Put $\delta_{ik}^{(n)}=\varepsilon_{ik}^{(n)}-\varepsilon_{i\,k+1}^{(n)}$, $\delta_{ik}'^{(n)}=\varepsilon_{ik}^{(n)}-\varepsilon_{i+1\,k}^{(n)}$, $\Delta_n=(\delta_{ik}^{(n)})$, $\Delta_n'=(\delta_{ik}'^{(n)})$.

By formula (4.1) we have

$$\|f_n(\cdot,\cdot)\|_{(r),(r)}=\sup_{\|x\|\leqslant 1}\varphi_{\Delta_n}(x). \tag{4.5}$$

We observe easily that the inequalities $\mu(\Delta_n)\leqslant c$ and $\mu(\Delta_n')\leqslant c$ imply $\vartheta(P_n)\leqslant 4c^2$; thus the supposition (4.4) implies $\overline{\lim\limits_{n\to\infty}}\,\mu(\Delta_n)+\overline{\lim\limits_{n\to\infty}}\,\mu(\Delta_n')=+\infty$. We may suppose without loss of generality that $\overline{\lim\limits_{n\to\infty}}\,\mu(\Delta_n)=+\infty$. Any perfect method having Mertens' property, we see that there exists a p such that $\omega(\Delta_n)\leqslant p$ for $n=1,2,\dots$ Hence by lemma 5 and (4.5) we get

$$\overline{\lim_{n\to\infty}}\,\|f_n(\cdot,\cdot)\|_{(r),(r)}=+\infty.$$

This is, however, impossible since by lemma 3 the sequence $\{\|f_n(\cdot,\cdot)\|_{(r),(r)}\}$ must be bounded. Thus, we have shown that $\overline{\lim\limits_{n\to\infty}}\,\vartheta(P_n)<+\infty$.

It is quite obvious that all the considerations of this paper are valid also for series with complex elements.

(Reçu par la Rédaction le 30. 11. 1947).

An example in Fourier series

by

A. ZYGMUND (Chicago).

1. Let $a_1, a_2, \dots$ be a sequence of real numbers such that $\sum a_k^2$ converges, and let $n_1, n_2, \dots$ be a sequence of positive integers such that $n_{k+1}/n_k\geqslant 3$ for all k. The partial products $p_k(x)$ of the infinite product

$$\prod_{\nu=1}^{\infty}(1+ia_\nu\cos n_\nu x) \tag{1}$$

are trigonometric polynomials,

$$p_k=\prod_{\nu=1}^{k}(1+ia_\nu\cos n_\nu x)=1+\sum_{\nu=1}^{\mu_k}\gamma_\nu\cos\nu x,\qquad \mu_k=n_k+\dots+n_2+n_1,$$

where the γ_ν are either real or purely imaginary, since all the terms obtained from multiplying out p_k are distinct. The passage from p_k to p_{k+1} consists in adding to p_k the polynomial $p_{k+1}-p_k$ whose all terms are of rank $>\mu_k$. Hence, making $k\to\infty$, we obtain, formally, a trigonometric series

$$1+\sum_{\nu=1}^{\infty}\gamma_\nu\cos\nu x, \tag{2}$$

the partial sums $s_n(x)$ of which have the property that $s_{\mu_k}=p_k$.

The series (2) may be said to *represent* the product (1).

Since

$$|p_k(x)|\leqslant\prod_{\nu=1}^{k}(1+a_\nu^2)^{\frac{1}{2}}\leqslant\prod_{\nu=1}^{\infty}(1+a_\nu^2)^{\frac{1}{2}}<+\infty,$$

a sequence of the partial sums of the series (2) is uniformly bounded. This shows that (2) is the Fourier series of a bounded function $f(x)$.

For all this, see also SALEM and ZYGMUND, **4**. The products (1) are the familiar F. RIESZ products modified by the insertion of the factor i.

2. Theorem 1. *The series* (2) *representing the product*

$$\prod_{k=1}^{\infty}\left(1+i\frac{\cos 3^k x}{k}\right) \tag{3}$$

has the partial sums uniformly bounded but divergent at a set of points which is of the second category (and so of the power of the continuum) in every interval.

The interest of Theorem 1 lies in the fact that in all previously known examples of Fourier series with uniformly bounded partial sums divergence takes place in sets of points which are at most denumerable. Theorem 1 naturally raises the problem of what may be said about the measure of the set of the points of divergence of the series representing (3).

Theorem 2. *If* $\sum_{k=1}^{\infty} a_k^2 < +\infty$, $n_{k+1}/n_k \geqslant 3$, *the series* (2) *representing* (1) *converges almost everywhere. If, in addition,* $\sum_{k=1}^{\infty}|a_k| = +\infty$, *the series* (2) *diverges in a set which is of the second category in every interval.*

Thus, in particular, the series (2) representing (3) converges almost everywhere.

The idea of the proof of the first part of Theorem 2 is not new, and was used, for somewhat similar purposes, elsewhere (see ZYGMUND, **6**). Owing, however, to the fact that the terms of the product (1) are complex-valued some modifications of the proof are indispensable.

3. To prove the uniform boundedness of the partial sums of the series representing (3) is a simple matter. For in this case

$$p_{k+1}(x)-p_k(x)=p_k(x)\,i\,(k+1)^{-1}\cos 3^{k+1}x = O(1/k),$$

since the p_k are uniformly bounded. If one takes into account that the nth Lebesgue constant is $O(\log n)$, and that the order

of the polynomial $p_{k+1}-p_k$ is $3^{k+1}+3^k+\ldots+3=O(3^k)$, one easily deduces that all the partial sums of all the differences $p_{k+1}-p_k$ are uniformly bounded. This together with the uniform boundednes of the p_k implies the uniform boundedness of all the partial sums of the series representing (3).

The divergence part of Theorem 1 is a special case of the second part of Theorem 2. To prove the latter we use the equation $1+z=\exp\{z+O(|z|^2)\}$ valid for small $|z|$. It gives

$$p_k(x)=\prod_1^k(1+i\,a_\nu\cos n_\nu x)=\exp\{i\sum_1^k a_\nu\cos n_\nu x\}\cdot\exp\{\sum_1^k O(a_\nu^2)\} \tag{4}$$

The last exponential tends, even uniformly in x, to a finite limit distinct from 0. On the other hand, it is known (see ZYGMUND, **7**), that if $\sum|a_k|=+\infty$, the partial sums of the lacunary series

$$\sum_{\nu=1}^{\infty} a_\nu \cos n_\nu x \tag{5}$$

(even if we only had $n_{\nu+1}/n_\nu \geqslant q > 1$) are unbounded at a set E of points which is everywhere dense. The set of points at which a series of continuous functions is unbounded is of the type G_δ and being everywhere dense in our case must necessarily be of the second category. Since $a_\nu \to 0$, the divergence of the p_k, and so also of the s_n, in E follows.

It may be added that the proof of the result, which we have just used, on the divergence of lacunary series (5) is particularly simple if $n_{k+1}/n_k \geqslant q$ where q is a number *greater* than 3 (see ZYGMUND, **7**, p. 77, or ZYGMUND, **5**, p. 130, Ex. 10). Thus the proof of Theorem 1 simplifies if in (3) we replace 3^k by 4^k, the more so that the proof of Theorem 2 is also simpler if q there is greater than 3 (see below).

4. We now pass to the proof of the first part of Theorem 2. We temporarily assume that

$$n_{k+1}/n_k \geqslant q > 3 \tag{6}$$

for all k. Let us set

$$\mu_k' = n_{k+1}-n_k-\ldots-n_2-n_1.$$

Since the ranks of all the terms of

$$p_{k+1}-p_k=i\alpha_{k+1}p_k\cos n_{k+1}x$$

are $\geqslant \mu_k'$, the terms of the series (2) all vanish for $\mu_k<\nu<\mu_k'$. By (6),

$$\mu_k'/\mu_k > n_{k+1}(1-q^{-1}-q^{-2}-\ldots)/n_k(1+q^{-1}+\ldots)\geqslant q-2>1$$

and so the series (2) has infinitely many gaps. The same holds for the series

$$\sum_{\nu=1}^{\infty}\gamma_\nu \sin n_\nu x, \tag{7}$$

conjugate to (2), whose partial sums we shall denote by $\breve{s}_n(x)$. Both (2) and (7) are Fourier series (the latter of the class L^2), and so are summable $(C,1)$ almost everywhere. Summability $(C,1)$ implies however that the partial sums s_{μ_k} and $\breve{s}_{\mu_k}$ corresponding to the beginnings of the gaps converge (see e. g. Zygmund, **5**, p. 251). In particular they are bounded almost everywhere.

Let us denote by $t_n(x)$ the partial sums of of the series

$$1+\sum_{\nu=1}^{\infty}\gamma_\nu e^{i\nu x} \tag{8}$$

by $t_n(x)$. Thus

$$t_n(x)=s_n(x)+i\breve{s}_n(x),\qquad s_n(x)=\frac{1}{2}\{t_n(x)+t_n(-x)\}. \tag{9}$$

The function

$$M(x)=\sup_k|t_{\mu_k}(x)|$$

is therefore finite almost everywhere.

Almost all points x have the property that $t_{\mu_k}(x)$ and $t_{\mu_k}(-x)$ both converge. Let us consider any such point x and let us set $M=\operatorname{Max}\{M(x),M(-x)\}$. Let A be so large that the following conditions are satisfied:

$$\Big|\sum_{\mu}^{\mu_{k-1}}\gamma_\nu e^{\pm i\nu x}\Big|\leqslant A,\qquad 1\leqslant\mu\leqslant\mu_{k-1}, \tag{10_{k-1}}$$

$$\Big|\sum_{\mu}^{\mu_i}\gamma_\nu e^{\pm i\nu x}\Big|\leqslant A-2M,\quad \mu_{i-1}<\mu\leqslant\mu_i;\ i=1,2,\ldots,k-1;\ \mu_0=0. \tag{11_{k-1}}$$

We shall show that if $|\alpha_k|\leqslant 1$ (which is certainly true for all k sufficiently large) and if A satisfies one more condition independent of k, then (10_{k-1}) and (11_{k-1}) imply (10_k) and (11_k).

Obviously,

$$s_{\mu_k}(x)=(1+\sum_{\nu=1}^{\mu_{k-1}}\gamma_\nu\cos n_\nu x)(1+i\alpha_k\cos n_k x)=$$

$$=s_{\mu_{k-1}}(x)+i\alpha_k\cos n_k x+\frac{1}{2}i\alpha_k\sum_{\nu=1}^{\mu_{k-1}}\gamma_\nu[\cos(n_k-\nu)x+\cos(n_k+\nu)x].$$

Since $n_k\pm\nu>0$,

$$t_{\mu_k}(x)=t_{\mu_{k-1}}+i\alpha_k e^{in_k x}+\frac{1}{2}i\alpha_k\sum_{\nu=1}^{\mu_{k-1}}\gamma_\nu[e^{i(n_k-\nu)x}+e^{i(n_k+\nu)x}]. \tag{12}$$

Let us consider separately the two cases

(a) $\mu_{k-1}<\lambda<n_k$, (b) $n_k\leqslant\lambda\leqslant\mu_k$.

In case (a), as seen from (12),

$$t_\lambda=t_{\mu_{k-1}},\qquad t_\lambda=t_{\mu_{k-1}}+\frac{1}{2}i\alpha_k\sum_{\nu=n_k-\lambda}^{\mu_{k-1}}\gamma_\nu e^{i(n_k-\nu)x}$$

according as $\lambda<n_k-\mu_{k-1}$ or $\lambda\geqslant n_k-\mu_{k-1}$. By (10_{k-1}), the absolute value of the last terms is $\leqslant\frac{1}{2}|\alpha_k|A$. In case (b),

$$t_{n_k}=t_{n_{k-1}}+i\alpha_k e^{in_k x},\qquad t_\lambda=t_{\mu_k}-\frac{1}{2}i\alpha_k\sum_{\lambda-n_k+1}^{\mu_{k-1}}\gamma_\nu e^{i(n_k+\nu)x}$$

for $\lambda>n_k$ and the absolute value of the last term is again $\leqslant\frac{1}{2}|\alpha_k|A$. Hence

$$|t_\lambda-t_{\mu_{k-1}}|\leqslant\frac{1}{2}|\alpha_k|(A+2),\qquad |t_\lambda-t_{\mu_k}|\leqslant\frac{1}{2}|\alpha_k|A \tag{13}$$

for $\mu_{k-1}<\lambda\leqslant n_k$ and $n_k<\lambda\leqslant\mu_k$ respectively (the addition of $|\alpha_k|$ on the right of the first inequality is actually needed for $\lambda=n_k$ only). In particular, assuming that $|\alpha_k|\leqslant 1$, we have

$$|t_\lambda|\leqslant M+\frac{1}{2}A+1\qquad\text{for } \mu_{k-1}<\lambda\leqslant\mu_k,$$

and obviously the result holds if x is replaced by $-x$.

Suppose now that A is so large that

$$2M+\frac{1}{2}A+1\leqslant A-2M.$$

Then, if $\mu_{k-1}<\mu\leqslant\mu_k$,

$$\Big|\sum_{\mu}^{\mu_k}\gamma_\nu e^{i\nu x}\Big|\leqslant|t_{\mu_k}|+|t_{\mu-1}|\leqslant M+M+\frac{1}{2}A+1\leqslant A-2M.$$

If $\mu_{j-1}<\mu\leqslant\mu_j$, $j<k$, then

$$\Big|\sum_{\lambda}^{\mu_k}\gamma_\nu e^{i\nu x}\Big|\leqslant\Big|\sum_{\lambda}^{\mu_j}\Big|+\Big|\sum_{\mu_j+1}^{\mu_k}\Big|\leqslant A-2M+2M=A.$$

Similarly, if x is replaced by $-x$. Thus (10_{k-1}) and (11_{k-1}) imply (10_k) and (11_k) and so they are valid for all k. Since $t_{\mu_k}(\pm x)$ converges, and since $a_k\to 0$, (13) implies that $t_\lambda(\pm x)$ converges. The same holds for $s_\lambda(x)$ on account of the second equation (9).

It remains to get rid of the assumption (6). Suppose that we only have $n_{k+1}/n_k\geqslant 3$. Let us split the product (1) into two, corresponding to ν even and odd respectively. For each of these subproducts the ratio of two successive n_ν is $\geqslant 9$, and so each of the subproducts converges almost everywhere. The same therefore holds for the whole product (1), so that the partial sums s_{μ_k} of (2) converge almost everywhere. In virtue of the well known theorem of Kuttner, the same holds for the partial sums $\check{s}_{\mu_k}$ of (7), and the rest of the proof remains unchanged (see Kuttner, **1**, or Hardy and Rogosinski, **2**). The result is stated there for the sequence of all the partial sums of a given trigonometric series, but it holds, with proof unchanged, for any fixed subsequence of the partial sums (see Marcinkiewicz and Zygmund, **3**).

5. Remarks. 1. Theorem 2 remains unchanged if in (1) we replace $a_\nu\cos n_\nu x$ by

$$a_\nu\cos n_\nu x+\beta_\nu\sin n_\nu x=\varrho_\nu\cos(n_\nu x+\theta_\nu),\qquad \varrho_\nu\geqslant 0,$$

provided $\sum_{\nu=1}^{\infty}\varrho_\nu^2<\infty$ in the first part of the theorem and $\sum_{\nu=1}^{\infty}\varrho_\nu=\infty$ in the second.

2. For the sake of completeness we shall briefly consider the behavior of (1) when $\sum_{\nu=1}^{\infty}a_\nu^2=\infty$. Let us also assume, for simplicity, that $a_\nu\to 0$. Then the product (1), and so also the series (2) representing it, diverges almost everywhere. For otherwise, in the formula

$$p_k(x)=\exp\Big\{i\sum_1^k a_\nu\cos n_\nu x+\frac{1}{2}\sum_1^k a_\nu^2\cos^2 n_\nu x+\sum_1^k O(|a_\nu|^3)\Big\}$$

the expression in curly brackets { } would tend to a finite limit in a set E of positive measure. We can even assume that the convergence is uniform. Integrating the expression in curly brackets over E and taking account of the fact that the numbers $\int_E e^{\pm inx}dx$ tend to zero, and that the sum of the squares of their moduli is finite, we easily obtain that $\sum_{\nu=1}^{\infty}a_\nu^2$ converges, contrary to assumption.

It is also easy to see that the resulting series (2) is not a Fourier-Lebesgue series. For then the partial sums s_n of (2) would converge in the mean of every order <1 (see e. g. Zygmund, **5**, p. 153). That would imply that a certain subsequence of the p_k converges almost everywhere, and an argument similar to the one just used would give $\sum_{\nu=1}^{\infty}a_\nu^2<\infty$.

University of Chicago,
Chicago, Ill., USA.

References

1. B. Kuttner, *A theorem on trigonometric series*, Journal of the London Math. Soc. 10 (1935), 131-135.
2. G. H. Hardy and W. Rogosinski, *Fourier series*.
3. J. Marcinkiewicz and A. Zygmund, *On the differentiability of functions and summability of trigonometric series*, Fundamenta Mathemat. 26 (1935), 1-41.
4. R. Salem and A. Zygmund, *On a theorem of Banach*, Proc. Nat. Acad., 1947, 293-295.
5. A. Zygmund, *Trigonometrical Series*, Monografie Matematyczne (1935).
6. A. Zygmund, *On lacunary trigonometric series*, Transactions American Math. Soc. 34 (1932), 435-446.
7. A. Zygmund, *Quelques théorèmes etc.*, Studia Math. 3 (1931). 77-91.

(Reçu par la Rédaction le 29. 3. 1948).