$$\frac{f(x_0)-fg(x_0)}{|f(x_0)-fg(x_0)|}=\frac{fg(x_0)-fgg(x_0)}{|fg(x_0)-fgg(x_0)|}$$

ist, was die im Satz 3 behauptete Lage des Tripels $f(x_0)$, $fg(x_0)$, $fgg(x_0)$ zum Ausdruck bringt.

Als Sonderfall des Satzes 3 für ebene Kontinua ist zu erwähnen:

***Satz 4.*** *Zerschneidet ein Kontinuum $X\subset R_2$ die Ebene $R_2$ nicht und ist $g\in X^X$, so gibt es ein $x_0\in X$ derart, daß die Punkte $x_0$, $g(x_0)$, $gg(x_0)$ auf einer Geraden, und zwar in dieser Reihenfolge liegen.*

In der Tat genügt es hierzu, im Satz 3 $f(x)=x$ für alle $x\in X$ zu setzen und die Äquivalenz zwischen dem Nichtzerschneiden der Ebene durch $X$ und dem Zusammenhange von $S_1^X$ [6]) zu verwenden.

Es sei noch bemerkt: wenn die Abbildung $g\in X^X$ involutorisch, d. h. wenn $gg(x)=x$ für alle $x\in X$ ist, so liefert der Satz 3 ein $x_0\in X$, wo $f(x_0)=fg(x_0)$, und der Satz 4 ein $x_0\in X$, wo $g(x_0)=x_0$ ist [7]).

[6]) K. Borsuk, Math. Ann. 106 (1932), S. 246; S. Eilenberg, Fund. Math. 26 (1936), S. 93 und S. 281 (der letzte Beweis ist der einfachste).

[7]) vgl. S. Eilenberg, Fund. Math. 25 (1936), S. 267—268.

# A problem on topological transformation of the plane.

By

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*Does there exist a topological (one-one, continuous) transformation $S$ of a plane into a plane for which the consecutive transforms of a point $M$*

$$SM,\ S^2M,\ S^3M,\ \dots \tag{1}$$

*form a set everywhere dense on the plane?* [1]).

Let $f(\varphi)$ be a continuous function with period $2\pi$ such that

$$\int_0^{2\pi} f(\varphi)\,d\varphi=0 \tag{2}$$

and that for no $\delta$ and no $n$ the function

$$F(\varphi,\delta,n)=f(\varphi)+f(\varphi+\delta)+\dots+f(\varphi+\overline{n-1}\,\delta) \tag{3}$$

is constant in an interval of values of $\varphi$.

We obviously have

$$\int_0^{2\pi} F(\varphi,\delta,n)\,d\varphi=0. \tag{4}$$

We shall use $n, m_1, n_1, m_2, \dots$ for integers and we shall assume the fractions $m_1/n_1$, $m_2/n_2, \dots$ to be irreducible.

In the course of defining the transformation $S$ we shall arrive at two sequences

$$\varepsilon_1, \varepsilon_2, \dots \quad\text{and}\quad k_1, k_2, \dots,$$

the first one of real positive numbers rapidly tending to zero and the second one of positive integers such that

$$k_{i+1}\,\varepsilon_{i+1}>2k_i\,\varepsilon_i \qquad \text{for all } i.$$

[1]) The problem has been put to me by S. Ulam.

It follows from (4) that the function $F(\varphi, \delta, n)$ may take positive and negative values as small as we please. Putting $\delta=\frac{m_1}{n_1}2\pi$ and taking a sufficiently small $\varepsilon_1>0$ we can find $\varphi_1$ such that

$$0<F\left(\varphi_1, \frac{m_1}{n_1}2\pi, n_1\right)=\varepsilon_1. \tag{5}$$

As $F(\varphi, \delta, n)$ is a continuous function of $\varphi$ and $\delta$, we conclude that given a positive integer $k_1$ there exist positive numbers $\tau_1, \alpha_1$ such that for

$$|\varphi|<\tau_1, \qquad |a|<\alpha_1 \tag{6}$$

the following inequalities are satisfied:

$$\left|F\left\{\varphi_1+\varphi, \left(\frac{m_1}{n_1}+a\right)2\pi, n\right\}-F\left(\varphi_1, \frac{m_1}{n_1}2\pi, n\right)\right|<\varepsilon_1^2 \quad \text{for } 0<n<k_1n_1, \tag{7}$$

$$k_1 n_1 \alpha_1<\min\left(\varepsilon_1^2, \frac{1}{n_1^2}\right). \tag{8}$$

Choose a rational $a_1$ such that

$$|a_1|<\frac{1}{2}\alpha_1, \qquad \frac{m_1}{n_1}+a_1=\frac{m_2}{n_2}, \tag{9}$$

where $n_2$ satisfies the inequalities

$$\frac{2\pi}{n_2}<\frac{1}{2}\tau_1 \qquad n_2>k_1n_1. \tag{10}$$

The function $F\left(\varphi_1+\varphi, \frac{m_2}{n_2}2\pi, n_2\right)$ is periodic with respect to $\varphi$ with period $2\pi/n_2$. Therefore we can find $\varphi_2$

$$0\leqq\varphi_2<2\pi/n_2 \tag{11}$$

and

$$\varepsilon_2<\varepsilon_1^2 \tag{12}$$

so that

$$F\left\{\varphi_1+\varphi_2, \frac{m_2}{n_2}2\pi, n_2\right\}=-\varepsilon_2. \tag{13}$$



Similarly to (6) we conclude that given a positive integer $k_2$ there exist positive numbers

$$\tau_2<\frac{1}{2}\tau_1, \qquad \alpha_2<\frac{1}{2}\alpha_1 \tag{14}$$

such that for

$$|\varphi|<\tau_2, \qquad |a|<\alpha_2 \tag{15}$$

the following inequalities are satisfied:

$$\left|F\left\{\varphi_1+\varphi_2+\varphi, \left(\frac{m_2}{n_2}+a\right)2\pi, n\right\}-F\left(\varphi_1+\varphi_2, \frac{m_2}{n_2}2\pi, n\right)\right|<\varepsilon_2^2 \quad \text{for } n\leqq k_2n_2, \tag{16}$$

$$k_2n_2\alpha_2<\min\left(\varepsilon_2^2, \frac{1}{n_2^2}\right). \tag{17}$$

Now again choose $a_2$

$$|a_2|<\frac{1}{2}\alpha_2 \tag{18}$$

$$\frac{m_2}{n_2}+a_2=\frac{m_3}{n_3}, \tag{19}$$

so that

$$\frac{2\pi}{n_3}<\frac{1}{2}\tau_2, \qquad n_3>k_2n_2. \tag{20}$$

Similarly to (11), (12), (13) we can choose

$$0\leqq\varphi_3<2\pi/n_3, \tag{21}$$

$$\varepsilon_3<\varepsilon_2^2 \tag{22}$$

so that

$$F\left(\varphi_1+\varphi_2+\varphi_3, \frac{m_3}{n_3}2\pi, n_3\right)=\varepsilon_3, \tag{23}$$

and so on.

Writing

$$\varphi_1+\varphi_2+\varphi_3+\ldots=\varphi_0, \qquad \lim_{i\to\infty}\frac{m_i}{n_i}=\theta,$$

we define the transformation $S$ by the equation

$$S\,r\,e^{i\varphi}=r\,e^{f(\varphi)}\,e^{i(\varphi+2\pi\theta)}. \tag{24}$$

We shall prove that the consecutive transforms of any point of the line $\varphi=\varphi_0$, except the origin, fill up densely the whole plane.

From (10), (11), (14), (20), (21) it follows that

$$|\varphi_0-\varphi_1|<\tau_1, \qquad |\varphi_0-\varphi_1-\varphi_2|<\tau_2, \qquad \dots \tag{25}$$

We have similarly

$$\left|\frac{m_1}{n_1}-\theta\right|<\alpha_1, \qquad \left|\frac{m_2}{n_2}-\theta\right|<\alpha_2, \qquad \dots \tag{26}$$

Let $M_0$ be the point $e^{i\varphi_0}$. We have

$$S^n M_0=e^{F(\varphi_0,2\theta\pi,n)}e^{(\varphi_0+2n\theta\pi)i}. \tag{27}$$

Consider the points

$$S^{kn_1}M_0 \qquad k=1,2,\dots,k_1 \tag{28}$$

$$S^{kn_2}M_0 \qquad k=1,2,\dots,k_2 \tag{29}$$

$$\dots\dots \qquad \dots\dots$$

We have

$$S^{kn_1}M_0=e^{F(\varphi_0,2\theta\pi,kn_1)}e^{(\varphi_0+kn_1\theta 2\pi)i}.$$

We shall denote further by $\beta$ any real number of modulus $<1$. By (5), (7), (8), (25), (26)

$$F(\varphi_0,\theta 2\pi,kn_1)=F\left(\varphi_1,\frac{m_1}{n_1}2\pi,kn_1\right)+\beta\varepsilon_1^2=k\varepsilon_1+\beta\varepsilon_1^2,$$

$$|kn_1\theta 2\pi|<\frac{2\pi}{n_1^2} \pmod{2\pi}.$$

Thus all the points (28) are in the angle $\left(\varphi_0-\frac{2\pi}{n_1^2},\ \varphi_0+\frac{2\pi}{n_1^2}\right)$ and their distances form the origin are $e^{k\varepsilon_1+\beta\varepsilon_1^2}$, $k=1,2,\dots,k_1$.

Similarly by (16), (17), (25), (26)

$$F(\varphi_0,\theta 2\pi,kn_2)=F\left(\varphi_1+\varphi_2,\frac{m_2}{n_2}2\pi,kn_2\right)+\beta\varepsilon_2^2=-k\varepsilon_2+\beta\varepsilon_2^2,$$

$$|kn_2\theta 2\pi|<\frac{2\pi}{n_2^2} \pmod{2\pi}.$$

Thus all the points (29) lie in the angle $\left(\varphi_0-\frac{2\pi}{n_2^2},\ \varphi_0+\frac{2\pi}{n_2^2}\right)$ and their distances from the origin are $e^{-k\varepsilon_2+\beta\varepsilon_2^2}$ $k=1,2,\dots,k_2$.



Going on this way and remembering the law of increase of $k_i\varepsilon_i$, we see that any point of the line $\varphi=\varphi_0$ is a limit point of the set of points $S^{kn_i}M_0$, $k=1,2,\dots,k_i$, $i=1,2,\dots$ and *a fortiori* of the set

$$S^n M_0 \qquad n=1,2,3,\dots \tag{30}$$

Applying the transformation $S^j$ to the set (30), we conclude that any point of the line

$$\varphi=\varphi_0+j\theta 2\pi \tag{31}$$

for any $j$ is also a limit point of the transformed set (30) and *a fortiori* of the set itself, since it includes its transformation. But the lines (31) forming an everywhere dense set on the plane, we conclude that the set (30) is everywhere dense on the whole of the plane, and thus *the transformation $S$ gives an answer in affirmative to our problem.*

Noticing that

$$S^n r e^{i\varphi_0}=r S^n e^{i\varphi_0},$$

we conclude that the set

$$S^n M \qquad n=0,1,2,\dots,$$

where $M$ is any point of the line $\varphi=\varphi_0$ is also everywhere dense on the plane.

It is easy to see that also transforms of any point of the plane, except the origin, have the same property of filling up densely the whole of the plane.

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