Theorem 4. *The cartesian product of a countable number of countable regular spaces is perfectly normal.*

Proof. Let $P = \mathfrak{P}_{n=1}^{\infty} P_n$, P_n being countable and regular. The spaces $P_1 \times \ldots \times P_n$ $(n=1,2,\ldots)$ are countable and regular, hence, as shown by Urysohn, perfectly normal and it suffices to apply theorem 2.

Example 2. If the spaces $P_1 \times \ldots \times P_n$ are completely normal, the space $\mathfrak{P}_{n=1}^{\infty} P_n$ need not be completely normal. Choosing $P_n = P_1$ for all n, where P_1 denotes the space of example 1, we may easily show (analogously as for $P_1 \times P_1$ in example 1) that $P_1 \times \ldots \times P_n$ are completely normal. On the other hand, the space $\mathfrak{P}_{n=1}^{\infty} P_n$, where $P_n = P_1$, is not perfectly normal, for its subspace P_1 is not. Hence $\mathfrak{P}_{n=1}^{\infty} P_n$ is not completely normal by theorem 3.

On Area and Length.

By

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1. This paper is concerned with intrinsic definitions of area and of length. Although the definitions are new, they are obtained by combining ideas which are quite familiar to anyone working in this field: the ideas of Banach [1, 2] which have been the basis of researches on area for twenty years [2, 14, 11, 12] and which consist in effect in introducing our intrinsic definitions in a special case (the case of a surface situated in a plane); and the well-known theory of measure of Carathéodory [5, 8]. Moreover the old definitions, based on simplicial approximations, have long been regarded as unsatisfactory: examples of space-filling curves which constitute surfaces of zero area though of positive volume have been known for forty years; the examples recently produced by Besicovith [3, 4] are even more conclusive.

The value of a particular definition however, depends mainly on its usefulness as a tool, and in this connection the Lebesgue-Fréchet definition of area has rendered great services. It has shown itself quite satisfactory for Lipschitzian surfaces (often misleadingly termed „rectifiable") and has led to important semi-continuity theorems in the Calculus of Variations. Above all, it has had sufficient depth to serve as background to Banach's fundamental methods already referred to.

The greater part of these results and methods remain when we adopt instead the present intrinsic definitions. We show in particular that the definitions agree for Lipschitzian surfaces. Moreover the new definitions are framed for the purpose of developing tools which are needed as a preliminary to the study of „generalized surfaces" in parametric form in the Calculus of Variations [1]), and such a theory would go much further than the existing semi-continuity theorems. Finally as regards the Banach methods, they play an even more fundamental part in the present theory, than they do in the older theory of area.

It should be stressed that the present definitions concern not only area, but also length, these two notions being treated symmetrically in the definitions and also linked together by relations which are based on the important theorem of Szpilrajn-Marczewski and Eilenberg [15, 7].

In the case of length, the classical definitions were so far considered satisfactory, but this is only because they were restricted to curves obtained by the mapping of a segment: the boundary of a surface given by the mapping of a general simply-connected plane domain has no length in the classical sense, because the boundary of this plane domain need not be homeomorphic with a Jordan curve. It is clearly desirable to possess a more general definition of length than the classical one.

Actually, the deepest part of this paper concerns length rather than area. We find it necessary to introduce two separate notions of length, which we term the intrinsic length and the boundary-length, and we obtain a general theorem connecting the two.

We conclude the paper with generalisations of the results of two recent notes [18, 19]. We extend to harmonic surfaces defined on an arbitrary simply-connected domain, the inequality $A<KLd$ between the area A, the diameter d, and the boundary length L. And we extend to surfaces of finite intrinsic area the lemma in the theory of surfaces, proved in [18] for polyhedra.

After the greater part of this paper had been written, I learned from Besicovitch that he intends to publish similar intrinsic definitions of area and to apply them to the Problem of Plateau. The final version of this paper has in consequence been strongly influenced by discussions with him during an all too short visit to England from the Cape. I take this opportunity of thanking him.

[1]) Nevertheless my researches in the Calculus of Variations have now led me to the view that both the intrinsic area and the Lebesgue-Fréchet area are required as tools in this subject. This is borne out by some results of T. Radó to appear in the Trans. Amer. Math. Soc. I take this opportunity of thanking Professor Radó for a number of interesting and important comments.

My debt to Mr H. D. Ursell is even greater: the whole theory of length is the outcome of joint researches on Carathéodory's theory of prime ends [6], which we are publishing elsewhere [16].

2. Multiple systems. In a (finite-dimensional Cartesian) [2]) space in which x is a variable point, a function $M(x)$ whose values are cardinal numbers will be said to define a „multiple system" M of which it is the „multiplicity function". For our purposes, the infinite values of $M(x)$ need not be distinguished from one another.

In the case in which $M(x)$ assumes only the values 0 and 1, we identify the notion of multiple system with that of set of points, and the multiplicity function reduces to the characteristic function: it takes the value 1 in the set, 0 outside.

We define further the sum and the product of multiple systems, by the sum and the product of the corresponding multiplicity functions, with the conventions that a product in which one factor vanishes is itself zero. In particular the product of a multiple system and a set of points is defined.

If $\mathfrak{X}$ is any additive class of sets E, we define its *extension to multiple systems* as the class of the multiple systems M such that, for each integer n, the class $\mathfrak{X}$ includes the set M_n of the points x for which $M(x)\geqslant n$.

If $\mu(E)$ is any measure defined for sets E of an additive class, we define its *extension* $\mu(M)$ to multiple systems, by the formula

$$\mu(M)=\sum_{n=1}^{\infty}\mu(M_n). \tag{2.1}$$

In particular we write $\mu^k(M)$ for the extension to multiple systems of the k-dimensional Caratheodory measure $\mu^k(E)$ of a set E; here $\mu^k(E)$ is the limit as $\varepsilon\to 0$ of the lower bound of the expression

$$c_k\cdot\sum_m (dE_m)^k$$

for decomposition $E=\sum E_m$ such that $dE_m<\varepsilon$, where dE_m is the diameter of E_m and where c_k is a constant $\left(c_1=1,\ c_2=\frac{\pi}{4}\right)$.

[2]) This restriction is not essential.

(2.2) (Szpilrajn-Marczewski and Eilenberg). *Let S_r denote the spherical shell $|x|=r$. Then there is a constant a_k such that*

$$\mu^{k+1}(M) \geqslant a_k \int_0^\infty \mu^k(M\cdot S_r)dr\,^{3}. \tag{2.3}$$

Proof. It is sufficient, by addition [3], in view of (2.1), to establish (2.3) in the case in which M is a set of points E. Writing $\mu^k(E)=c_k L^k(E)$ we obtain the required result with $a_k=c_{k+1}/c_k$ [4]), from Theorem I of [7].

3. The Banach value-system M. Suppose that a continuous function $f(w)$, whose values are points of x-space, is given at the points w of a plane closed set W_0, or more generally, at those of an absolute F_σ set, i. e. of an enumerable sum of closed compact sets of any metric space [5]).

We denote by $M(x;f;W)$ the number (finite or $+\infty$) of points w of a subset W of W_0 which satisfy the condition $f(w)=x$; or, in other words, the upper bound of the number of points of the finite subsets H_x of W which consist of points w for which $f(w)=x$.

We write $M(x)$ for $M(x;f;W_0)$ and M and M_W for the systems whose multiplicity functions are $M(x)$ and $M(x;f;W)$. We term M and M_W the *Banach value-system* of $f(w)$ in W_0 and in W respectively.

We observe [5]) that for an expanding sequence of sets $W^{(n)}\subset W_0$,

$$M(x;f;\textstyle\sum W^{(n)}) = \lim_n M(x;f;\, W^{(n)}), \tag{3.1}$$

and that, except possibly at the points x for which $M(x)=\infty$,

$$M(x;f;W_0-W)=M(x)-M(x;f;W). \tag{3.2}$$

Moreover, we can take over from Banach [2, p. 226, Th. 1] the general lines of the proof of the following result:

(3.3) *The Banach multiplicity function $M(x)$ is measurable* (B).

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[3]) The integral is an upper Lebesgue integral: this allows us to integrate term by term a series of non-negative functions of r.

[4]) It is probable that this value of a_k can be increased, and that the best value of a_1 is 1 instead of $c_2/c_1=\pi/4$.

[5]) We do not require any hypothesis about the nature of W_0 in (3.1) and (3.2).

Proof. Given $\varepsilon>0$, we express W_0 as a finite or enumerable sum of disjoint F_σ sets W_n of diameter $<\varepsilon$, which we can take to be, for instance, common parts of W_0 with the differences of a corresponding number of spheres of diameter $<\varepsilon$ which cover W_0.

The set E_n of the values of $f(w)$ for w in W_n, is an F_σ set — it is an enumerable sum of sets of values of $f(w)$ assumed in closed compact subsets of W_n.

Denote by $M_\varepsilon(x)$ the sum of the characteristic functions of the sets E_n — it is immaterial to us that it depends on other things besides ε. Clearly $M_\varepsilon(x)$ is measurable (B), in fact it is the limit of an ascending sequence of upper semi-continuous functions; and since the W_n are disjoint, we have

$$M_\varepsilon(x) \leqslant M(x). \tag{3.4}$$

On the other hand, if x is fixed and N is any finite number $\leqslant M(x)$, we have for sufficiently small ε

$$N \leqslant M_\varepsilon(x) \tag{3.5}$$

because, reverting to the notation of the definition of $M(x;f;W)$, there is an H_x consisting of N points, and these must lie in N different sets W_n if ε is less than the mutual distances of these points.

From (3.4) and (3.5), $M(x)$ is the limit of $M_\varepsilon(x)$ as $\varepsilon\to 0$, and is therefore measurable (B).

We conclude the paragraph with the following results which follow trivially from the definitions and from (3.1), (3.2), (3.3) when we substitute in (3.2) W_1 and $M(x;f;W_1)$ for W_0 and $M(x)$, and in (3.3) any closed subset W of W_1 in place of W_0:

(3.6) *Let $\mu(E)$ denote a Carathéodory measure for sets E of points x; and let W_1 denote a Borel subset of W_0 such that $M(x;f;W_1)$ is measurable (B), and write E_1 for the set of x for which $M(x;f;W_1)=+\infty$. Then if $\mu(E_1)=0$, the function of x $M(x;f;W)$ is measurable (μ) whenever W is a Borel subset of W_1, and the expression $\mu(M_W)$ is then an absolutely additive function of this Borel subset W.*

In particular this is so if $\mu(M_{W_1})<\infty$ or again if E_1 is empty. In the latter case $M(x;f;W)$ is measurable (B) whenever W is a Borel subset W_1.

4. Classical length and area. Although we shall not base our definitions of intrinsic area and length directly of the ideas of Banach, which have just been developed, but rather on a modification of the Banach ideas due to Morrey [10], this is convenient point at which to discuss the classical definitions of length of a curve and of area of a Lipschitzian surface.

(4.1) *In the case in which W_0 is a segment, $\mu^1(M)$ is the classical length of the curve represented by $f(w)$.*

Proof. Let W_0 be the segment $a \leqslant w \leqslant b$ of the axis of real numbers, and let E be the set of the points x of our curve, and let L be the classical length. As a consequence of the definition of Carathéodory's linear measure, we see that

$$L \geqslant \mu^1(E) \geqslant |f(b)-f(a)|. \tag{4.2}$$

Now divide W_0 into half-open or closed intervals (α, β) which can be taken for the sets W_n of the proof of (3.3). Since the corresponding set E_n differs only by one point from the image of a segment, it follows from (4.2) by addition for the various W_n, that the expression,

$$L_\varepsilon = \mu^1(M_\varepsilon) = \sum \mu^1(E_n)$$

—in which M_ε is the system with the multiplicity function $M_\varepsilon(x)$— lies between L and $\sum |f(\beta)-f(\alpha)|$.

From (3.4), we see that

$$\sum |f(\beta)-f(\alpha)| \leqslant \mu^1(M)$$

and therefore that

$$L \leqslant \mu^1(M).$$

While from Fatou's theorem, as $\varepsilon \to 0$ by a sequence,

$$\int M(x)\,d\mu^1 \leqslant \liminf \int M_\varepsilon(x)\,d\mu^1$$

i. e.

$$\mu^1(M) \leqslant \liminf \mu^1(M_\varepsilon) \leqslant L,$$

and this completes the proof, the substance of which goes back to Banach [2, p. 228, Th. 2].

We pass on to the proof of the following result:

(4.3) ***Theorem.*** *Suppose $f(w)$ Lipschitzian on a bounded closed plane set W_0. Then for any Borel subset $W \subset W_0$,*

$$\mu^2(M_W) = \iint_W |J|\,du\,dv \tag{4.4}$$

where J is the vector whose components are the Jacobians of the pairs of components of $f(w)$ with respect to the components u, v of w.

We recall that $f(w)$ is termed Lipschitzian on W_0 if there exists a constant K such that

$$|f(w')-f(w'')| \leqslant K \cdot |w'-w''| \tag{4.5}$$

for every pair of points w', w'' of W_0.

We shall divide the proof of (4.3) into several stages.

Denoting by $f(W)$ the set of the values x which are assumed by $f(w)$ when $w \in W$, we first observe that

$$\mu^2[f(W)] \leqslant K \cdot \mu^2(W) \tag{4.6}$$

so that in particular $\mu^2[f(W)] = 0$ when $\mu^2(W) = 0$. Here and in the sequel, the constant K has a value which depends on the context. The proof of (4.6) follows easily from the definition of Carathéodory 2-dimensional measure, and may be left to the reader.

We shall make use also of the following elementary lemma:

(4.7) *There is a function ε_N of N only, which tends to 0 as $N \to \infty$, and which has the property that we can cover any elliptic disc, possibly reducing to a segment or to a point, by not more than N sets of points in such a manner that*

$$\frac{\pi}{4} \sum d^2 - A \leqslant \varepsilon_N D^2$$

where d denotes the diameter of a covering set, D that of the disc, and A the area of the disc.

Proof of (4.7). We may suppose trivially that $D=1$, and it is sufficient to prove that given $\varepsilon > 0$ we can cover the elliptic disc (now possibly reducing to a segment) by not more than $N(\varepsilon)$ sets in such a manner that

$$\frac{\pi}{4} \sum d^2 - A < \varepsilon,$$

where $N(\varepsilon)$ is a function of ε only.

To this effect we first cover the given elliptic disc Δ of diameter 1 by an infinity of circles as covering sets, in such a manner that

$$\frac{\pi}{4}\sum d^2 - \Delta < \frac{1}{2}\varepsilon;$$

we can do this, for instance, by using Vitali's theorem to cover almost all Δ by disjoint interior circles, and then covering the rest of by further circles for which

$$\frac{\pi}{4}\sum d^2 < \frac{1}{2}\varepsilon.$$

By Borel's covering theorem, there exist a finite number $N(\varepsilon,\Delta)$ of the above two types of covering circles, which cover Δ. Now take a finite series of elliptic discs Δ_ν of diameter 1, each contained in the following and differing from it in area by at most $\frac{1}{2}\varepsilon$ in such a manner that every elliptic disc of diameter 1 can be so placed as to be contained in a certain disc Δ_ν of the series, and at the same time to contain the preceding disc $\Delta_{\nu-1}$. Writing $N(\varepsilon)$ for the greatest of the numbers $N(\varepsilon,\Delta_\nu)$, we cover Δ by the covering circles of Δ_ν and find that

$$\frac{\pi}{4}\sum d^2 - \Delta < \frac{1}{2}\varepsilon + \mu^2(\Delta_\nu - \Delta) \leqslant \varepsilon.$$

This completes the proof of (4.7) since the number of the covering circles is at most $N(\varepsilon)$.

Proceeding with the proof of Theorem (4.3), we recall that a Lipschitzian $f(w)$ has almost everywhere partial derivatives f_u and f_v which are the coefficients of a total differential df. This is proved, for instance in Saks [13, Chap. IX, Th. 14.2 p. 311; see also Th. 14.5 p. 312, and Th. 12.2 p. 300]; the fact that the values of $f(w)$ are vectors clearly makes no difference. We could also suppose $f(w)$ Lipschitzian in a rectangle, or in the whole plane, by replacing it by a suitable continuation.

We choose any $\varepsilon > 0$ and we express almost all the Borel set W as a finite sum of disjoint Borel subsets W', in each of which the differential exists and its coefficients, the partial derivatives of f, have oscillation less than $\frac{1}{3}\varepsilon$.

Almost every point of W is point of density of just one of the W', and for every circle C of sufficiently small radius $\varrho(C) < \varepsilon$ with such a point as centre, we have

$$\mu^2(C\cdot W') > (1-\varepsilon)\cdot\mu^2(C),$$

and for w in $C\cdot W$ and in particular for w in $C\cdot W'$

$$|f(w) - g(w)| < \varepsilon\varrho(C),$$

where $g(w)$ is a linear function of the components u, v of w, whose constant partial derivatives in C nowhere exceed in magnitude the partial derivatives of f in $C\cdot W'$.

By Vitali's theorem, we can cover all but plane measure ε of W by a finite number of such disjoint circles C_n. We write Ω_n for the corresponding finite number of the sets $C_n W'$, Ω for $\Sigma\Omega_n$, and $g_n(w)$ for the linear function $g(w)$ corresponding to the circle C_n. We have

$$\mu^2(W-\Omega) < \varepsilon + \sum \mu^2(C - \Omega_n) < \varepsilon + \frac{\varepsilon}{1-\varepsilon}\sum \mu^2(\Omega_n) < K\varepsilon.$$

This being so, we cover, by (4.7), each elliptic disc $g_n(C_n)$ by at most N sets in the manner explained, the integer N being chosen for small ε to be, for instance, the integral part of ε^{-1}, so that the expressions $\varepsilon\sqrt{N}$, $\varepsilon^2 N$ and ε_N become less than some $\delta(\varepsilon)$ which tends to 0 with ε. By increasing the diameters d of the covering sets to $d^* = d + 2\varepsilon\varrho(C_n)$, we cover all points distant less than $\varepsilon\varrho(C_n)$ from $g_n(C_n)$, and therefore we cover $f(\Omega_n)$. We observe that

$$\begin{aligned}\sum (d^{*2} - d^2) &< K\cdot N\varepsilon^2\varrho(C_n)^2 + K\cdot\sqrt{N}\varepsilon\varrho(C_n)\sqrt{\sum d^2}\\ &< K\cdot\delta(\varepsilon)\cdot[A_n + \varrho(C_n)^2]\end{aligned}$$

where A_n is the area of the elliptic disc $g_n(C_n)$ which is at most

$$K\cdot\varepsilon\varrho(C_n)^2 + \iint_{\Omega_n} |J|\,du\,dv.$$

We thus find that

$$\frac{\pi}{4}\sum d^{*2} < \iint_{\Omega_n} |J|\,du\,dv + K\cdot B_n$$

where $\sum B_n < K\cdot\delta(\varepsilon)$; moreover each d^* is clearly at most $K\varepsilon$.

By covering in this way the various $f(\Omega_n)$, and by covering further the set $f(W-\Omega)$ — whose 2-dimensional measure is small on account of (4.6) — by sets for which the sum of the squares of the diameters is at most $K\varepsilon$, we obtain a covering of $f(W)$ by sets whose diameters d' are at most $K\varepsilon$ and satisfy a relation of the type

$$\frac{\pi}{4}\sum d'^2 < \iint_{\Omega} |J|\, du dv + K\cdot\delta(\varepsilon),$$

where $\delta(\varepsilon)\to 0$ as $\varepsilon\to 0$.

In this we increase the double integral by taking it over W, and we make $\varepsilon\to 0$. We obtain

$$\mu^2[f(W)] \leqslant \iint_W |J|\, du dv. \tag{4.8}$$

This being so, we apply the inequality (4.8) to the subsets of W obtained by a subdivision of W into disjoint F_σ sets W_m of diameter less than ε together with a set of plane measure 0. Ignoring this last set, for which the relevant expressions both vanish, we obtain by adding together the inequalities derives from (4.8) by the substitution of W_m for W, and using the notation of paragraph 3,

$$\int M_\varepsilon(x)\, d\mu^2 \leqslant \iint_W |J|\, du dv;$$

here $M_\varepsilon(x)\to M(x)$ monotonely as $\varepsilon\to 0$ and therefore the left-hand side tends to the corresponding integral of $M(x)$, i. e. to $\mu^2(M_W)$; consequently

$$\mu^2(M_W) \leqslant \iint_W |J|\, du dv.$$

To prove Theorem (4.3) it remains to be shown that this inequality reduces to an equality. We need only show this in the case $W=W_0$, since the two sides are additive functions of the set W and since their difference, which is of constant sign, is therefore greatest in absolute value when $W=W_0$. Moreover, as already stated, we may and we shall suppose that W_0 is a rectangle.

Thus we have now only to establish in the case in which W is replaced by a rectangle R, the reverse inequality

$$\mu^2(M_R) \geqslant \iint_R |J|\, du dv.$$

Moreover, we need clearly only prove the weaker inequality

$$\mu^2(M_R) \geqslant \iint_W |J|\, du dv \tag{4.9}$$

where the integral is now taken over a smaller range, consisting of a set W (as we shall now term it) of points of R at which $|J|$ exceeds a certain fixed positive constant k, such that f is differentiable at every point of W.

Proceeding as earlier in this proof, given $\varepsilon>0$ we express this set W as a finite sum of disjoint Borel sets W' in each of which the partial derivatives of f have oscillation less than ε; and we again cover all but plane measure ε of W by a finite number of disjoint circular discs C_n contained in R and of radius $\varrho(C_n)<\varepsilon$, such that if w_n denotes the centre of C_n, W'_n denotes that one of the sets W' which contains the point w_n, and $g(w)$ denotes in C_n the linear function agreeing with f at w_n and having there the same differential, we have

$$\mu^2(C_n\cdot W'_n) > (1-\varepsilon)\mu^2(C_n)$$
$$|f(w)-g(w)| < \varepsilon\cdot\varrho(C_n) \qquad \text{for } w \text{ in } C_n.$$

Moreover we observe that if $h(w)$ denotes the orthogonal projection of $f(w)$ on the plane of the elliptic (non-singular) disc $g(C_n)$ for w in C_n, then a fortiori

$$|h(w)-g(w)| < \varepsilon\cdot\varrho(C_n) \qquad \text{for } w \text{ in } C_n. \tag{4.10}$$

Finally, if J_n denotes the value of J at w_n, we have $|J-J_n|<K\varepsilon$.

This being so, let E_n denote the set of points of the elliptic disc $g(C_n)$ whose distance from the bounding ellipse is at least $\varepsilon\varrho(C_n)$. Clearly the plane measure of $g(C_n)-E_n$ is at most $K\varepsilon\varrho(C_n)^2$. Thus

$$\begin{aligned}\iint_W |J|\, du dv &< K\varepsilon + \sum_n \iint_{W'_n C_n} |J|\, du dv\\ &< K\varepsilon + \sum_n \iint_{W'_n C_n} (|J_n| + K\varepsilon)\, du dv\\ &< K\varepsilon + \sum_n \iint_{C_n} (|J_n| + K\varepsilon)\, du dv\\ &= K\varepsilon + K\varepsilon + \sum_n \mu^2[g(C_n)]\\ &< K\varepsilon + K\varepsilon + K\varepsilon\sum_n \varrho(C_n)^2 + \sum_n \mu^2[E_n];\end{aligned}$$

and moreover

$$\mu^2(M_R) \geqslant \sum_n \mu^2[f(C_n)] \geqslant \sum_n \mu^2[h(C_n)].$$

Hence to prove (4.9), and so to complete the proof of Theorem (4.3), it is sufficient to show that $h(C_n)$ contains E_n. This last result follows, in the case $E_n \neq 0$, from the fact that by (4.10), the topological index, at any point z_0 of E_n, of the mapping

$$z = g(w) + t\cdot[h(w) - g(w)] \quad \text{where} \quad 0 \leqslant t \leqslant 1$$

is continuous in t and so constant and equal to unity.

5. The Morrey value-system M^*. We shall now modify the ideas introduced in paragraph 3, by using the methods of Morrey [10], see also T. Radó [11]. We shall suppose that the continuous function $f(w)$ whose values are points of x-space, is defined at the points w of a *bounded* plane closed set W_0.

Two points of W_0 will be termed *equivalent* if they lie on a same connected subset of W_0 throughout which $f(w)$ is constant. The set of all the points equivalent to a given point will be termed an *element* of W_0; it is either a single point or a continuum.

A set whose common part with W_0 is a sum of elements will be termed a *whole set*.

(5.1) *In any neighbourhood of an element ω there is a whole closed set K which contains a further neighbourhood of ω.*

Proof. Let x be the constant value of $f(w)$ on ω and let H_n be the closed subset of W_0 consisting of the points w for which $|f(w) - x| \leqslant 1/n$. Let Ω_δ be the continuum consisting of the points distant from ω by not more than δ, where δ is chosen so small as function of n, that $\Omega_\delta W_0 \subset H_n$: this is possible by continuity.

We denote by K_n the maximal continuum (or saturated continuum according to Janiszewski: it exists by [9] p. 21, Th. II) such that

$$\Omega_\delta \subset K_n \subset \Omega_\delta + H_n.$$

We may suppose that δ is a decreasing function of n, in which case Ω_δ and H_n contract as n increases, and therefore K_n contracts. The common part ΠK_n is thus a continuum ([9] p. 20, Th. I) contained in $W_0 + \Omega_\delta$, and therefore in W_0 since $\delta \to 0$ as $n \to \infty$; it is also contained in H_n for all n, and this requires $f(w)$ to be constant on it with the value x. Since ΠK_n contains ω, this is only possible if

$$\Pi K_n = \omega.$$

Hence for large enough n, the set $K = K_n$, which contains a neighbourhood of ω, can be made to lie in any previously given neighbourhood of ω.

It only remains to verify that K is a whole set. To see this, let ω' be any element for which $K\omega' \neq 0$. Then $\omega' \subset H_n$ since H_n is whole, and therefore the set $K + \omega'$ is a continuum containing Ω_δ and contained in $\Omega_\delta + H_n$, so that, by the maximal property, $K + \omega' \subset K$, i. e. $K + \omega' = K$. This requires ω' to be contained in K, and so completes the proof.

We are now in a position to proceed with definitions analogous to those of paragraph 3. We denote by E_x^* any finite subset of W_0 which consists of non-equivalent points w for which $f(w) = x$.

For any subset W of W_0, we denote by

$$M^*(x; f; W)_{\mathrm{mod}\, W_0} \quad \text{or simply by} \quad M^*(x; f; W),$$

the upper bound of the number of points of the various E_x^* contained in W. We write $M^*(x)$ for $M^*(x; f; W_0)$.

We observe that for any expanding sequence of whole subsets of W_0

$$M^*(x; f; \sum_n W^{[n]}) = \lim_n M^*(x; f; W^{[n]}) \tag{5.2}$$

and that for a whole subset W of W_0

$$M^*(x; f; W_0 - W) = M^*(x) - M^*(x; f; W) \tag{5.3}$$

except possibly at the points x for which $M^*(x) = \infty$.

Furthermore we have the following result:

(5.4) *The Morrey multiplicity function $M^*(x)$ is measurable* (*B*).

Proof. We proceed as in the proof of (3.3). Given $\varepsilon > 0$, we can — on account of (5.1) and of Borel's covering theorem — cover W_0 by a finite number of whole closed sets K each of which has all its points at a distance $< \varepsilon$ from an element. By forming the differences of the sets $K \cdot W_0$, we can therefore express W_0 as a finite sum of disjoint whole F_σ sets W_n such that every pair of points of W_n are distant $< \varepsilon$ from an element.

The set E_n of the values of $f(w)$ for w in W_n is an F_σ set. We denote by $M_\varepsilon^*(x)$ the sum of the caracteristic functions of the sets E_n, and we observe that $M_\varepsilon^*(x)$ is measurable (*B*) and that

$$M_\varepsilon^*(x) \leqslant M^*(x). \tag{5.5}$$

On the other hand, if x is fixed and N is any finite number $\leqslant M^*(x)$, we have for sufficiently small ε

$$N \leqslant M^*_\varepsilon(x) \tag{5.6}$$

because there must be an E^*_x consisting of N points, and these must lie in N different sets W_n if ε is sufficiently small, since no two points of E^*_x can be simultaneously at an arbitrarily small distance from an element [6]).

From (5.5) and (5.6), $M^*(x)$ is the limit of $M^*_\varepsilon(x)$ as $\varepsilon \to 0$, and is therefore measurable (B).

We denote by $\mathfrak{W}$ the smallest additive class of whole subsets of W_0 which includes all whole F_σ subsets of W_0. (An additive class of sets is one which includes among its members the difference of any two members and the sum of any enumerable selection of members).

We write M^* and M^*_W for the systems whose multiplicity functions are $M^*(x)$ and $M^*(x; f; W)$: we term these systems the *Morrey value-systems* of $f(w)$ in W_0 and in W respectively.

We have then the following result:

(5.7) *Let $\mu(E)$ be a Carathéodory measure in x-space, and suppose that $\mu(M^*) < \infty$. Then*

$$M^*(x; f; W)$$

is measurable (μ) whenever W belongs to $\mathfrak{W}$; and $\mu(M^*_W)$ is an absolutely additive function of the sets W of $\mathfrak{W}$.

Proof. In the case in which W is a whole closed set, the function $M^*(x; f; W)$ is measurable (B) [and therefore measurable (μ)] since we can take W for the set W_0 of (5.4). Hence by (5.2) and (5.3), remembering that $\mu(E) = 0$ when E is the set of the values of x for which $M^*(x) = \infty$, the assertions of (5.7) must hold good in the case in which W belongs to the subclass $\mathfrak{W}_1$ of $\mathfrak{W}$ consisting of the smallest additive class of whole sets which includes all whole closed subsets of W_0.

[6]) This is really another application of (5.1): if w and w' are distant less than ε from an element ω^*_ε then the element ω which contains w has a neighbourhood Ω_η (where $\eta \to 0$ as $\varepsilon \to 0$) which contains w'. If this is the case for all $\varepsilon > 0$, we see that $w' \in \omega$, i. e. that w and w' are equivalent.

To complete the proof it is sufficient to show that $\mathfrak{W}_1$ is identical with $\mathfrak{W}$, i. e. that every whole F_σ subset of W_0 belongs to $\mathfrak{W}_1$ [7]). This is the case if we show that a whole F_σ set is an enumerable sum of whole closed sets.

Since a whole set, which is an enumerable sum of closed sets W, is also an enumerable sum of the sets W', where each W' consists of all the points w equivalent to points of one of the W, it is enough to show that each W' is closed.

We may suppose $W \subset W_0$. Let w'_n be a sequence of points of W' with the limit w'. We have to show that w' belongs to W'. Let w_n be a point of W equivalent to w'_n and let w be a limit of the sequence w_n. Then w and w' are simultaneously at an arbitrarily small distance from the element which contains w_n and w'_n (for some large n). Hence by a remark already made [8]), the point w' is equivalent to the point w of W, and therefore w' belongs to W'.

This completes the proof.

(5.8) (i) *If $\mu(M) < \infty$, then $\mu(M) = \mu(M^*)$.*

(ii) *If W_0 is a segment, $\mu^1(M) = \mu^1(M^*)$.*

(iii) *If $f(w)$ is Lipschitzian, $\mu^2(M) = \mu^2(M^*)$.*

These results are immediate: (i) follows from the fact that $M(x)$ and $M^*(x)$ can only differ at points at which the former is infinite; (ii) from the fact that on a segment the elements which do not reduce to single points constitute an enumerable aggregate; (iii) follows from (i) and from Theorem (4.3).

We conclude this paragraph by observing that the notions we have introduced are invariant under far more general transformations than those used in the classical theory of curves and surfaces.

Let us say that the continuous mapping $f(w)$ defined on W_0 is transformed „approximately topologically" into a continuous mapping $g(z)$ defined on Z_0 of the z-plane, if there exist a whole subset W of W_0 and a whole subset Z of Z_0 such that

(a) outside W, the function $f(w)$ takes at most an enumerable number of values x,

[7]) We leave unsolved the question whether $\mathfrak{W}$ is the class of whole Borel subsets.

[8]) See footnote [6]).

(b) outside Z, the function $g(z)$ takes at most an enumerable number of values x,

(c) there is a (1,1)-correspondence between the elements ω in W for the function $f(w)$, and the elements ζ in Z for the function $g(z)$, such that

$$f(w) = g(z)$$

whenever w and z belong to corresponding elements ω and ζ.

Since the corresponding pair of multiplicity functions differ for at most an enumerable number of values of x, we see that

(5.9) *Morrey value-systems, derived from one another by an approximately topological transformation, have the same measure relative to any Carathéodory measure which vanishes for enumerable sets* [9].

6. Boundary value-systems. Even the general type of transformation considered at the end of the preceding paragraph does not cover the case of the transformation of boundaries caused by conformal representation of the interiors of two simply-connected domains. It is therefore necessary that we should introduce yet a third kind of value-system, for which the fundamental elements of the w-plane are neither points, nor *maximal* continua of constancy.

The place of these fundamental elements is taken by entities introduced by Carathéodory [6], which are termed prime ends. We shall assume familiarity with Carathéodory's theory. As we wish to establish certain relations between the two types of fundamental entities, we shall continue to use the word „element" in its former sense. We shall retain also the notation and definitions of the preceding paragraph.

We shall suppose in addition that W_0 is a finite *continuum*. Its complementary domains are then simply-connected in the complex plane of w: we shall suppose that there is a selection of them, not necessarily including all the complementary domains if there are several of them, but including at least one domain, such that the function $f(w)$ which is given and continuous on W_0, is constant on each prime end of each domain of the selection.

[9]) It is easily seen that two representations of surfaces which are „equivalent in the sense of Kerékjártó" [20, p. 765] can be obtained from one another by an approximately topological transformation. In particular, the same is therefore true of representations which are equivalent in the sense of Fréchet. The intrinsic area defined in paragraph 7 is thus independent of the selected representation of a Fréchet surface.

The symbol D will be used to denote any domain belonging to the given selection. We define $N(x; f; D)$ as the number of prime ends of D throughout which $f(w)$ assumes the constant value x; the multiple system $N(D)$, whose multiplicity function is $N(x;f;D)$ will be termed the boundary value-system of f for the complementary domain D.

(6.1) *Invariance Theorem for boundary value-systems. Conformal transformation of D into the unit circle of the z-plane defines a correspondence between the prime ends of D and the points of the circumference of the unit circle, such that, if $g(z)$ denotes for $|z|=1$ the value of $f(w)$ on the prime end corresponding to z, we have*

(i) *$g(z)$ is continuous for $|z|=1$;*

(ii) *$N(x; f; D)$ is the Banach multiplicity function, for the Banach value-system of $g(z)$ on the circumference $|z|=1$;*

(iii) *if either f or g be continued into the interior of the relevant domain, then the given conformal transformation provides a continuation of the other, such that if one function is uniformly continuous so is the other.*

With regard to the above statement, it must be understood that in (iii) if D is the infinite complementary domain, continuity at the point at infinity of the w-plane is suitably interpreted.

Proof of (6.1). We denote by $w=\varphi(z)$, a conformal representation of D into the unit circle $|z|<1$. We choose any continuation of $f(w)$ into D which is uniformly continuous in the closure of D; such a continuation exists as is well-known, we denote it by the same functional symbol. We denote further by $P(\theta)$ the prime end of D corresponding to the point $z=e^{i\theta}$, and we write $g(e^{i\theta})$ for the constant value of $f(w)$ for w in $P(\theta)$. We shall show first that

(6.2) *As $z\to e^{i\theta}$ from within the circle, $f[\varphi(z)]\to g(e^{i\theta})$.*

Carathéodory [6, p. 350, Satz XIII] shows that if a sequence of points z interior to the unit circle tends to $e^{i\theta}$, then the corresponding sequence of the points $\varphi(z)$ tends to the prime end $P(\theta)$ and vice-versa. Given any such a sequence of z, there is a subsequence z_n so that the points $w_n=\varphi(z_n)$ tend to a limit w, and this point w lies on $P(\theta)$ by Carathéodory's theorem. Therefore the corresponding value of $f(w)$ is $g(e^{i\theta})$ by definition, and this value is, by continuity of f, the limit of the sequence $f(w_n)=f[\varphi(z_n)]$. Hence every subsequence of the points $f[\varphi(z)]$ where $z\to e^{i\theta}$ from inside the circle, contains another subsequence tending to $g(e^{i\theta})$, and this is only possible if $[f\varphi(z)]\to g(e^{i\theta})$. Thus (6.2) is proved.

From (6.2) we deduce at once (i) of Theorem (6.1), since if $\theta\to\theta_0$ we can find an interior point z depending on θ and tending to $e^{i\theta_0}$ such that $f[\varphi(z)]-g(e^{i\theta})$ tends to 0.

Moreover, to complete the proof of (iii) of Theorem (6.1), it is enough to show that if $z=\psi(w)$ is the inverse conformal transformation, and $g(z)$ is continued into the interior of the unit circle so as to be uniformly continuous, then

(6.3) *As a point w_n of D tends to a boundary point w, the value $g[\psi(w_n)]$ tends to $f(w)$.*

In proving (6.3), we may suppose that $\psi(w_n)$ tends to a limit, by taking a subsequence if necessary. This limit cannot be an interior point, for otherwise, by continuity $w_n=\varphi[\psi(w_n)]$ would tend to an interior point of D contrary to our hypothesis. Denoting the limit by $e^{i\theta}$, Carathéodory's theorem requires w to lie in $P(\theta)$, so that $\operatorname{Lim} g[\psi(w_n)]=g(e^{i\theta})=f(w)$ and (6.3) is established.

To complete the proof of (6.1), we need only observe that the statement (ii) follows at once from the definitions.

As a corollary, it follows from (3.3) that

(6.4) *$N(x;f;D)$ is measurable (B).*

We shall now give some important relations connecting the expressions $N(x;f;D)$ and $\sum_D N(x;f;D)$ with Morrey value-systems. We write $W_n(D)$ for the sum of the elements of W_0 meeting at least n prime ends of D, and W_n for the sum of the elements of W_0 meeting at least n prime ends of the same or different complementary domains D of the given selection. Then

(6.5) *The following relations hold except perhaps at enumerably many values of x,*

(a) $N(x;f;D)=M^*[x;f;W_1(D)]+M^*[x;f;W_2(D)]$,

(b) $\sum_D N(x;f;D)=M^*(x;f;W_1)+M^*(x;f;W_2)$,

and the functions which occur in them are all measurable (B).

Proof. As regards measurability, we observe that by a well-known theorem (Janiszewski [9] Theorem I):

(6.6) *$W_1(D)$ is closed.*

Hence

(6.7) $W_1=\sum_D W_1(D)$ *is an F_σ set.*

We observe also that $W_2(D)$ and W_2 are expressible as enumerable sums of closed sets of the type $W(U,V)$, where this last expression denotes the sum of the elements of W_0 each of which meets two given closed bounded sets U, V [10]). Thus

(6.8) *$W_2(D)$ and W_2 are F_σ sets.*

The expressions which occur on the right-hand sides are therefore measurable (B) on account of (5.4), (5.2), and of the fact, established generally in the course of the proof of (5.7) but here obvious directly, that the sets concerned are enumerable sums of *whole* closed sets.

Finally, (6.5) (a) and (b) follow at once from

(6.9) *$W_3(D)$ and W_3 are enumerable sums of elements at most.*

And this last assertion is a corollary of Ursell and Young [16] Theorem I, since prime ends are here subsets of elements. This completes the proof.

7. Intrinsic area, intrinsic length, boundary length. As the traditional distinction between a curve and a surface is only partly relevant in a definition of length or area, in view of well-known examples of curves of positive area, we prefer to use the neutral term „map", or more precisely „continuous map": we shall mean by it the continuous mapping $f(w)$ of a finite plane continuum W_0. The same mapping applied to a subset W of W_0 defines a submap on W, and we shall suppose that W is whole (mod W_0).

We term *intrinsic area* of a continuous map, or of a submap, the 2-dimensional measure of the Morrey value-system defined by the multiplicity function $M^*(x;f;W_0)$, or $M^*(x;f;W)$. Similarly its *intrinsic length* is the corresponding 1-dimensional measure.

[10]) Boundaries of subdomains determined by „rational" cross-cuts. See [16].

It is, however, necessary to introduce a second notion of length which applies to boundaries only, and which does not in general agree with the intrinsic length. The difference between the two notions is a natural one: thus in evaluating the intrinsic length of a figure which consists of a circumference together with a radius, we add the length of the radius to that of the circumference; but we must add *twice* the length of the radius if we require the length of the figure *as a boundary* of the interior domain.

In order to apply the theory of the preceding paragraph, we require the following result:

(7.1) *Suppose that the submap S defined by $f(w)$ on the fixed subset W has finite intrinsic length $L^*(S)$; and let D be any complementary domain of W_0 whose boundary is included in W. Then the function $f(w)$ is constant on each prime end of D.*

This is a Theorem of Ursell and Young [16, or the abstract in Bull. Amer. Math. Soc. **54** (1948)].

We can now define our second notion of length. Let S_0 be the given continuous map, and let D be a complementary domain of W_0. If there exists no submap S of finite intrinsic length, defined by $f(w)$ on a whole subset W including the boundary of D, then we say that the boundary of S_0 on D has infinite boundary-length. If, on the other hand, at least one such submap S exists, then by (7.1) the function $f(w)$ defines a boundary value-system for the complementary domain D, and the 1-dimensional measure of this boundary value-system is defined to be the boundary-length (frontier-length)

$$L_{fr}(S_0; D)$$

of the boundary of S_0 on D. We shall sometimes write it $L_{fr}(f,D)$ or simply $L_{fr}(D)$.

Applying (6.5) (b) to the selection of domains D whose boundaries lie in the fixed set W, we obtain

Theorem (7.2). *Suppose that the submap S of S_0 defined by $f(w)$ on the subset W has finite intrinsic length $L^*(S)$; then, for the complementary domains D of W_0 whose boundaries lie in W we have*

$$\sum_D L_{fr}(S_0; D) \leqslant 2 \cdot L^*(S).$$

This follows at once from (6.5) (b), since an enumerable set of x does not affect the 1-dimensional measure, if we observe that the right-hand side of (6.5) (b) cannot exceed $2M^*(x;f;W)$ because $W_2 \subset W_1 \subset W$, each of these sets being whole.

The theorem just proved may be again illustrated by considering the figure consisting of a circumference together with a radius, in the case of the function $f(w)=w$. The circumference is the boundary of the outer domain complementary to the figure, while the boundary of the inner complementary domain consists of the circumference together with twice the radius: the sum of the boundary-lengths for the two complementary domains is thus here just twice the intrinsic length of the figure.

Theorem (7.2) should also be compared with the results of Ważewski [17] which connect Carathéodory measure with the length of a curve passing through a given set of points. We note that boundary-length is a notion which may be expressed in terms of the classical length of a closed curve: in fact, by (6.1) (ii), if $g(z)$ is defined as there, we see on account of (4.1) that

(7.3) *If $L_{fr}(S_0;D)$ is finite, it is the classical length of the closed curve described by $g(z)$ as z describes the circumference of the unit circle.*

With the help of (7.3) we may now extend a theorem of [19] to harmonic surfaces defined on arbitrary simply-connected bounded plane domains. If $f(w)$ is defined as a continuous function on the boundary of a domain D and is constant on each prime end of D, we term its *harmonic interpolation* in D the continuous function in $\bar{D}$ which is harmonic in D and which agrees with f on the boundary; the existence of such a function follows at once from (6.1) (iii) where we suppose g continued into the interior of the unit circle by means of the Poisson integral.

(7.4) *If $L_{fr}(f,D)$ is finite and has the value L, and if A denotes the classical area (or the Banach area, or the intrinsic area) of the surface defined on $\bar{D}$ by the harmonic interpolation of f, then*

$$A < KLd,$$

where K is an absolute constant, and where d is the diameter of the set of values of f on the boundary of D.

It is enough, in view of the results of [19], to verify that A is like L unaltered when we pass by conformal representation to the corresponding functions defined on the unit circle. Now the Banach area of the part of our surface which corresponds to the boundary of D is zero, while that corresponding to D is, by definition, unaltered when we make a (1,1)-transformation of D into another domain. Moreover, since D is the sum of an expanding sequence of closed sets on which our harmonic interpolation is Lipschitzian, and since the same holds in the unit circle, it follows from Theorem (4.3) that the Banach area is in both cases equal to the classical area (expressed as a double integral over the *interior* of the relevant domain). Finally the identity of these expressions with the intrinsic area follows from (5.8) (i). This completes the proof.

8. Generalisation of a „lemma in the Theory of Surfaces" [18]. We consider a surface $\mathcal{S}$, i. e. a continuous map defined by $f(w)$ on a rectangle R of the w-plane, and we suppose that $\mathcal{S}$ has finite intrinsic area A, and that the boundary-curve of $\mathcal{S}$, defined by $f(w)$ on the boundary of R, lies in a sphere S_a of radius a and centre the origin. We denote by Ω_r the maximal sub-continuum of R which includes the boundary of R and is such that $f(w)$ lies in S_r; i. e. $|f(w)|\leqslant r$ for all w of Ω_r. We denote by W_r the set of the points of Ω_r for which $|f(w)|=r$, by Σ_r the submap defined by $f(w)$ on W_r and by $L^*(\Sigma_r)$ its intrinsic length.

Clearly $L^*(\Sigma_r)\leqslant(M^*\cdot S_r)$, where M^* is the Morrey value-system defined by $\mathcal{S}$. Hence (2.3) leads to

$$A\geqslant a_1\int_a L^*(\Sigma_r)\,dr. \tag{8.1}$$

Given $\varepsilon>0$, it follows that if we choose $b=a\cdot\exp(1/\varepsilon)$ there must exist a value $r<b$ such that $r\cdot L^*(\Sigma_r)\leqslant\varepsilon A/a_1$.

This being so, let W be the set obtained by adding to Ω_r for this particular value of r, those of its complementary domains in R in which the oscillation of $f(w)$ is less than $b-r$; we denote by D the remaining complementary domains in R. There can only be a finite number of such D: in fact D must either have a boundary on which the oscillation of $f(w)$ exceeds $\frac{1}{2}(b-r)$, or else contain (by uniform continuity) at least one circle of a certain fixed radius; the latter case can only occur for a finite number of D at most, and, since $L^*(\Sigma_r)<\infty$, the same is true of the former case.

We denote by $f_\varepsilon(w)$ the function equal to $f(w)$ in W and to its harmonic interpolation in each D; it defines on R a surface $\mathcal{S}_\varepsilon$ situated in the sphere S_b.

The two surfaces differ on R at points at which the functions $f(w)$ and $f_\varepsilon(w)$ take different values, and at points which are contained in different maximal continua throughout which they take the same constant value. All such points clearly lie in $\Sigma D+\Sigma B$, where B is the sum of the elements of $\mathcal{S}$ which meet the boundary of D. Since however, each $B\subset W_r$, the image-set $f(B)=f_\varepsilon(B)$ has length at most $L^*(\Sigma_r)$ and therefore zero area. Moreover, the diameter of the submap of $\mathcal{S}_\varepsilon$ on D is at most $2r$, and its Banach area is therefore at most

$$K\cdot 2r\cdot L_{fr}(D)$$

by (7.4), where $L_{fr}(D)$ denotes the relevant boundary-length. Hence, using Theorem (7.2), the sum of these areas corresponding to the various D is at most

$$2K\cdot r\cdot L^*(\Sigma_r)\leqslant(2K/a_1)\cdot\varepsilon\cdot A.$$

From these facts we may conclude as follows:

Theorem (8.2). *There is an absolute constant K such that any surface $\mathcal{S}$ of finite intrinsic area A and whose boundary lies in a sphere of radius a, can be so modified as to lie in a sphere of radius $b=a\cdot\exp(1/\varepsilon)$, the modification being made by replacing its parametric representation $f(w)$ by its harmonic interpolation in a finite number of simply-connected domains interior to the fundamental rectangle R, in such a manner that the total area of these harmonic portions of the new surface together with any points corresponding to elements which have been altered by tha above modifications, is at most $K\cdot\varepsilon\cdot A$.*

9. In the case of a surface represented parametrically on a rectangle, there is agreement between intrinsic area, Banach area, Lebesgue-Fréchet area, and the classical area given by a double integral, whenever we are given a Lipschitzian representation, or again, when we have a non-parametric representation which is absolutely continuous in the sense of Tonelli: this follows at once from (4.3) together with (5.8), and from the results of Besicovitch [3] together with what is well-known. It is emphasized further by Besicovitch that such agreement can be expected only in very special circumstances.

All these definitions are at the present time useful tools of Analysis: the intrinsic area, besides being more satisfactory from the point of view of pure theory, has already led us to applications of its own in the preceding paragraphs. This makes it all the more important that we should study further the relationship between the various definitions of area, particularly from the point of view of inequalities between them.

In this connection we prove the following result:

(9.1) *Suppose that the continuous vector function $f(w)$ has partial derivatives f_u, f_v almost everywhere in the rectangle R of the plane of $w=u+iv$. Then the intrinsic area of the surface $x=f(w)$ represented parametrically on R is not less than the double integral*

$$\iint_R \{(f_u)^2\cdot(f_v)^2-(f_u\cdot f_v)^2\}^{1/2}\,du\,dv.$$

Preliminary reduction. The integrand will again be written for brevity $|J|$, where J is the vector Jacobian used in (4.3). We observe that the double integral is unaffected by replacing its range R by the subset Q of R in which the integrand exists and is not zero. With the notation of paragraph 5, where R takes the place of W_0, it is thus sufficient to prove that there is a whole subset W of Q with the same plane measure as Q, such that $\mu^2(M_W^*)\geqslant\iint_W |J|\,du\,dv$; in fact we shall establish this relation with the sign of equality. This will be the case if we prove:

(9.2) *There is a sequence of disjoint Borel subsets W_n of Q such that*
(i) *each element R which meets W_n reduces to a single point,*
(ii) *f is Lipschitzian in each W_n,*
(iii) *the set $Q-\sum W_n$ has plane measure zero.*

For by (ii) and (4.3) this would imply $\mu^2(M_{W_n})=\iint_{W_n}|J|\,du\,dv$, and by addition $\mu^2(M_W^*)=\iint_W |J|\,du\,dv$, where $W=\sum W_n$, since (i) implies $M_{W_n}=M_{W_n}^*$.

Before proving (9.2), we shall establish a lemma which is implicitely contained in a proof by Saks [13, p. 301] of a theorem on approximate differentiability:

(9.3) *In order that a finite function $f(w)$ which is measurable on a plane set Q, be almost everywhere on Q approximately derivable with respect to each of the variables u and v* (where $w=u+iv$), *it is necessary and sufficient that almost all points of Q be contained in the sum of a sequence of measurable sets R_n in each of which f is Lipschitzian.*

Proof of lemma (9.3). It is clearly enough to prove the necessity of this condition. We may further suppose, by the theorem proved in Saks [13, p. 300, Th. 12.1], that f is approximately differentiable almost everywhere in Q. We denote, for each positive integer n, by R_n the set of the points w' of Q such that, for every square interval J of diameter $dJ\leqslant 2/n$ which contains w', at least 3/4 of the measure of J is occupied by points w which satisfy the condition $|f(w)-f(w')|\leqslant n\cdot dJ$. (We follow closely the notation of Saks: J is thus a square, and no longer a Jacobian).

Clearly almost all point of Q lie in the sum of the sets R_n. It remains to show that f is Lipschitzian in each R_n.

To this effect, let w' and w'' be any two points of R_n distant less than $1/n$; and denote by J the square interval which circumscribes the circle with w' and w'' as extremities of a diameter. Since J has diameter less than $2/n$, it contains a point w for which neither of the differences $|f(w)-f(w')|$ and $|f(w)-f(w'')|$ can exceed $n\cdot dJ$ — in fact 1/2 of J consists of such points — and this is only possible if $|f(w')-f(w'')|\leqslant 2n\cdot dJ\leqslant 4n\cdot|w'-w''|$.

In the case in which f has on Q a finite upper bound M, this completes the proof of (9.3), the Lipschitz constant in R_n being at most the greater of the numbers $4n$ and $2M\cdot n$. The general case follows at once by means of a preliminary dissection of Q into a series of sets in each of which f is bounded.

Proof of (9.2). Following Kuratowski, we term hereditary, a property of a set, if every subset of this set possesses it also. We shall express a subset of Q which comprizes almost all points of Q, as the sum of a sequence of measurable sets R_n with certain hereditary properties. One such property may, by our lemma (9.3), be taken to be: f is Lipschitzian in R_n.

We remark that if each R_n of such a dissection of Q has hereditary properties P and is almost entirely covered by a sequence of sets with further hereditary properties P', then there is a new dissection of Q in which the new sets R_n have the properties P' in addition to the properties P. In particular, we can and shall replace any dissection of Q by another for which the sets R_n are disjoint and closed.

On account of a well-known result due to Rademacher, we may suppose that the Lipschitzian function which agrees with f in R_n has, at every point of R_n, a total differential whose coefficients are the partial derivatives of f. (See for instance [13, p. 311, Th. 14.2]).

Now in Q, and therefore in R_n, the expression $(f_u)^2\cdot(f_v)^2-(f_u\cdot f_v)^2$ does not vanish, whence neither does $f_u\cdot\cos\theta+f_v\cdot\sin\theta$ for any θ. From the existence of a total differential, it follows that we have an inequality of the form

(9.4) $|f(w)-f(w')|>\dfrac{|w-w'|}{m}$ *where* m *is an integer depending on* w *only, whenever* w *is a point of* R_n *and* w' *a sufficiently near point of* R_n.

Also, simply because the partial derivatives of f exist at each point of Q and therefore of R_n, we have an inequality of the form

(9.5) $|f(w)-f(w+h)|<p\cdot|h|$ *where* p *is an integer depending on* w *only, whenever* w *is a point of* R_n *and* h *is real or pure imaginary, and sufficiently small. (The point* $w+h$ *need not belong to* R_n*).*

From (9.4) and (9.5), it follows that each R_n can be covered by a sequence of sets with the hereditary property:

(9.6) *There exist positive constants* a, b, c *such that for every pair of points* w, w' *of the set, and for every real or pure imaginary* h *whose modulus is less than* c, *we have*

$$|f(w)-f(w')|>a\cdot|w-w'| \quad \textit{and} \quad |f(w)-f(w+h)|<b\cdot|h|.$$

It follows that we can choose the sequence of closed disjoint subsets R_n of Q so that each set R_n has the property (9.6), and in addition f is Lipschitzian on it, in such a manner that $\sum R_n$ covers almost all points of Q.

To prove (9.2) it is now sufficient to show that these hypotheses on the R_n, imply:

(9.7) *No point of density of* R_n *can be situated on a continuum of constancy of* f *which does not reduce to that single point.*

For if we prove this, then, if W_n is a Borel subset of R_n with the same plane measure as R_n, the sequence of sets W_n has the properties stated in (9.2).

To prove (9.7), suppose the contrary, and let w' be a point of density of R_n situated on such a continuum of constancy. Then for all small r there is a point w'' of this continuum, situated on the perimeter of the square interval of centre w' and side $2r$. We denote by w a point of R_n, if any, situated on the side of this square which contains w'', or on one of the two relevant sides if w'' is a corner.

We may suppose $r<c$, so that if $h=w''-w$, the number h is real or pure imaginary and we have $|h|<c$. Since f takes the same constant value at the points w' and $w''=w+h$, it follows from (9.6) that

$$a\cdot r\leqslant a\cdot|w-w'|<|f(w)-f(w')|=|f(w)-f(w+h)|<b\cdot|h|.$$

Thus a segment of length $r\cdot a/b$ of the perimeter of the given square contains no point of R_n. This implies that the mean density of R_n on the perimeter of any small square interval of centre w' has an upper bound less than unity: since the same is then true of the mean density *in* any such square, this contradicts the definition of w' as a point of density.

This establishes (9.7) and so completes the proof of (9.2) and therefore of (9.1).

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