Corollary 14. *If m and n are any two natural numbers, not both zero, having d as their greatest common divisor, and if $m\cdot\mathfrak{p}=n\cdot\mathfrak{q}$, then there is a cardinal $\mathfrak{r}$ such that $\mathfrak{p}=\frac{n}{d}\cdot\mathfrak{r}$ and $\mathfrak{q}=\frac{m}{d}\cdot\mathfrak{r}$.*

Proof: By Theorem 9 (with $m=d$), the hypothesis implies

$$\frac{m}{d}\cdot\mathfrak{p}=\frac{n}{d}\cdot\mathfrak{q}.$$

Hence, the natural numbers $\frac{m}{d}$ and $\frac{n}{d}$ being relatively prime, the conclusion follows by Theorem 13.

Corollary 14 clearly comprehends both Theorem 9 and Theorem 13 as particular cases.

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Retraction properties for normal Hausdorff spaces[1]).

By

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1. Introduction. The idea of a retract was formulated by K. Borsuk [1] and retraction properties for separable metric spaces were developed by K. Borsuk, N. Aronszajn [3], and others. This theory has played a prominent role in investigations concerned with separable metric spaces. It is the purpose of this paper to develop a similar theory for normal Hausdorff spaces.

Section two will be devoted to results of a preliminary nature. In the next section we give definitions of absolute retract and absolute neighborhood retract for normal Hausdorff spaces. These concepts are characterized with the aid of the Tychonoff cube. We show in section four that the topological product of any set of absolute retracts or any finite set of absolute neighborhood retracts is an absolute retract or an absolute neighborhood retract, respectively. In the next section, under certain conditions, we prove that the union of two absolute retracts or absolute neighborhood retracts is again an absolute retract or an absolute neighborhood retract, respectively, in a restricted sense.

Section six is concerned with an extension of Borsuk's well-known theorem [4, p. 86] on the extension of continuous maps into a n-sphere. Borsuk's theorem states: If C is a closed subset of a separable metric space X, then for any continuous map $f:(X\times0)\cup(C\times(0,1))\rightarrow N$, where N is a n-sphere or more generally a separable metric absolute neighborhood retract, there exists an extension F of f over $X\times(0,1)$, such that $F:X\times(0,1)\rightarrow N$. We prove this theorem for a compact Hausdorff space X, substituting a retract B of any absolute retract A for the point 0 and the unit

[1]) A Dissertation accepted by the University of Pennsylvania in fulfillment of the research requirement for the Degree of Doctor of Philosophy.

interval (0,1), respectively. Moreover, N and A are assumed to be an absolute neighborhood retract and an absolute retract, respectively, in the sense of the generalized definitions as given in this paper. We also prove another extension theorem in which the results are of a similar nature; however, the emphasis of the hypothesis is differently placed. Here we assume simply that N is a topological space, but that C is a closed neighborhood retract of an absolute neighborhood retract X.

In section seven we show that any absolute retract has the fixed point property, and that for an absolute neighborhood retract any null-homotopic continuous map has a fixed point.

Finite dimensional separable metric absolute neighborhood retracts have been characterized by K. Borsuk [2, p. 240]. By exhibiting an example, we show in section eight that Borsuk's result does not lend itself in a natural way to a characterization of finite dimensional absolute neighborhood retracts as defined in this paper. The study of retraction properties is contineud in section nine where other results are formulated.

The author wishes to express his deep appreciation to Professor A. D. Wallace for his helpful advice and constant encouragement.

2. Retracts. The following conventions are used throughout this paper. We abbreviate normal Hausdorff space by „NH space". The words „map" and „transformation" are used in the sense of „continuous correspondence".

The theorems in this section hold for any Hausdorff space, and will be needed later in this paper. With the exception of (2.2), the proofs of the corresponding theorems for separable metric spaces as given by K. Borsuk [1] will apply directly to these theorems.

(2.1) ***Definition.*** *Given the sets A and B such that $B\subset A$, we say that f is a map retracting A onto B provided f is defined and continuous on A, $f(A)\subset B$, and $f(x)=x$ for every $x\in B$. If such a map exists for the set B, then B is called a retract of A.*

Depending strongly on the Hausdorff separation axiom we obtain the following result.

(2.2) ***Theorem.*** *Every retract of a set is relatively closed in that set.*

In addition we have:

(2.3) ***Theorem.*** *If B is a retract of A and the set A has the fixed point property with respect to maps of A into A, then B also has the fixed point property with respect to maps of B into B.*

(2.4) ***Definition.*** *Given the sets P and P_1 such that $P\subset P_1$ and a map $f\colon P\to Q$, we call the map g an extension of the map f over P_1 relative to Q provided $g\colon P_1\to Q$ and $g(x)=f(x)$ for all $x\in P$.*

(2.5) ***Theorem.*** *If a map f defined on a set P admits an extension g over a set P_1 relative to a set A, then f admits an extension over P_1 relative to every retract B of A which contains $f(P)$.*

3. Absolute Retracts and Absolute Neighborhood Retracts. (3.1) ***Definition.*** *Given the sets A and B such that $B\subset A$, we say that B is a neighborhood retract of A provided there exists an open set U such that $B\subset U\subset A$ and such that B is a retract of U.*

(3.2) ***Definition.*** *A space A is called an absolute neighborhood retract provided it is a compact Hausdorff space and for every topological image A_1 of A, such that A_1 is contained in a NH space M, we have A_1 is a neighborhood retract of M.* We abbreviate „absolute neighborhood retract" by „ANR" or „ANR set".

(3.3) We denote by ΠI_α the topological product of any arbitrary number of topological spaces, where each I_α denotes a topological space. For $x\in\Pi I_\alpha$ we have $x=\{x_\alpha\}$ where $x_\alpha\in I_\alpha$ for all α. In order to introduce a topology into ΠI_α, we define neighborhood of a point $x=\{x_\alpha\}$ as follows. Consider any finite number of indices $\alpha_1,\dots,\alpha_n$ and the corresponding spaces $I_{\alpha_1},\dots,I_{\alpha_n}$. Let S_{α_i} be any neighborhood of the point x_{α_i} in I_{α_i} for $i=1,\dots,n$. Then $y=\{y_\alpha\}$ is an element of the neighborhood $U\{\alpha_1,\dots,\alpha_n;S_{\alpha_1},\dots,S_{\alpha_n}\}$ of $x=\{x_\alpha\}$ provided $y_{\alpha_i}\in S_{\alpha_i}$ for $i=1,\dots,n$. If each I_α is the unit interval, then the space ΠI_α is called a Tychonoff cube [6].

(3.4) ***Theorem.*** *A necessary and sufficient condition for a set to be an ANR is that it be homeomorphic to a closed neighborhood retract of some Tychonoff cube.*

Proof. Necessity. Let A be an ANR. Since A is a compact Hausdorff space, we can map A topologically into some Tychonoff cube T [5, p. 29]. Let $h(A)=A_1$ where h is a homeomorphism and A_1 is a subset of T. Since T is compact [6, p. 763], we have T is a NH space and therefore by (3.2) A_1 is a neighborhood retract of T.

In virtue of the continuity of h and the compactness of A, we have A_1 is compact and therefore closed in T.

Sufficiency. Let $h(A)=A_1$ where h is a homeomorphism and A_1 is a closed neighborhood retract of some Tychonoff cube T. Consider any other homeomorphic image A_2 of A such that A_2 is contained in a NH space M. Let $k(A)=A_2$ where k is a homeomorphism. T is a Tychonoff cube and hence compact [6, p. 763]. Therefore A_1 is compact and since $kh^{-1}(A_1)=A_2$, we have A_2 is compact and hence closed in M. We now apply Tietze's Extension Theorem [5] to the map hk^{-1}: $A_2 \to T$ and obtain an extension f of hk^{-1} over M relative to T. Since A_1 is a neighborhood retract of T, there exist an open set $U_1 \supset A_1$ and a retracting map r such that r: $U_1 \to A_1$. Now $f(M)\cap U_1$ is an open subset of $f(M)$. Hence $f^{-1}[f(M)\cap U_1]$ is an open subset of M and clearly $f^{-1}[f(M)\cap U_1]\supset A_2$. The map $kh^{-1}rf$ retracts the open set $f^{-1}[f(M)\cap U_1]$ onto A_2 because:

$$kh^{-1}rf\{f^{-1}[f(M)\cap U_1]\}=kh^{-1}(A_1)=k(A)=A_2$$

and for $x \in A_2$ we have

$$kh^{-1}rf(x)=kh^{-1}r[hk^{-1}(x)]=kh^{-1}[hk^{-1}(x)]=x$$

since $f(x)=hk^{-1}(x)$ and $hk^{-1}(x) \in A_1$.

(3.5) ***Definition.*** *A space A is called an absolute retract provided it is a NH space and for every topological image A_1 of A, such that A_1 is contained in a NH space M, we have A_1 is a retract of M.* We abbreviate „absolute retract" by „AR" or „AR set".

(3.6) ***Theorem.*** *A necessary and sufficient condition for a set to be an AR is that it be homeomorphic to a retract of some Tychonoff cube.*

This result may be verified by a proof entirely analogous to that given for (3.4). We simply remark that in the proof of this theorem the Tychonoff cube T takes the place of the open set U_1 which appears in the proof of (3.4) and M takes the place of the open set $f^{-1}[f(M)\cap U_1]$.

Since every Tychonoff cube is compact [6], we obtain from (3.6) the

(3.7) ***Corollary.*** *Every AR is compact.*

From the definitions (3.2) and (3.5) we have immediately the following result.

(3.8) *The property of being either an ANR or an AR is a topological invariant.*

(3.9) ***Theorem.*** *If a set is a closed neighborhood retract of an ANR, then the set is an ANR.*

Proof. Let B be a closed neighborhood retract of an ANR set A. By (3.4) A is homeomorphic to a closed neighborhood retract A_1 of some Tychonoff cube T. Let $h(A)=A_1$ where h is a homeomorphism. Then there exist open sets U_1 in T and V in A and retracting maps r and f such that $U_1\supset A_1$, $V\supset B$ and r: $U_1\to A_1$, f: $V\to B$. Consider the set G consisting of all $x \in U_1$ such that $h^{-1}r(x) \in V$. Let $h(B)=B_1$. Clearly $G\supset B_1$. Now $h^{-1}r$ maps U_1 onto A and V is open in A. Therefore $(h^{-1}r)^{-1}(V)$ is open in U_1 and hence open in T. But $(h^{-1}r)^{-1}(V)=G$ and hence G is open in T. The map $hfh^{-1}r$ retracts the open set G onto B_1 because:

$$hfh^{-1}r(G)\subset B_1$$

and for $b_1 \in B_1$ we have

$$hfh^{-1}r(b_1)=hfh^{-1}(b_1)=hf[h^{-1}(b_1)]=hh^{-1}(b_1)=b_1, \text{ since } h^{-1}(b_1) \in B$$

implies $f[h^{-1}(b_1)]=h^{-1}(b_1)$. By (3.2) A is compact and since B is closed in A and $h(B)=B_1$, we have B_1 is compact and therefore closed in T.

Thus B_1 is a closed neighborhood retract of T. By (3.4) B is an ANR.

(3.10) ***Theorem.*** *If a set is a retract of an AR, then the set is an AR.*

We can verify this theorem by a proof which parallels that given for (3.9) in all important details, except that we use (2.2) to show that the retract is a normal space.

4. Topological Product. (4.1) ***Lemma.*** *A necessary and sufficient condition for a set A to be an ANR is that A be a compact Hausdorff space and that any map f defined on a closed subset P of a normal space P_1 such that $f(P)\subset A$, admits an extension over some open subset V of P_1 relative to A where V contains P.*

Necessity. By (3.4) we have $h(A)=A_1$ where h is a homeomorphism and A_1 is a closed neighborhood retract of some Tychonoff cube T. Let U_1 be an open subset of T and r be a retracting map such that r: $U_1\to A_1$. Since P is a closed subset of the normal space P_1

and $hf(P)\subset T$, by Tietze's Extension Theorem [5] there exists an extension g of hf over P_1 relative to T. Now the set $U_1\cap g(P_1)$ is open in $g(P_1)$ and hence $V=g^{-1}[U_1\cap g(P_1)]$ is open in P_1. Clearly $V\supset P$ since for $p\in P$, we have $g(p)=hf(p)\in U_1$. The map $h^{-1}rg$ is an extension of f over V relative to A because since $g(V)=U_1\cap g(P_1)$ we have $h^{-1}rg(V)\subset A$ and for $p\in P$ we have

$$h^{-1}rg(p)=h^{-1}r[hf(p)]=h^{-1}[hf(p)]=f(p).$$

Sufficiency. Let $h(A)=A_1$ where h is a homeomorphism and A_1 is a subset of an NH space M. Now A is compact by hypothesis and therefore A_1 is closed in M. Since $h^{-1}(A_1)=A$, by hypo thesis there exists an open set V containing A_1 and an extension H of h over V relative to A. The map hH retracts V onto A_1 because $hH(V)\subset A_1$ and for $a_1\in A_1$ we have $hH(a_1)=h[h^{-1}(a_1)]=a_1$. By (3.2) A is an ANR set.

(4.2) ***Theorem.*** *If the sets $A_1,\ldots,A_n$ are ANR sets, then the topological product ΠA_i is an ANR.*

Proof. Since each A_i is a compact Hausdorff space, we have ΠA_i is a compact Hausdorff space [5]. Consider any closed subset P of a normal space P_1 and any map f defined on P such that $f(P)\subset\Pi A_i$. For any $p\in P$ we have $f(p)=\{f_1(p),\ldots,f_n(p)\}$ where $f_i(p)\in A_i$ for $i=1,\ldots,n$. Since A_i is an ANR and $f_i\colon P\to A_i$, by the necessity of (4.1) there exist an open subset V_i of P_1 such that $V_i\supset P$ and an extension F_i of f_i over V_i relative to A_i for $i=1,\ldots,n$. Clearly $V_1\cap\ldots\cap V_n\supset P$ and $V_1\cap\ldots\cap V_n$ is open in P_1. We define a map $F\colon V_1\cap\ldots\cap V_n\to\Pi A_i$ by $F(v)=\{F_1(v),\ldots,F_n(v)\}$ for $v\in V_1\cap\ldots\cap V_n$. F is an extension of f over $V_1\cap\ldots\cap V_n$ relative to ΠA_i because $F(V_1\cap\ldots\cap V_n)\subset\Pi A_i$ and for $p\in P$ we have

$$F(p)=\{F_1(p),\ldots,F_n(p)\}=\{f_1(p),\ldots,f_n(p)\}=f(p).$$

By the sufficiency of (4.1) ΠA_i is an ANR.

Using a proof which is entirely analogous to that used in (4.1) we obtain the

(4.3) ***Lemma.*** *A necessary and sufficient condition for a set A to be an AR is that A be a compact Hausdorff space and that any map f defined on a closed subset P of a normal space P_1 such that $f(P)\subset A$, admits an extension over P_1 relative to A.*

By (4.3) and a proof which parallels that given for (4.2) in important details we obtain the

(4.4) ***Theorem.*** *If $\{A_\alpha\}$ is a collection of sets where each A_α is an AR, then the topological product ΠA_α is an AR.*

5. Sum Theorems. (5.1) ***Theorem.*** *Let C be an ANR such that $C=A\cup B$ where A and B are closed in C and $A\cap B$ is a neighborhood retract of C, then both A and B are ANR.*

K. Borsuk's proof [2, p. 226] of this result for a separable metric ANR can be applied directly to this theorem.

We call a space *perfectly normal* provided all its subsets are normal.

(5.2) ***Definition.*** *A space A is called a restricted ANR (restricted AR) provided it is a compact Hausdorff space (NH space) and for every topological image A_1 of A, such that A_1 is contained in a perfectly normal Hausdorff space M, we have A_1 is a neighborhood retract of M (retract of M).*

We shall need the following result which is due to P. Urysohn:

(5.3) *If the space M is perfectly normal and A and B are closed subsets of M, then there exist open sets U_1 and U_2 such that $U_1\supset A-B$, $U_2\supset B-A$ and $U_1\cap U_2=0$.*

(5.4) ***Theorem.*** *Let $C=A\cup B$ where $A\cap B$ is a neighborhood retract of C and A and B are ANR sets, then C is a restricted ANR set.*

Proof. Let $h(C)=C_1$ where h is a homeomorphism and C_1 is contained in a perfectly normal Hausdorff space M. Let $h(A)=A_1$ and $h(B)=B_1$. Since A and B are ANR sets, we have both A_1 and B_1 are compact and hence closed in M. By (5.3) there exists an open set U in M such that

$$A_1-B_1\subset U\subset\overline{U}\subset M-(B_1-A_1)=(M-B_1)\cup(A_1\cap B_1).$$

We let

$$P=\overline{U}\cup(A_1\cap B_1)\quad\text{and}\quad Q=(M-U)\cup(A_1\cap B_1).$$

We notice that P,Q and $P\cap Q$ are closed in M and that $A_1\subset P$, $B_1\subset Q$, $P\cup Q=M$ and $A_1\cap P\cap Q=B_1\cap P\cap Q=A_1\cap B_1$.

Since $A \cap B$ is a neighborhood retract of C and $A \cap B \subset A \subset C$, we have $A \cap B$ is a neighborhood retract of A and hence by (3.9) $A \cap B$ is an ANR. Thus there exist an open subset V of M such that $A_1 \cap B_1 \subset V$ and a retracting map r such that $r\colon V \to A_1 \cap B_1$. Applying the normality of M, there exists an open subset W of M such that $A_1 \cap B_1 \subset W \subset \overline{W} \subset V$. Since $A_1 \cap B_1 \subset P \cap Q \cap \overline{W} \subset V$, r retracts $P \cap Q \cap \overline{W}$ onto $A_1 \cap B_1$.

We define

$$f_1(x) = r(x), \quad \text{for} \quad x \in P \cap Q \cap \overline{W},$$
$$f_1(x) = x, \quad \text{for} \quad x \in A_1,$$

and

$$f_2(x) = r(x), \quad \text{for} \quad x \in P \cap Q \cap \overline{W},$$
$$f_2(x) = x, \quad \text{for} \quad x \in B_1.$$

Both $P \cap Q \cap \overline{W}$ and A_1 are closed in $(P \cap Q \cap \overline{W}) \cup A_1$ and $r(x) = x$ for $x \in A_1 \cap (P \cap Q \cap \overline{W}) = A_1 \cap B_1$. Hence f_1 is continuous. Similarly f_2 is continuous.

By (3.8) A_1 is an ANR. Hence noting that $(P \cap Q \cap \overline{W}) \cup A_1$ is closed in P, we apply (4.1) and obtain a set Z_1 open in P and such that $(P \cap Q \cap \overline{W}) \cup A_1 \subset Z_1$ and an extension g_1 of f_1 such that $g_1\colon Z_1 \to A_1$. Similarly we obtain a set Z_2 open in Q and such that $(P \cap Q \cap \overline{W}) \cup B_1 \subset Z_2$ and an extension g_2 of f_2 such that $g_2\colon Z_2 \to B_1$.

Since $A_1 \cap (P - Z_1) = 0$ and P is normal, there exists a set S_1 open in P such that $A_1 \subset S_1 \subset \overline{S}_1 \subset Z_1$. Similarly there exists a set S_2 open in Q such that $B_1 \subset S_2 \subset \overline{S}_2 \subset Z_2$. Using $\complement W$ to denote the complement of W, we define

$$K(x) = g_1(x), \quad \text{for} \quad x \in \overline{S}_1 - (P \cap Q \cap \complement W)$$
$$K(x) = g_2(x), \quad \text{for} \quad x \in \overline{S}_2 - (P \cap Q \cap \complement W).$$

Both $\overline{S}_1 - (P \cap Q \cap \complement W)$ and $\overline{S}_2 - (P \cap Q \cap \complement W)$ are closed in $\{\overline{S}_1 - (P \cap Q \cap \complement W)\} \cup \{\overline{S}_2 - (P \cap Q \cap \complement W)\}$ and $g_1(x) = g_2(x)$ for $x \in \{\overline{S}_1 - (P \cap Q \cap \complement W)\} \cap \{\overline{S}_2 - (P \cap Q \cap \complement W)\} \subset P \cap Q \cap \overline{W}$.

Hence $K(x)$ is continuous.

Let $R_1 = S_1 - (P \cap Q \cap \complement W)$ and $R_2 = S_2 - (P \cap Q \cap \complement W)$. Clearly R_1 is open in P and R_2 is open in Q. Moreover K retracts $R_1 \cup R_2$ onto $A_1 \cup B_1$ because

$$A_1 \cup B_1 \subset R_1 \cup R_2 \subset \{\overline{S}_1 - (P \cap Q \cap \complement W)\} \cup \{\overline{S}_2 - (P \cap Q \cap \complement W)\}.$$

Now $R_1 = Y_1 \cap P$ and $R_2 = Y_2 \cap Q$ where Y_1 and Y_2 are open subsets of M. Also $Y_1 \cap (P - Q) \subset R_1$, $Y_2 \cap (Q - P) \subset R_2$ and

$$Y_1 \cap Y_2 = Y_1 \cap Y_2 \cap (P \cup Q) \subset R_1 \cup R_2.$$

Hence

$$A_1 \cup B_1 = \{(A_1 \cup B_1) \cap (P - Q)\} \cup \{(A_1 \cup B_1) \cap (Q - P)\} \cup \{(A_1 \cup B_1) \cap P \cap Q\} \subset$$
$$\subset \{Y_1 \cap (P - Q)\} \cup \{Y_2 \cap (Q - P)\} \cup \{Y_1 \cap Y_2\} \subset R_1 \cup R_2.$$

Now the set $X = \{Y_1 \cap (P - Q)\} \cup \{Y_2 \cap (Q - P)\} \cup \{Y_1 \cap Y_2\}$ is open in M and we have $A_1 \cup B_1 \subset X \subset R_1 \cup R_2$. Therefore K retracts X onto $A_1 \cup B_1 = C_1$.

Using methods similar to those used in (5.1) and (5.4) we arrive at the following two results.

(5.5) ***Theorem.*** *Let C be an AR such that $C = A \cup B$ where A and B are closed in C and $A \cap B$ is a retract of C, then both A and B are AR.*

(5.6) ***Theorem.*** *Let $C = A \cup B$ where $A \cap B$ is a retract of C and A and B are AR sets, then C is a restricted AR set.*

6. Extension of Borsuk's Theorem. (6.1) ***Lemma.*** *Let C be a subset of a Hausdorff space X. In the product space $X \times A$ where A is a compact Hausdorff space, let U be an open set containing $C \times A$. Then there exists an open set V in X containing C such that $V \times A$ is contained in U.*

In virtue of the compactness of A, we may apply the proof given by Hurewicz and Wallman [4, p. 86] to the above lemma.

(6.2) ***Definition.*** *We say that a set B is a deformation retract of a set X provided there exists a map r retracting X onto B such that r is homotopic to the identity map.*

(6.3) ***Lemma.*** *If B is a retract of an AR set A, then B is a deformation retract of A. Moreover, there exists a deformation G mapping $A \times (0,1)$ onto A such that the points of $B \times (0,1)$ are fixed.*

Proof. Let r be a map which retracts A onto B. We define a map g such that $g\colon A \times 0 \cup B \times (0,1) \cup A \times 1 \to A$ by

$$g(a,0) = r(a), \quad \text{for} \quad a \in A$$
$$g(b,t) = b, \quad \text{for} \quad b \in B \quad \text{and} \quad t \in (0,1)$$
$$g(a,1) = a, \quad \text{for} \quad a \in A.$$

Since $A\times 0$, $B\times(0,1)$ and $A\times 1$ are closed and the three definitions of g agree on all common domains, g is well defined and continuous. $A\times 0\cup B\times(0,1)\cup A\times 1$ is a closed subset of the normal space $A\times(0,1)$ so that we may apply (4.3) and obtain an extension G of g over $A\times(0,1)$ relative to A. Clearly G is the required deformation having the property that $G(b,t)=b$ for $b\in B$ and $t\in(0,1)$.

(6.4) ***Extension of Borsuk's Theorem*** [4, p. 86]. *Let C be a closed subset of a compact Hausdorff space X and let B be a retract of an AR set A. Then for any map f such that*

$$f\colon (X\times B)\cup(C\times A)\to N,$$

where N is an ANR, there exists an extension F of f over $X\times A$ such that $F\colon X\times A\to N$.

Proof. Since $(X\times B)\cup(C\times A)$ is closed, by (4.1) there exists an open subset U of $X\times A$ such that $U\supset(X\times B)\cup(C\times A)$ and an extension f' of f over U relative to N. By (6.1) there exists an open subset V of X such that $C\subset V$ and $V\times A\subset U$. Clearly f' is defined over $(X\times B)\cup(V\times A)$. Observing that C and $\mathcal{C}V$ are disjoint closed sets, we apply Urysohn's Lemma and obtain a continuous real function $p(x)$ defined over X such that $0\leqslant p(x)\leqslant 1$ for all $x\in X$, $p(x)=1$ for $x\in C$, and $p(x)=0$ for $x\in\mathcal{C}V$. By (6.3) there exists a retracting map r such that $r\colon A\to B$ and a deformation G such that $G\colon A\times(0,1)\to A$, $G(a,0)=r(a)$ for $a\in A$, $G(b,t)=b$ for $b\in B$ and $t\in(0,1)$, and $G(a,1)=a$ for $a\in A$.

Consider the map $F\colon X\times A\to N$ defined by

$$F(x,a)=f'[x,G(a,p(x))] \quad\text{for}\quad x\in X \quad\text{and}\quad a\in A.$$

Now f' is defined over $V\times A$ so that clearly F is well defined for $x\in V$ and $a\in A$. For $x\in\mathcal{C}V$ and $a\in A$ we have

$$G(a,p(x))=G(a,0)=r(a)\in B.$$

Recalling that f' is defined over $X\times B$, we observe that F is also well defined for $x\in\mathcal{C}V$ and $a\in A$. Moreover F is clearly continuous on $X\times A$.

We now show that F agrees with f on $(X\times B)\cup(C\times A)$. For $x\in X$ and $b\in B$ we have

$$F(x,b)=f'[x,G(b,p(x))]=f'(x,b)=f(x,b).$$

For $c\in C$ and $a\in A$ we have

$$F(c,a)=f'[c,G(a,p(c))]=f'[c,G(a,1)]=f'(c,a)=f(c,a).$$

(6.5) ***Theorem.*** *Let C be a closed neighborhood retract of an ANR set X. Let B be a retract of an AR set A and let $X\times A$ be perfectly normal. Then for any map f such that*

$$f\colon (X\times B)\cup(C\times A)\to R,$$

where R is any topological space, there exists an extension F of f over $X\times A$ such that $F\colon X\times A\to R$.

Proof. By (3.9) C is an ANR, and hence by (4.2) $C\times A$ is an ANR. By (3.10) B is an AR, and hence by (4.2) $X\times B$ is an ANR. Now $(X\times B)\cap(C\times A)=C\times B$ an ANR set. Hence $(X\times B)\cup(C\times A)$ is a restricted ANR by (5.4). Since $X\times A$ is perfectly normal, there exist an open subset U of $X\times A$ such that $(X\times B)\cup(C\times A)\subset U$ and a retracting map r such that $r\colon U\to(X\times B)\cup(C\times A)$. The map fr is an extension of f over U relative to R. By (6.1) there exists a subset V of X such that V is open in X, $C\subset V$, and $V\times A\subset U$. This proof may now be completed by an argument which parallels that given for (6.4).

7. Fixed Point Property. (7.1) ***Theorem.*** *If A is an AR, then every transformation which maps A into A has a fixed point.*

Proof. By (3.6) we have $h(A)=A_1$ where h is a homeomorphism and A_1 is a retract of some Tychonoff cube. Since every Tychonoff cube has the fixed point property [7, p. 770], we have by (2.3) that A_1 has this property. Moreover the fixed point property is a topological invariant. Since h is a homeomorphism, this completes the proof.

We say that a map f is null-homotopic provided f is homotopic to a constant map.

(7.2) ***Theorem.*** *If A is an ANR, and f is a null-homotopic map of A into A, then f has a fixed point.*

Proof. According to (3.4), A is homeomorphic to a closed neighborhood retract of some Tychonoff cube T. We lose no generality by assuming A is contained in T. Now f is homotopic to a map g where $g\colon A\to A$ and $g(a)=a'\in A$ for all $a\in A$. Hence there exists a map k such that $k\colon A\times(0,1)\to A$, $k(a,0)=f(a)$ for all $a\in A$, and $k(a,1)=g(a)$ for all $a\in A$.

We define a map K such that $K\colon (T\times 1)\cup(A\times(0,1))\to A$ by

$$K(x,1)=a',\quad\text{for all}\quad x\in T,$$
$$K(x,t)=k(x,t)\quad\text{for all}\quad x\in A\quad\text{and}\quad t\in(0,1).$$

$K(x,t)$ is continuous because both $T\times 1$ and $A\times(0,1)$ are closed in their union $(T\times 1)\cup(A\times(0,1))$ and for

$$(x,t)\in(T\times 1)\cap(A\times(0,1))=A\times 1$$

we have $k(x,t)=a'$. Since A is an ANR, we may now apply (6.4), and obtain an extension K' of K over $T\times(0,1)$ relative to A.

We define a map G such that $G\colon T\to A$ by

$$G(x)=K'(x,0), \quad \text{for all} \quad x\in T.$$

Now G is an extension of f over T relative to A because for any $a\in A$ we have

$$G(a)=K'(a,0)=K(a,0)=k(a,0)=f(a).$$

Since T has the fixed point property [7, p. 770], there exists some element p of T such that $G(p)=p$. This implies p is an element of A because $G\colon T\to A$. Since G is an extension of f, we have $f(p)=p$.

8. A Comparison. K. Borsuk characterized the concept of a separable metric ANR set for finite dimensional spaces with the following result [2, p. 240]: For finite dimensional sets, ANR sets can be characterized as compact, locally contractile [2, p. 235], metrisable spaces. In view of this result, it is natural to conjecture that for finite dimensional spaces, ANR sets as defined in this paper should admit a characterization as compact, locally contractile, Hausdorff spaces. Indeed we shall prove that any ANR is a compact, locally contractile, Hausdorff space. However, the suggested characterization is impossible. We shall show this by exhibiting a set which is a finite dimensional, compact, locally contractile, Hausdorff space, but which is not an ANR.

(8.1) ***Lemma.*** *If A is a retract of B and B is locally contractile at a point $p\in A$, then A is locally contractile at p.*

Borsuk's proof [2, p. 237] of the above lemma for separable metric spaces will hold here unchanged.

(8.2) ***Theorem.*** *Any ANR is a compact, locally contractile, Hausdorff space.*

Proof. If A is an ANR, then A is a compact Hausdorff space by definition.

By (3.4), A is homeomorphic to a closed neighborhood retract A_1 of some Tychonoff cube T. Let $h(A)=A_1$ where h is a homeomorphism, and let U be the open subset of T which retracts onto A_1. It is easy to show that any open subset of a Tychonoff cube is locally contractile. Hence U is locally contractile and by (8.1) A_1 is locally contractile.

Consider any $p\in A$, and any neighborhood V of p. Let $h(p)=p_1\in A_1$. Since p_1 is an element of the open set $h(V)$ and A_1 is locally contractile, there exist a neighborhood W of p_1 and a map $f\colon W\times(0,1)\to h(V)$ such that for $w\in W$, we have $f(w,0)=w$ and $f(w,1)=q\in h(V)$. Now $p\in h^{-1}(W)\subset V$, and $h^{-1}(W)$ is open in A. We define a map $F\colon h^{-1}(W)\times(0,1)\to V$ by

$$F(x,t)=h^{-1}f(h(x),t) \quad \text{for} \quad x\in h^{-1}(W) \quad \text{and} \quad t\in(0,1).$$

For $x\in h^{-1}(W)$, we have

$$F(x,0)=h^{-1}f(h(x),0)=h^{-1}h(x)=x$$

and

$$F(x,1)=h^{-1}f(h(x),1)=h^{-1}(q)\in V.$$

(8.3) ***Corollary.*** *Any ANR is locally connected.*

(8.4) *We now construct a compact, locally contractile, Hausdorff space of dimension one, which is not an ANR.*

Let $X=(0,1)$ and $Y=(0,1)$ and let $Q=X\times Y$. We topologize Q in the following manner. For a point (x_1,y_1) with $0<y_1<1$, we define neighborhood to mean all points (x_1,y) such that $y_1-e<y<y_1+e$ where e is any positive real number such that $y_1-e\geqslant 0$ and $y_1+e\leqslant 1$. For a point $(x_1,1)$ a neighborhood consists of all points (x_1,y) such that $y>1-e$ where $0<e\leqslant 1$. The two types of neighborhoods defined above we shall call linear neighborhoods.

For a point $(x_1,0)$ where $0<x_1<1$ a neighborhood consists of all points (x,y) such that $0\leqslant x_1-e<x<x_1+e\leqslant 1$ and such that $x\neq x_1$ and also all points (x_1,y) except for any closed set of points (x_1,y) where $0<y_2\leqslant y\leqslant y_3\leqslant 1$. Neighborhoods for the points $(0,0)$ and $(1,0)$ are the same as the kind last defined except for being one sided. The last two types of neighborhoods we shall call rectangular neighborhoods.

In virtue of the rectangular neighborhoods for points of the form $(x,0)$ and the compactness of the unit interval $(0,1)$, it is clear that Q is a compact Hausdorff space. Using the compactness of Q and a covering definition of dimension for normal Hausdorff spaces in a recent paper by E. Hemmingsen [8, p. 496, definition 2.1], it can be shown without difficulty that Q is of dimension one.

We shall now show that Q is locally contractile. Consider any point (x_1,y_1) such that $y_1 \neq 0$. Any neighborhood U of (x_1,y_1) is a linear neighborhood and is contractile in itself in virtue of the map $f\colon U\times(0,1)\to U$ where

$$f(x_1,y,t)=(x_1,ty_1+(1-t)y), \quad \text{for} \quad (x_1,y,t)\in U\times(0,1).$$

Consider any point $(x_1,0)$ and any neighborhood V of $(x_1,0)$. V is a rectangular neighborhood and consists of all points (x,y) such that $x\neq x_1$ and $0\leqslant x_1-\epsilon<x<x_1+\epsilon\leqslant 1$ and also all points (x_1,y) except for some closed set consisting of all points (x_1,y) such that $0<y_1\leqslant y\leqslant y_2\leqslant 1$. Delete from V the points (x_1,y) for all y such that $y_1\leqslant y\leqslant 1$. The remaining points of V form a neighborhood V_1 of $(x_1,0)$. We shall show V_1 is contractile in V. Define a map $g_1\colon V_1\times(0,\frac{1}{2})\to V_1$ by

$$g_1(x,y,t)=(x,(1-2t)y), \quad \text{for} \quad (x,y,t)\in V_1\times(0,\tfrac{1}{2}),$$

and a map $g_2\colon V_1\times(\frac{1}{2},1)\to V_1$ by

$$g_2(x,y,t)=((2t-1)x_1+2(1-t)x,0), \quad \text{for} \quad (x,y,t)\in V_1\times(\tfrac{1}{2},1).$$

Define a map $f\colon V_1\times(0,1)\to V_1$ by

$$f(x,y,t)=g_1(x,y,t), \quad \text{for all } t \text{ such that } 0\leqslant t\leqslant\tfrac{1}{2},$$
$$f(x,y,t)=g_2(x,y,t), \quad \text{for all } t \text{ such that } \tfrac{1}{2}\leqslant t\leqslant 1.$$

Clearly f is continuous since both $V_1\times(0,\frac{1}{2})$ and $V_1\times(\frac{1}{2},1)$ are closed in their union $V_1\times(0,1)$ and for a point

$(x,y,\frac{1}{2})\in V_1\times(0,\frac{1}{2})\cap V_1\times(\frac{1}{2},1)$ we have $g_1(x,y,\frac{1}{2})=(x,0)=g_2(x,y,\frac{1}{2})$.

We observe that

$$f(x,y,0)=(x,y),$$

and

$$f(x,y,1)=(x_1,0).$$

Hence V_1 is contractile in itself and therefore in V.

We shall now show that Q is not an ANR. Assume Q is an ANR. In virtue of the linear neighborhoods for points of the form (x,y) where $y\neq 0$, clearly Q contains uncountably many disjoint open subsets. By (3.4) we have $h(Q)=Q_1$ where Q_1 is a neighborhood retract of some Tychonoff cube T. Since $h^{-1}\colon Q_1\to Q$ is a map and for a map the inverse of an open set is open, it is evident that Q_1 contains uncountably many disjoint open subsets. There exist an open subset U of T and a retracting map r such that $r\colon U\to Q_1$. Clearly U contains uncountably many disjoint open subsets and hence so does T because U is open in T. But this is a contradiction because no Tychonoff cube can contain uncountably many disjoint open subsets. Thus Q is not an ANR.

9. Further Results. Using (6.4) and the proof given by H. Sammelson [9, p. 448] for separable metric spaces, we obtain for NH spaces:

(9.1) ***Fox's Theorem.*** *Let A be an ANR and let B be a deformation retract of A, then there exists a deformation mapping $A\times(0,1)$ into A such that the points of $B\times(0,1)$ are fixed.*

We can link together the concepts of ANR set and AR set by the

(9.2) ***Theorem.*** *If A is an ANR and some element p of A is a deformation retract of A, then A is an AR.*

Proof. By (3.4) A is homeomorphic to a closed neighborhood retract of some Tychonoff cube T. We lose no generality by assuming A is contained in T. Since p is a deformation retract of A, there exists a map f such that $f\colon A\times(0,1)\to A$, $f(a,1)=a$ for $a\in A$, and $f(a,0)=r(a)$ for $a\in A$ where r is a retracting map defined by $r(A)=p$.

We define a map F such that $F\colon (T\times 0)\cup(A\times(0,1))\to A$ by

$$F(x,0)=p, \quad \text{for} \quad x\in T,$$
$$F(x,t)=f(x,t), \quad \text{for} \quad x\in A \quad \text{and} \quad t\in(0,1).$$

F is continuous because both $T\times 0$ and $A\times(0,1)$ are closed in their union $(T\times 0)\cup(A\times(0,1))$, and for

$$(x,t)\in(T\times 0)\cap(A\times(0,1))=A\times 0$$

we have $f(x,t)=r(x)=p$. Since A is an ANR, we can now apply (6.4), and obtain an extension F' of F over $T\times(0,1)$ relative to A.

We define a map G such that $G\colon T\to A$ by

$$G(x)=F'(x,1), \quad \text{for} \quad x\in T.$$

Now G retracts T onto A because for any $a\in A$ we have

$$G(a)=F'(a,1)=F(a,1)=f(a,1)=a.$$

Hence by (3.6) A is an AR.

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What paths have length?

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In the classical theory, the length of the curve $y=f(x)$ $(a\leqslant x\leqslant b)$ is determined by computing the integral $\int_a^b \sqrt{1+f'^2(x)\,dx}$. Geometrically, this means that in determining the length of an arc we really compute the area of a plane domain. The length of the circular arc $y=\sqrt{1-x^2}$ $(0\leqslant x\leqslant b)$ is the area of the plane domain $(0\leqslant x\leqslant b,\ 0\leqslant y\leqslant 1/\sqrt{1-x^2})$. If the arc happens to be a quarter of a circle, the domain is not even bounded.

In a series of previous papers [1]), the author has developed a more geometric approach to the problem based on the definition of the length of a path as the limit of the lengths of inscribed polygons which get indefinitely dense in the path. This length was studied in spaces of increasing generality. For instance, when applied to vector spaces our results comprise not only Finsler spaces but spaces with locally Minkowskian metrics in which the indicatrices (or unit spheres) are positive in some directions and negative or zero in others. On each stage we formulated sufficient conditions

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