La propriété (σ) entraîne évidemment la propriété (λ); d'autre part, il résulte des propositions (i) et (ii) que

(iii) *Si* $\aleph_1 = \mathfrak{c}$, *la propriété* (λ) *n'entraîne pas la propriété* (σ).

3. E étant un espace métrique, appelons *mesure dans* E chaque fonction finie [13]) non négative d'ensemble borelien dans E, absolument additive et s'annulant pour les ensembles composés d'un seul point. Considérons la propriété suivante:

(β) Chaque mesure dans E s'annule identiquement.

On sait que la propriété (β) satisfait à la condition 2° (voir n° 1) et qu'il existe un ensemble linéaire de puissance $\aleph_1$ jouissant de la propriété (β) [14]). Par conséquent, il résulte de 1 (i) que

(i) *Si* $\aleph_1 = \mathfrak{c}$, *il existe un ensemble de dimension n situé dans* $\mathcal{E}^{n+1}$ (*de même qu'un ensemble de dimension dénombrable dans* $\mathcal{H}_0$ *et un autre de dimension indénombrable dans* $\mathcal{H}$) *qui jouit de la propriété* (β).

Considérons enfin la propriété suivante d'un ensemble E:

(C) Il existe pour chaque suite $\{a_n\}$ de nombres positifs une décomposition $E = E_1 + E_2 + \dots$ telle que $\delta(E_n) < a_n$ $(n = 1, 2, \dots)$.

On sait que chaque ensemble jouissant de la propriété (C) jouit également de la propriété (β) [15]) et qu'il est de dimension 0 [16]). Or, nous voyons d'après (i) que

(ii) *Si* $\aleph_1 = \mathfrak{c}$, *la propriété* (β) *n'entraîne pas la propriété* (C) [17]).

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[13]) Cette prémisse est essentielle. P. ex. la mesure linéaire d'ensembles situés dans un carré n'est pas une mesure dans ce sens.

[14]) W. Sierpiński et E. Szpilrajn, *Remarque sur le problème de la mesure*, Fund. Math. 26 (1936), pp. 256—261.

[15]) Théorème de M. Poprougénko. Cf. E. Szpilrajn, *Remarques sur les fonctions complètement additives*, Fund. Math. 22 (1934), p. 311 et Sierpiński-Szpilrajn, l. c.

[16]) Car la propriété (C) entraîne la mesure linéaire nulle et celle-ci entraîne la dimension 0. Cf. E. Szpilrajn, *La mesure et la dimension*, Fund. Math. 28 (1937), p. 85.

[17]) Une autre démonstration de cette proposition se trouve dans l'article de M. E. Szpilrajn, *Sur les ensembles et les fonctions absolument mesurables* (en polonais), C. R. de la Soc. des Sciences et des Lettres de Varsovie 30 (1937).

Some theorems on orthogonal systems.

By

J. Marcinkiewicz and A. Zygmund (Wilno).

1. Let $\{\varphi_n(x)\}_{n=1,2,\dots}$ be an arbitrary system of functions orthonormal in an interval (a, b), that is

$$\int_a^b \varphi_m \varphi_n \, dx = 0 \qquad (m, n = 1, 2, \dots)$$

$$\int_a^b \varphi_n^2 \, dx = 1 \qquad (n = 1, 2, \dots).$$

(For simplicity we restrict ourselves to the case of real functions φ_n.)

From the last equations it follows, in particular, that all the functions $\varphi_n(x)$ are of the class $L^2(a, b)$. If therefore we consider the Fourier coefficients

$$c_n = \int_a^b f \varphi_n \, dx \qquad (n = 1, 2, \dots) \tag{1.1}$$

of any function f, with respect to the system $\{\varphi_n\}$, we must, in the general case, assume that $f \in L^2(a, b)$. Otherwise, the integrals (1.1) may not exist.

In the case when the functions φ_n are uniformly bounded, the integrals (1.1) exist for every $f \in L(a, b)$. In this case, a number of results have been proved about the coefficients c_n. These results generalize the well-known Bessel's inequality and the Riesz-Fischer theorem [1]).

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[1]) Cf. F. Riesz [1]; the results are reproduced in Kaczmarz and Steinhaus [1] and in Zygmund [1].

In the present paper we intend to consider a slightly more general case, viz. when the functions φ_n satisfy, for a certain $\nu>2$, the inequalities

$$(1.2)\qquad \left(\int_a^b |\varphi_n|^\nu \, dx\right)^{1/\nu} \leqslant M_n, \qquad (n=1,2,\ldots)$$

where the M_n are finite numbers. When $\nu=\infty$, the left-hand side of (1.2) becomes the essential upper bound of $|\varphi_n|$ [1]).

By $\mu=\mu(\nu)$ we shall denote throughout the number satisfying the equation

$$1/\mu+1/\nu=1.$$

Hence, following the usual notation, $\mu=\nu'$. It is plain that $1\leqslant\mu<2$. Under the hypotheses (1.2), the integrals (1.1) exist for $f\in L^\mu(a,b)$.

Theorem 1. *Let*

$$(1.3)\qquad \mu\leqslant p\leqslant 2,$$

and let q satisfy the equation

$$(1.4)\qquad \mu/p+(2-\mu)/q=1.$$

Then, under the conditions (1.2),

$$(1.5)\qquad \left(\sum_{n=1}^{\infty} M_n^{2-q}|c_n|^q\right)^{1/q} \leqslant \left(\int_a^b |f|^p \, dx\right)^{1/p}.$$

Theorem 2. *Let*

$$(1.6)\qquad 1\leqslant p\leqslant 2,$$

and let q be given by the equation

$$(1.7)\qquad (2-\mu)/p+\mu/q=1.$$

If the series $\sum M_n^{2-p}|c_n|^p$ and $\sum |c_n|^2$ both converge, then there is a function $f\in L^q(a,b)$ satisfying (1.1) *and such that*

$$(1.8)\qquad \left(\int_a^b |f|^q \, dx\right)^{1/q} \leqslant \left(\sum_{n=1}^{\infty} M_n^{2-p}|c_n|^p\right)^{1/p}.$$

If $\nu=\infty$ (that is $\mu=1$) and $M_1=M_2=\ldots=M$, Theorems 1 and 2 reduce to F. Riesz's well-known theorems.

[1]) Cf. also Rosskopf [1].

If we use M. Riesz's convexity theorems [1]), the proof of Theorems 1 and 2 does not differ essentially from that of F. Riesz's, theorems. Nevertheless, for the sake of completeness we give the proof here, starting with Theorem 1.

In the first place we observe that $p=2$ implies $q=2$. Hence (1.5) is true for $p=2$. If $p=\mu$, that is $q=\infty$, the inequality (1.5) is also true. It may then be written

$$\operatorname*{Max}_n |c_n| \leqslant M_n \left(\int_a^b |f|^\mu \, dx\right)^{1/\mu},$$

and this inequality is a consequence of (1.1) and (1.2). Let

$$M_{\alpha,\beta}=\operatorname*{Max}_f \left(\sum \left|\frac{c_n}{M_n}\right|^{1/\beta} M_n^2\right)^{\beta} \Bigg/ \left(\int_a^b |f|^{1/\alpha} \, dx\right)^{\alpha},$$

where $f\in L^{1/\alpha}(a,b)$. We have shown that

$$M_{1/2,1/2}\leqslant 1, \qquad M_{1/\mu,0}\leqslant 1.$$

By M. Riesz's convexity theorem, $\log M_{\alpha\beta}$ is a convex function on the segment joining the points $(1/2,1/2)$ and $(1/\mu,0)$. (It will be noted that this segment lies in the triangle $0\leqslant\alpha\leqslant 1$, $0\leqslant\beta\leqslant\alpha$). In particular, $M_{\alpha\beta}\leqslant 1$ for every point α,β of that segment. The equation of the segment is (1.4) if we replace there $1/p$ by α and $1/q$ by β. This completes the proof of Theorem 1.

Theorem 1 remains valid, and its proof unaffected, if the interval (a,b) is infinite. We observe that, if $f\in L^p(a,b)$ and p satisfies (1.3), the integrals (1.1) exist. For $\varphi_n\in L^\nu(a,b)$ and $\varphi_n\in L^2(a,b)$. Hence, by Hölder's inequality, $\varphi_n\in L^{p'}(a,b)$ if $2\leqslant p'\leqslant\nu$. This gives the integrability of the product $f\varphi_n$ over (a,b).

We now pass to the proof of Theorem 2. Let

$$(1.9)\qquad f(x)=\operatorname*{l.i.m.}_{N\to\infty}\sum_{n=1}^{N} c_n\varphi_n(x),$$

where l. i. m. denotes limit in measure. By the Riesz-Fischer theorem, if $\sum c_n^2$ converges, then $f(x)$ exists and belongs to L^2. Moreover, f satisfies (1.8) for $p=2$.

[1]) See M. Riesz [1], or Zygmund [1], 192 sqq.

If $p=1$, the relation (1.7) gives $q=\mu/(\mu-1)=\nu$, and (1.8) takes the form

$$\left(\int_a^b |f|^\nu\, dx\right)^{1/\nu} \leqslant \sum_{n=1}^{\infty} M_n |c_n|.$$

In order to prove this inequality, we observe that if f is defined by (1.9), then

$$|f| \leqslant \sum |c_n \varphi_n| = \sum (|c_n|\, M_n)^{1/\mu} (|c_n|\, M_n)^{1/\nu} \frac{|\varphi_n|}{M_n}$$
$$\leqslant \left(\sum |c_n|\, M_n\right)^{1/\mu} \left(\sum |c_n|\, M_n \frac{|\varphi_n|^\nu}{M_n^\nu}\right)^{1/\nu}.$$

We have not yet proved that f exists. Let $S(x)$ denote the right-hand side of the last inequality. If the integral of $S^\nu(x)$ over (a, b) is finite, then the series $\sum c_n \varphi_n(x)$ is convergent (indeed absolutely convergent) almost everywhere, so that $f(x)$ exists. But, on account of (1.2),

$$\left(\int_a^b S^\nu(x)\, dx\right)^{1/\nu} \leqslant \left(\sum |c_n|\, M_n\right)^{1/\mu} \left(\sum |c_n|\, M_n\right)^{1/\nu} = \sum |c_n|\, M_n.$$

Hence (1.8) is established for $p=1$.

Let

(1.10) $$M_{\alpha,\beta} = \operatorname*{Max}_{\{c_n\}} \left(\int_a^b |f|^{1/\beta}\, dx\right)^\beta \Bigg/ \left(\sum_{n=1}^{\infty} \left|\frac{c_n}{M_n}\right|^{1/\alpha} M_n^2\right)^\alpha,$$

where we take into consideration all the sequences $\{c_n\}$ for which the denominator of the ratio is finite. The class of such sequences will be denoted by $\mathfrak{S}_\alpha$. If $1/\alpha=2$ and $1/\beta=2$, or if $1/\alpha=1$ and $1/\beta=\nu$, we mean by f the function (1.9). We have then

(1.11) $$M_{1/2,1/2} \leqslant 1, \qquad M_{1,1/\nu} \leqslant 1.$$

Let l denote the segment joining the points $(1/2, 1/2)$ and $(1, 1/\nu)$. (l lies in the triangle $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant \alpha$). By M. Riesz's convexity theorem, the linear operation which transforms the sequence $\{c_n/M_n\}$ into a function f, and which is so far defined for $\{c_1\}\in\mathfrak{S}_1$ and $\{c_n\}\in\mathfrak{S}_{1/2}$, may be extended to every $\mathfrak{S}_\alpha$ where $1/2 \leqslant \alpha \leqslant 1$. If the point (α, β) lies on l, the number $M_{\alpha\beta}$ defined above is finite; moreover, $\log M_{\alpha\beta}$ is convex on l. This, in connection with (1.11) gives $M_{\alpha\beta} \leqslant 1$ for the points on l. Hence (1.8) is established.

It remains to show that we have (1.1). Let $s_N = c_1\varphi_1 + \ldots + c_N\varphi_N$. The linear operation which we consider is additive. The relation (1.8) applied to $f-s_N$ gives

$$\left(\int_a^b |f-s_N|^q\, dx\right)^{1/q} \leqslant \left(\sum_{n=N+1}^{\infty} |c_n|^p M_n^{2-p}\right)^{1/p},$$

i. e. $\{s_n\}$ tends to f in the mean. Hence f is again defined by (1.9).

The condition of the convergence of the series $\sum |c_n|^2$ has not so far been used. Taking it now into account, we see that the function f defined by (1.9) belongs to L^2, and that

$$\int_a^b |f-s_N|^2\, dx \to 0 \qquad \text{as } N\to\infty.$$

Hence, if $N>n$

$$\int_a^b f\varphi_n\, dx = \int_a^b (f-s_N)\, \varphi_n\, dx + c_n = c_n,$$

for

$$\left|\int_a^b (f-s_N)\, \varphi_n\, dx\right| \leqslant \left(\int_a^b |f-s_N|^2\, dx\right)^{1/2} \left(\int_a^b \varphi_n^2\, dx\right) \to 0 \quad \text{as } N\to\infty.$$

This gives (1.1), and so completes the proof Theorem 2.

2. It may be asked how far the hypothesis of the convergence of *both* the series $\sum |c_n|^p M_n^{2-p}$ and $\sum |c_n|^2$ is indispensable for the truth of Theorem 2.

It is plain that if the numbers M_n are bounded below by a positive constant, the inequality $\sum |c_n|^p M_n^{2-p} < \infty$, implies $\sum |c_n|^p < \infty$, and so also $\sum |c_n|^2 < \infty$. Hence the condition of the convergence of the series $\sum |c_n|^2$ may be omitted in the statement of Theorem 2.

This is, in particular, the case when the interval (a,b) is finite. For then, by the well-known theorem on the mean values of integrals,

$$\left(\frac{1}{b-a}\int_a^b |\varphi_n|\, dx\right)^{1/\nu} \geqslant \left(\frac{1}{b-a}\int_a^b |\varphi_n|^2\, dx\right)^{1/2} = \frac{1}{(b-a)^{1/2}},$$

i. e. $M_n \geqslant (b-a)^{1/\nu - 1/2}$.

If the interval (a, b) is infinite, the situation is different, for then we may have $M_n \to 0$. In that case, it is possible to find a sequence $\{c_n\}$ such that

(2.1) $$\sum |c_n|^p M_n^{2-p} < \infty, \qquad \sum |c_n|^2 = \infty.$$

The first of these relations implies, as we have seen, the existence of a function f, defined by (1.9), and satisfying (1.8). But if we try to prove that f fulfils (1.1), we encounter difficulties.

The function f belongs to L^q, so that the existence of the integrals (1.1) is certain only in the case when the φ_n belong to the conjugate class $L^p(a,b)$, where $p=q/(q-1)<2$. But the relation $\varphi_n \epsilon L^p(a,b)$ is not necessarily true. The functions φ_n belong to $L^2(a,b)$ as well as to $L^\nu(a,b)$. Hence they belong to every $L^s(a,b)$, provided that $2 \leqslant s \leqslant \nu$. Since (a,b) is infinite, this does not necessarily imply $\varphi_n \epsilon L^p(a,b)$ for $p<2$, and so the existence of the Fourier coefficients of f is not assured.

We shall now show that there exists an orthonormal system $\{\varphi_n\}$, and a sequence $\{c_n\}$ satisfying (2.1), such that if f is defined by (1.9), one at least of the products $f\varphi_n$ is not integrable.

Let $r_n(x)$, where $n=2,3,\dots$, by the sequence of Rademacher's periodic functions, i. e.

$$r_n(x)=\operatorname{sign}\sin(2^{n-1}\pi x) \qquad (-\infty<x<\infty;\ n=2,3,\dots).$$

Let $\{\varepsilon_n\}_{n=2,3,\dots}$ be a sequence of positive numbers tending to 0 and such that

$$\sum_{n=2}^{\infty} \varepsilon_n^{(2-p)/2} < \infty. \tag{2.2}$$

For $n=2,3,\dots$, we put

$$\varphi_n(x)=C_n\sqrt{\frac{\varepsilon_n}{m^{1+\varepsilon_n}}}\,r_n(x) \qquad (m-1\leqslant |x|<m;\ m=1,2,\dots), \tag{2.3}$$

where the numbers C_n are so chosen that the integral of φ_n^2 over $(-\infty,\infty)$ is equal to 1. Hence $C_n \to 2^{-1/2}$. In particular the C_n are bounded below by a number $A>0$.

We add that the φ_n $(n=2,3,\dots)$ are odd functions.

Let

$$\Phi(x)=\sum_{n=2}^{\infty}\varphi_n(x).$$

The series on the right converges uniformly, as is seen from the inequality $|\varphi_n|\leqslant C_n\varepsilon_n^{1/2}$ and from (2.2). Let

$$h(x)=1 \text{ if } \Phi(x)\geqslant 0, \qquad h(x)=-1 \text{ if } \Phi(x)<0.$$

Let $\{\delta_m\}_{m=1,2,\dots}$ be a sequence of positive numbers which will be defined presently and such that $2\sum\delta_m^2=1$. Let

$$\varphi_1(x)=\delta_m h(x) \qquad (m-1\leqslant x<m;\ m=1,2,\dots),$$
$$\varphi_1(-x)=\varphi_1(x) \qquad (x\geqslant 0).$$

It is easy to see that the sequence $\varphi_1(x), \varphi_2(x), \dots$ is orthonormal on $(-\infty,\infty)$ (Since for $n=2,3,\dots$ the products $\varphi_1\varphi_n$ are odd functions of x, their integrals over $(-\infty,\infty)$ vanish).

Let $c_1=c_2=\dots=1$, and let correspondingly $f(x)=\varphi_1(x)+\varphi_2(x)+\dots$ On account of (2.2) and (2.3), we may suppose that the series $\sum|c_n|^p M_n^{2-p}$ converges. We shall show that the integral $\int_0^\infty f\varphi_1\,dx$ does not exist. (This will imply the non-existence of $\int_{-\infty}^{\infty} f\varphi_1\,dx$). It is sufficient to show that

$$\int_0^\infty \Phi(x)\,\varphi_1(x)\,dx=\infty, \tag{2.4}$$

where $\Phi(x)=f(x)-\varphi_1(x)=\varphi_2(x)+\varphi_3(x)+\dots$ The latter integral is equal to

$$\sum_{m=1}^{\infty}\delta_m\int_{m-1}^{m}\left|\sum_{n=2}^{\infty}C_n\sqrt{\frac{\varepsilon_n}{m^{1+\varepsilon_n}}}\,\varphi_n(x)\right|dx. \tag{2.5}$$

If is well-known that if $a_2, a_3, \dots$ are arbitrary numbers, then

$$\int_0^1\left|\sum_{n=2}^{\infty}a_n\varphi_n(x)\right|dx \geqslant B\left(\int_0^1\left|\sum_{n=2}^{\infty}a_n\varphi_n(x)\right|^2dx\right)^{1/2}=B\left(\sum_{n=2}^{\infty}|a_n|^2\right)^{1/2},$$

where $B>0$ is an absolute constant [1]). Hence the sum (2.5) is not less than

$$AB\sum_{m=1}^{\infty}\delta_m\left(\sum_{n=2}^{\infty}\frac{\varepsilon_n}{m^{1+\varepsilon_n}}\right)^{1/2}. \tag{2.6}$$

Now observe that the series

$$\sum_{m=1}^{\infty}\left(\sum_{n=2}^{\infty}\frac{\varepsilon_n}{m^{1+\varepsilon_n}}\right)=\sum_{n=2}^{\infty}\left\{\sum_{m=1}^{\infty}\frac{\varepsilon_n}{m^{1+\varepsilon_n}}\right\}$$

diverges, since the sums in curly brackets tend to 1 as $n\to\infty$. It follows that we can find a sequence $\{\delta_m\}$ such that $2\sum\delta_m^2=1$, and yet (2.5) is infinite. Hence, we have (2.4).

3. In this paragraph, we generalize Paley's theorems on orthogonal series. Unlike in the case of Theorems 1 and 2, the changes which have now to be made in the proofs, are not quite trivial, and the reason of this is easy to see. Let us take, for example, Theorem 3 which follows, and which reduces to a theorem of Paley if $\nu=\infty$ and $M_1=M_2=\dots=M$ [2]). Paley proved his theorem first

[1]) Cf. e. g. Zygmund [1], 129.

[2]) See Paley [1], or Zygmund [1], 202 sqq.

in the case of q integral and greater than 3, and then interpolated by means of convexity theorems. This could be done, since the theorem is obvious for $q=2$, and every $q>2$ is contained in an interval $(2, q_0)$ where q_0 is an integer greater than 3.

In our case it is plain that we must restrict ourselves to the values of q not exceeding ν. Otherwise, the function f may not belong to L^q (Take the simple case $c_1=1, c_2=c_3=\ldots=0$). If $\nu>4$, and if the exponent q is integral and satisfies the inequality

$$4 \leqslant q < \nu,$$

the proof of Theorem 3 follows by an argument which is a modification of Paley's argument. An application of convexity theorems gives (3.3) for every q such that

$$2 \leqslant q \leqslant q_0 < \nu,$$

where q_0 is the largest integer $<\nu$. If $\nu<4$, the previous argument is not sufficient. Nor can it be applied if ν exceeds 4, but is not an integer. The case $q_0 \leqslant q < \nu$ remains then open.

Theorem 3. *Suppose that*

$$2 \leqslant q < \nu,$$

and that the series

$$\sum_{n=1}^{\infty} |c_n|^q M_n^{\frac{\nu}{\nu-2}(q-2)} n^{\frac{\nu-1}{\nu-2}(q-2)} \tag{3.1}$$

converges, where the M_n are defined by (1.2). *Suppose moreover that*

$$M_1 \leqslant M_2 \leqslant \ldots \leqslant M_n \leqslant \ldots \tag{3.2}$$

Then there is a function $f \epsilon L^q(a,b)$ satisfying (1.1), *and such that*

$$\left(\int_a^b |f|^q\, dx\right)^{1/q} \leqslant A_{q,\nu} \left(\sum_{n=1}^{\infty} |c_n|^q M_n^{\frac{\nu}{\nu-2}(q-2)} \cdot n^{\frac{\nu-1}{\nu-2}(q-2)}\right)^{1/q}. \tag{3.3}$$

Here $A_{q,\nu}$ depends on q and ν only. If by $A_{q,\nu}$ we mean the least number satisfying (3.3) *for all sequences $\{c_n\}$, then*

$$A_{q,\nu} \leqslant A \frac{\nu-2}{\nu-q} q, \tag{3.4}$$

where A is an absolute constant.

Theorem 4. *Suppose that*

$$\mu < p \leqslant 2,$$

that $f \epsilon L^p(a,b)$, and that we have (3.2). *Then the Fourier coefficients c_n of f, with respect to $\{\varphi_n\}$, satisfy the inequality*

$$\left(\sum_{n=1}^{\infty} |c_n|^p M_n^{\frac{\nu}{\nu-2}(p-2)} n^{\frac{\nu-1}{\nu-2}(p-2)}\right)^{1/p} \leqslant B_{p,\nu} \left(\int_a^b |f|^p\, dx\right)^{1/p}. \tag{3.5}$$

The constant $B_{p,\nu}$ depends on p and ν only. Moreover

$$B_{p,\nu} = A_{q,\nu}, \tag{3.6}$$

if p and q are connected by the equation $1/p+1/q=1$.

The order of the functions φ_n within the system $\{\varphi_n\}$ is plainly irrelevant. If, for example, $M_n \to \infty$, we may change this order so as to have (3.2).

4. The proof of Theorem 3 will be based on the following

Lemma 1. *Let*

$$\Phi_m(x) = \sum_{k=2^{m-1}}^{2^m-1} c_k \varphi_k(x), \qquad C_m = \sum_{k=2^{m-1}}^{2^m-1} |c_k|^q M_k^{\frac{\nu}{\nu-2}(q-2)} k^{\frac{\nu-1}{\nu-2}(q-2)} \tag{4.1}$$

for $m=1,2,\ldots$. *Then, if*

$$4\nu/(\nu+2) \leqslant q < \nu \tag{4.2}$$

we have

$$\int_a^b |\Phi_m \Phi_n|^{q/2}\, dx \leqslant C_m^{1/2} C_n^{1/2} 2^{-\frac{1}{2}\cdot\frac{\nu-q}{\nu-2}|m-n|}. \tag{4.3}$$

In order to simplify the notation, we supress the range of summation in sums, and suppose that k varies between 2^{m-1} and 2^m-1, and l between 2^{n-1} and 2^n-1. We also suppose that $m \leqslant n$. Then, by Hölder's inequality,

$$|\Phi_m|^{q/2} \leqslant \left(\sum |c_k|^{1/\mu} M_k^{1/\mu} \cdot |c_k|^{1/\nu} M_k^{-1/\mu} |\varphi_k|\right)^{q/2} \leqslant \left(\sum |c_k| M_k\right)^{q/2\mu} \left(\sum |c_k| M_k^{-(\nu-1)} |\varphi_k|^\nu\right)^{q/2\nu}.$$

Using Hölder's inequality again, we have

$$\text{(4.7)}\quad \begin{aligned}\int_a^b |\Phi_m\Phi_n|^{q/2}dx &\leqslant \Big(\sum{}'|c_k|M_k\Big)^{\frac{q}{2\nu}}\Big(\int_a^b\sum{}'|c_k|M_k^{-(\nu-1)}|\varphi_k|^{\nu}dx\Big)^{\frac{q}{2\nu}}\Big(\int_a^b|\Phi_n|^{\frac{q\nu}{2\nu-q}}\Big)^{\frac{2\nu-q}{2\nu}}\\ &\leqslant \Big(\sum{}'|c_k|M_k\Big)^{q/2}\Big(\int_a^b|\Phi_n|^{q_1}dx\Big)^{q/2q_1},\end{aligned}$$

where

$$q_1 = q\nu/(2\nu-q).$$

We may write

$$\sum{}'|c_k|\,M_k=\sum{}'|c_k|\,M_k^{\frac{\nu}{\nu-2}\cdot\frac{q-2}{q}}k^{\frac{q-2}{q}\cdot\frac{\nu-1}{\nu-2}}\cdot M_k^{\frac{2(\nu-q)}{q(\nu-2)}}k^{-\frac{q-2}{q}\cdot\frac{\nu-1}{\nu-2}},$$

so that, by Hölder's inequality,

$$\text{(4.5)}\quad \Big(\sum{}'|c_k|\,M_k\Big)^{q/2}\leqslant C_m^{1/2}\Big(\sum{}'M_k^{\frac{2}{q-1}\frac{\nu-q}{\nu-2}}k^{-\frac{q-2}{q-1}\cdot\frac{\nu-1}{\nu-2}}\Big)^{\frac{q-1}{2}}.$$

The inequalities (4.2) imply that

$$2\leqslant q_1<\nu.$$

Let

$$\text{(4.6)}\quad (2-\mu)/p_1+\mu/q_1=1,$$

so that

$$\text{(4.7)}\quad p_1=q(\nu-2)/\nu(q-2),\qquad 1-p_1=-2(\nu-q)/\nu(q-2).$$

From Theorem 2, we obtain

$$\text{(4.8)}\quad \Big(\int_a^b|\Phi_n|^{q_1}dx\Big)^{q/2q_1}\leqslant\Big(\sum c_l^{p_1}M_l^{2-p_1}\Big)^{q/2p_1}.$$

In view of (4.7) and (4.8) we may write

$$\text{(4.9)}\quad \begin{aligned}\sum c_l^{p_1}M_l^{2-p_1}&=\sum c_l^{p_1}M_l^{\frac{\nu}{\nu-2}(q-2)\frac{p_1}{q}}l^{\frac{\nu-1}{\nu-2}(q-2)\frac{p_1}{q}}\cdot M_l^{-\frac{2(\nu-q)}{\nu(q-2)}}l^{-\frac{\nu-1}{\nu-2}(q-2)\frac{p_1}{q}}\\ &\leqslant C_n^{p_1/q}\Big(\sum M_l^{-\frac{2(\nu-q)}{\nu(q-2)}\cdot\frac{q}{q-p_1}}l^{-\frac{\nu-1}{\nu-2}(q-2)\frac{p_1}{q-p_1}}\Big)^{\frac{q-p_1}{q}}\end{aligned}$$

$$\text{(4.10)}\quad \Big(\int_a^b|\Phi_n|^{q_1}dx\Big)^{q/2q_1}\leqslant C_n^{1/2}\Big(\sum M_l^{-\frac{2(\nu-q)}{\nu(q-2)}\frac{q}{q-p_1}}l^{-\frac{\nu-1}{\nu-2}(q-2)\frac{p_1}{q-p_1}}\Big)^{\frac{q-p_1}{2p_1}}.$$

We now suppose that $m<n$; to the case $m=n$ we shall return a little later. From (3.2) we have

$$M_k\leqslant M_{2m-1}\leqslant M_{2n-1}\leqslant M_l,$$

and so, applying (4.4), (4.5), and (4.10),

$$\text{(4.11)}\quad \int_a^b|\Phi_m\Phi_n|^{q/2}dx\leqslant C_m^{1/2}C_n^{1/2}\Big(\sum k^{-\frac{q-2}{q-1}\cdot\frac{\nu-1}{\nu-2}}\Big)^{\frac{q-1}{2}}\Big(\sum l^{-\frac{\nu-1}{\nu-2}(q-2)\frac{p_1}{q-p_1}}\Big)^{\frac{q-p_1}{2p_1}}.$$

For any integer $r>0$ and any $\alpha>0$ we have

$$\text{(4.12)}\quad \sum_{j=2^{r-1}}^{2^r-1}j^{-\alpha}\leqslant(2^{r-1})^{-\alpha}2^{r-1}=2^{(r-1)(1-\alpha)}.$$

We apply this inequality to the sums on the right of (4.11); the inequality (4.3), for $m<n$, turns out to be a consequence of (4.11).

Suppose now that $m=n$. In view of Theorem 2, we may write

$$\text{(4.13)}\quad \int_a^b|\Phi_m|^q dx\leqslant\Big(\sum{}'|c_k|^pM_k^{2-p}\Big)^{q/p}.$$

Here p is given by (1.7). Therefore

$$p=\frac{q(\nu-2)}{q\nu-q-\nu},\qquad 2-p=\frac{\nu(q-2)}{q\nu-q-\nu}.$$

In other words,

$$2-p=\frac{\nu(q-2)}{(\nu-2)}\cdot\frac{p}{q}.$$

If follows that

$$\begin{aligned}\sum{}'|c_k|^pM_k^{2-p}&=\sum{}'|c_k|^pM_k^{\frac{\nu(q-2)}{\nu-2}\cdot\frac{p}{q}}k^{(q-2)\frac{\nu-1}{\nu-2}\frac{p}{q}}\cdot k^{-(q-2)\frac{\nu-1}{\nu-2}\frac{p}{q}}\\ &\leqslant\Big(\sum{}'|c_k|^qM_k^{\frac{\nu(q-2)}{\nu-2}}k^{\frac{\nu-1}{\nu-2}(q-2)}\Big)^{p/q}\Big(\sum k^{-(q-2)\frac{\nu-1}{\nu-2}\frac{p}{q-p}}\Big)^{(q-p)/q}\\ &=C_m^{p/q}\Big(\sum k^{-1}\Big)^{(q-p)/q}\leqslant C_m^{p/q}.\end{aligned}$$

From this and (4.13) we conclude that

$$\int_a^b|\Phi_m|^q dx\leqslant C_m,$$

so that (4.3) is true also for $m=n$.

Remark. If

$$2 \leqslant q \leqslant 4\nu/(\nu+2), \tag{4.14}$$

the inequality (4.3) must be replaced by

$$\int_a^b |\Phi_m \Phi_n|^{q/2} dx \leqslant C_m^{1/2} C_n^{1/2}\, 2^{-\frac{1}{4}\frac{\nu}{\nu-2}(q-2)|m-n|}. \tag{4.15}$$

For we have

$$\int_a^b |\Phi_m \Phi_n|^{q/2} dx \leqslant \left(\int_a^b |\Phi_m|^{q_1}\right)^{q/2q_1} \left(\int_a^b \Phi_n^2\right)^{q/4},$$

where $q_1 = 2q/(4-q)$. The condition (4.14) implies that $2 \leqslant q_1 \leqslant \nu$. Hence the last two integrals may be estimated by means of Theorem 2. We omit the calculations here, since the inequality (4.15) will not be required later.

5. We shall now prove Theorem 3, beginning with the case of q satisfying (4.2). We write

$$\lambda_q = \frac{1}{2}\cdot\frac{\nu-q}{\nu-2},$$

and suppose that

$$r-1 \leqslant q \leqslant r, \tag{5.1}$$

where r is an integer $\geqslant 3$. Let S denote the sum of the series (3.1).

Arguing as in (4.9), we find

$$\sum_{l=1}^{\infty} c_l^2 \leqslant S^{2/q} \left(\sum_{l=1}^{\infty} M_l^{-\frac{2\nu}{\nu-2}} l^{-\frac{2(\nu-1)}{\nu-2}}\right)^{(q-2)/q}.$$

From (3.2) we see that all the M_l exceed a positive constant. It follows that the last series converges. Hence, if S is finite, so is $\sum c_l^2$. By the Riesz-Fischer theorem there is a function $f \in L^2$, satisfying (1.1), and we have only to prove that f satisfies (3.3). The series $\sum c_n \varphi_n$ converges in the mean to f. We may write

$$\int_a^b |f|^q dx \leqslant \int_a^b \Big(\sum_m |\Phi_m|\Big)^q dx = \int_a^b \Big(\sum_m |\Phi_m|\Big)^{\frac{q}{r}\cdot r} dx \leqslant \int_a^b \Big(\sum_m |\Phi_m|^{\frac{q}{r}}\Big)^r dx,$$

since, by (5.1), $q/r \leqslant 1$. Let

$$|\Phi_m|^{q/r} = \Psi_m.$$

Then

$$\int_a^b |f|^q dx \leqslant \int_a^b \Big(\sum_m \Psi_m\Big)^r dx \leqslant \int_a^b \Big(\sum_{m_1 \ldots m_r} \Psi_{m_1}\Psi_{m_2}\ldots\Psi_{m_r}\Big) dx, \tag{5.2}$$

where the m_i vary independently from 1 to ∞.

We may write

$$\int_a^b \Psi_{m_1}\Psi_{m_2}\ldots\Psi_{m_r}\, dx = \int_a^b \prod_{\substack{i,k=1\\ i\neq k}}^{r} (\Psi_{m_i}\Psi_{m_k})^{1/2(r-1)} dx.$$

Let $R = r(r-1)$. An application of Hölder's inequality with R exponents, each equal to R, gives

$$\int_a^b \Psi_{m_1}\ldots\Psi_{m_r}\, dx \leqslant \prod_{\substack{i,k=1\\ i\neq k}}^{r} \left\{\int_a^b (\Psi_{m_i}\Psi_{m_k})^{r/2} dx\right\}^{1/R} = \prod_{\substack{i,k=1\\ i\neq k}}^{r} \left\{\int_a^b |\Phi_{m_i}\Phi_{m_k}|^{q/2} dx\right\}^{1/R}$$

$$\leqslant \prod_{\substack{i,k=1\\ i\neq k}}^{r} C_{m_i}^{1/2R} C_{m_k}^{1/2R}\, 2^{-\lambda_q|m_i-m_k|/R}$$

$$= \prod_{i=1}^{r} C_{m_i}^{1/r} \left\{\prod_{\substack{k=1\\ k\neq i}}^{r} 2^{-2\lambda_q|m_i-m_k|/R}\right\},$$

in view of the lemma. We take into account (5.2) and apply Hölder's inequality again. Then

$$\int_a^b |f|^q dx \leqslant \sum_{m_1,\ldots,m_r} \left\{\prod_{i=1}^{r} C_{m_i}^{1/r} \prod_{\substack{k=1\\ k\neq i}}^{r} 2^{-2\lambda_q|m_i-m_k|/R}\right\}$$

$$\leqslant \prod_{i=1}^{r} \left\{\sum_{m_1,\ldots,m_r} C_{m_i} \prod_{\substack{k=1\\ k\neq i}}^{r} 2^{-2\lambda_q|m_i-m_k|/(r-1)}\right\}^{1/r}.$$

The sum in curly brackets may be first summed with respect to $m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_r$, and then with respect to m_i. In this way we easily obtain

$$\int_a^b |f|^q dx \leqslant \Big(\sum_{m=1}^{\infty} C_m\Big)\Big(\sum_{n=-\infty}^{\infty} 2^{-2\lambda_q|n|/(r-1)}\Big)^{r-1}. \tag{5.3}$$

Let ξ be any positive number less than 1. Then

$$\sum_{n=-\infty}^{\infty} 2^{-|n|\xi} < \frac{2}{1-2^{-\xi}} < \frac{4}{\xi \log 2} < \frac{8}{\xi}.$$

We put $\xi=2\lambda_q/(r-1)$. From the definition of λ_q we see that $0<\xi<1$. Hence, observing that $2\lambda_q<1$,

$$\left\{\sum_{n=-\infty}^{\infty} 2^{-2\lambda_q|n|/(r-1)}\right\}^{r-1} \leqslant \left\{\frac{8(r-1)}{2\lambda_q}\right\}^{r-1} \leqslant \left\{\frac{8q}{2\lambda_q}\right\}^{q} = \left(\frac{8(\nu-2)}{\nu-q}\,q\right)^{q}.$$

From this and (5.3) we deduce the inequalities (3.3) and (3.4), provided that

$$q^* \leqslant q<\nu, \qquad \text{where} \qquad q^*=4\nu/(\nu+2).$$

We have $\nu-q^*=\nu(\nu-2)/(\nu+2)$ so that $1<(\nu-2)/(\nu-q^*)<2$. From the inequality

$$A_{q,\nu}<8\frac{\nu-2}{\nu-q}\,q \qquad (q^*\leqslant q<\nu)$$

we obtain, in particular, that

$$A_{q^*,\nu}<8\,\frac{\nu-2}{\nu-q^*}\,q^*<16q^*\leqslant 64.$$

Let $c(x)$ be a function equal to $c_k M_k^{\nu/(\nu-2)}\,k^{(\nu-1)/(\nu-2)}$ for $x=k$ $(k=1,2,\dots)$ and otherwise quite arbitrary. Let $\psi(x)$ be the function constant in the interior of the intervals $(k-1,k)$, and having a jump $M_k^{-2\nu/(\nu-2)}\,k^{-2(\nu-1)/(\nu-2)}$ at the point $x=k$. Then

$$\sum |c_k|^q\, M_k^{\frac{\nu}{\nu-2}(q-2)}\, k^{\frac{\nu-1}{\nu-2}(q-2)} = \int |c(x)|^q\, d\psi(x),$$

where the integral is a Lebesgue-Stieltjes integral. The function $f(x)=\sum c_n\varphi_n$ is a linear transformation of $c(x)$. By M. Riesz's convexity theorem,

$$\log A_{q,\nu}=\log \operatorname*{Max}_{c}\left\{\left(\int |f|^q\,dx\right)^{1/q}\Big/\left(\int |c(x)|^q d\psi(x)\right)^{1/q}\right\}$$

is a convex function of $1/q$ in the interval $1<q<\nu$. We have $A_{2,\nu}\leqslant 1$, $A_{q^*,\nu}\leqslant 64$. Hence $A_{q,\nu}\leqslant 64$ for $2\leqslant q\leqslant q^*$. In particular $A_{q,\nu}$ is finite, so that (3.3) is established for $2\leqslant q<\nu$. If $2\leqslant q\leqslant q^*$, then

$$A_{q,\nu}\leqslant 64\leqslant 32q=32\,\frac{\nu-q}{\nu-q}\,q\leqslant 32\,\frac{\nu-2}{\nu-q}\,q.$$

This proves (3.4) for $2\leqslant q<\nu$ [1]. The value $A=32$ is of course too generous, but we are not interested in the best possible value for A.

6. Theorem 4 may be deduced from Theorem 3 by a familiar argument. Let us fix an integer $N>0$, and let

$$\gamma_n=|c_n|^{p-1}\operatorname{sign} c_n\cdot M_n^{\frac{\nu}{\nu-2}(p-2)}\, n^{\frac{\nu-1}{\nu-2}(p-2)} \qquad (n=1,2,\dots,N),$$

$$g=\sum_{n=1}^{N}\gamma_n\varphi_n.$$

Then

$$\sum_{n=1}^{N}|c_n|^p M_n^{\frac{\nu}{\nu-2}(p-2)}\, n^{\frac{\nu-1}{\nu-2}(p-2)}=\sum_{n=1}^{N} c_n\gamma_n=\int_a^b f\,g\,dx \tag{6.1}$$
$$\leqslant\left(\int_a^b |f|^p\,dx\right)^{1/p}\left(\int_a^b |g|^q\,dx\right)^{1/q}$$

On account of Theorem 3,

$$\left(\int_a^b |g|^q\,dx\right)^{1/q}\leqslant A_{q,\nu}\left\{\sum_{n=1}^{N}|\gamma_n|^q\, M_n^{\frac{\nu}{\nu-2}(q-2)}\, n^{\frac{\nu-1}{\nu-2}(q-2)}\right\}^{1/q}$$
$$=A_{q,\nu}\left\{\sum_{n=1}^{N}|c_n|^p\, M_n^{\frac{\nu}{\nu-2}(p-2)}\, n^{\frac{\nu-1}{\nu-2}(p-2)}\right\}^{1/q}.$$

From this and (6.1) we easily derive (3.5), in a slightly modified form: the sum on the left is taken from $n=1$ to $n=N$, and not from 1 to ∞. The exact formula (3.5), with $B_{p,\nu}=A_{q,\nu}$, follows on making $N\to\infty$.

A special case of Theorems 3 and 4 deserves attention, viz. when the integrals (1.2) are bounded. We may then suppose that

$$M_1=M_2=\dots=M. \tag{6.2}$$

Let $c_1^*, c_2^*, \dots, c_n^*, \dots$ denote the sequence $|c_1|, |c_2|, \dots$ rearranged in descending order of magnitude. Since we may change the order of the functions φ_n within the sequence $\{\varphi_n\}$, we have

[1] For the case $\nu=\infty$ cf. Verblunsky [1].

Theorem 3'. *If, for $2\leqslant q<\nu$, the series*

$$\sum_{n=1}^{\infty} c_n^{*q}\, n^{\frac{\nu-1}{\nu-2}(q-2)} = \sum_{n=1}^{\infty} c_n^{*q}\, n^{\frac{q-2}{2-\mu}} \tag{6.3}$$

converges, then there is a function $f\in L^q(a,b)$ satisfying (1.1) *and such that*

$$\left(\int_a^b |f|^q\,dx\right)^{1/q} \leqslant A_{q,\nu}\, M^{\frac{\nu}{\nu-2}\cdot\frac{q-2}{q}} \left\{\sum_{n=1}^{\infty} c_n^{*q}\, n^{\frac{\nu-1}{\nu-2}(q-2)}\right\}^{1/q}.$$

Theorem 4'. *If $\mu<p\leqslant 2$, and $c_1, c_2, \ldots$ are the Fourier coefficients of an $f\in L^p$, then*

$$\left\{\sum_{n=1}^{\infty} c_n^{*p}\, n^{\frac{\nu-1}{\nu-2}(p-2)}\right\}^{1/p} \leqslant B_{p,\nu}\, M^{\frac{\nu}{\nu-2}\cdot\frac{(2-p)}{p}} \left(\int_a^b |f|^p\,dx\right)^{1/p}.$$

The constants $A_{q,\nu}$ and $B_{p,\nu}$ are the same as those in Theorems 3 and 4.

Let S denote the sum of the series (6.3) and $\mathfrak{S}$ the sum of the series $\sum|c_n|^p$, where p and q satisfy (1.7). It is not difficult to see that

$$S^{1/q} \leqslant 2^{\frac{3}{2-\mu}}\, \mathfrak{S}^{1/p}. \tag{6.4}$$

In the first place,

$$q=\frac{p\mu}{p-2+\mu}, \qquad \frac{q-2}{2-\mu}+1=\frac{q-\mu}{2-\mu}=\frac{\mu}{p-2+\mu}=\frac{q}{p}\geqslant 1. \tag{6.5}$$

Now,

$$\begin{aligned} S &= \sum_{N=0}^{\infty}\sum_{n=2^N}^{2^{N+1}-1} c_n^{*q}\, n^{\frac{q-2}{2-\mu}} \leqslant 2^{\frac{q-2}{2-\mu}} \sum_{N=0}^{\infty} c_{2^N}^{*q}\, 2^{Nq/p} \\ &\leqslant 2^{\frac{q-2}{2-\mu}} \left(\sum_{N=0}^{\infty} c_{2^N}^{*p}\, 2^N\right)^{q/p} = 2^{\frac{q-2}{2-\mu}+\frac{q}{p}} \left(c_1^{*p} + \sum_{N=1}^{\infty} c_{2^N}^{*p}\, 2^{N-1}\right)^{q/p} \\ &\leqslant 2^{\frac{2q-3}{2-\mu}} \left(c_1^{*p} + \sum_{N=1}^{\infty}\sum_{n=2^{N-1}}^{2^N-1} c_n^{*p}\right)^{q/p} \leqslant 2^{\frac{2q-3}{2-\mu}} \left(2\sum_{n=1}^{\infty} c_n^{*p}\right)^{q/p}, \end{aligned}$$

which easily gives (6.4). Theorem 2, in the special case (6.2), turns out to be a consequence of Theorem 3', although in a less precise

form: into the right-hand side of (1.8) we have to introduce a numerical factor $A_{q,\nu}2^{3/(2-\mu)}$, which depends on q and ν only. A similar connexion may be established between Theorems 4' and 1 [1]).

7. Theorem 5. *Let $q>2$ and suppose that the series* (3.1) *converges. The numbers M_n are supposed to satisfy* (3.2). *Then the series*

$$\sum_{n=1}^{\infty} c_n\varphi_n. \tag{7.1}$$

converges almost everywhere for every rearrangement of its terms. If

$$\sum_{n=1}^{\infty} c_{n_k}\varphi_{n_k} \tag{7.2}$$

is any rearrangement of the series (7.1) *and if*

$$S^*(x)=\operatorname*{Max}_k\left|\sum_{i=1}^{k} c_{n_i}\varphi_{n_i}(x)\right|,$$

then

$$\left\{\int_a^b S^{*q}(x)\,dx\right\}^{1/q} \leqslant A^*_{q,\nu}\sum_{n=1}^{\infty}|c_n|^q\, M_n^{(q-2)\frac{\nu}{\nu-1}}\, n^{(q-2)\frac{\nu-1}{\nu-2}}, \tag{7.3}$$

where $A^*_{q,\nu}$ depends on q and ν only.

For $\nu=\infty$, $M_1=M_2=\ldots=M$, the theorem was established by Paley [2]). The convergence almost everywhere of the series (7.2) is an easy consequence of (7.3), and so it is sufficient to prove the latter inequality.

Let S_m, where $m=1,2,\ldots$, denote the series the terms of which coincide with the terms of the series (7.2) at the places where $2^{m-1}\leqslant n_k\leqslant 2^m-1$; the remaining terms of the series S_m are equal to zero. Let $S_{m,k}(x)$ be the k-th partial sum of the series S_m, and let

$$S_m^*(x)=\operatorname*{Max}_k|S_{m,k}(x)|.$$

Then

$$S^*(x)\leqslant S_1^*(x)+S_2^*(x)+\ldots \tag{7.4}$$

[1]) Cf. Paley [1], or Zygmund [1], p. 206.
[2]) Paley [1].

Lemma 2. Let

$$G(t)=\max_{1\le m\le 2^N-1}\Big|\sum_{n=1}^{m}d_n\varphi_n(t)\Big|.$$

Then, if $q>2$,

$$\Big(\int_a^b G^q(x)\,dx\Big)^{1/q}\le K_{q,\nu}\,2^{N\frac{\nu-1}{\nu-2}\frac{q-2}{q}}\Big({\sum_{n=1}^{2^N}}'|d_n|^q M_n^{\frac{\nu}{\nu-2}(q-2)}\Big)^{1/q}$$

where $K_{q,\nu}$ *depends only on* q *and* ν.

The proof which follows is almost identical with the proof of a lemma of Paley's paper. Let

$$\Psi_{\lambda,m}(x)={\sum_{n=(m-1)2^{N-\lambda}+1}^{m2^{N-\lambda}}}'d_n\varphi_n(x)\qquad(0\le\lambda\le N;\ 1\le m\le 2^\lambda),$$

$$\Psi_\lambda(x)=\max_{1\le m\le 2^\lambda}|\Psi_{\lambda,m}(x)|.$$

Hence

$$G(x)\le\sum_{\lambda=0}^{N}\Psi_\lambda(x).$$

Now, in virtue of Theorem 3,

$$\int_a^b\Psi_\lambda^q(x)\,dx\le\sum_{m=1}^{2^\lambda}\int_a^b|\Psi_{\lambda,m}(x)|^q\,dx$$

$$\le\sum_{m=1}^{2^\lambda}A_{q,\nu}^q\,2^{\frac{\nu-1}{\nu-2}(q-2)(N-\lambda)}{\sum_{n=(m-1)2^{N-\lambda}+1}^{m2^{N-\lambda}}}'|d_n|^q M_n^{\frac{\nu}{\nu-1}(q-2)}$$

$$=A_{q,\nu}^q\,2^{\frac{\nu-1}{\nu-2}(q-2)(N-\lambda)}{\sum_{n=1}^{2^N}}'|d_n|^q M_n^{\frac{\nu}{\nu-1}(q-2)}.$$

By Minkowski's inequality

$$\Big(\int_a^b G^q(x)\,dx\Big)^{1/q}\le\sum_{\lambda=0}^{N}\Big(\int_a^b\Psi_\lambda^q(x)\,dx\Big)^{1/q}$$

$$\le A_{q,\nu}\sum_{\lambda=0}^{N}2^{\frac{\nu-1}{\nu-2}(N-\lambda)\frac{q-2}{q}}\cdot\Big\{{\sum_{n=1}^{2^N}}'|d_n|^q M_n^{\frac{\nu-1}{\nu-2}(q-2)}\Big\}^{1/q}.$$

This gives Lemma 2 with

$$K_{q,\nu}=\frac{2^{\frac{\nu-1}{\nu-2}\frac{q-2}{q}}}{2^{\frac{\nu-1}{\nu-2}\frac{q-2}{q}}-1}A_{q,\nu}\le C\frac{q}{q-2}A_{q,\nu},$$

where C is an absolute constant.

Lemma 3. *For every* $q>2$

$$\int_a^b(S_m^*S_n^*)^{q/2}\,dx\le L_{q,\nu}C_m^{1/2}C_n^{1/2}\,2^{-\lambda_{q,\nu}|m-n|},\tag{7.5}$$

where $L_{q,\nu}$ *and* $\lambda_{q,\nu}$ *depend on* q *and* ν *only, and the* C*'s are defined by* (4.1).

Let $0<\theta<1$ be a number which will be fixed later. Then

$$\int_a^b S_m^{*q/2}S_n^{*q/2}\,dx=\int_a^b S_m^{*q(1-\theta)/2}S_m^{*\theta q/2}S_n^{*q/2}\,dx\le\tag{7.6}$$

$$\le\Big(\int_a^b S_m^{*q}\,dx\Big)^{(1-\theta)/2}\Big(\int_a^b S_m^{*q\theta/(1+\theta)}S_n^{*q/(1+\theta)}\,dx\Big)^{(1+\theta)/2}=AB,$$

say. Let k vary between 2^{m-1} and 2^m-1. On account of Lemma 2,

$$A\le\Big(\int_a^b S_m^{*q}\,dx\Big)^{(1-\theta)/2}\le K_{q,\nu}^{(1-\theta)q/2}\Big\{2^{(m-1)\frac{\nu-1}{\nu-2}(q-2)}\sum|d_k|^q M_k^{\frac{\nu}{\nu-2}(q-2)}\Big\}^{\frac{1-\theta}{2}}\tag{7.7}$$

$$\le K_{q,\nu}^{(1-\theta)q/2}C_m^{(1-\theta)/2}.$$

On the other hand,

$$S_m^*(x)\le\sum|c_k|\,|\varphi_k|\le\big(\sum|c_k|M_k\big)^{1/\mu}\big(\sum|c_k|M_k^{-(\nu-1)}|\varphi_k|^\nu\big)^{1/\nu}.$$

Thence, applying Hölder's inequality, we deduce

$$B\le\Big(\sum|c_k|M_k\Big)^{q\theta/2\mu}\Big\{\int_a^b\Big(\sum|c_k|M_k^{-(\nu-1)}|\varphi_k|^\nu\Big)^{q\theta/\nu(1+\theta)}S_n^{*q/(1+\theta)}\,dx\Big\}^{(1+\theta)/2}\tag{7.8}$$

$$\le\Big(\sum|c_k|M_k\Big)^{q\theta/2\mu}\Big\{\int_a^b\Big(\sum|c_k|M_k^{-(\nu-1)}|\varphi_k|^\nu\Big)dx\Big\}^{q\theta/2\nu}\Big(\int_a^b S_n^{*q_1}\,dx\Big)^{q/2q_1}$$

$$\le\Big(\sum|c_k|M_k\Big)^{q\theta/2}\Big(\int_a^b S_n^{*q_1}\,dx\Big)^{q/2q_1}=B_1B_2,$$

say, where we write, for short,

$$q_1 = \frac{q\nu}{\nu + \theta(\nu - q)}. \tag{7.9}$$

On account of (4.5)

$$B_1 = \Big(\sum |c_k|\, M_k\Big)^{q\theta/2} \leqslant C_m^{\theta/2} \Big(\sum{}' M_k^{\frac{2}{q-1}\frac{\nu-q}{\nu-2}} k^{-\frac{q-2}{q-1}\frac{\nu-1}{\nu-2}}\Big)^{\frac{q-1}{2}\theta}. \tag{7.10}$$

Let us assume that $q_1 > 2$; we see that this is true if for θ we take a sufficiently small value. Moreover $q_1 \leqslant q < \nu$. If the suffix l varies between 2^{n-1} and $2^n - 1$, we may write (cf. Lemma 2)

$$B_2 = \Big(\int_a^b S_n^{*q_1}\, dx\Big)^{q/2q_1} \leqslant K_{q_1,\nu}^{q/2}\, 2^{n\frac{\nu-1}{\nu-2}\frac{q_1-2}{q_1}\cdot\frac{q}{2}} \Big\{\sum{}' |c_l|^{q_1} M_l^{\frac{\nu}{\nu-2}(q_1-2)}\Big\}^{q/2q_1}.$$

To the last sum we apply Hölder's inequality so as to introduce the expression C_n. From this and the inequalities (7.6), (7.7), (7.8) and (7.10) we obtain

$$\int_a^b S_m^{*q/2} S_n^{*q/2}\, dx \leqslant K_{q,\nu}^{(1-\theta)q/2} K_{q_1,\nu}^{q/2} C_m^{1/2} C_n^{1/2}\, 2^{n\frac{\nu-1}{\nu-2}\frac{q_1-2}{q_1}\frac{q}{2}}$$

$$\times \Big\{\sum M_k^{\frac{2}{q-1}\frac{\nu-q}{\nu-2}} k^{-\frac{q-2}{q-1}\frac{\nu-1}{\nu-2}}\Big\}^{\frac{q-1}{2\theta}} \Big\{\sum M_l^{-\frac{2\nu}{\nu-2}} l^{-\frac{\nu-1}{\nu-2}(q-2)\frac{q}{q-q_1}}\Big\}^{\frac{(q-q_1)}{2q_1}}.$$

If $m < n$, we obtain, using (3.2)

$$\int_a^b (S_m^* S_n^*)^{q/2}\, dx \leqslant K_{q,\nu}^{(1-\theta)/2} K_{q_1,\nu}^{q/2} C_m^{1/2} C_n^{1/2}\, 2^{-(n-m)\frac{\nu-q}{\nu-2}\frac{\theta}{2}}.$$

Once the value $\theta = \theta_{q,\nu}$ has been fixed, q_1 becomes a function of q and ν, and the last inequality gives (7.5). For $m = n$, the inequality (7.5) is an easy consequence of Lemma 2.

We shall not investigate the order of the constants $L_{q,\nu}$ and $\lambda_{q,\nu}$.

We are now in a position to establish Theorem 5. We start with the inequality (7.4) and apply to it the argument of the first part of section 5 of this paper. The argument is now even a little simpler, since the inequality (7.5) is established for all q interior to the interval $(2, \nu)$ and so no interpolation by means of convexity theorems is necessary. The expressions S_m^* play now the same part as the functions Φ_m in section 5. Instead of (5.3), we obtain an analogous inequality, with $|f|$ replaced by S^*, and λ_q replaced by $\lambda_{q,\nu}$; moreover, on the right-hand side will appear the factor $L_{q,\nu}$. This is, in a different notation, the inequality (7.3). Theorem 5 is thus established.

8. If in Theorems 3 and 4 we interchange the *rôle* of the function f and of the coefficients c_n, we obtain theorems which are also true. These new theorems will be stated in a moment.

Let $\{\varphi_n\}$ be an orthonormal system in an interval (a, b). We assume for simplicity that (a, b) is of the form $(0, A)$, where $0 < A \leqslant \infty$. This is no restriction of generality, unless (a, b) is of the form $(-\infty, \infty)$. It will be convenient to suppose that $A = \infty$. This we can always do, for if $A < \infty$, we may extend the definitions of f and of the φ_n, by putting $f = 0$, $\varphi_1 = \varphi_2 = \ldots 0$ in the interval (A, ∞).

For any given $f(x)$, we denote by $f^*(x)$ the function equimeasurable with $|f(x)|$ in the interval $(0, \infty)$ and non-increasing. The theorems we intend to prove may be stated as follows[1]).

Theorem 6. *Let $q \geqslant 2$, and let $f(x)$ be defined in $(0, \infty)$. The Fourier coefficients c_n of f with respect to $\{\varphi_n\}$ satisfy the inequality*

$$\Big(\sum_{n=1}^{\infty}{}' |c_n|^q M_n^{2-q}\Big)^{1/q} \leqslant \widetilde{A}_{q,\mu}\Big(\int_0^{\infty} f^{*q} x^{(q-2)/\mu}\, dx\Big)^{1/q}. \tag{8.1}$$

The coefficient $\widetilde{A}_{q,\mu}$ depends on q and μ only and satisfies the inequality

$$\widetilde{A}_{q,\mu} \leqslant Aq/(2-\mu), \tag{8.2}$$

where A is an absolute constant.

Theorem 7. *Let $1 \leqslant p \leqslant 2$ and suppose that the series $\sum |c_n|^p M_n^{2-p}$ and $\sum |c_n|^2$ are both finite. There is then a function f satisfying* (1.1) *and such that*

$$\Big(\int_0^{\infty} f^{*p} x^{(p-2)/\mu}\, dx\Big)^{1/p} \leqslant \widetilde{B}_{p,\mu}\Big(\sum_{n=1}^{\infty} |c_n|^p M_n^{2-p}\Big)^{1/p} \tag{8.3}$$

where $\widetilde{B}_{p,\mu}$ depends on p and μ only. If $1/p + 1/q = 1$, we have

$$\widetilde{B}_{p,\mu} = \widetilde{A}_{p,\mu}. \tag{8.4}$$

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[1]) For the case $\nu = \infty$ cf. Verblunsky [1] and Zygmund [1], 208.

Theorems 6 and 7 hold, *a fortiori*, if we replace there f^* by $|f|$, but the argument we apply below seems to require the use of f^* even for the proof of the weaker results just mentioned. The reader will also note that the numbers M_n are not supposed to satisfy (3.2).

We add that Theorems 6 and 7 hold in the case when the system $\{\varphi_n\}$ is orthonormal in the interval $(-\infty, \infty)$. By f^* we must then mean the function which is equimeasurable with $|f|$, even, and non-increasing in $(0, \infty)$. The integral in (8.1) (and similarly in (8.3)) must then be replaced by $\int\limits_{-\infty}^{\infty} f^{*q} |x|^{(q-2)/\mu} dx$. The proofs undergo but little change.

Finally, we observe that if $f^{*q} x^{(q-2)/\mu} \epsilon L(0, \infty)$, the Fourier coefficients of f with respect to the functions φ_n exist. This follows, in particular, from the argument with which we prove (8.1), but can also be established independently. Applying Hölder's inequality to the integrals $\int\limits_0^h f^{*\mu} dx$ and $\int\limits_h^\infty f^{*2} dx$ so as to introduce the integral $\int f^{*q} x^{(q-2)/\mu} dx$, we obtain at once that $f^* \epsilon L^\mu(0, h)$, $f^* \epsilon L^2(h, \infty)$, whatever $h > 0$. In particular, if E_1 and E_2 denote respectively the sets where $|f| \geqslant 1$ and $|f| < 1$, $|f|^\mu$ is integrable over E_1, $|f|^2$ integrable over E_2. Since $\varphi_n \epsilon L^\nu(0, \infty)$, the product $f\varphi_n$ is integrable over E_1. Similarly, the relation $\varphi_n \epsilon L^2(0, \infty)$ implies the integrability of $f\varphi_n$ over E_2. Hence $f\varphi_n \epsilon L(0, \infty)$, and our assertion is established.

Lemma 4. *Let $f(x)$ be a function non negative and non increasing in $(0, \infty)$. Then, for any pair of integers r and s $(-\infty < r, s < \infty)$, and any $q \geqslant 4$,*

$$(8.5)\qquad \sum_{n=1}^{\infty} M_n^{2-q} \left| \int\limits_{2^r}^{2^{r+1}} f\varphi_n dx \right|^{q/2} \left| \int\limits_{2^s}^{2^{s+1}} f\varphi_n dx \right|^{q/2} \leqslant 2^q C_{r-1}^{1/2} C_{s-1}^{1/2} 2^{-|r-s|\left(\frac{1}{\mu}-\frac{1}{2}\right)},$$

where

$$C_r = \int\limits_{2^r}^{2^{r+1}} f^q x^{(q-2)/\mu} dx.$$

This lemma is an analogue of Lemma 1 $\left(\text{observe that } \frac{1}{\mu} - \frac{1}{2} = \frac{\nu - 2}{2\nu}\right)$ but now we do not suppose the inequality $q < \nu$ be to satisfied.

Let $I_{r,n}$ denote the integral of $f\varphi_n$ over $(2^r, 2^{r+1})$. First of all,

$$|I_{r,n}| \leqslant f(2^r) \int\limits_{2^r}^{2^{r+1}} |\varphi_n| dx \leqslant f(2^r) \left(\int\limits_{2^r}^{2^{r+1}} |\varphi_n|^\nu dx \right)^{1/\nu} 2^{r/\mu} \leqslant f(2^r) M_n 2^{r/\mu}.$$

We write $|I_{r,n}|^{q/2} |I_{s,n}|^{q/2} = |I_{r,n}|^{q/2} |I_{s,n}|^{(q-4)/2} |I_{s,n}|^2$. To the first two factors on the right we apply the previous inequality, and observe that, by Bessel's inequality, $|I_{s,1}|^2 + |I_{s,2}|^2 + \dots$ does not exceed the integral of $|f|^2$ over $(2^s, 2^{s+1})$. Hence the sum in (8.5) does not exceed

$$(8.6)\qquad \{f(2^r) 2^{r/\mu}\}^{q/2} \{f(2^s) 2^{s/\mu}\}^{(q-4)/2} f^2(2^s) 2^s.$$

On the other hand, we have

$$(8.7)\qquad f(2^r) 2^{r-1} \leqslant \int\limits_{2^{r-1}}^{2^r} f dx = \int\limits_{2^{r-1}}^{2^r} f x^{(q-2)/\mu q} x^{-(q-2)/\mu q} dx \leqslant C_{r-1}^{1/q} 2^{(r-1)\left(\frac{q-1}{q} - \frac{q-2}{q\mu}\right)},$$

since, for $\alpha > 0$,

$$\int\limits_{2^{r-1}}^{2^r} x^{-\alpha} dx \leqslant 2^{r-1} 2^{-(r-1)\alpha} = 2^{(r-1)(1-\alpha)}.$$

Applying to (8.6) the inequality (8.7), and the corresponding inequality for $f(2^s) 2^{s-1}$, we obtain (8.5). The lemma is thus established.

We shall now prove Theorem 6, starting with the case when f is non-negative and non increasing, and $q = 4, 5, \dots$ The argument will be similar to that of the first part of section 5. We write

$$c_n = \int\limits_0^\infty f\varphi_n dx = \sum_{m=-\infty}^{\infty} \int\limits_{2^m}^{2^{m+1}} f\varphi_n dx = \sum_{m=-\infty}^{\infty} c_n^m,$$

say. Let $Q = q(q-1)$, $\lambda = \frac{1}{\mu} - \frac{1}{2}$, and let $S(r, s)$ denote the sum in (8.5). Then

$$\sum_{n=1}^{\infty} |c_n|^q M_n^{2-q} \leqslant \sum_{n=1}^{\infty} M_n^{q-2} \left(\sum_{m=-\infty}^{\infty} |c_n^m| \right)^q = \sum_{m_1 \dots m_q} \sum_{n=1}^{\infty} |c_n^{m_1} \dots c_n^{m_q}|.$$

By Hölder's inequality

$$\sum_{n=1}^{\infty} M_n^{2-q} |c_n^{m_1} \dots c_n^{m_q}| \leqslant \prod_{\substack{i,k=1 \\ i \neq k}}^{q} \{S^{q/2}(m_i, m_k)\}^{1/Q} \leqslant 2^q \prod_{i=1}^{q} C_{m_i - 1}^{1/q} \left\{ \prod_{\substack{k=1 \\ k \neq i}}^{q} 2^{-2\lambda |m_i - m_k|/Q} \right\}.$$

Finally we obtain the inequality (analogous to (5.3))

$$\sum_{n=1}^{\infty} M_n^{2-q}|c_n|^q \leqslant \Big(\sum_{m=-\infty}^{\infty} C_m\Big)\Big\{\sum_{n=-\infty}^{\infty} 2^{-\lambda|n|(q-1)}\Big\}^{q-1}.$$

This is the inequality (8.1) which we have thus established for f non-negative and non-increasing, and for $q=4,5,\dots$ For these values of q we have also proved (8.2).

We now pass to the proof of (8.1) for general f, and $q=4,5,\dots$ Instead of (8.1) it is sufficient to establish the apparently weaker inequalities

$$\sum_{n=1}^{N} M_n^{2-q}|c_n|^q \leqslant \tilde{A}_{q,\mu}\int_0^\infty f^{*q}x^{(q-2)/\mu}\,dx \qquad (N=1,2,\dots) \tag{8.8}$$

for from this we obtain (8.1) on making $N\to\infty$.

We make the following remark. Let us suppose that (8.8) is established for every function $f_k(x)$ of a sequence $\{f_k\}$ converging almost everywhere to f. Let us also suppose that $\lim_k c_n^k = c_n$, for $n=1,2,\dots$, where c_n^k denotes the Fourier coefficients of f_k with respect to φ_n. Then (8.8) holds also for the function f, provided that

$$\int_0^\infty f_k^{*q}x^{(q-2)/\mu}\,dx \to \int_0^\infty f^{*q}x^{(q-2)/\mu}\,dx \qquad (k\to\infty). \tag{8.9}$$

We add that if $f_k(x)\to f(x)$ almost everywhere, then $f_k^*(x)\to f^*(x)$, except, perhaps, at the at most enumerable set of points of discontinuity of f^*.

(*a*) Now, (8.8) is certainly true if f is a non-negative step-function assuming only a finite number of values. For in this case we may rearrange the order of the intervals of constancy of f so as to obtain a non-increasing f. At the same time we transform the system $\{\varphi_n\}$ into another orthonormal system. Hence, in the case considered, the theorem is a consequence of the results already established.

(*b*) Let f be bounded and vanishes outside some interval $(0, A)$. We have then $f(x)=\lim f_k(x)$, $c_n=\lim_k c_n^k$, where the f_k are of the form (*a*). We also have (8.9). Hence (8.8) holds in case (*b*).

(*c*) Let f be an arbitrary positive function. Then $f(x)=\lim f_k(x)$, where the f_k form a non-decreasing sequence and have the property (*b*). By Lebesgue's theorem on the integration of monotone sequences we have (8.9). By the same theorem, $\lim_k c_n^k = c_n$ (we consider separately the sets of points where $\varphi_n\geqslant 0$ and $\varphi_n<0$). Hence (8.8) is true again.

(*d*) In the general case, $f=f_1-f_2$ where $f_1=\mathrm{Max}\,(0,f)$, and $f_2=\mathrm{Max}\,(0,-f)$. Correspondingly $c_n=c_n'-c_n''$, and by Minkowski's inequality

$$\Big\{\sum M_n^{2-q}|c_n|^q\Big\}^{1/q} \leqslant \Big\{\sum M_n^{2-q}|c_n'|^q\Big\}^{1/q} + \Big\{\sum M_n^{2-q}|c_n''|^q\Big\}^{1/q}$$
$$\leqslant \tilde{A}_{q,\mu}\Big(\int_0^\infty f_1^{*q}x^{(q-2)/\mu}\,dx\Big)^{1/q} + \tilde{A}_{q,\mu}\Big(\int_0^\infty f_2^{*q}x^{(q-2)/\mu}\,dx\Big)^{1/q} \leqslant$$
$$\leqslant 2\tilde{A}_{q,\mu}\Big(\int_0^\infty f^{*q}x^{(q-2)/\mu}\,dx\Big).$$

Hence (8.1) is proved for general f and $q=4,5,\dots$ In order to extend the result to the remaining values of q, we replace, first, f^* by $|f|$ in (8.1). To the new inequality (which holds for $q=2$ also) we may apply M. Riesz's convexity theorem. This theorem gives, in addition, (8.2). If we now wish to obtain (8.1) (with f^*, and not $|f|$, on the right), we consider, as before, the cases (*a*), (*b*), (*c*), (*d*). In each of these cases, the same argument as before, permits us to replace $|f|$ by f^* [1]). Theorem 6 is thus completely established.

9. It remains to prove Theorem 7. For f we take the function (1.9). Let us fix $N>0$ and put $f_N=c_1\varphi_1+\dots+c_N\varphi_N$. Let $q=p/(p-1)$ We verify that

$$\Big\{\int_0^\infty f_N^{*p}x^{(p-2)/\mu}\,dx\Big\}^{1/p} = \mathop{\mathrm{Max}}_g \int_0^\infty f_N^* g\,dx$$

for all $g\geqslant 0$ with $\Big\{\int_0^\infty g^q x^{(q-2)/\mu}\,dx\Big\}^{1/q}\leqslant 1$. It is even sufficient to restrict $g(x)$ to the domain of step functions vanishing for large x. A moment's consideration shows that

$$\int_0^\infty f_N^* g\,dx = \int_0^\infty f_N\gamma\,dx,$$

[1]) Cf. also Zygmund [1], p. 209.

where the absolute value of the function $\gamma(x)=\gamma(x;\,g,\,N)$ is equimeasurable with g. Let $d_1, d_2, \ldots$ be the Fourier coefficients of γ. Then

$$(9.1)\qquad \left(\int_0^\infty f_N^{*p}\, x^{(p-2)/\nu}\, dx\right)^{1/p} = \operatorname*{Max}_g \int_0^\infty f_N\,\gamma\, dx = \operatorname*{Max}_g \left|\sum_1^N{}' c_n\, d_n\right|$$

$$\leqslant \operatorname{Max}\left(\sum_1^N{}' |c_n|^p M_n^{2-p}\right)^{1/p}\left(\sum_1^N{}' |d_n|^q M_n^{2-q}\right)^{1/q}$$

$$\leqslant \left(\sum_1^N{}' |c_n|^p M_n^{2-p}\right)^{1/q} \operatorname{Max}\left\{\widetilde{A}_{q,\nu}\left(\int_0^\infty \gamma^{*q}\, x^{(q-2)/\nu}\, dx\right)\right\}^{1/q}$$

$$= \left(\sum_1^N{}' |c_n|^p M^{2-p}\right)^{1/p} \widetilde{A}_{q,\nu} \operatorname{Max}\left(\int_0^\infty g^{*q}\, x^{(q-2)/\nu}\, dx\right)^{1/q}$$

$$\leqslant \left(\sum_1^N{}' |c_n|^p M_n^{2-p}\right)^{1/p} \widetilde{A}_{q,\nu} \operatorname{Max}\left(\int_0^\infty g^q\, x^{(q-2)/\nu}\, dx\right)^{1/q}$$

$$\leqslant \widetilde{A}_{q,\nu}\left(\sum_1^N{}' |c_n|^p M_n^{2-p}\right)^{1/p}.$$

On account of the condition $\sum|c_n|^2<\infty$, there is a sequence $\{f_{N_k}(x)\}$ which tends almost everywhere to $f(x)$. Hence $f^*_{N_k}(x)$ tends almost everywhere to $f^*(x)$. Comparing the extreme terms of (9.1) and putting $N=N_k$ we obtain (8.3) and (8.4) by an application of Fatou's well-known lemma. Thus Theorem 7 is established.

Similarly as in the second part of § 6, we may prove that, Theorems 6 and 7 generalize (except for a numerical factor) Theorems 1 and 2.

There is also a theorem which stands to Theorem 6 in the same relation as Theorem 5 to Theorem 3. The rôle of the function $S^*(x)$ is played by the numbers

$$C_n^* = \operatorname*{Max}_{\xi,\eta}\left|\int_\xi^\eta f\,\varphi_n\, dx\right| \qquad (0<\xi<\eta<\infty).$$

We finally add that there is a number of theorems which may be looked upon as intermediate results between Theorems 1 and 2 on the one hand, and Theorems 3, 4, 6, 7 on the other. For $\nu=\infty$ this type of theorem was first studied by Hardy and Littlewood [1]), and their argument can be applied without essential change to the case $\nu<\infty$. We shall not consider this subject here.

[1]) Hardy and Littlewood [1]; cf. also Zygmund [1], 234.

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