ $(\nu=0, 1, \ldots, 2n)$ takes the same values as $h(x)$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} I_n[x-u, g_u]\, du = s_n(x, f) \text{ [3])}. \tag{32}$$

[1]) M. Riesz [1]; Zygmund [2], pp. 149, 165.

[2]) For the second inequality (30) see M. Riesz [2]. The first inequality (30) may be obtained by considering the n-th partial sum of $K_{2n}(x)$, where $K_n(x)$ denotes Fejér's kernel, and making use of (29). The first inequality will not be used in the sequel.

[3]) Marcinkiewicz [2].

We give the proof, which is very simple. Let $D_n(t)$ denote the Dirichlet kernel (22). Then

$$(33)\qquad I_n[x-u, g_u] = \frac{1}{\pi}\int_0^{2\pi} g_u(t)\, D_n(x-u-t)\, d\varphi_{2n+1}(t).$$

Integrating the right-hand side with respect to u, and inverting the order of integration (this is permissible, since the integral in (33) is but a finite sum) we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} I_n[x-u, g_u]\, du = \frac{1}{\pi}\int_0^{2\pi} d\varphi_{2n+1}(t)\frac{1}{2\pi}\int_0^{2\pi} g_u(t)\, D_n(x-u-t)\, du =$$
$$= \frac{1}{\pi}\int_0^{2\pi} d\varphi_{2n+1}(t)\frac{1}{2\pi}\int_0^{2\pi} f(u+t)\, D_n(x-u-t)\, du =$$
$$= \frac{1}{\pi}\int_0^{2\pi} d\varphi_{2n+1}(t)\frac{1}{2\pi}\int_0^{2\pi} f(u)\, D_n(x-u)\, du = \frac{s_n(x,f)}{2\pi}\int_0^{2\pi} d\varphi_n(t) = s_n(x,f).$$

From (32) we have, by Jensen's inequality,

$$|s_n(x,f)|^p \leqslant \frac{1}{2\pi}\int_0^{2\pi} |I_n[x-u, g_u]|^p\, du.$$

We integrate this relation with respect to x, and invert the order of integration:

$$\int_0^{2\pi} |s_n(x,f)|\ dx \leqslant \frac{1}{2\pi}\int_0^{2\pi} du\int_0^{2\pi} |I_n[x-u, g_u]|^p\, dx =$$
$$= \frac{1}{2\pi}\int_0^{2\pi} du\int_0^{2\pi} |I_n[x, g_u]|^p dx \leqslant \frac{B_p^p}{2\pi}\int_0^{2\pi} du\int_0^{2\pi} |I_n[x, g_u]|^p\, d\varphi_{2n+1}(x) =$$
$$= \frac{B_p^p}{2\pi}\int_0^{2\pi} du\int_0^{2\pi} |f(u+x)|^p\, d\varphi_{2n+1}(x) = B_p^p\int_0^{2\pi} |f(u)|^p\, du.$$

This completes the proof of Theorem 8.

6. We shall now show that Theorem 2 is false for $p=1$ and $p=\infty$. For let $S(x)$ denote the Dirichlet kernel $D_n(x)$. Since $D_n(x)$ vanishes at the points $2\pi\nu/(2n+1)$, where $\nu=1, 2, \dots 2n$, we have

$$\int_0^{2\pi} |S|\, d\varphi_{2n+1} = \pi.$$

On the other hand, the integral of $|S(x)|$ over $(0, 2\pi)$ ("Lebesgue's constant") is an unbounded function of n.

In order to verify that Theorem 2 is false for $p=\infty$, we consider the trigonometrical polynomial $S(x)$, of order n, which at the points $2\pi\nu/(2n+1)$ $(\nu=0, 1, \dots, 2n)$ takes the values $(-1)^\nu$. That is

$$S(x) = \frac{2}{2n+1}\sum_{\nu=0}^{2n} (-1)^\nu D_n(x-x_\nu).$$

It is easy to see that

$$S\left(-\frac{\pi}{2n+1}\right) = \frac{1}{2n+1}\sum_{\nu=0}^{2n} \frac{1}{\sin\frac{\pi}{2n+1}\left(\nu+\frac{1}{2}\right)},$$

and the right-hand side of this equation is unbounded with n.

7. *Lemma* $\mathfrak{d}$. *Every trigonometrical polynomial $S(x)$ satisfies, for $p>1$, the inequality*

$$(34)\qquad \left\{\int_0^{2\pi} |\bar{S}(x)|^p\, dx\right\}^{1/p} \leqslant \bar{R}_p \left\{\int_0^{2\pi} |S(x)|^p\, dx\right\}^{1/p},$$

where $\bar{R}_p$ is a constant depending on p only.

This is a special case of M. Riesz's well-known theorem. The inequality (7) follows at once from (34) and from the inequality

$$\left\{\int_0^{2\pi} |\bar{S}|^p\, d\varphi_{2n+1}\right\}^{1/p} \leqslant (\pi p+1)^{1/p}\left\{\int_0^{2\pi} |\bar{S}(x)|^p\, dx\right\}^{1/p}$$

established in (21). For $\bar{A}_p$ we may take $(\pi p+1)^{1/p}\bar{R}_p$.

A similar argument enables us to deduce (8) from Theorem 2 and the inequality (34). Such a deduction would however give a rather imperfect estimate for the constant $\bar{B}_p$, and we prefer to use an argument similar to (27).

There is a function g such that

$$\int_0^{2\pi}|g|^{p'}\,dx=1,\qquad \int_0^{2\pi}\overline{S}\,g\,dx=\left\{\int_0^{2\pi}|\overline{S}|^{p}\,dx\right\}^{1/p}.$$

Hence

$$(35)\qquad \left(\int_0^{2\pi}|\overline{S}|^{p}\,dx\right)^{1/p}=\int_0^{2\pi}\overline{S}\,g\,dx=\int_0^{2\pi}\overline{S}\,s_n[g]\,dx$$

$$=-\int_0^{2\pi}S\,\bar{s}_n[g]\,dx=-\int_0^{2\pi}S\,\bar{s}_n[g]\,d\varphi_{2n+1}$$

$$\leqslant\left(\int_0^{2\pi}|S|^{p}\,d\varphi_{2n+1}\right)^{1/p}\left(\int_0^{2\pi}|\bar{s}_n[g]|^{p'}\,d\varphi_{2n+1}\right)^{1/p'}$$

$$\leqslant A\left(\int_0^{2\pi}|S|^{p}\,d\varphi_{2n+1}\right)^{1/p}\left(\int_0^{2\pi}|\bar{s}_n[g]|^{p'}\,dx\right)^{1/p'}$$

$$\leqslant A\overline{R}_{p'}\left(\int_0^{2\pi}|S|^{p}d\varphi_{2n+1}\right)^{1/p}\left(\int_0^{2\pi}|g|^{p'}dx\right)^{1/p'}=A\overline{R}_{p'}\left(\int_0^{2\pi}|S|^{p}\,d\varphi_{2n+1}\right)^{1/p}.$$

This gives (8) with

$$(36)\qquad \overline{B}_p=A\overline{R}_{p'}.$$

If for $\overline{B}_p$ we take the least possible value, then we have the nequality

$$(37)\qquad \overline{R}_p\leqslant\overline{B}_p\leqslant A\overline{R}_p\qquad (p>1),$$

analogous to (31). The second inequality follows from (36) and the equation $\overline{R}_p=\overline{R}_{p'}$. The first inequality may be obtained in the same way as the first inequality in (31). We need only observe that the formula (32) holds if we replace in it $I_n[x-u,g_u]$ and $s_n(x,f)$ by $\bar{I}_n[x-u,g_u]$ and $\bar{s}_n(x,f)$ respectively (in the formula (33) we shall have to replace I_n by $\bar{I}_n$, and D_n by $\overline{D}_n$, where $\overline{D}_n$ is the conjugate Dirichlet kernel). Since $\overline{R}_p$ satisfies an inequality

$$(38)\qquad \overline{K}p\leqslant\overline{R}_p\leqslant\overline{L}p\qquad (p\geqslant 2),$$

similar to (30), we obtain from (30), (31), (37) and (38) that

$$(39)\qquad \alpha p\leqslant B_p\leqslant\beta p,\qquad \alpha p\leqslant\overline{B}_p\leqslant\beta p\qquad (p\geqslant 2),$$

where α and β are absolute positive constants.

8. In order to prove the first part of Theorem 4, we observe that, for any real u,

$$\exp u\leqslant 2\cosh u=2\sum_{\nu=0}^{\infty}\frac{u^{2\nu}}{(2\nu)!}.$$

Let us replace here u by $\lambda|S|$, or by $\lambda|\overline{S}|$, and integrate the result over the interval $0\leqslant x\leqslant 2\pi$. Using the inequalities (6), (8) and (39), and applying Stirling's formula for $(2\nu)!$, we obtain (10) and (11), with $\lambda_0=1/\beta e$.

9. The proof of Theorem 5 will be based on the Young inequality

$$uv\leqslant u\log^{+}u+e^{v}\ [1]\qquad (u\geqslant 0,\ v\geqslant 0)$$

and on the following

Lemma ε. *If* $|g(x)|\leqslant 1$, *then*

$$\int_0^{2\pi}\exp|s_n[g]|\,dx<\gamma,\qquad \int_0^{2\pi}\exp|\bar{s}_n[g]|\,dx<\gamma,$$

where γ *is an absolute constant* [2].

We start from the formula

$$(40)\qquad \int_0^{2\pi}|S|\,dx=\int_0^{2\pi}S(x)\,g(x)\,dx,\qquad \text{where}\quad |g(x)|=1.$$

Let β be a constant which will be fixed in a moment. We have

$$(41)\qquad \int_0^{2\pi}|S|\,dx=\int_0^{2\pi}S(x)\,s_n[g]\,dx=\int_0^{2\pi}S(x)\,s_n[g]\,d\varphi_{2n+1}$$

$$\leqslant\int_0^{2\pi}\left|\frac{S}{\beta}\right|\log^{+}\left|\frac{S}{\beta}\right|d\varphi_{2n+1}+\int_0^{2\pi}\exp|\beta\,s_n[g]|\,d\varphi_{2n+1}.$$

It is easy to see that the inequality (12) will have been established when we have shown that the last integral in (41) does not exceed an absolute constant. The equation $\exp u=1+u+\frac{1}{2}u^2+\dots$ and

[1] See *e.g.* Hardy, Littlewood and Pólya [1], p. 61.

[2] See Zygmund [3], or Zygmund [2], p. 164.

the inequality (5) give

$$\int_0^{2\pi} \exp|\beta s_n[g]|\, d\varphi_{2n+1} \leqslant \int_0^{2\pi} \exp|A\beta s_n[g]|\, dx.$$

In view of Lemma ε, the last integral is less than γ if only $A\beta \leqslant 1$. This completes the proof of (12). The inequality (13) may be established in the same way.

10. The proof of Theorem 6 will be based on the following proposition of Kolmogoroff:

Lemma ζ. *Let $f(x)$ be any L-integrable function of period 2π, and let $\bar{f}(x)$ denote the function conjugate to $f(x)$. Then, for $0<\mu<1$,*

(42) $$\int_0^1 |\bar{f}(x)|^\mu dx \leqslant K_\mu \Big(\int_0^{2\pi} |f(x)|\, dx\Big)^\mu.$$

where K_μ depends on μ only. The product $(1-\mu)K_\mu$ does not exceed an absolute constant K [1]).

If x is different from the points of discontinuity of the function $\varphi_{2n+1}(\theta)$, we have the formula

(43) $$S(x) = \frac{1}{\pi}\int_0^{2\pi} S(\theta)\, D_n(x-\theta)\, d\varphi_{2n+1}(\theta) =$$

$$= \frac{\sin nx}{\pi}\int_0^{2\pi} S(\theta)\cos n\theta \frac{d\varphi_{2n+1}(\theta)}{2\operatorname{tg}\frac{1}{2}(x-\theta)} - \frac{\cos nx}{\pi}\int_0^{2\pi} S(\theta)\sin n\theta \frac{d\varphi_{2n+1}(\theta)}{2\operatorname{tg}\frac{1}{2}(x-\theta)} +$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} S(\theta)\cos\tfrac{1}{2}(x-\theta)\, d\varphi_{2n+1}(\theta).$$

Let $g_i(x)$, where $i=1, 2, 3$, denote respectively the last three terms on the right of (43). In view of the inequality

$$|S(x)|^\mu \leqslant |g_1(x)|^\mu + |g_2(x)|^\mu + |g_3(x)|^\mu \qquad (0<\mu<1),$$

the inequality (14) will have been established when we have shown that

(44) $$\int_0^{2\pi} |g_i(x)|^\mu dx \leqslant K_\mu \Big(\int_0^{2\pi} |S(x)|\, d\varphi_{2n+1}(x)\Big)^\mu \qquad (i=1, 2, 3).$$

[1]) See Kolmogoroff [1], or Zygmund [2] p. 150. In the rest of this paragraph, we shall denote by K_μ any constant depending on μ only and such that $(1-\mu)K_\mu$ is bounded.

From the relation

$$|g_3(x)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |S(\theta)|\, d\varphi_{2n+1}(\theta)$$

follows the inequality (44) for $i=3$ (with $K_\mu = (2\pi)^{1-\mu}$). We may therefore confine our attention to the cases $i=1$ and $i=2$, e. g. to the former, the proof in both cases being exactly the same. Let

$$I(x) = \frac{1}{\pi}\int_0^{2\pi} S(\theta)\cos n\theta \frac{d\varphi_{2n+1}(\theta)}{2\operatorname{tg}\frac{1}{2}(x-\theta)}.$$

It is enough to show that

(45) $$\int_0^{2\pi} |I(x)|^\mu dx \leqslant K_\mu \Big(\int_0^{2\pi} |S(\theta)|\, d\varphi_{2n+1}(\theta)\Big)^\mu.$$

We write

(46) $$\begin{cases} \alpha_\nu = S(\theta_\nu)\cos n\theta_\nu \\ f(\theta) = \alpha_\nu \quad \text{for} \quad \theta_\nu \leqslant \theta < \theta_{\nu+1} \qquad (\nu = 0, 1, \ldots, 2n) \\ \Delta(x) = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta)\frac{d\theta}{2\operatorname{tg}\frac{1}{2}(x-\theta)} - \frac{2}{2n+1}\sum_{\nu=0}^{2n}\frac{\alpha_\nu}{2\operatorname{tg}\frac{1}{2}(x-\theta_\nu)}. \end{cases}$$

The last equation may be written

(47) $$I(x) = \bar{f}(x) - \Delta(x).$$

The first equation (46) gives

$$\int_0^{2\pi} |f(\theta)|\, d\theta \leqslant \int_0^{2\pi} |S(\theta)|\, d\varphi_{2n+1}(\theta).$$

Hence, in view of (42)

(48) $$\int_0^{2\pi} |\bar{f}(\theta)|^\mu d\theta \leqslant K_\mu \Big(\int_0^{2\pi} |S(\theta)|\, d\varphi_{2n+1}\Big)^\mu.$$

Let us suppose that $\theta_k < x < \theta_{k+1}$. Then, if

(49) $$\xi_\nu(x) = \frac{1}{\pi}\int_{\theta_\nu}^{\theta_{\nu+1}} \Big\{\frac{1}{2\operatorname{tg}\frac{1}{2}(x-\theta)} - \frac{1}{2\operatorname{tg}\frac{1}{2}(x-\theta_\nu)}\Big\} d\theta,$$

we have

$$(50)\quad |\Delta(x)|\leqslant\sum_{\nu=0}^{2n}|\alpha_\nu\,\xi_\nu(x)|=\sum_\nu{}'|\alpha_\nu\,\xi_\nu(x)|+|\alpha_{k-1}\,\xi_{k-1}(x)|+|\alpha_k\xi_k(x)|+$$
$$+|\alpha_{k+1}\,\xi_{k+1}(x)|=\delta_0(x)+\delta_1(x)+\delta_2(x)+\delta_3(x),$$

say. The dash $'$ denotes that the terms with the indices $\nu=k-1,k,k+1$ are omitted in summation.

In what follows, we shall denote by $A_1, A_2, \ldots$ positive absolute constants. It is easily seen that

$$\delta_0(x)\leqslant\frac{A_1}{(2n+1)^2}\sum_\nu{}'\frac{|\alpha_\nu|}{\sin^2\dfrac{x-\theta_\nu}{2}}\qquad(\theta_k<x<\theta_{k+1}).$$

Hence

$$\int_{\theta_k}^{\theta_{k+1}}\delta_0(x)\,dx\leqslant\frac{A_2}{2n+1}\sum_\nu{}'\frac{|\alpha_\nu|}{(\nu-k)^2},$$

$$\int_0^{2\pi}\delta_0(x)\,dx\leqslant\frac{A_3}{2n+1}\sum_{\nu=0}^{2n}|\alpha_\nu|\leqslant\frac{A_3}{2\pi}\cdot\int_0^{2\pi}|S(\theta)|\,d\varphi_{2n+1},$$

$$(51)\quad\left(\frac{1}{2\pi}\int_0^{2\pi}\delta_0^\mu(x)\,dx\right)^{1/\mu}\leqslant\frac{1}{2\pi}\int_0^{2\pi}\delta_0(x)\,dx\leqslant\frac{A_3}{4\pi^2}\int_0^{2\pi}|S(\theta)|\,d\varphi_{2n+1},$$

For $\theta_k<x<\theta_{k+1}$ we have

$$(52)\qquad |\xi_{k-1}(x)|<\frac{1}{\pi}\left|\int_{\theta_{k-1}}^{\theta_k}\frac{d\theta}{2\,\mathrm{tg}\,\frac{1}{2}(\theta-x)}\right|\leqslant\frac{2}{2n+1}\cdot\frac{1}{x-\theta_k}$$

$$(53)\qquad |\xi_{k+1}(x)|<\frac{1}{\pi}\left|\int_{\theta_{k+1}}^{\theta_{k+2}}\frac{d\theta}{2\,\mathrm{tg}\,\frac{1}{2}(\theta_{k+1}-x)}\right|\leqslant\frac{2}{2n+1}\cdot\frac{1}{\theta_{k+1}-x}.$$

On the other hand, we have the following inequalities, where the integrals are taken in the principal value sense:

$$\left|\int_{\theta_k}^{\theta_{k+1}}\frac{d\theta}{2\mathrm{tg}\,\frac{1}{2}(x-\theta)}\right|<\frac{2\pi}{2n+1}\cdot\frac{1}{x-\theta_k},\qquad\text{if}\quad\theta_k<x\leqslant\tfrac{1}{2}(\theta_k+\theta_{k+1})$$

$$\left|\int_{\theta_k}^{\theta_{k+1}}\frac{d\theta}{2\mathrm{tg}\,\frac{1}{2}(x-\theta)}\right|<\frac{2\pi}{2n+1}\cdot\frac{1}{\theta_{k+1}-x},\qquad\text{if}\quad\tfrac{1}{2}(\theta_k+\theta_{k+1})\leqslant x<\theta_{k+1}.$$

The formula (49) therefore gives

$$(54)\qquad |\xi_k(x)|<\frac{2}{2n+1}\left(\frac{2}{x-\theta_k}+\frac{1}{\theta_{k+1}-x}\right)\qquad(\theta_k<x<\theta_{k+1}).$$

From the inequalities (52), (53), and (54) we obtain

$$\int_{\theta_k}^{\theta_{k+1}}\delta_1^\mu(x)\,dx\leqslant\frac{A_4}{1-\mu}\,\frac{|\alpha_{k-1}|^\mu}{2n+1},$$

$$\int_{\theta_k}^{\theta_{k+1}}\delta_2^\mu(x)\,dx\leqslant\frac{A_4}{1-\mu}\,\frac{|\alpha_k|^\mu}{2n+1},$$

$$\int_{\theta_k}^{\theta_{k+1}}\delta_3^\mu(x)\,dx\leqslant\frac{A_4}{1-\mu}\,\frac{|\alpha_{k+1}|^\mu}{2n+1}.$$

This gives, for $i=1,2,3,$

$$(55)\qquad\int_0^{2\pi}\delta_i^\mu(x)\,dx\leqslant\frac{A_4}{1-\mu}\cdot\frac{1}{2n+1}\sum_{\nu=0}^{2n}|\alpha_\nu|^\mu$$
$$\leqslant\frac{A_4}{1-\mu}\left(\frac{1}{2n+1}\sum_{\nu=0}^{2n}|\alpha_\nu|\right)^\mu\leqslant\frac{A_4}{1-\mu}\left(\int_0^{2\pi}|S(\theta)|\,d\varphi_{2n+1}\right)^\mu.$$

Finally, from (47), (50), (48), (51), and (55) we deduce

$$|I(x)|^\mu\leqslant|\bar{f}(x)|^\mu+|\Delta(x)|^\mu\leqslant|f(x)|^\mu+\sum_{i=0}^{3}\delta_i^\mu(x),$$

$$\int_0^{2\pi}|I(x)|^\mu\,dx\leqslant\int_0^{2\pi}|\bar{f}(x)|^\mu\,dx+\sum_{i=0}^{3}\int_0^{2\pi}\delta_i^\mu(x)\,dx\leqslant\frac{A_5}{1-\mu}\left(\int_0^{2\pi}|S(x)|\,d\varphi_{2n+1}\right)^\mu.$$

This proves (45), and so completes the proof of the inequality (14).

The proof of the inequality (15) is analogous. The use is made of the formula

$$(56)\qquad\bar{S}(x)=\frac{1}{\pi}\int_0^{2\pi}S(\theta)\,\bar{D}_n(\theta-x)\,d\varphi_{2n+1}(\theta),$$

where

$$\bar{D}_n(u)=\sin u+\sin 2u+\ldots+\sin nu=\frac{\cos\frac{1}{2}u-\cos(n+\frac{1}{2})u}{2\sin\frac{1}{2}u}$$

is the expression conjugate to Dirichlet's kernel D_n. We leave the details of the proof to the reader.

It is of some interest to observe that the suffix $2n+1$ in Theorems 2—6 may be replaced by any integer $m>2n$. For m odd this follows from the fact that, if the polynomial S is of order n, it may also be considered as a polynomial of any order greater than n. For m even the proofs are exactly the same as in the case $m=2n+1$. If is sufficient to make use of the general equation (25).

11. Let s_n and σ_n denote respectively the partial sums and the first arithmetic means of the series $\sum_{\nu=0}^{\infty} u_\nu$, i. e. $s_n=\sum_{\nu=0}^{n} u_\nu$, $\sigma_n=\sum_{\nu=0}^{n}\left(1-\frac{\nu}{n+1}\right)u_\nu$.

Consider, for $k>0$, the expression

$$\sigma_{n,k}=\frac{1}{k}\left\{(n+k)\,\sigma_{n+k-1}-n\,\sigma_{n-1}\right\}= \tag{57}$$

$$=\frac{1}{k}\sum_{\nu=0}^{n+k}(n+k-\nu)\,u_\nu-\frac{1}{k}\sum_{\nu=0}^{n}(n-\nu)\,u_\nu=\sum_{\nu=0}^{n}u_\nu+\frac{1}{k}\sum_{\nu=n+1}^{n+k}(n+k-\nu)\,u_\nu.$$

Hence we obtain $\sigma_{n,k}$ by adding to s_n a linear combination of $k-1$ terms with indices exceeding n. A particularly simple form has the expression $\sigma_{n,n}=2\sigma_{2n-1}-\sigma_{n-1}$ [1]).

Suppose that σ_n tends to a finite limit g. If $n\to\infty$ and $k\geqslant n\varepsilon$, where ε is an arbitrary but fixed positive constant, then $\sigma_{n,k}\to g$.

Thus the formula (57) gives us a method of summation, at least as strong as the first arithmetic mean. The interest which this method may present is due to the fact that the first $n+1$ terms of the series (56) enter into $\sigma_{n,k}$ with coefficients equal to 1.

If $u_0+u_1+\ldots$ is the Fourier series of a function f, the expression (57) will be denoted by $\sigma_{n,k}(x,f)$, or by $\sigma_{n,k}[f]$.

It is well known that, in general, the partial sums $s_n(x,f)$ of the Fourier series of a function f do not represent the function very well. In particular, $s_n(x,f)$ may diverge at some points for f continuous, or diverge everywhere for f integrable. From the preceding remarks, however, it follows that if we complete $s_n(x,f)$ by a group of terms with indices exceeding n, we obtain an expression, $\sigma_{n,k}(x,f)$, with represents $f(x)$ at least as well as the Fejér

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[1]) The expressions $\sigma_{n,k}$ have already been considered (for a different purpose) by de la Vallée Poussin [1], 33 sqq.

means do. Thus, if f is continuous, then $\sigma_{n,k}(x,f)$ tends uniformly to $f(x)$. If f is only integrable, $\sigma_{n,k}(x,f)$ tends almost always to $f(x)$; moreover $\sigma_{n,k}(x,f)$ tends in mean to $f(x)$, etc. (we always assume that $k\geqslant n\varepsilon$).

The following remarks will be useful later:

$$\textit{If}\quad |f(x)|\leqslant M,\quad \textit{then}\quad |\sigma_{n,k}[f]|\leqslant\left(1+\frac{2n}{k}\right)M. \tag{58}$$

$$\textit{If}\quad \int_0^{2\pi}|f(x)|\,dx\leqslant M,\quad \textit{then}\quad \int_0^{2\pi}|\sigma_{n,k}[f]|\,dx\leqslant\left(1+\frac{2n}{k}\right)M. \tag{59}$$

These remarks follow from (57) and from the well known facts that, if $|f|\leqslant M$, then $|\sigma_n[f]|\leqslant M$, and if $\int_0^{2\pi}|f|\,dx\leqslant M$, then $\int_0^{2\pi}|\sigma_n[f]|\,dx\leqslant M$.

We are now in a position to prove Theorem 7. Suppose that $g(x)$ satisfies the conditions (40). We observe that

$$\int_0^{2\pi}S\,g\,dx=\int_0^{2\pi}S\,s_n[g]\,dx=\int_0^{2\pi}S\,\sigma_{n,k}[g]\,dx=\int_0^{2\pi}S\,\sigma_{n,k}[g]\,d\varphi_{2n+k}.$$

In view of (58), we obtain

$$\int_0^{2\pi}|S|\,dx\leqslant\left(1+\frac{2n}{k}\right)\int_0^{2\pi}|S|\,d\varphi_{2n+k},$$

that is the formula (16).

In order to prove (17), we notice that

$$\operatorname*{Max}_x|S(x)|=\operatorname{Sup}\int_0^{2\pi}S\,g\,dx,\qquad \text{where}\quad \int_0^{2\pi}|g|\,dx=1.$$

But we have again

$$\int_0^{2\pi}S\,g\,dx=\int_0^{2\pi}S\,s_n[g]\,dx=\int_0^{2\pi}S\,\sigma_{n,k}[g]\,dx=\int_0^{2\pi}S\,\sigma_{n,k}[g]\,d\varphi_{2n+k}.$$

Hence

$$\operatorname*{Max}_x|S(x)|\leqslant\operatorname*{Max}_\nu\left|S\left(\frac{2\pi\nu}{2n+k}\right)\right|\int_0^{2\pi}|\sigma_{n,k}[g]|\,d\varphi_{2n+k}.$$

$$\leqslant\left(1+\frac{2n}{k}\right)\operatorname*{Max}_\nu\left|S\left(\frac{2\pi\nu}{2n+k}\right)\right|,$$

which proves (17).

The inequalities (16) and (17) are limiting cases of the following inequality

$$\left(\int_0^{2\pi}|S|^p\,dx\right)^{1/p}\leqslant\left(1+\frac{2n}{k}\right)\left(\int_0^{2\pi}|S|^p\,d\varphi_{2n+k}\right)^{1/p}\quad (k=1,2,\ldots),$$

where p is any number $\geqslant 1$. The proof of this general inequality is similar to the proof of (16).

12. The following theorem on conjugate polynomials is an analogue of known theorems on conjugate functions [1]).

Theorem 9. *Let $S(\theta)$ be any trigonometrical polynomial of order n, and $\bar{S}(\theta)$ the conjugate polynomial. Then*

$$\text{(58)}\qquad \left(\int_0^{2\pi}|\bar{S}|^p\,d\varphi_{2n+1}(\theta)\right)^{1/p}\leqslant L_p\left(\int_0^{2\pi}|S|^p\,d\varphi_{2n+1}(\theta)\right)^{1/p}\qquad (p>1)$$

$$\text{(59)}\qquad \int_0^{2\pi}|\bar{S}|\,d\varphi_{2n+1}(\theta)\leqslant M\int_0^{2\pi}|S|\log^+|S|\,d\varphi_{2n+1}(\theta)+N$$

$$\text{(60)}\qquad \int_0^{2\pi}|\bar{S}|^\mu\,d\varphi_{2n+1}(\theta)\leqslant K_\mu\left(\int_0^{2\pi}|S|\,d\varphi_{2n+1}(\theta)\right)^\mu.$$

Here L_p depends exclusively on p, K_μ exclusively on μ, and M, N are absolute constants. The product $(1-\mu)K_\mu$ is bounded for $0<\mu<1$.

The inequality (58) is a consequence of (8) and of the inequality (5) applied to $\bar{S}$. We may therefore put $L_p=A\bar{B}_p$. Similarly (59) follows from the formulae (13) and (5). The inequality (60) would be a consequence of (15), if we could prove, for at least one value of μ, $0<\mu<1$, the inequality

$$\int_0^{2\pi}|S|^\mu\,d\varphi_{2n+1}\leqslant C\int_0^{2\pi}|S|^\mu\,dx.$$

We do not know whether the latter inequality is true, and so we must apply a different argument. Our proof will be similar to that of Theorem 6. By K_μ we shall denote throughout any function of μ such that $(1-\mu)\,K_\mu=O(1)$. Absolute constants will be denoted by $C, C_1, \ldots$

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[1]) See M. Riesz [1], or Zygmund [2], 147 sqq., where also full references are given.

We start from the formula (56). For $\theta_k\leqslant x<\theta_{k+1}$ it may be written

$$\text{(61)}\quad \bar{S}(x)=\frac{2}{2n+1}\sum_{\nu=k}^{k+1}S(\theta_\nu)\bar{D}_n(x-\theta_\nu)+\frac{2}{2n+1}\sum_{\nu=0}^{2n}{}' S(\theta_\nu)\bar{D}_n(x-\theta_\nu)=U(x)+V(x),$$

say. The dash $'$ will denote throughout that the terms with indices k and $k+1$ are omitted in summation. We observe that $\bar{D}_n(0)=0$, $|\bar{D}_n(u)|\leqslant n$. It follows that $|U(\theta_k)|\leqslant|S(\theta_{k+1})|$, and so

$$\text{(62)}\quad \int_0^{2\pi}|U(\theta)|^\mu\,d\varphi_{2n+1}\leqslant\int_0^{2\pi}|S(\theta)|^\mu\,d\varphi_{2n+1}\leqslant(2\pi)^{1-\mu}\left(\int_0^{2\pi}|S(\theta)|\,d\varphi_{2n+1}\right)^\mu.$$

From (61) and (62) we see that (60) will have been established, when we have shown that

$$\text{(63)}\qquad \int_0^{2\pi}|V(\theta)|^\mu\,d\varphi_{2n+1}\leqslant K_\mu\left(\int_0^{2\pi}|S(\theta)|\,d\varphi_{2n+1}\right)^\mu.$$

It is easy to verify that

$$V(x)=\frac{2}{2n+1}\sum_\nu{}'\frac{S(\theta_\nu)}{2\,\mathrm{tg}\,\frac12(x-\theta_\nu)}+\frac{1}{2n+1}\sum_\nu{}' S(\theta_\nu)\sin n(x-\theta_\nu)-$$
$$-\frac{2\cos nx}{2n+1}\sum{}'\frac{S(\theta_\nu)\cos n\theta_\nu}{2\,\mathrm{tg}\,\frac12(x-\theta_\nu)}+\frac{2\sin nx}{2n+1}\sum{}'\frac{S(\theta_\nu)\sin n\theta_\nu}{2\,\mathrm{tg}\,\frac12(x-\theta_\nu)}=$$
$$=V_0(x)+V_1(x)-\sin nx\,V_2(x)+\cos nx\,V_3(x),$$

say. The formula (63) is a consequence of the inequalities

$$\text{(64)}\qquad \int_0^{2\pi}|V_i(x)|^\mu\,d\varphi_{2n+1}\leqslant K_\mu\left(\int_0^{2\pi}|S|\,d\varphi_{2n+1}\right)^\mu,\qquad (i=0,1,2,3),$$

which we now intend to prove. The case $i=1$ is obvious. It is sufficient to prove one of the cases $i=0,2,3$, for the proofs of the remaining are similar.

Let us take, for example, $i=2$. We assume that $\theta_k\leqslant x<\theta_{k+1}$. We have, then, by turns

$$|V_2(\theta_k)|^\mu\leqslant|V_2(\theta_k)-V_2(x)|^\mu+|V_2(x)|^\mu,$$

$$\frac{2\pi}{2n+1}|V_2(\theta_k)|^\mu\leqslant\int_{\theta_k}^{\theta_{k+1}}|V_2(\theta_k)-V_2(x)|^\mu\,dx+\int_{\theta_k}^{\theta_{k+1}}|V_2(x)|^\mu\,dx,$$

$$(65)\quad \int_0^{2\pi} |V_2(\theta)|^\mu \, d\varphi_{2n+1} \leqslant \sum_{k=0}^{2n} \int_{\theta_k}^{\theta_{k+1}} |V_2(\theta_k) - V_2(x)|^\mu \, dx + \int_0^{2\pi} |V_2(x)|^\mu \, dx = A + B,$$

say. For $\theta_k \leqslant x < \theta_{k+1}$,

$$|V_2(\theta_k) - V_2(x)| \leqslant C \sum_{\nu=0}^{2n}{}' \frac{|S(\theta_\nu)|}{(k-\nu)^2}.$$

Hence

$$(66)\quad A \leqslant 2\pi C^\mu \sum_{k=0}^{2n} \frac{1}{2n+1} \left(\sum_\nu{}' \frac{|S(\theta_\nu)|}{(k-\nu)^2} \right)^\mu \leqslant$$

$$\leqslant 2\pi C^\mu \left(\sum_{k=0}^{2n} \frac{1}{2n+1} \sum_\nu{}' \frac{|S(\theta_\nu)|}{(k-\nu)^2} \right)^\mu \leqslant$$

$$\leqslant C_1 \left(\frac{1}{2n+1} \sum_{\nu=0}^{2n} |S(\theta_\nu)| \right)^\mu \leqslant C_1 \left(\int_0^{2\pi} |S| \, d\varphi_{2n+1} \right)^\mu.$$

Let us now consider the function $I(x)$ appearing in § 10. We have

$$I(x) = V_2(x) + \frac{2}{2n+1} \sum_{\nu=k}^{k+1} \frac{S(\theta_\nu)}{2 \operatorname{tg} \frac{1}{2}(x-\theta_\nu)} = V_2(x) + W(x),$$

say. It is not difficult to verify that

$$\int_0^{2\pi} |W(x)|^\mu \, dx \leqslant K_\mu \left(\int_0^{2\pi} |S| \, d\varphi_{2n+1} \right)^\mu.$$

From this, the inequality $|V_2|^\mu \leqslant |I|^\mu + |W|^\mu$, and the inequality (45) we conclude that

$$(67)\quad B = \int_0^{2\pi} |V_2(x)|^\mu \, dx \leqslant K_\mu \left(\int_0^{2\pi} |S| \, d\varphi_{2n+1} \right).$$

The relations (65), (66), and (67) prove (64) in the case $i=2$. This completes the proof of Theorem 9.

13. The theorems which we have established for trigonometrical polynomials have their analogues for ordinary polynomials

$$(68)\quad P(z) = c_0 + c_1 z + \ldots + c_n z^n.$$

For example,

Theorem 10. *Every polynomial $P(z)$ of the form* (68) *satisfies the following inequalities*:

$$(69)\quad \left(\int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \right)^{1/p} \leqslant B_p^* \left(\int_0^{2\pi} |P(e^{i\theta})|^p \, d\varphi_{n+1}(\theta) \right)^{1/p} \qquad (p>1),$$

$$(70)\quad \int_0^{2\pi} |P(e^{i\theta})| \, d\theta \leqslant M^* \int_0^{2\pi} |P(e^{i\theta})| \log^+ |P(e^{i\theta})| \, d\varphi_{n+1}(\theta) + N^*,$$

$$(71)\quad \int_0^{2\pi} |P(e^{i\theta})|^\mu \, d\mu \leqslant K_\mu^* \left(\int_0^{2\pi} |P(e^{i\theta})| \, d\varphi_{n+1}(\theta) \right)^\mu \qquad (0<\mu<1).$$

The inequality (69) corresponds to the inequality (6), and may be easily deduced from the latter if $n=2k$ is an even integer. For then

$$|P(e^{i\theta})| = \left| \sum_{\nu=0}^{n} c_\nu e^{i\nu\theta} \right| = \left| \sum_{\nu=-k}^{k} c_{\nu+k} e^{i\nu\theta} \right| = |S(\theta)|,$$

where $S(\theta) = c_0 e^{-ik\theta} + \ldots + c_n e^{ik\theta}$ is a trigonometrical polynomial (not necessarily real) of order k. The formula (69) follows from the inequality

$$(72)\quad \left(\int_0^{2\pi} |S(\theta)|^p \, d\theta \right)^{1/p} \leqslant B_p \left(\int_0^{2\pi} |S(\theta)|^p \, d\varphi_{2k+1} \right)^{1/p},$$

and in the case considered we may put $B_p^* = B_p$.

It is plain that in (72) we may replace φ_{2k+1} by φ_{2k+2}.

If $n=2k+1$ is an add integer, the argument is a little more complicated and gives a bigger value for B_p^*. We write

$$P(z) = a_0 + zQ(z)$$

where $Q(z)$ is a polynomial of order $2k$. Now,

$$\left(\frac{1}{2\pi}\int_0^{2\pi}|P|^p d\theta\right)^{1/p} \leqslant |a_0| + \left(\frac{1}{2\pi}\int_0^{2\pi}|Q|^p d\theta\right)^{1/p}$$

$$\leqslant |a_0| + B_p\left(\frac{1}{2\pi}\int_0^{2\pi}|Q|^p d\varphi_{2k+2}\right)^{1/p}$$

$$= |a_0| + B_p\left(\frac{1}{2\pi}\int_0^{2\pi}|Q|^p d\varphi_{n+1}\right)^{1/p}$$

$$\leqslant (1+B_p)\,|a_0| + B_p\left(\frac{1}{2\pi}\int_0^{2\pi}|P|^p d\varphi_{n+1}\right)^{1/p}.$$

On the other hand,

$$|a_0| = \left|\frac{1}{2\pi}\int_0^{2\pi}P(e^{i\theta})\,d\varphi_{n+1}\right| \leqslant \left(\frac{1}{2\pi}\int_0^{2\pi}|P|^p d\varphi_{n+1}\right)^{1/p}.$$

This and the previous inequalities give us (69), with $B_p^* = 2B_p+1$.

The proof of (70) is similar to that of (69), if we observe that the function $\chi(u) = u\log^+ u$ $(u \geqslant 0)$ is (i) convex, (ii) non-decreasing so that

$$\chi(a+b) \leqslant \chi(2a) + \chi(2b) \qquad (a, b \geqslant 0),$$

and (iii) satisfies the inequality $u \leqslant \chi(u) + e$.

Applying Jensen's inequality, we deduce that

$$\chi(|a_0|) \leqslant \frac{1}{2\pi}\int_0^{2\pi}\chi(|P(e^{i\theta})|)\,d\varphi_{n+1}.$$

The details of the proof we leave to the reader.

In the proof of (71) we use the inequality $(a+b)^\mu \leqslant a^\mu + b^\mu$, if $a \geqslant 0$, $b \geqslant 0$, $0 < \mu < 1$.

Theorem 7 can also be extended to the case of ordinary polynomials. If $P(z)$ is of the form (68), then (cf. (16))

$$\int_0^{2\pi}|P(e^{i\theta})|\,d\theta \leqslant A_{n,k}\int_0^{2\pi}|P(e^{i\theta})|\,d\varphi_{n+k} \qquad (k \geqslant 1), \tag{72}$$

where $A_{n,k}$ is bounded if the ratio n/k is bounded. If n is even, we may put $A_{n,k} = \left(1+\frac{n}{k}\right)$; otherwise, $A_{n,k} = \left(1+\frac{n+1}{k-1}\right)$ for $(k \geqslant 2)$. (That we may put $A_{n,1} = n+1$ is obvious.) A similar extension holds for (17).

14. We conclude this chapter by a few remarks on the coefficients of trigonometrical polynomials.

Let $\dots c_{-1}, c_0, c_1, \dots$ be the complex Fourier coefficients of a function $f(\theta)$. In the theory of trigonometrical series there is a number of theorems connecting the values of the sums $\sum |c_\nu|^\alpha$ with those of the integrals $\int_0^{2\pi}|f|^\beta\,d\theta$, where α and β are certain constants. These results hold, of course, when f is a trigonometrical polynomial S, of order n. From the theorems established in this chapter, and in particular from Theorems 1 and 2, we see that in this special case we may roughly speaking, replace the integral $\int_0^{2\pi}|f|^\beta d\theta$ by the integral $\int_0^{2\pi}|f|^\beta\,d\varphi_m(\theta)$, where is m a function of n.

In this passage, however, generally something is lost, and the results obtained in this way are less precise than the original ones. To take an example, let us suppose that the Fourier series of f is a power series in $e^{i\theta}$. It is well known that then

$$\sum_{\nu=1}^{\infty}\frac{|c_\nu|}{\nu} \leqslant \frac{1}{2}\int_0^{2\pi}|f(\theta)|\,d\theta, \tag{73}$$

and that the factor $\frac{1}{2}$ on the right cannot be replaced by anything smaller [1]).

Let us now assume that $f(\theta) = P(e^{i\theta})$, where $P(z)$ is of the form (68). From (73) and from the formula (72) where we take, for example $k=n$, we obtain at once

$$\sum_{\nu=1}^{n}\frac{|c_\nu|}{\nu} \leqslant A\int_0^{2\pi}|P(e^{i\theta})|\,d\varphi_{2n}(\theta),$$

where A is an absolute constant. But what is the least possible value of this constant. we are unable to decide.

A similar situation occurs in most cases. One of the few exceptions is the following proposition, which is an analogue of the very well known Hausdorff-Young inequalities.

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[1]) Hardy and Littlewood [1], Fejér [1]; Zygmund [2], p. 158.

Theorem 11. *If* $S(\theta)=\sum_{\nu=-n}^{n}{}' c_\nu e^{i\nu\theta}$ *is a trigonometrical polynomial of order* n, *and if* $1\leqslant p\leqslant 2$, $p'=p/(p-1)$, *then*

$$\left(\frac{1}{2\pi}\int_0^{2\pi}|S(\theta)|^{p'}d\varphi_{2n+1}\right)^{1/p'}\leqslant\left(\sum_{\nu=-n}^{n}{}'|c_\nu|^p\right)^{1/p},$$

$$\left(\sum_{\nu=-n}^{n}{}'|c_\nu|^{p'}\right)^{1/p'}\leqslant\left(\frac{1}{2\pi}\int_0^{2\pi}|S(\theta)|^{p}d\varphi_{2n+1}\right)^{1/p}.$$

The proof is exactly the same as that of the Hausdorff-Young theorem.

The first inequality is obvious for $p=1$ and $p=2$, and the general case follows by an application of M. Riesz's convexity theorem [1]). The same argument applies to the second inequality (which may also be deduced from the first) if we observe that the values $S(\theta)$ takes at the points of discontinuity of the function φ_{2n+1} are quite arbitrary, and define the polynomial uniquely.

CHAPTER II.

1. Let $f(t)$ be a function integrable L and of period 2π. Let $D_n(t)$ denote Dirichlet's kernel. The expression

$$I_n(x,f)=I_n[f]=\frac{1}{\pi}\int_0^{2\pi}f(t)\,D_n(x-t)\,d\varphi_{2n+1}(t) \tag{1}$$

represents the trigonometrical polynomial of order $\leqslant n$ which at the points $x_\nu=2\pi\nu/(2n+1)$ $(\nu=0,1,\ldots,2n)$ takes the same values as the function f. (This follows *e. g.* from the fact that $D(x_0)=n+\frac{1}{2}$, $D(x_\nu)=0$ for $\nu>0$). Besides (1) we shall consider the expression

$$I_{n,u}(x,f)=I_{n,u}[f]=\frac{1}{\pi}\int_0^{2\pi}f(t)\,D_n(x-t)\,d\varphi_{2n+1}(t-u), \tag{2}$$

where u is a parameter. This expression is the trigonometrical polynomial, of order $\leqslant n$, which takes the values $f(x_\nu+u)$ at the points $x_\nu+u$.

---

[1]) M. Riesz [2]; Zygmund [2], p. 190 sqq.

From (1) and (2) we see that

$$I_{n,u}(x,f)=\frac{1}{\pi}\int_0^{2\pi}f(t+u)\,D_n(x-u-t)\,d\varphi_{2n+1}(t)=I_n(x-u,g), \tag{2a}$$

where $g(t)=g_u(t)=f(t+u)$ [1]).

In this chapter we shall prove a few theorems about the behaviour of the expressions $I_{n,u}(x,f)$ for different values of u.

Theorem 1. (i) *If* $f\epsilon L^p$, $p>1$, *then, for* $n\to\infty$,

$$\int_0^{2\pi}\int_0^{2\pi}|I_{n,u}(x,f)-f(x)|^p\,dx\,du\to 0. \tag{3}$$

(ii) *If* $|f|\log^+|f|\,\epsilon\, L$, *we have* (3) *with* $p=1$.

(iii) *If* $f\epsilon L$, *then*

$$\int_0^{2\pi}\left[\int_0^{2\pi}|I_{n,u}(x,u)-f(x)|^\mu dx\right]^{1/\mu}du\to 0\quad \textit{for every}\quad 0<\mu<1. \tag{4}$$

For the proof of part (i), we shall need the inequality

$$\int_0^{2\pi}\int_0^{2\pi}|I_{n,u}(x,f)|^p\,du\,dx\leqslant 2\pi B_p^p\int_0^{2\pi}|f(x)|^p\,dx, \tag{5}$$

where B_p is the constant of Theorem 2 of Chapter I. For

$$\int_0^{2\pi}|I_{n,u}(x,f)|^p\,dx\leqslant B_p^p\int_0^{2\pi}|I_{n,u}(x,f)|^p\,d\varphi_{2n+1}(x-u)=B_p^p\int_0^{2\pi}|f(x+u)|^p d\varphi_{2n+1}(x).$$

Integrating this inequality with respect to u, and observing that the last integral is but a finite sum we obtain (5).

Now, in order to prove (3), we write $f=f_1+f_2$, where f is a trigonometrical polynomial, and

$$\int_0^{2\pi}|f_2(x)|^p\,dx<\varepsilon.$$

Then

$$I_{n,u}[f]-f=I_{n,u}[f_1]-f_1+I_{n,u}[f_2]-f_2=I_{n,u}[f_2]-f_2, \tag{6}$$

---

[1]) The function $f(x)$ need not be defined at every point. If f is defined for almost every x, the polynomials $I_{n,u}[f]$ are defined for almost all u.

provided n exceeds the order of the polynomial f_1. For such n, the left hand side of (3) is equal to

$$(7)\qquad \int_0^{2\pi}\int_0^{2\pi} |I_{n,u}[f_2]-f_2(x)|^p\,dx\,du \leqslant$$

$$\leqslant 2^p \int_0^{2\pi}\int_0^{2\pi} |I_{n,u}[f_2]^p\,dx\,du + 2^p\cdot 2\pi \int_0^{2\pi} |f_2(x)|^p\,dx \leqslant 2^p\cdot 2\pi\,(B_p^p+1)\,\varepsilon.$$

Since ε may be arbitrarily small, part (i) of Theorem 2 follows.

The proof of part (ii) is analogous. We start with the inequality

$$\int_0^{2\pi}\int_0^{2\pi} |I_{n,u}(x,f)|\,dx\,du \leqslant 2\pi A \int_0^{2\pi} |f|\log^+|f|\,dx + 2\pi B,$$

where A and B are taken from Theorem 5 of Chapter I. Applying this inequality to kf, where k is an arbitrary positive constant, we obtain

$$\int_0^{2\pi}\int_0^{2\pi} |I_{n,u}(x,f)|\,dx\,du \leqslant 2\pi A \int_0^{2\pi}\int_0^{2\pi} |f|\log^+|kf|\,dx + 2\pi B/k.$$

We take k so large that $2\pi B/k<\varepsilon$, and then, having fixed k, we write $f=f_1+f_2$, where f_1 is a trigonometrical polynomial, and f_2 is such that

$$\int_0^{2\pi} |f_2|\,dx < \varepsilon \qquad \text{and} \qquad \int_0^{2\pi} |f_2|\log^+|kf_2|\,dx < \varepsilon \;\;{}^{1}).$$

Arguing as in (6) and (7), we obtain

$$\int_0^{2\pi}\int_0^{2\pi} |I_{n,u}[f]-f(x)|\,dx\,du \leqslant 2\pi A\,\varepsilon + \varepsilon + 2\pi\,\varepsilon,$$

which proves part (ii) of Theorem 1.

The proof of part (iii) we leave to the reader, observing only that the place of (5) will be taken by the inequality

$$\int_0^{2\pi}\Big[\int_0^{2\pi} |I_{n,u}[f]|^\mu\,dx\Big]^{1/\mu} du \leqslant 2\pi\, C^{1/\mu} \int_0^{2\pi} |f(x)|\,dx,$$

which follows from the inequality (14) of Chapter I.

[1]) For f_1 we may take a Fejér mean, with index sufficiently high, of the Fourier series of f. Cf. e. g. Zygmund [2], p. 83 sqq.

2. Theorem 2. *For every $f\in L^p$, $p>1$, there is a sequence of integers $n_1<n_2<\dots$, such that, for almost every u,*

$$(8)\qquad \int_0^{2\pi} |I_{n_k,u}[f]-f(x)|^p\,dx \to 0 \qquad (k\to\infty).$$

Similarly if $|f|\log^+|f|\in L$, we have (8) with $p=1$. If $f\in L$, the relation (8) holds for every $0<p<1$.

We shall restrict ourselves to the case of $f\in L^p$. For the proof it is sufficient to observe that the sequence of functions

$$\chi_n(u) = \int_0^{2\pi} |I_{n,u}[f]-f(x)|^p\,dx$$

tends to 0 in the mean. Hence there is a sequence $\{n_k\}$ such that $\chi_{n_k}(u)\to 0$ for almost all u.

It follows in particular that, for almost all u,

$$\varliminf_{n\to\infty} \int_0^{2\pi} |I_{n,u}[f]-f(x)|^p\,dx = 0 \;\;{}^{1}).$$

3. We shall see later that we cannot replace here $\varliminf$ by lim. For the proof of this fact we shall use the following

Theorem 3. *There is a function F integrable L, of period 1 and such that for no value of u does the expression*

$$\xi_n(u,F) = \frac{1}{n}\sum_{i=0}^{n-1} F\Big(u+\frac{i}{n}\Big)$$

tend to the integral

$$(9)\qquad \int_0^1 F(x)\,dx.$$

In order to grasp the meaning of the theorem we observe that

(a) If F is continuous, then $\xi_n(u,F)$ tends to the value (9) for all u, even uniformly in u.

(b) If $F\in L$, and n_k is any increasing sequence of indices such that n_{k+1} is divisible by n_k $(k=1,2,\dots)$, then $\xi_{n_k}(u,F)$ tends to (9) for almost all u [2]).

[1]) It may be shown that for the sequence $\{n_k\}$ of Theorem 2 we may take a *fixed* sequence, independent of f. For example, we may put $n_k=\frac{1}{2}(3^k-1)$. To this problem we shall return on another occasion.

[2]) Jessen [1]. Theorem 3 answers the problem propounded by Jessen, *loc. cit.*

We define the function $F(x)$ by the following conditions

$$(10)\qquad \begin{cases} \text{(i)} & F(u+1)=F(u) \\ \text{(ii)} & F(u)=|u|^{-\frac{1}{2}}\log\dfrac{1}{|u|} \qquad \text{for} \quad |u|\leqslant\dfrac{1}{2}. \end{cases}$$

In the proof of the theorem we shall use the following well-known fact from the theory of continued fractions: for every irrational u there is a sequence of fractions $\{p_i/q_i\}_{i=1,2,\ldots}$ such that $q_1<q_2<\ldots$, and that

$$(11)\qquad \left|u-\frac{p_i}{q_i}\right|<\frac{1}{q_i^2} \qquad (i=1,2,\ldots).$$

From (10) we see that $F(u)>2q_i\log q_i$ for $|u|<q_i^{-2}$. From this and from (11) it follows that

$$\xi_{q_i}(u,F)>2\log q_i.$$

Hence, if u is irrational, then

$$(12)\qquad \overline{\lim_{q\to\infty}}\,\xi_q(u,F)=+\infty.$$

If we wish to have this relation satisfied also for all rational u, we may proceed as follows. Let $r_1, r_2, \ldots$ be the sequence of all the rational points of the interval $(0,1)$. Modifying the definition of F at the points r_n, we put $F(r_n)=n$. In particular, the new F is finite everywhere. Then (12) holds for every u, and Theorem 3 is established.

The function F defined above belongs to the class L^p for every $p<2$. The problem whether there exists a function F satisfying the conditions of Theorem 3 and integrable in a power $\geqslant 2$ (in particular a bounded function), remains open.

Theorem 3 is not entirely sufficient for our purposes. We shall now prove

4. ***Theorem 3'.*** *There is an integrable function F, of period 1, and such that*

$$\overline{\lim_{n\to\infty}}\,\xi_{2n+1}(u,F)=\infty$$

for almost all u.

We shall show first that in the interval $(0,1)$ there is a set H, of positive measure, with the following property. For every $u\in H$ there is a sequence of fractions, $\{p_i/q_i\}_{i=1,2,\ldots}$ *with odd denominators*, and such that

$$(13)\qquad \left|u-\frac{p_i}{q_i}\right|<\frac{4}{q_i^2} \qquad (i=1,2,\ldots).$$

For suppose that the set H of points satisfying the above condition is of measure 0. Hence, in view of the theorem (stated in the proof of Theorem 3) on the approximation by rational numbers, there is a set H_1 of measure 1, contained in the interior of the interval $(0,1)$, and such that for every $u\in H_1$ we have (11), where the q_i are even. Let H_2 denote the set H_1 translated to the interval $(1,2)$. Then, if $v\in H_2$, there is a sequence of fractions $r_i/2s_i$ satisfying the inequalities

$$(14)\qquad \left|v-\frac{r_i}{2s_i}\right|<\frac{1}{4s_i^2}.$$

We may plainly suppose that the numbers r_i are odd and that $1<r_i/2s_i<2$. Let H_3 be the set obtained from H_2 by the transformation $u=1/v$. From (15) we obtain

$$\left|u-\frac{2s_i}{r_i}\right|<\frac{1}{4s_i^2},$$

and since $r_i<4s_i$, our assertion is established.

Let H^* denote the set of points $u+\dfrac{\alpha}{\beta}$, where $u\in H$ and α/β are arbitrary fractions with odd denominators. Since H is of positive measure, almost every point of the interval $(0,1)$ belongs to H^*. To every $u\in H^*$ corresponds a sequence of fractions, $\{p_i/q_i\}_{i=1,2,\ldots}$ with odd denominators, satisfying the inequality

$$\left|u-\frac{p_i}{q_i}\right|<\frac{4\beta^2}{q_i^2} \qquad (i=1,2,\ldots).$$

Here β depends on u only. It is easy to see that the function $F(u)$ defined by (10) fulfils Theorem 3'. The proof is the same as that of Theorem 3.

5. The argument used in the proof of Theorem 3′ yields a slightly more general result.

Let $\varphi(n)$ be a decreasing sequence of positive numbers satisfying the following wo conditions:

(i) *For every integral $\beta > 0$, $\varphi(n) = O(\varphi(\beta n))$.*

(ii) *For almost every real θ there is a sequence of fractions $\{p_i/q_i\}_{i=1,2,\dots}$ $(q_1 < q_2 < \dots)$ such that*

$$\left|\theta - \frac{p_i}{q_i}\right| < \varphi(q_i). \tag{15}$$

Let, moreover, $ax+b$ be an arbitrary linear form with integral coefficients. Then, to almost every real θ corresponds a sequence of fractions p_i/q_i satisfying the relation

$$\left|\theta - \frac{p_i}{q_i}\right| = O(\varphi(q_i))$$

and having the denominators q_i of the form

$$q_i = ax_i + b \qquad (i = 1, 2, \dots; \quad x_i \text{ integral}).$$

We may assume that $b \neq 0$. Let $K(\lambda, \mu)$ denote the set of irreducible fractions of the form

$$\frac{ax+\lambda}{ay+\mu}, \tag{16}$$

where x and y are arbitrary integers, λ is one of the numbers $0, 1, \dots, a-1$, and μ one of the numbers $1, 2, \dots, a-1$. The number of the sets $K(\lambda, \mu)$ is finite. Hence there exist two integers λ_0, μ_0, and a set S of positive measure, such that for every $\theta \in S$ we have (15), where the irreducible fractions p_i/q_i are of the form (16), with $\lambda = \lambda_0$, $\mu = \mu_0$. If is plain that the greatest common divisor of the numbers λ_0, μ_0, a is equal to 1, for otherwise the fractions $(ax+\lambda_0)/(ay+\mu_0)$ would be reducible, contrary to assumption. It follows that there exist three integers α, β, γ such that

$$\alpha\mu_0 + \beta\lambda_0 + \gamma a = 1.$$

The fractions $\dfrac{\alpha}{\beta} + \dfrac{ax+\lambda_0}{ay+\mu_0}$ are therefore of the form $\dfrac{ax+1}{ay\beta+\mu_0\beta}$, and in a set of positive measure give approximation $\leqslant \varphi(ay+\mu_0) = O(\varphi(ay\beta+\mu_0\beta))$. By inversion we prove that the points of a set S_1 of positive measure can be approximated to by fractions $\dfrac{p}{ax+1}$ with an error $O(\varphi(ax+1))$.

The fractions r/s, where s is of the form $am+1$, are everywhere dense. It follows that the fractions

$$\frac{r}{s} + \frac{p}{ax+1}$$

which are also of the form $p/(ax+1)$, approximate to almost every θ with an error $O(\varphi(ax+1))$. Hence the fractions

$$\frac{1}{b} + \frac{p}{ax+1} = \frac{p'}{ay+b}$$

approximate to almost every θ with an error $O(\varphi(ay+b))$. This completes the proof.

6. The following proposition completes Theorem 2.

Theorem 4. *For every $p \geqslant 1$, there is an $f \in L^p$ such that*

$$\overline{\lim_{n\to\infty}} \int_0^{2\pi} |I_{n,u}[f]|^p\, dx = \infty \quad \textit{for almost all } u.$$

Let $f = F^{1/p}$, where F is the function of Theorem 3′, but constructed on the interval $(0, 2\pi)$ instead of $(0, 1)$. Then $f \in L^p$. In view of Theorem 1 of Chapter I,

$$\int_0^{2\pi} |I_{n,u}[f]|^p\, dx \geqslant \frac{1}{A} \int_0^{2\pi} |I_{n,u}[f]|^p\, d\varphi_{2n+1}(x-u) =$$

$$= \frac{1}{A} \int_0^{2\pi} |f(x+u)|^p\, d\varphi_{2n+1}(x) = \frac{1}{A} \int_0^{2\pi} F(x+u)\, d\varphi_{2n+1}(x).$$

It is now sufficient to apply Theorem 3′.

7. ***Theorem 5.*** *There is an integrable function $f(x)$ of period 2π such that the sequence of the polynomials $I_{n,u}[f]$ diverges at almost all points x, u of the square $0 \leqslant x \leqslant 2\pi$, $0 \leqslant u \leqslant 2\pi$.*

We put

$$f(x) = |x|^{-3/4} \qquad \textit{for} \quad |x| \leqslant \pi.$$

This function has only one point where it is infinite. In every interval not containing that point, $f(x)$ is of bounded variation.

In virtue of Theorem 3′, for almost every u there is a sequence of fractions p_i/q_i, with q_i odd, such that

$$\left|u + 2\pi\frac{p_i}{q_i}\right| \leqslant \frac{C}{q_i^2}. \tag{17}$$

Here C is a constant which depends only on u. We fix such a u, and write $x' = x - u$. The function $\sin\frac{1}{2}(2n+1)t$ vanishes at the points of discontinuity of $\varphi_{2n+1}(t)$. Hence from the formula (2a) we obtain

$$I_{\frac{1}{2}(q_i-1),u}(x, f) = \sum_{\nu=0}^{q_i-1} \frac{\sin\frac{1}{2}q_i x'}{q_i} (-1)^\nu f(u+t_\nu) \frac{1}{\sin\frac{1}{2}(x'-t_\nu)}, \tag{18}$$

where $t_\nu = 2\pi\nu/q_i$. This formula is valid if x does not coincide with any of the points t_ν.

In particular we suppose that $0<x<2\pi$. We also suppose that $0<x'<2\pi$. We can therefore find a $\delta>0$ such that

(19) $$0<2\delta\leqslant x\leqslant 2\pi-2\delta<2\pi, \qquad \delta\leqslant x'\leqslant 2\pi-\delta.$$

The sum in (18) we split into three sums

(20) $$I_{\frac{1}{2}(q_i-1,u)}(x,f)=\sum{}'+\sum{}''+\sum{}''',$$

where $\sum', \sum'', \sum'''$ correspond respectively to the points t_ν belonging to the intervals $(0, x'-\delta)$, $(x'-\delta, x'+\delta)$, $(x'+\delta, 2\pi)$. The sum $\sum''$ may be considered as Lagrange's interpolating polynomial formed for the function $g(t)$ which is equal to $f(u+t)$ in the interval $x'-\delta\leqslant t\leqslant x'+\delta$, and to 0 elsewhere (mod 2π). When t runs through the interval $(x'-\delta, x'+\delta)$, the function $g(t)$ takes on the same values as the function $f(x+\tau)$ in the interval $-\delta\leqslant\tau\leqslant\delta$. Hence, in view of (19), $g(t)$ is of bounded variation, and, by a well-known theorem [1]),

(21) $$\sum{}''=O(1).$$

The term $\nu=p_i$ of the sum (18) belongs either to $\sum'$ or to $\sum'''$, e. g. to the former. From (17) we see that the absolute value of this term exceeds

$$\frac{|\sin\frac{1}{2}q_i x'|}{C^{3/4}}q_i^{1/2}.$$

From (17) we also see that, for $k=1, 2, \ldots$,

$$\left|u+\frac{2\pi(p_i\pm k)}{q_i}\right|\geqslant\frac{2\pi k}{q_i}-\frac{C}{q_i^2}>\frac{k}{q_i},$$

provided i is large enough. Hence

(22) $$\left|\sum{}'\right|\geqslant|\sin\tfrac{1}{2}q_i x'|\left\{\frac{q_i^{1/2}}{C^{3/4}}-\frac{1}{q_i\sin\frac{1}{2}\delta}\sum_{k=1}^{q_i}\left(\frac{q_i}{k}\right)^{3/4}\right\}\geqslant\frac{q_i^{1/2}}{2C^{3/4}}|\sin\tfrac{1}{2}q_i x'|,$$

for $i\geqslant i_0$. Similarly

(23) $$\left|\sum{}'''\right|\leqslant|\sin\tfrac{1}{2}q_i x'|\frac{1}{q_i\sin\frac{1}{2}\delta}\sum_{k=1}^{q_i}\left(\frac{q_i}{k}\right)^{3/4}=O(1).$$

[1]) See e. g. Jackson [1].

It is well-known that

$$\overline{\lim_{i\to\infty}}|\sin\tfrac{1}{2}q_i x'|=1$$

for almost all x', that is for almost all x [1]). From this and from the formulae (20), (21), (22), (23) we infer that

$$\overline{\lim_{i\to\infty}}|I_{\frac{1}{2}(q_i-1),u}(x,f)|=\infty.$$

for almost all x. This completes the proof of Theorem 5.

8. Besides the polynomials I_n, we may consider Jackson's polynomials

$$J_n(x,f)=J_n[f]=\int_0^{2\pi}f(t)\,K_n(x-t)\,d\varphi_{n+1}(t),$$

where

$$K_n(t)=\frac{1}{2(n+1)}\left(\frac{\sin(n+1)\frac{t}{2}}{\sin\frac{t}{2}}\right)^2$$

is Fejér's kernel. The polynomial J_n takes at the points of discontinuity of φ_{n+1} the same values as the function f, but, besides, the derivative J_n' vanishes at those points. It is well-known that, if f is continuous, the $J_n[f]$ tend uniformly to f, a property which is not shared by the polynomials $I_n[f]$. If is therefore not unnatural to expect, that also in the case of f discontinuous the behaviour of the J_n is definitely much better than that of the I_n. But this is not so.

Let

$$J_{n,u}(x,f)=J_{n,u}[f]=\int_0^{2\pi}f(t)K_n(x-t)\,d\varphi_{n+1}(t-u).$$

Theorems 1 and 2 of this chapter remain valid if we replace in them I_n by J_n. The new results are rather trivial and follow from the most elementary properties of Fejér's kernel (Unlike in the case of the I_n, Theorem 2 of Chapter I is not necessary for the proof). We shall however prove that also the negative results of Theorem 4 and 5 hold for the J_n.

[1]) Cf. e. g. Zygmund [2], p. 269.

Theorem 4'. *For every $p \geqslant 1$, there is an $f \epsilon L^p$, such that*

$$\overline{\lim_{n\to\infty}} \int_0^{2\pi} |J_{n,u}[f]|^p\, dx = \infty \tag{24}$$

for all u.

The proof is similar to that of Theorem 4. Let $f = F^{1/p}$, where F is the function defined in § 4, but constructed on $(0, 2\pi)$. Then

$$\int_0^{2\pi} |J_{n,u}[f]|^p\, dx \geqslant \frac{1}{A} \int_0^{2\pi} |J_{n,u}[f]|^p\, d\varphi_{n+1}(x-u) =$$

$$= \frac{1}{A} \int_0^{2\pi} |f(x+u)|^p\, d\varphi_{n+1}(x) = \frac{1}{A} \int_0^{2\pi} F(x+u)\, d\varphi_{n+1}(x),$$

which proves (24).

Theorem 5'. *There is an integrable f such that the sequence $\{J_{n,u}[f]\}_{n=1,2,\dots}$ diverges at almost all points of the square $0 \leqslant x \leqslant 2\pi$, $0 \leqslant u \leqslant 2\pi$.*

This theorem corresponds to Theorem 5, but its proof is simpler than that of Theorem 5. We only note that the terms in Jackson's sums will be positive, and so the whole sum will be large if at least one of the terms is large.

9. The following proposition generalizes Lemma γ of Chapter I.

Theorem 6. *Let*

$$f(x) \sim \sum_{s=-\infty}^{\infty} \gamma_s e^{isx}, \tag{25}$$

and suppose for simplicity that f is real, that is $\gamma_{-s} = \bar{\gamma}_s$. Suppose, moreover, that the series $\sum_{\lambda=1}^{\infty} \frac{|\gamma_\lambda|}{\lambda}$ converges. (This is certainly true if $f \epsilon L^p$, where $p > 1$). Then, if $s_n(x)$ denotes the n-th partial sum of the Fourier series of f, and I is any interval, whose length will be denoted by $|I|$, then

$$\lim_{n\to\infty} \left\{ \frac{1}{|I|} \int_I I_{n,u}(x, f)\, du - s_n(x) \right\} = 0. \tag{26}$$

This shows that if the expression $I_{n,u}(x, f)$ tends uniformly to a limit g, when u remains within I, the Fourier series of f converges at the point x to the value g.

The formula (2) gives

$$I_{n,u}(x, f) = \sum_{k=-n}^{n} \gamma_k^{(n)}(u)\, e^{ikx}, \tag{27}$$

where

$$\gamma_k^{(n)}(u) = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, e^{-ikt}\, d\varphi_{2n+1}(t-u) = \frac{1}{2\pi} \int_0^{2\pi} f(t+u)\, e^{-ik(t+u)}\, d\varphi_{2n+1}(t). \tag{28}$$

Since, in view of (25),

$$f(t+u)\, e^{-ik(t+u)} \sim \sum_{s=-\infty}^{\infty} \gamma_s\, e^{i(s-k)(t+u)},$$

the formula (28) yields the well known relation

$$\gamma_k^{(n)}(u) \sim \sum_{\nu=-\infty}^{\infty} \gamma_{(2n+1)\nu+k}\, e^{i(2n+1)\nu u}. \tag{29}$$

Integrating this over I, we obtain

$$\int_I \gamma_k^{(n)}(u)\, du = \gamma_k |I| + 2\theta \sum_\nu{}' \left| \frac{\gamma_{(2n+1)\nu+k}}{(2n+1)\nu} \right| \tag{30}$$

$$= \gamma_k |I| + 4\theta_1 \sum_\nu{}' \left| \frac{\gamma_{(2n+1)\nu+k}}{(2n+1)\nu+k} \right|,$$

where θ and θ_1 do not exceed 1 in absolute value, and the dash $'$ signifies that the term $\nu = 0$ is omitted in summation. From (27) and (30) we deduce the inequality

$$\left| \frac{1}{|I|} \int_I I_{n,u}(x, f)\, du - s_n(x) \right| \leqslant \frac{8}{|I|} \sum_{\lambda=n+1}^{\infty} \frac{|\gamma_\lambda|}{\lambda},$$

which proves (26).

10. We conclude the chapter by a few minor remarks.

Let the complex Fourier coefficients of an $f \epsilon L^2$ be γ_s. Integrating the formula

$$\frac{1}{2\pi} \int_0^{2\pi} |I_{n,u}[f]|^2\, dx = \sum_{k=-n}^{n} |\gamma_k^{(n)}(u)|^2$$

with respect to u and taking account of (27) and (29), we obtain

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} |I_{n,u}[f]|^2\, du\, dx = \sum_{\lambda=-\infty}^{\infty} |\gamma_\lambda|^2.$$

If $s_n(x)$ is the n-th partial sum of the Fourier series of f, then, observing that $I_{n,u}[f]-s_n=I_{n,u}[f-s_n]$, we have

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}|I_{n,u}(x,f)-s_n(x)|^2\,du\,dx=2\sum_{\lambda=n+1}^{\infty}{}' |\gamma_\lambda|^2.$$

Thence we easily deduce the inequality

$$\sum_{n=0}^{\infty}{}'\int_0^{2\pi}\int_0^{2\pi}|I_{n,u}-s_n|^2\,du\,dx\leqslant C\sum_{\lambda=1}^{\infty}{}'\lambda\,|\gamma_\lambda|^2,$$

where C is an absolute constant. Similarly, if $n_{k+1}/n_q>q>1$ and $n_1\geqslant 1$,

$$\sum_{k=1}^{\infty}\int_0^{2\pi}\int_0^{2\pi}|I_{n_k,u}-s_{n_k}|^2\,du\,dx\leqslant C_q\sum_{\lambda=2}^{\infty}{}'|\gamma_\lambda|^2\log\lambda,$$

with C_q depending on q only.

The first of these inequalities shows that, if $\sum'\lambda\,|\gamma_\lambda|^2<\infty$, then, for almost every translation u, the sequence $I_{n,u}(x,f)$ converges to $f(x)$ almost everywhere in the interval $0\leqslant x\leqslant 2\pi$. The same may be said of the lacunary sequence $I_{n_k,u}[f]$, provided that the series $\sum|\gamma_\lambda|^2\log\lambda$ converges.

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Sur la géométrisation des types d'ordre dénombrable[1]).

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D'après les théorèmes classiques de la Théorie des Ensembles, on peut faire correspondre d'une façon bien déterminée aux types d'ordre d'ensembles dénombrables ordonnés certains ensembles de nombres réels: c'est que chaque ensemble ordonné dénombrable est semblable à un sous-ensemble de l'ensemble des nombres rationnels rangés selon leur grandeur (et même, à une infinité de tels ensembles); et qu'en outre, chaque ensemble composé de nombres rationnels peut être remplacé par un nombre réel, notamment, par le nombre réel qui lui vient correspondre dans la correspondance biunivoque entre la famille de tous les ensembles composés de nombres rationnels et l'ensemble de tous les nombres réels.

Nous allons réaliser cette interprétation géométrique des types d'ordre dénombrable à l'aide de la méthode suivante, due à M. Lebesgue et qui paraît être la plus simple possible [2]).

Imaginons d'abord l'ensemble des nombres rationnels de l'intervalle 01 rangé en une suite infinie bien déterminée (composée d'éléments différents)

(1) $$r_1,\ r_2,\ \dots\ r_n,\ \dots$$

Soit t un élément de l'ensemble $\mathfrak{C}$ non-dense de Cantor:

$$t=\frac{t^1}{3}+\frac{t^2}{9}+\frac{t^3}{27}+\dots\qquad (t^n=0 \text{ ou } 2).$$

[1]) Présenté à la Soc. Pol. de Math., Section de Varsovie, le 9. X. 1936.

[2]) Journal de Math. 1905 (chap. VIII).