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On the asphericity of regions in a 3-sphere.

By

J. H. C. Whitehead (Oxford).

1. This note arises out of an attempt to answer two questions proposed by S. Eilenberg[1]), namely:

1. *pour quelles courbes simples fermées $\Omega_0 \subset S^3$ l'ensemble $S^3 - \Omega_0$ est asphérique?*
2. *pour quels couples $\Omega_1, \Omega_2 \subset S^3$ de courbes simples fermées disjointes l'ensemble $S^3 - (\Omega_1 + \Omega_2)$ est asphérique?*

I first show how Reidemeister's theory[2]) of "Homotopiekettenringe" can be applied to the study of the first question in case Ω_0 is a polygonal knot, which I will call k, and in **4** I show how the methods of **2** and **3** can be applied to the study of similar questions. Some examples are given in **5**, and **6** contains an "addition theorem" with an application to the study of knots and linkages. Taking $M^3 = S^3$ it follows from theorem 6, in **7**, that the hypothesis "$S^3 - k$ is aspherical", k being any polygonal knot in a 3-sphere S^3, implies the algebraic analogue of Dehn's lemma[3]) for circuits in S^3 (i.e. it implies that, if k bounds a singular 3-cell without singularities on the boundary, then $\pi_1(S^3 - k)$ is cyclic.) The final section is an appendix on the group ring of an "indexed" group[4]).

[1]) Fund. Math., **28** (1937), p. 241. We recall that a space X, is called aspherical (W. Hurewicz, Proc. Akad. Amsterdam, **39** (1936), p. 215) if all the (additive) higher homotopy groups $\pi_n(X)$, $n > 1$, reduce to zero. $\pi_1(X)$ is the (multiplicative) fundamental group, which need not reduce to 1.

[2]) See: Abh. Math. Sem. Hamburg, **10** (1934), p. 211; Journal für die r. u. a. Math., **173** (1935), p. 164, and other papers.

[3]) Math. Annalen, **69** (1910), p. 147. There is a gap in Dehn's argument (at the top of p. 151) which has not yet been filled. See also E. Pannwitz, Math. Annalen, **108** (1933), p. 629 (§3), and I. Johansson, Math. Annalen, **110** (1934), p. 312 and **115** (1938), p. 658.

[4]) J. W. Alexander, Trans. American Math. Soc., **30** (1928), 290.

I wish to express my gratitude to Dr. Eilenberg for many valuable suggestions, which have led, among other things, to great improvements in theorems 5 and 6 (the proof of theorem 5 is due mainly to him).

2. Let $k \subset R^3$ be any oriented, polygonal knot in $S^3 = R^3 + w$, where R^3 is Euclidean 3-space and w is an ideal point at infinity. Let R^2 be any plane which does not meet k, and let V be a point which is separated from k by R^2 and which is in general position relative to k. Let C be the singular 2-cell swept out by the recti- linear segment VP as P describes k. Let $P_1, \dots, P_n$ be the points in which k pierces C, arranged so that P_i and $P_{i+1} (P_{n+1} = P_1)$ are consecutive and the cyclic order $P_1, \dots, P_n$ determines a positive orientation of k ($n > 2$ if k is knotted and we may assume that $n > 2$ even if k is unknotted). Let T^3, an open set, be a thin tubular neughbourhood of k, whose (polyhedral) boundary $\dot{T}^3$, cuts C in simple circuits $\bar{f}_1, \dots, \bar{f}_n$ enclosing $P_1, \dots, P_n$; let Q_i be the point in which $\bar{f}_i$ meets VP_i, and let $\bar{e}_i = Q_i Q_{i+1}$ be that segment of the circuit $C \cdot \dot{T}^3$ which does not contain the other points Q_j. Let e_i and f_i be the 1-cells $\bar{e}_i - (Q_i + Q_{i+1})$ and $\bar{f}_i - Q_i$, let E_i^2 be the 2-cell on $\dot{T}^3$ whose point-set frontier is $\bar{f}_i + \bar{e}_i + \bar{f}_{i+1}$, and let F_i^2 be the 2-cell on C whose frontier consists of [5]) VQ_i, e_i and VQ_{i+1}, together with f_j and VQ_j for every j such that P_j lies in the 2-element swept out by VQ as Q describes $\bar{e}_i$. Then the frontier of the 3-cell $E^3 = S^3 - (\bar{T}^3 + C)$ consists of

$$\sum_{i=1}^{n} (E_i^2 + F_i^2 + e_i + f_i + VQ_i).$$

We now find a set of generators and relations for $G = \pi_1(S^3 - T^3)$, which can be written down by a well known rule [6]) from the projection of k on R^2 from V, taking V to be "underneath" R^2. Let us describe as the positive side of F_i^2 that side from which the segment VQ seems to be moving anti-clockwise as Q describes $\bar{e}_i$ positively (i.e. from Q_i to Q_{i+1}). With any point in E^3 as base point, let a_i $(i = 1, \dots, n)$ be the element of G corresponding to an oriented

[5]) We shall use VQ_i to stand both for the open 1-cell and for the (rectilinear) segment consisting of this 1-cell and its end points, V and Q_i. It will always be clear from the context which is meant.

[6]) See, for example, K. Reidemeister, *Knotentheorie*, Berlin (1932), pp. 50 and 53.

circuit in $S^3 - T^3$ which pierces C just once, in F_i^2, the arrow of orientation pointing from the negative to the positive side of F_i^2. Then as is well known, G is generated by $a_1, \dots, a_n$, subject to the relations

$$a_{j_i}^{\varepsilon_i} a_i = a_{i-1} a_{j_i}^{\varepsilon_i} \qquad (i = 1, \dots, n; a_0 = a_n), \tag{2.1}$$

where $f_i \subset \bar{F}_{j_i}^2$ and $\varepsilon_i = +1$ if e_i lies on the positive side of $F_{j_i}^2$, $\varepsilon_i = -1$ if e_i lies on the negative side of $F_{j_i}^2$. Let $\mathfrak{R} = \mathfrak{R}(G)$ be the integral group ring of G, the elements of $\mathfrak{R}$ being linear forms

$$n_1 \beta_1 + n_2 \beta_2 + \dots,$$

where $\beta_i \in G$ and $n_1, n_2, \dots$ are rational integers, all but a finite number of which are zero.

Two remarks concerning "Homotopiekettenringe": First the r-chains in $\widetilde{K}$, the universal covering complex of a complex K, will be written as

$$C_1^r \xi_1 + C_2^r \xi_2 + \dots + C_p^r \xi_p, \qquad \xi_i \in \mathfrak{R}\{\pi_1(K)\},$$

with the coefficients ξ_i on the right of the basis elements $C_1^r, \dots, C_p^r$. Secondly we recall a rule which, for the purposes of reference, we shall call the *fundamental rule*. Let E^r be any cell in K and let s_1 and s_2 be two oriented segments leading from a point O, in K, to a point in E^r. Let a be the element in $\pi_1(K)$, with base point O, corresponding to the oriented circuit $s_1 - s_2$. As usual, let a point $\widetilde{P}$ in $\widetilde{K}$ be defined by a class of segments in K joining O to an arbitrary point P, two segments being in the same class if, and only if, one is deformable into the other with O and P held fixed throughout the deformation. Let $\widetilde{E}_i^r$ be the cell in $\widetilde{K}$ corresponding to E^r and the segment s_i $(i = 1, 2)$. Then the fundamental rule is

$$\widetilde{E}_1^r = \widetilde{E}_2^r a.$$

We have seen that $S^3 - T^3$ carries a complex K, consisting of a 3-cell E^3, 2-cells E_i^2, F_i^2, 1-cells e_i, f_i and VQ_i, and vertices V, Q_i. Let $\widetilde{K}$ be the universal covering complex of K; let $O \in E^3$ be the base point of $G = \pi_1(K)$ and $\widetilde{O}$ the point in $\widetilde{K}$ corresponding to the unit element G. As a basis for the r-chains $(r = 1, 2, 3)$ in $\widetilde{K}$, with coefficients in $\mathfrak{R}$, we take:

(1) ($r=3$) the 3-cell $\mathcal{E}^3$, covering E^3, which contains $\widetilde{O}$,

(2) (r=2) the 2-cells $\mathcal{E}_i^2$, on $\dot{\mathcal{E}}^3$ covering E_i^2, together with the 2-cells $\mathcal{F}_i^2$, on $\dot{\mathcal{E}}^3$ covering the positive sides of F_i^2,

(3) ($r=1$) the 1-cells $\widetilde{e}_i$ incident with both $\mathcal{E}_i^2$ and $\mathcal{F}_i^2$ and the 1-cells $\widetilde{f}_i$, on $\dot{\mathcal{E}}_i^2$ covering f_i. Thus, if $\mathcal{E}_i^2$, seen from inside $\mathcal{E}^3$, is visualized as a long thin rectangle with an arrow pointing lengthwise, in accordance with the orientation of k, $\widetilde{e}_i$ is the long side on the left of the arrow and $\widetilde{f}_i$ is the short side at the beginning of the arrow. The remaining 1-cells, and the 0-cells of the basis may be chosen arbitrarily.

We now fix the orientations of the various cells. Let $abcd$ be a simplex in E^3, of which the face abc is in F_i^2 and d is on the positive side of F_i^2. Then the positive orientation of E^3 is the one determined by the even permutations of $abcd$ if the cyclic order abc, when seen from d, determines a clockwise rotation [7]). The positive orientation of F_i^2 is to be the one determined by the cyclic order abc. The same applies to E_i^2, the simplex abc being in this case in E_i^2 and the vertex d inside E^3. Thus $\mathcal{F}_i^2$ and $\mathcal{E}_i^2$ are positively related to $\mathcal{E}^3$, and it follows from the fundamental rule that

$$\dot{\mathcal{E}}^3=\sum_{i=1}^{n}\{\mathcal{E}_i^2+\mathcal{F}_i^2(1-a_i)\}. \tag{2.2}$$

The segments $\bar{e}_i$ have already been oriented, Q_i being the first and Q_{i+1} the last point for each value of i. Thus $\widetilde{e}_i$ is positively related to $\mathcal{E}_i^2$, negatively related to $\mathcal{F}_i^2$, and is not on the boundary of any other 2-cell in the basis. By the fundamental rule

$$\dot{\mathcal{E}}_i^2=\widetilde{e}_i(1-a_i)+\dots,\qquad \dot{\mathcal{F}}_i^2=-\widetilde{e}_i+\dots \tag{2.3}$$

The circuits $\bar{f}_i$ are to be oriented in the sense $Q_i^-Q_iQ_i^+$, where $Q_i^-Q_iQ_i^+$ is an oriented arc of $\bar{f}_i$ passing from the negative to the positive side of F_i^2. Thus $\widetilde{f}_i$ is negatively related to $\mathcal{E}_i^2$. The remaining orientations may be assigned arbitrarily.

Now let

$$\Gamma^2=\sum_{i=1}^{n}(\mathcal{E}_i^2\,\xi_i+\mathcal{F}_i^2\,\eta_i) \tag{2.4}$$

be any 2-chain in K.

[7]) This may be taken as the definition of "clockwise".

Then

$$\dot{\Gamma}^2=\sum_{i=1}^{n}\widetilde{e}_i\{(1-a_i)\,\xi_i-\eta_i\}+\dots,$$

whence, if Γ^2 is a cycle,

$$\eta_i=(1-a_i)\,\xi_i. \tag{2.5}$$

In finding the coefficient of $\widetilde{f}_i$ in $\dot{\Gamma}^2$ we distinguish between two cases. First let E_i lie on the positive side of $F_{j_i}^2$. Then

$$\dot{\mathcal{E}}_i^2=-\widetilde{f}_i+\dots,\qquad \dot{\mathcal{E}}_{i-1}^2=\widetilde{f}_i a_{j_i}+\dots,\qquad \dot{\mathcal{F}}_{j_i}^2=\widetilde{f}_i+\dots,$$

and $\widetilde{f}_i$ is not on the boundary of any other 2-cell in the basis. If Γ^2 is a cycle it follows from these equations and from (2.5) that

$$-\xi_i+a_{j_i}\xi_{i-1}+(1-a_{j_i})\,\xi_{j_i}=0,$$

or

$$\xi_i-\xi_{j_i}=a_{j_i}(\xi_{i-1}-\xi_{j_i}). \tag{2.6}$$

On the other hand if E_i^2 is on the negative side of $F_{j_i}^2$ we have

$$\dot{\mathcal{E}}_i=-\widetilde{f}_i+\dots,\qquad \dot{\mathcal{E}}_{i-1}^2=\widetilde{f}_i a_{j_i}^{-1}+\dots,\qquad \dot{\mathcal{F}}_{j_i}^2=-\widetilde{f}_i a_{j_i}^{-1}+\dots,$$

whence, if Γ^2 is a cycle,

$$-\xi_i+a_{j_i}^{-1}\xi_{i-1}-a_{j_i}^{-1}(1-a_{j_i})\,\xi_{j_i}=0,$$

or

$$\xi_i-\xi_{j_i}=a_{j_i}^{-1}(\xi_{i-1}-\xi_{j_i}). \tag{2.7}$$

The equations (2.6) and (2.7) may be combined into

$$\xi_i-\xi_{j_i}=a_{j_i}^{\varepsilon_i}(\xi_{i-1}-\xi_{j_i}), \tag{2.8}$$

where, as in (2.1), $\varepsilon_i=+1$ or -1 according as e_i lies on the positive or negative side of $F_{j_i}^2$.

The equations (2.8) are obviously satisfied if $\xi_1=\dots=\xi_n=\xi$, for any value of ξ. Such a set of solutions will be described as *trivial*, any other as *non-trivial*.

Theorem 1. *Any 2-cycle in $\widetilde{K}$ is of the form*

$$\Gamma^2=\sum_{i=1}^{n}\{\mathcal{E}_i^2+\mathcal{F}_i^2(1-a_i)\}\,\xi_i, \tag{2.9}$$

where $\xi_1,\dots,\xi_n$ satisfy (2.8), *and $\Gamma^2\sim 0$ if and only if $\xi_1=\dots=\xi_n$. Conversely, the 2-chain given by* (2.9) *is a cycle if $\xi_1,\dots,\xi_n$ satisfy* (2.8).

It follows from (2.5) that any 2-cycle in $\tilde{K}$ is of the form (2.9), and we have proved that if Γ^2, given by (2.9), is a cycle, then $\xi_1, \dots, \xi_n$ satisfy (2.8). If $\xi_1 = \dots = \xi_n = \xi$, say, we have $\Gamma^2 = (\mathcal{S}^3 \xi)^\bullet$, and since $\mathcal{S}^3$ is a basis for the 3-chains in $\tilde{K}$, any bounding 2-cycle is of this form, and is therefore given by (2.9) with $\xi_1 = \dots = \xi_n$.

It remains to prove that Γ^2, given by (2.9), is a cycle provided $\xi_1, \dots, \xi_n$ satisfy (2.8). Let the basis for the 1-chains in $\tilde{K}$ be completed by segments $\tilde{V}\tilde{Q}_1, \dots, \tilde{V}\tilde{Q}_n$, covering $VQ_1, \dots, VQ_n$, and let the co-efficient of $\tilde{V}\tilde{Q}_i$ in $\dot{\Gamma}^2$ be ζ_i. The co-efficients of $\tilde{e}_i$ in $\dot{\Gamma}^2$ are zero, and, in consequence of (2.8), so are the co-efficients of $\tilde{f}_i$. Therefore the co-efficient of $\tilde{Q}_i$ in $\ddot{\Gamma}^2 = (\dot{\Gamma}^2)^\bullet$ is $\pm\zeta_i$. But $\ddot{\Gamma}^2 = 0$. Therefore $\zeta_i = 0$, whence $\dot{\Gamma}^2 = 0$, and the theorem is established.

It is obvious that $\pi_r(S^3 - k) = \pi_r(S^3 - T^3) = \pi_r(K)$ for all values of r, and well known that $\pi_r(K) = \pi_r(\tilde{K})$ if $r > 1$. Since $\pi_1(\tilde{K}) = 1$ it follows from a theorem due to Hurewicz [8]) that $\pi_2(\tilde{K}) = \beta_2(\tilde{K})$, where $\beta_n(X)$ is the n^{th} homology group of X. Since $\beta_r(\tilde{K}) = 0$ if $r > 2$ it follows from the same theorem that if $\beta_2(\tilde{K}) = 0$, then $\pi_r(\tilde{K}) = \beta_r(\tilde{K}) = 0$ for all values of r, and hence that $\pi_r(K) = 0$ if $r > 1$. Therefore we have

Theorem 2. *The space $S^3 - k$ is aspherical if, and only if, the equations* (2.8) *imply $\xi_1 = \dots = \xi_n$.*

There is a simple rule for writing down the equations (2.8), possibly with trivial modifications, from the projection of the knot on R^2 from V, which is to be regarded as "below" the plane R^2. Just one equation corresponds to each crossing, and if s_j is the upper segment at any crossing, s_h and s_i the lower segments ($h = i-1$ or $i+1$), and if s_i is the one which, seen from above, is on the left of s_j, positively described, then the corresponding equation is

$$\xi_i - \xi_j = a_j(\xi_h - \xi_j), \tag{2.10}$$

irrespective of whether $h = i-1$ or $i+1$.

3. ***Theorem 3.*** *Any one of the equations* (2.8) *is satisfied in consequence of the others.*

It is enough to prove that Γ^2, given by (2.9), is a cycle in consequence of equations (2.8) with $i = 1, \dots, r-1, r+1, \dots, n$, say with

[8]) Proc. Akad. Amsterdam, **38** (1935), p. 522.

$i = 1, \dots, n-1$. The co-efficients of $\tilde{e}_1, \dots, \tilde{e}_n$ in $\dot{\Gamma}^2$ are zero for any values of $\xi_1, \dots, \xi_n$, and the co-efficients of $\tilde{f}_1, \dots, \tilde{f}_{n-1}$ are zero in consequence of (2.8) with $i = 1, \dots, n-1$. As in the last part of theorem 1 the co-efficients of $\tilde{V}\tilde{Q}_1, \dots, \tilde{V}\tilde{Q}_{n-1}$ in $\dot{\Gamma}^2$ are zero. Therefore the co-efficient of V in $\ddot{\Gamma}^2$ is $\pm\zeta_n$, where ζ_n is the co-efficient of $\tilde{V}\tilde{Q}_n$. Therefore $\zeta_n = 0$, and the co-efficients in $\dot{\Gamma}^2$ of all the 1-cells, except possibly $\tilde{f}_n$, are zero.

Let $\dot{\Gamma}^2 = \tilde{f}_n \varrho$ and let $\tilde{Q}_n$ be taken as the common end point, of $\tilde{e}_n$ and $\tilde{f}_n$. Then it follows from the fundamental rule that

$$\dot{\tilde{f}}_n = \tilde{Q}_n(a_n - 1).$$

Therefore $\ddot{\Gamma}^2 = \tilde{Q}_n(a_n - 1)\varrho$, whence $(a_n - 1)\varrho = 0$. Making multiplication commutative in G we see that a_n is not of finite order. It follows from an argument in **8** below that $a_n - 1$ is not a 0-divisor, and hence that $\varrho = 0$. Therefore Γ^2 is a cycle and the theorem is established.

It follows from theorem 1 that the 2-cycles in $\tilde{K}$ are in a (1—1) correspondence, given by (2.9), with the sets of solutions of (2.8), the bounding cycles corresponding to the trivial sets of solutions. Thus we may think of solutions $\xi_1, \dots, \xi_n$, of (2.8), as "co-ordinates" of the cycle (2.9). In each homology class there is a unique cycle, namely $\Gamma^2 - \dot{\mathcal{S}}^3 \xi_m$, whose m^{th} co-ordinate vanishes. Combining these observations with theorem 3 we have;

Theorem 4. *The 2-dimensional homology classes of $\tilde{K}$ are in* (1—1) *correspondence with the sets of solutions of the equations*

$$\xi_\lambda - \xi_{j_\lambda} = a_{j_\lambda}^{\varepsilon_\lambda}(\xi_{\lambda-1} - \xi_{j_\lambda}), \qquad \xi_m = 0, \tag{3.1}$$

where $\lambda = 1, \dots, r-1, r+1, \dots, n$ $(\xi_0 = \xi_n)$.

There is a close formal analogy between the equations (3.1), in which we take $m = r = n$ to simplify the notation, and the equations from which Alexander's polynomial is derived. For write $a_n = t$, $a_i = a_i t$. Then the relations (2.1), expressed in terms of $a_1, \dots, a_{n-1}, t$, become

$$(a_{j_i} t)^{\varepsilon_i} a_i t = a_{i-1} t (a_{j_i} t)^{\varepsilon_i}, \qquad a_n = 1.$$

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If $\varepsilon_i=1$ we have $a_{j_i} t a_i = a_{i-1} t a_{j_i}$, or

$$a_{1j_i} a_{0i} = a_{1\,i-1} a_{0j_i}, \tag{3.2}$$

where $a_{pl}=t^{-p} a_l t^p$. Making multiplication commutative between a_{pi} and a_{qj} $(i,j=1,\dots,n-1;\ p,q=0,\pm 1,\dots)$, replacing commutative multiplication by commutative addition, and 1 by 0, and writing $a_{pj}=x^p a_j$, (3.2) becomes

$$a_i + x a_{j_i} = a_{j_i} + x a_{i-1} \quad \text{or} \quad a_i - a_{j_i} = x(a_{i-1} - a_{j_i})$$

with $a_n=0$. A similar process leads to

$$a_i - a_{j_i} = x^{-1}(a_{i-1} - a_{j_i})$$

if $\varepsilon_i=-1$. It is well known that each of the relations (2.1) is implied by the rest of them, and, omitting the one for which $i=n$, what may be called Alexander's relations take the form

$$a_\lambda - a_{j_\lambda} = x^{\varepsilon_\lambda}(a_{\lambda-1} - a_{j_\lambda}), \qquad a_n=0, \tag{3.3}$$

where $\lambda=1,\dots,n-1$ $(a_0=a_n=0)$.

If multiplication is made commutative in G, and hence in $\mathfrak{R}$, and if ξ_λ is replaced by a_λ, and a_i by x, the equations (3.1) become (3.3). If $a_1, a_2, \dots, a_n$ are treated as unknown variables the determinant of the co-efficients in (3.3) is $\pm x^p \Delta(x)$, where $\Delta(x)$ is Alexander's polynomial. Since[9] $\Delta(x) \neq 0$ any solutions of (3.1) must reduce to $(0,\dots,0)$ when multiplication is made commutative.

Except for the paragraph containing theorem 2 everything we have said remains valid if $\mathfrak{R}(G)$ is replaced by $\mathfrak{R}(G/G_0)$, where G_0 is any invariant sub-group of G, and $\widetilde{K}$ by a regular covering of K with G_0 as its fundamental group. Taking G_0 to be the commutator group of G, it follows from the last paragraph that, in this case[10], $\beta_2(\widetilde{K})=0$.

4. Let the knot k in **1** be replaced by a finite, but not necessarily connected, semi-linear graph g. Then the residual space S^3-g may be studied by similar methods to those described in **1**. The star C, with V as its centre, is defined as before. Instead of a solid torus T^3 we have a 3-dimensional manifold bounded by one or more surfaces

[9]) $\Delta(1)=1$ (Alexander, loc. cit., p. 299).

[10]) This is also implicit in Eilenberg's results (loc. cit. theorems 1 and 4).

of arbitrary genus. Here, as in **1**, it may be shown that E^3 is a 3-cell (i.e. a homeomorphic image of R^3). But this is unnecessary, since we only need to know that $\pi_1(E^3)=1$, $\beta_2(E^3)=0$, which is obvious. The only essential differences are those which arise when g contains multiple vertices, that is to say vertices which are incident with more than two edges (rejecting the case where $\beta_1(g)=0$, we may assume that every vertex of g is incident with at least two edges). First we assign either one of the two possible orientations to each "string" in g, where a string means either a line whose end-points, but none of whose internal points, are multiple vertices, or a circuit containing at most one multiple vertex. Then, projecting g on R^2 from V (as before V is to be below R^2) the generators of $G=\pi_1(S^3-g)$ are defined as in the case of a knot or linkage, except that the segments $s_1,\dots,s_n$, corresponding to the generators $a_1,\dots,a_n$, may terminate either at a crossing or at a multiple vertex. To each crossing corresponds a relation of the form (2.1) and to a multiple vertex P, a relation of the form

$$a_{i_1}^{\varepsilon_1} \dots a_{i_p}^{\varepsilon_p} = 1, \tag{4.1}$$

where $s_{i_1},\dots,s_{i_p}$ are the segments incident with P, and $\varepsilon_i=\pm 1$, according as P is the first or last point of s_i.

The manifold M^2, corresponding to $\dot{T}^3$, may be cut up into cells E_i^2 by the method used in **1**. If $E_{i_1}^2,\dots,E_{i_p}^2$ are those which correspond to the segments $s_{i_1},\dots,s_{i_p}$ meeting at a multiple vertex P, we take the 1-cells of $\dot{E}_{i_\lambda}^2$ in the neighbourhood of P to be segments t_λ and $t_{\lambda+1}$ $(t_{p+1}=t_1)$, joining the two points in which VP meets M^2 near P, and such that $\bar{E}_{i_\lambda}^2$ meets $\bar{E}_{i_{\lambda+1}}^2$ in $t_{\lambda+1}$. If the orientations are such that each cell of M^2 is positively related to E^3 we can attach a unique set of co-ordinates $\xi_1,\dots,\xi_n$ to each 2-cycle in $\widetilde{K}$, where $\xi_1,\dots,\xi_n$ satisfy equations defined as follows:

(1) to each crossing corresponds an equation of the form (2.10),

(2) to each multiple point corresponds an equation of the form $\xi_{i_1}=\dots=\xi_{i_p}$, where $i_1,\dots,i_p$ mean the same as in (4.1).

There is no difficulty in extending theorems 1, 3 and 4 to the situation considered here, and theorem 2 is valid if the discarded equation is one which corresponds to a crossing.

5. *Examples*:

1. The equations (2.10) are:

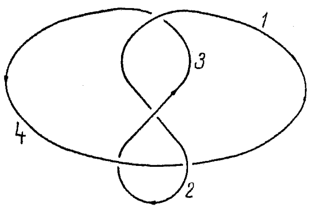

Fig. 1.

(1) $\xi_1-\xi_3=\alpha_3(\xi_2-\xi_3)$,

(2) $\xi_3-\xi_4=\alpha_4(\xi_2-\xi_4)$,

(3) $\xi_3-\xi_1=\alpha_1(\xi_4-\xi_1)$,

(4) $\xi_1-\xi_2=\alpha_2(\xi_4-\xi_2)$,

to which we may add $\xi_4=0$.

From (2) and (4), with $\xi_4=0$, we have

$$\xi_3=\alpha_4\xi_2, \qquad \xi_1=(1-\alpha_2)\xi_2.$$

Substituting in (3) we have

$$(\alpha_4+\alpha_2-1)\xi_2=\alpha_1(\alpha_2-1)\xi_2,$$

or $\varrho\xi_2=0$, where $\varrho=1-\alpha_1-\alpha_2-\alpha_4+\alpha_1\alpha_2$. With a notation explained in **8** below, we may take $\delta(\alpha_i)=1$, in which case $\varrho=1+\sigma$, where every term in σ has a positive degree. Therefore it follows from an argument in **8** that ϱ is not a 0-divisor. Therefore $\xi_2=0$, whence $\xi_1=\xi_3=0$ and the residual space is aspherical.

2. The equations are

(1) $\xi_2-\xi_3=\alpha_3(\xi_1-\xi_3)$,

(2) $\xi_2-\xi_4=\alpha_4(\xi_1-\xi_4)$,

(3) $\xi_3-\xi_5=\alpha_5(\xi_4-\xi_5)$,

(4) $\xi_4-\xi_1=\alpha_1(\xi_5-\xi_1)$,

(5) $\xi_5-\xi_3=\alpha_3(\xi_6-\xi_3)$,

(6) $\xi_3-\xi_1=\alpha_1(\xi_6-\xi_1)$,

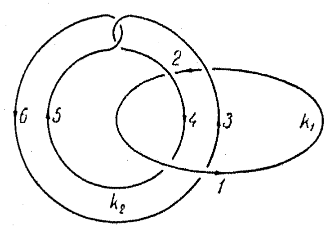

Fig. 2.

to which we may add $\xi_1=0$.

From (4) and (6) we have $\xi_4=\alpha_1\xi_5$, $\xi_3=\alpha_1\xi_6$. Substituting in (3) and (5), we have

$$\alpha_1\xi_6-\xi_5=\alpha_5(\alpha_1-1)\xi_5, \qquad \xi_5-\alpha_1\xi_6=\alpha_3(1-\alpha_1)\xi_6.$$

From the second of these we have

$$\xi_5=(\alpha_1+\alpha_3-\alpha_3\alpha_1)\xi_6,$$

and adding the second to the first

$$\alpha_5(\alpha_1-1)\xi_5+\alpha_3(1-\alpha_1)\xi_6=0.$$

Therefore $\varrho\xi_6=0$, where $\varrho=\alpha_5(\alpha_1-1)(\alpha_1+\alpha_3-\alpha_3\alpha_1)+\alpha_3(1-\alpha_1)$. Writing $\delta(\alpha_i)=1$ we have $\varrho=\alpha_3+\sigma$, where each term in σ is of higher degree than the first. Therefore ϱ is not a 0-divisor and $\xi_6=0$. Therefore $\xi_3=\xi_4=\xi_5=0$, and from (1), with $\xi_1=0$, we have $\xi_2=0$. Therefore the residual space is aspherical.

3. The equations are

(1) $\xi_2-\xi_3=\alpha_3(\xi_1-\xi_3)$,

(2) $\xi_3-\xi_5=\alpha_5(\xi_2-\xi_5)$,

(3) $\xi_4-\xi_8=\alpha_8(\xi_3-\xi_8)$,

(4) $\xi_5-\xi_2=\alpha_2(\xi_4-\xi_2)$,

(5) $\xi_1=\xi_5=\xi_6=\xi_{10}$,

(6) $\xi_7-\xi_9=\alpha_9(\xi_6-\xi_9)$,

(7) $\xi_8-\xi_4=\alpha_4(\xi_7-\xi_4)$,

(8) $\xi_9-\xi_{10}=\alpha_{10}(\xi_8-\xi_{10})$,

(9) $\xi_{10}-\xi_7=\alpha_7(\xi_9-\xi_7)$,

to which we may add $\xi_{10}=0$.

From (5), with $\xi_{10}=0$, we have $\xi_1=\xi_5=\xi_6=0$, and from (1) and (2)

$$\xi_2=(1-\alpha_3)\xi_3,$$

$$\xi_3=\alpha_5\xi_2=\alpha_5(1-\alpha_3)\xi_3.$$

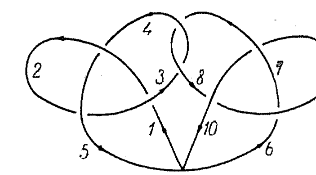

Fig. 3.

Writing $\delta(\alpha_i)=1$ we see that $1-\alpha_5(1-\alpha_3)$ is not a 0-divisor, and hence that $\xi_3=0$. Therefore $\xi_2=0$, and from (4), (3) (since $1-\alpha_8$ is not a 0-divisor), (7) and (8) we have successively $\xi_4=0$, $\xi_8=0$, $\xi_7=0$ and $\xi_9=0$. Therefore the residual space is aspherical.

6. Let $P=P_1+P_2$, $P_{12}=P_1\cdot P_2$, where P, P_1 and P_2 are connected polyhedra, and

1. let $\pi_r(P_i)=0$ $(i=1,2;\ r=2,\dots,n)$,
2. if $n>2$ let $\pi_s(P_{12})=0$ $(s=2,\dots,n-1)$ [11],
3. let any circuit in P_{12} which is homotopic to a point in P_1 or in P_2 be homotopic to a point in P_{12}. If P_{12} is connected we shall express this by saying that $\pi_1(P_{12})$ is a sub-group of $\pi_1(P_1)$ and of $\pi_1(P_2)$.

Theorem 5. *Under these conditions* $\pi_r(P)=0$ $(r=2,\dots,n)$.

[11]) Though $\pi_n(X)$ is only defined, in general, if X is connected, $\pi_n(X)=0$ will have the obvious meaning whether X is connected or not.

First let P_{12} be connected. Then it follows from the third condition that $\pi_1(P)$ is a free product [12]), with identified sub-groups, of $\pi_1(P_1)$ and $\pi_1(P_2)$. Therefore $\pi_1(P_i)$ and $\pi_1(P_{12})$ are sub-groups of $\pi_1(P)$. Let $\widetilde{P}$ be a universal covering space of P and let P_i^* and P_{12}^* be the sub-spaces of $\widetilde{P}$ covering P_i and P_{12}. Since $\pi_1(P_i)$ and $\pi_1(P_{12})$ are sub-groups of $\pi_1(P)$ each component, $\widetilde{P}_i$ or $\widetilde{P}_{12}$, of P_i^* or P_{12}^*, is a universal covering space of P_i or P_{12}. Therefore $\pi_\varrho(\widetilde{P}_i)=0$ and $\pi_\sigma(\widetilde{P}_{12})=0$, whence [13]) $\beta_\varrho(P_i^*)=0$ and $\beta_\sigma(P_{12}^*)=0$ for $\varrho=1,\dots,n$ and $\sigma=1,\dots,n-1$. It follows from a known theorem [14]) that $\beta_r(\widetilde{P})=0$, whence $\pi_r(\widetilde{P})=\pi_r(P)=0$ $(r=2,\dots,n)$. Thus the theorem is established in case P_{12} is connected.

If P_{12} is not connected let t be a (connected) tree with exactly one point in each component of P_{12}, and no other point in P. Let $Q=P+t$, $Q_i=P_i+t$ and $Q_{12}=Q_1\cdot Q_2=P_{12}+t$. Then Q_{12} is connected. On comparing the universal covering spaces of P and Q it obvious that if $\pi_r(Q)=0$ then $\pi_r(P)=0$ $(r=2,\dots,n)$. Similarly it follow from the first two conditions on P_i and P_{12} that $\pi_r(Q_i)=0$ for $r=2,\dots,n$, and $\pi_s(Q_{12})=0$ for $s=2,\dots,n-1$ if $n>2$. In consequence of the third condition satisfied by P_{12} it follows without difficulty that $\pi_1(Q_{12})$ is a sub-group [15]) of $\pi_1(Q_1)$ and of $\pi_1(Q_2)$. Therefore it follows from what we have already proved that $\pi_r(Q)=\pi_r(P)=0$ and the theorem is established.

Corollary. *If P_1, P_2 and P_{12} are aspherical so is P, subject to the third condition of theorem 5.*

It follows from the corollary to theorem 5 that the residual spaces of many knots and linkages obtained from others by a process which I have described elsewhere as doubling [16]) are aspherical. For let T_1^3 be a tubular neighbourhood of the knot k_1 in the linkage illustrated by fig. 2, **5**, and let $P_1=S^3-(T_1^3+k_2)$. We have seen that $S^3-(k_1+k_2)$, and hence that P_1 is aspherical. Moreover

[12]) H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Leipzig (1934), p. 177. For a definition of free products with identified sub-groups see K. Reidemeister, *Einführung in die Kombinatorische Topologie*, Brunswick (1932), p. 41.

[13]) Hurewicz, Proc. Akad. Amsterdam, **38** (1935), p. 522.

[14]) P. Alexandroff and H. Hopf, *Topologie I*, Berlin (1935), p. 294.

[15]) $\pi_1(Q_i)$ is the free product of $\pi_1(P_i)$ and a free group. If X is any component of P_{12}, $\pi_1(X)$ is a sub-group of $\pi_1(P_i)$ and hence of $\pi_1(Q_i)$.

[16]) Journal of the L. M. S., **12** (1937), p. 63. The process of doubling is illustrated by fig. 2 and fig. 4. The knot k' in fig. 4 (next page) is obtained by doubling k_1 in fig. 2, p. 158.

M. H. A. Newman [17]) and I have proved that $\pi_1(T_1^3)$ is a sub-group of $\pi_1(P_1)$. Therefore, if T^3 is a tubular neighbourhood of a knot k in a linkage $k+L$, such that $P_2=S^3-(T^3+L)$ is aspherical and $\pi_1(\dot{T}^3)$ is a sub-group of $\pi_1(P_2)$, then $S^3-(k'+L)$ is aspherical, where k' is a knot obtained by doubling k. For example, taking $L+k$ to be the linkage k_1+k_2 itself (fig. 2), with $k=k_1$, the region $S^3-(k'+k_2)$ is aspherical, where $k'+k_2$ is the linkage indicated by fig. 4. Similarly the residual spaces of all the linkages considered in the joint paper by Newman and myself are aspherical.

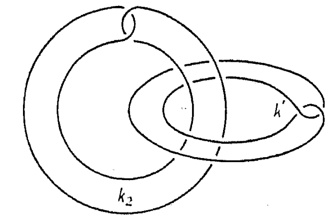

Fig. 4.

Eilenberg [18]) has observed that if $L\subset S^3$ is a linkage such that S^3-L is aspherical, the asphericity of S^3-L expresses some kind of interlinking between the component circuits of L. The preceding paragraph shows that $S^3-(k+k')$ may be aspherical even though k and k' are not feebly linked (faiblement enlacées), or even n-linked [19]) (n-enlacées) for a given value of n. This suggests that $S^3-(k+k')$ may be aspherical provided k is not contained in any (non-singular) 3-element which does not meet k'. More generally, using F to denote a closed set in S^3, we may ask;

Is S^3-F aspherical unless $F=F_1+F_2$, where $F_1\neq 0$, $F_2\neq 0$, and F_1 is contained in a 3-element which does not meet F_2?

This is equivalent to the question [20]):

If U is a open set in S^3 is $\pi_2(U)=0$ provided every non-singular, polyhedral 2-sphere in U bounds a 3-element in U?

To conclude this section I will show that an affirmative answer to this question is implied by an affirmative answer in case S^3-U is a recti-linear graph. First, if $U=S^3$ it is well known that $\pi_2(U)=0$,

[17]) Quarterly Journal of Math. (Oxford), **8** (1937), p. 14 (theorem 2).

[18]) Fund. Math., **28** (1937), p. 242.

[19]) Eilenberg, Fund. Math., **29** (1937), p. 118 et seq.

[20]) This question has been answered in the affirmative by Eilenberg on the assumption that U and S^3-U are connected and $\pi_1(U)$ is an infinite cyclic group. (Fund. Math., **28** (1937), p. 238, theorem 1).

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and if $U \neq S^3$ we may assume that $S^3 - U$ is a polyhedron. For let $f(S^2) \subset U$ be any map of a 2-sphere in U. Then $S^3 - U$ is contained in a polyhedron P^3, which does not meet $f(S^2)$, and is such that every component of P^3 contains a component of $S^3 - U$. If $f(S^2)$ is homotopic to a point in $U_1 = S^3 - P^3$ it is homotopic to a point in U, since $U_1 \subset U$. Moreover, any non-singular 2-sphere in U_1 bounds a 3-element in U_1, assuming that U has this property. For a 2-sphere in U_1 bounds a 3-element E^3 in U, and if E^3 were to meet P^3 it would contain a component cf P^3, and hence a component of $S^3 - U$. So we may take $U = S^3 - P$ in the first place, where P is a polyhedron. By a familiar process of contracting 3-simplexes of P which have 2-simplexes in common with $\overline{U}$ we may replace P by a 2-dimensional polyhedron. So we suppose $S^3 - U$ to be a 2-dimensional, simplicial complex P^2. It is easy to see that the effect on its homotopy type of replacing U by $U + A$, where A is the interior of a 2-simplex in P^2, is the same as the effect of replacing U by $U + s$, where s is a segment with its end points, but no inner point, in U. On comparing the universal covering spaces of U and $U + s$ it is clear that $\pi_2(U) = 0$ if $\pi_2(U + s) = 0$. Moreover it is easily proved that if any non-singular 2-sphere in U bounds in U (and therefore bounds a 3-element [21]) in U the same in true of $U + A$. Therefore we may remove the 2-simplexes of P^2, leaving the graph composed of the edges and vertices of P^2, and it follows from the hypothesis that $\pi_2(U) = 0$ (whence U is aspherical, since we are assuming that $U \neq S^3$).

7. Let k be a polygonal circuit in a (polyhedral) connected, 3-dimensional manifold M^3, which may be open (i.e. an infinite, unbounded polyhedron), closed or bounded. Let T^3 be a tubular neighbourhood of k in M^3, assuming that k is internal to M^3 if the latter is bounded, and let m be an oriented "meridian" circuit on $T^2 = \dot{T}^3$ (i.e. m bounds a 2-element in $\overline{T}^3$ which cuts k in a single point). Let l be a simple circuit on T^2 which does not bound in $\overline{T}^3$ and cuts m in a single point O. Taking O as the base point of $G = \pi_1(M^3 - T^3)$, let α be the element of G corresponding to l and β the element corresponding to m.

Theorem 6. *If $M^3 - T^3$ is aspherical and if $\alpha = 1$, then G is cyclic, $\pi_1(M^3) = 1$ and M^3 is closed.*

[21]) Cf. J. W. Alexander, Proc. Nat. Academy of Sciences, **10** (1924), p. 6.

Let $\widetilde{K}$ be the universal covering space of $K = M^3 - T^3$. First, I say that, since $\alpha = 1$, the sub-group of G generated by β is infinite, whether K is aspherical or not. For if β were of finite order each component of $\dot{\widetilde{K}}$ covering T^2 would be a torus, which is impossible since [22]) $\beta_1(\widetilde{K}) = 0$. Therefore β is not of finite order and, since $\alpha = 1$, the part of $\dot{\widetilde{K}}$ covering T^2 consists of one or more cylinders C_i $(i = 1, 2, \ldots)$.

Let $\widetilde{l} \subset C_1$ be a circuit covering l, let $\widetilde{O}$ be the image of O on $\widetilde{l}$ and let Γ_0^2 be a chain in $\widetilde{K}$ bounded by $\widetilde{l}$. Let γ be any element in G and $\widetilde{O}\widetilde{O}_1$ an oriented segment in $\widetilde{K}$ whose image in K is a circuit which represents the element γ. If $\widetilde{O}_1 \epsilon C_1$ we may take $\widetilde{O}\widetilde{O}_1$ to be on C_1, since $\pi_1(\widetilde{K}) = 1$, and it follows that γ is a power (positive, negative or zero) of β.

Eilenberg has proved a lemma [23]) which, with trivial alterations in the wording, may be stated as follows: Let X_0 and Y be compact sets in $\widetilde{K}$ and let $X_m = X_0 \tau^m$, where τ is an element, not of finite order, in the group of covering transformations (Deckbewegungsgruppe). Then there is a positive N such that $X_m \cdot Y = 0$ if $\pm m > N$. Taking $X_0 = \Gamma_0^2$, $Y = \widetilde{O}\widetilde{O}_1$ and τ to be a translation of C_1 into itself, we find 2-chains Γ_{-N}^2, Γ_N^2 which do not meet $\widetilde{O}\widetilde{O}_1$, and whose boundaries bound a band $B^2 \subset C_1$, containing $\widetilde{O}$. Since K is aspherical there is a chain Γ^3 such that

$$\dot{\Gamma}^3 = \Gamma_{-N}^2 + B^2 + \Gamma_N^2 \quad (\text{mod } 2).$$

Since $B^2 \subset \dot{\widetilde{K}}$ the closure of $\widetilde{K} - \Gamma^3$ meets Γ^3 in a sub-set [24]) of $\Gamma_{-N}^2 + \Gamma_N^2$. Therefore points in $\Gamma^3 - (\Gamma_{-N}^2 + \Gamma_N^2)$ are separated from points in $\widetilde{K} - \Gamma^3$ by $\Gamma_{-N}^2 + \Gamma_N^2$, and since $\Gamma_{-N}^2 + \Gamma_N^2$ does not meet the segment $\widetilde{O}\widetilde{O}_1$ the point $\widetilde{O}_1$ lies in Γ^3. Since $\widetilde{O} \epsilon \dot{\widetilde{K}}$ and τ is a topological transformation, $\widetilde{O}_1 \epsilon \dot{\widetilde{K}}$. Therefore $\widetilde{O}_1 \epsilon \dot{\Gamma}^3$, since a point on the boundary of an n-dimensional manifold cannot be internal to an n-chain in the manifold. Therefore $\widetilde{O}_1 \epsilon B^2 \subset C_1$, γ is a power of β and G is the cyclic group generated by β.

Since G is generated by β and m bounds a 2-cell in M^3 it follows that $\pi_1(M^3) = 1$.

[22]) H. Kneser, Gött. Nach., 1925, p. 128.

[23]) Fund. Math., **28** (1937), p. 236.

[24]) This is a proper sub-set only if Γ_0^2, and hence $\Gamma_{\pm N}^2$ have 2-cells in $\dot{\widetilde{K}}$.

Let E^2 be a 2-cell in K bounded by l. Since $\pi_2(K)=0$, a singular 2-sphere which covers T^2 (which we now assume to be oriented) with degree 1, and E^2 with degree 0, bounds a singular 3-cell in K. Therefore there is a finite chain $C^3 \subset K$ such that[25]) $\dot{C}^3 = T^2$, and $C^3 - T^3$ is a finite, non-zero 3-cycle on M^3. Therefore M^3 is closed and the theorem is established.

8. We conclude with same remarks on a group ring $\mathfrak{R}=\mathfrak{R}(G)$. Let G_0 be a sub-group of G and $\mathfrak{R}_0$ the group ring of G_0. Let one element be selected from each residue class $G_0\beta$ in G. Then each element in G has a unique representation in the form $\gamma\beta_i$, where $\gamma \epsilon G_0$ and $\beta_1, \beta_2, \ldots$ are the selected elements. Therefore each element in $\mathfrak{R}$ has a unique representation of the form $\varrho_1\beta_1 + \varrho_2\beta_2 + \ldots$, where $\varrho_i \epsilon \mathfrak{R}_0$. That is to say, $\mathfrak{R}$ is a modulus with coefficients in $\mathfrak{R}_0$ and $\beta_1, \beta_2, \ldots$ as linearly independent basis elements. If $\varrho(\varrho_1\beta_1 + \varrho_2\beta_2 + \ldots) = 0$ it follows that $\varrho\varrho_i = 0$ $(i=1,2,\ldots)$ and we have our first result:

If $\varrho \epsilon \mathfrak{R}_0$ is not a 0-divisor in $\mathfrak{R}_0$ it is not a 0-divisor in $\mathfrak{R}$.

In particular, if $\alpha \epsilon G$ is not of finite order no non-zero polynomial $f(\alpha)$, in which negative exponents are allowed, is a 0-divisor.

In the case of a knot k, each element β in $G=\pi_1(S^3-k)$ has a "degree" $\delta(\beta)$, given by $\delta(\beta)=L(s,k)$, where s is a circuit representing β and $L(s,k)$ is the looping co-efficient of s and k[26]). Moreover $\delta(1)=0$ and $\delta(\beta_1\beta_2)=\delta(\beta_1)+\delta(\beta_2)$ (i.e. $\beta \to \delta(\beta)$ is a homomorphism of G on the additive group of integers). An element $n_1\beta_1 + \ldots + n_r\beta_r$ of $\mathfrak{R}$ will be described as *homogeneous of degree m* if $\delta(\beta_1)=\ldots=\delta(\beta_r)=m$, and any element of $\mathfrak{R}$ may be written in the form

$$\eta = \eta_p + \ldots + \eta_q, \tag{8.1}$$

where η_m is homogeneous of degree m $(m=p,\ldots,q)$ and $p<\ldots<q$. It will be convenient to regard the zero element of $\mathfrak{R}$ as homogeneous of all degrees. Now let η be given by (8.1) with $\eta_p \neq 0$, $\eta_q \neq 0$ and let $\zeta = \zeta_r + \ldots + \zeta_s$ $(r<\ldots<s; \zeta_r \neq 0, \zeta_s \neq 0)$, where ζ_m is homogeneous of degree m. Then

$$\eta\zeta = \eta_p\zeta_r + \ldots + \eta_q\zeta_s, \tag{8.2}$$

[25]) Alternatively, K has the same homotopy type as a graph, in this case a circle, since it is aspherical and $\pi_1(K)$ is a free group (Eilenberg, Annals of Math., **38** (1937), p. 656). Therefore $\beta_2(K)=0$.

[26]) Cf. Alexander (loc. cit.).

the degree of each remaining term being greater than $p+r$ and less than $q+s$. If a sum of homogeneous elements in $\mathfrak{R}$, of different degrees, is zero it is obvious that each element is zero. Therefore $\eta\zeta=0$ implies $\eta_p\zeta_r=\eta_q\zeta_s=0$, and we have the result[27])

If η, given by (8.1) is a 0-divisor, so are η_p and η_q.

Similar remarks apply to a linkage, the degree of any element in $G=\pi_1\{S^3-(k_1+\ldots+k_p)\}$ being the exponent of the corresponding element in some cyclic factor group of the homology group

$$\beta_1\{S^3-(k_1+\ldots+k_p)\}.$$

If each element of G has a degree we may imbed $\mathfrak{R}$ in the ring $\mathfrak{R}^*$, consisting of all linear forms, finite or infinite, $n_1\beta_1+n_2\beta_2+\ldots$, which satisfy the conditions[28])

(1) *$\delta(\beta_i) \geqslant p$, where p does not depend on i,*

(2) *only a finite number of the elements $\beta_1, \beta_2, \ldots$ have any given degree.*

We recall that an element η in any ring with a unit element 1 is called a right unit (or left unit) if there is an element η' (or η'') such that $\eta'\eta=1$ (or $\eta\eta''=1$). If η is both a right and a left unit (i. e. if $\eta'\eta=\eta\eta''=1$) it follows from the associative law that $\eta'\eta\eta''=\eta'=\eta''$, and η' is called the inverse, η^{-1}, of η. The elements in the ring which have an inverse obviously constitute a multiplicative group. Returning to the ring $\mathfrak{R}^*$:

If η_p is a right unit, or left unit, or has an inverse, so does $\eta=\eta_p+\eta_{p+1}+\cdots$

For if $\zeta\eta_p=1$ it follows from an equation similar to (4.2) that we may suppose $\zeta=\zeta_{-p}$ to be homogeneous of degree $-p$. Then

$$(1+\theta_1+\ldots)\zeta_{-p}(\eta_p+\eta_{p+1}+\ldots)= \\ =(1+\theta_1+\ldots)(1+\eta_1^*+\ldots)=1 \qquad (\eta_q^*=\zeta_{-p}\eta_{p+q})$$

provided $\theta_q=-(\theta_{q-1}\eta_1^*+\ldots+\eta_q^*)$. Therefore $\eta'\eta=1$, where

$$\eta'=\eta'_{-p}+\eta'_{-p+1}+\ldots, \quad \text{with} \quad \eta'_{-p}=\zeta_{-p}, \quad \eta'_{-p+q}=\theta_q\zeta_{-p}.$$

Similarly, if η_p is a left unit so is η, whence η has an inverse if η_p has an inverse.

[27]) Cf. W. Magnus, Math. Annalen, **111** (1935), p. 259.

[28]) Cf. a forthcoming paper by G. Higman (Journ. London Math. Soc.).

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The preceding results have an amusing formal consequence, valid for any ring $\mathfrak{R}$, with a unit element 1. Let $\eta_1, \eta_2, \ldots$ be an infinite sequence of elements in $\mathfrak{R}$, with repetitions allowed. Then $\eta'_q = \eta''_q$, where η'_q and η''_q are defined by the recurrence formulae, $\eta'_0 = \eta''_0 = 1$ and

$$\eta'_q = -(\eta'_{q-1}\eta_1 + \eta'_{q-2}\eta_2 + \ldots + \eta_q),$$
$$\eta''_q = -(\eta_1\eta''_{q-1} + \eta_2\eta''_{q-2} + \ldots + \eta_q).$$

This is true for any ring since it is true for the ring which is freely generated by $\eta_0 = 1, \eta_1, \eta_2, \ldots$, with infinite sums allowed, provided no product $\pm\eta_{m_1}\ldots\eta_{m_n}$ is repeated infinitely many times. For if a degree, given by $\delta(\pm\eta_{m_1}\ldots\eta_{m_n}) = m_1 + \ldots + m_n$, is assigned to each product, only a finite number of terms in such a sum can have the same degree. It follows from induction on q that η'_q and η''_q are homogeneous of degree q and, as before, that $\eta'\eta = \eta\eta'' = 1$, where

$$\eta = 1 + \eta_1 + \eta_2 + \ldots, \qquad \eta' = 1 + \eta'_1 + \eta'_2 + \ldots, \qquad \eta'' = 1 + \eta''_1 + \eta''_2 + \ldots$$

Therefore $\eta' = \eta''$, whence $\eta'_q = \eta''_q$.

Balliol College, Oxford, England.

On the relation between the fundamental group of a space and the higher homotopy groups.

By

Samuel Eilenberg (Warszawa).

1. $\mathcal{Y}$ will denote a separable, connected metric space locally connected in dimensions $0, 1, \ldots, n$ [1]). Given a compact metric space $\mathcal{X}$, the continuous functions $f(\mathcal{X}) \subset \mathcal{Y}$ with the distance formula

$$|f_0 - f_1| = \sup_{x \in \mathcal{X}} |f_0(x) - f_1(x)|$$

form a metric space $\mathcal{Y}^{\mathcal{X}}$.

Given two points $x_0 \in \mathcal{X}$ and $y_0 \in \mathcal{Y}$ the equation $f(x_0) = y_0$ defines a closed subset $\mathcal{Y}^{\mathcal{X}}(x_0, y_0)$ of $\mathcal{Y}^{\mathcal{X}}$.

I will denote the closed interval $[0,1]$ by $\mathcal{J}$ and $\mathcal{X} \times \mathcal{J}$ will stand for the cartesian product of $\mathcal{X}$ and $\mathcal{J}$. Two functions $f_0, f_1 \in Y^X$ will be called *homotopic* if there is a function $g \in \mathcal{Y}^{\mathcal{X} \times \mathcal{J}}$ such that

$$f_0(x) = g(x, 0), \quad f_1(x) = g(x, 1) \quad \text{for all} \quad x \in \mathcal{X}.$$

If also

$$g(x_0, t) = y_0 \quad \text{for all} \quad t \in \mathcal{J},$$

we say that $f_0, f_1 \in \mathcal{Y}^{\mathcal{X}}(x_0, y_0)$ are *homotopic rel.* (x_0, y_0).

2. Let $\mathcal{X}$ be a polyhedron and X a subpolyhedron of $\mathcal{X}$. It is well known that $T = \mathcal{X} \times (0) + X \times \mathcal{J}$ is a retract of $\mathcal{X} \times \mathcal{J}$ and therefore that

(2.1) *Every* $f \in \mathcal{Y}^T$ *has an extension* $f' \in \mathcal{Y}^{\mathcal{X} \times \mathcal{J}}$ [2]).

It follows immediately from (2.1) that

(2.2) *Given two homotopic functions* $f_0, f_1 \in \mathcal{Y}^X$ *and an extension* $f'_0 \in \mathcal{Y}^{\mathcal{X}}$ *of* f_0, *there is an extension* $f'_1 \in \mathcal{Y}^{\mathcal{X}}$ *of* f_1 *homotopic to* f'_0 [2]).

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[1]) C. Kuratowski, Fund. Math. **24** (1935), p. 269.

[2]) See for instance P. Alexandroff und H. Hopf, *Topologie I*, Berlin 1935, p. 501.