2. Quant à la notation, le symbole $\prod_{j=1}^{n}$ est entendu par l'auteur au sens de la Logique: *pour tout* $j=1,2,\dots,n$.

On prouve facilement que la convergence dans l'espace métrique définie au nº **10** est équivalente dans l'espace L (nº **16**) à la convergence asymptotique, et que $L_1^* = L$.

3. Il est à remarquer que la condition (A_3) formulée déjà par Daniell dans un Mémoire peu connu[2]) avait été ensuite souvent omise[3]), ce qui n'était pas juste. Mazurkiewicz, auquel ledit travail était problement inconnu, a introduit lui-même cette condition et prouvé que la non-décroissance des distributrices, appliquée parfois au lieu de (A_3), n'est pas suffisante.

4. L'interprétation de la notion de *EVA* au nº **16** peut être généralisée: au lieu de la mesure de Lebesgue dans l'intervalle $0 \leqslant t \leqslant 1$, on peut employer une mesure abstraite normée, σ-additive. De plus, il résulte du théorème de Daniell-Kolmogoroff[4]) que chaque *EVA* peut être interpreté de cette manière. Ceci permet de formuler le résultat principal de ce travail comme un théorème concernant les classes de fonctions mesurables par rapport à une mesure abstraite. Ceci permet aussi de parvenir au résultat de Mazurkiewicz sur une autre voie, à savoir à l'aide du théorème d'après lequel la mesure de Lebesgue est, dans un sens, universelle pour toutes les mesures séparables[5]). J'aborderai ce sujet ailleurs.

Il est cependant à remarquer que, pour les probabilistes, les distributrices des variables sont préférables aux variables mêmes[6]) et le jeu de pile et face est préférable à l'espace des fonctions mesurables. Sous cet aspect l'énoncé de Mazurkiewicz semble être particulièrement réussi.

E. M.

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[2]) P. J. Daniell, *Integrals in an infinite number of dimensions*, Annals of Math. (2), **20** (1918-9), p. 281-8, et *Functions of limited variation in an infinite number of dimensions*, Annals of Math. (2), **21** (1919-20), p. 30-38.

[3]) Cette condition ne figure ni dans le livre de M. Kolmogoroff, cité ci-dessus (voir en particulier p. 18), ni dans *Random variables and probability distributions* de M. H. Cramér (Cambridge 1937). On la trouve dans le livre plus récent de M. Cramér: *Mathematical methods of statistics* (Princeton 1946), p. 79.

[4]) Kolmogoroff, l. c., p. 27.

[5]) Cf. p. ex. la communication de E. Marczewski, *Sur l'isomorphie des mesures séparables*, Colloquium Mathematicum **1** (1947), p. 39-40 et les travaux y cités.

[6]) Voir p. ex. P. Lévy, *Théorie de l'addition de variables aléatoires*, Paris 1937, p. XVI et J. L. Doob, *Probability in function space*, Bull. Amer. Math. Soc. **53** (1947), p. 15-30, en particulier p. 15.

Complementary domains of continuous curves[1]).

By

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1. ***Introduction.*** Suppose that space is metric, compact, connected, and locally connected. It is known that each pair of points can be joined by an arc. If the space is locally topologically equivalent to a subset of the plane, then each complementary domain of this arc has property S. In this paper we show that the arc may be so chosen that its complementary domains have property S even if the space is not locally planar.

Much of this paper is devoted to the development of theorems regarding partitionings. The results regarding the complements of continuous curves are applications of these results. Although finite coverings by open sets could be used to prove these results, once the theorems regarding partitionings are demonstrated, partitionings seem to be a more efficient method of accomplishing them. It is hoped that partitionings will be useful in other connections. As pointed out in sections 4 and 5, problems handled by other methods can sometimes be treated effectively by using partitionings. In fact, partitionings provided a means for showing [2,3] that a compact continuous curve (locally connected continuum) can be convexified. The study of partitionings throws light on the structure of continuous curves and is of interest aside from its applications.

We shall use the following definitions.

Property S. A set M has *property* S if for each positive number ϵ, M is the sum of a finite number of connected subsets each of diameter less than ϵ.

Uniformly locally connected. A set M is *uniformly locally connected* if for each positive number ϵ there is a positive number $\delta(\epsilon)$ such that each pair of points of M at a distance apart of less than $\delta(\epsilon)$ belong to a connected subset of M of diameter less than ϵ.

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[1]) Presented to the American Mathematical Society, November 26, 1948.

Distance. We denote the distance between two points p,q by $D(p,q)$. If p and q belong to a connected set M, $E(M;p,q)$ denotes the greatest lower bound of the diameters of all connected subsets of M containing $p+q$. We note that E is a metric for M that preserves its topology if M is locally connected. Also, M is uniformly locally connected under E if M has property S under D.

Partitioning. A finite collection G of mutually exclusive connected open subsets of M is a *partitioning* of M if the sum of the elements of G is dense in M. If each element of G is of diameter less than ϵ, G is an *ϵ-partitioning*. If each element of G has property S, it is an *S partitioning*. (Moise used the expression grille-decomposition instead of partitioning. See [2,3]).

Brick Partitioning. An S partitioning G of M is a *brick partitioning* if:

(a) each domain containing a point of M which is a limit point of each of two elements of G also contains a point of M which is a limit point of each of these same two elements of G but of no other element of G,

(b) each element of G is uniformly locally connected under $E(M;x,y)$,

(c) each boundary point in M of an element of G is a boundary point of another element of G.

Refinement. A collection H is a refinement of the collection G if each element of H is a subset of an element of G.

Partition Chain. A partition chain is a finite collection $C=[c_1,c_2,\ldots,c_n]$ of connected and uniformly locally connected domains such that:

(a) $\bar{c}_i$ intersects $\bar{c}_j$ if and only if $i=j-1$, j, or $j+1$,

(b) c_i is regular, that is $c_i=\bar{c}_i-F(\bar{c}_i)$ [$F(A)$ denotes the boundary of A],

(c) $\bar{c}_i\cdot\bar{c}_{i+1}$ contains a point which is not a limit point of $F(\bar{C}^*)$ (C^* denotes the sum of the elements of C).

If the point p belongs to one end element of C and the point q to the other, C is called a chain from p to q. If each element of C is of diameter less than ϵ, C is an ϵ-partition chain.

Circular Partition Chain. The definition is the same as the above except that in (a) we use the convention that $n+1=1$.

Run Straight Through. A partition chain E is said to *run straight through* the partition chain C if E is a refinement of C and E is such that if two elements of it are subsets of the same element of C, then each element of E between them is a subset of this same element of C.

2. *Brick Partitioning.* The elements of a brick partitioning G of space are packed in something like brick; that is, for each pair of elements of G whose boundaries intersect each other, there is a point on their common boundary that is not a boundary point of any other element of G. An advantage of such a partitioning is that two adjacent elements of G may be consolidated into one element so that the resulting partitioning is also a brick partitioning.

In proving the theorems of this section, we make use of Theorem 4 of *Partitioning a Set* [2] which states that for each positive number ϵ any set with property S can be S ϵ-partitioned.

Theorem 1. *Suppose M is a connected domain with property S and B is a subset of the boundary of M. Then for each positive number ϵ there is an S partitioning $[g_0,g_1,\ldots,g_n]$ of M such that the diameter of g_i $(i=1,2,\ldots,n)$ is less than ϵ, $\bar{g}_0$ intersects $M\cdot\bar{g}_i$, and $E(M+B;B,g_j)\geqslant 0$, the inequality holding if and only if $j=0$.*

Proof. Let H be an S ϵ-partitioning of M and T be a connected subset of M such that T intersects each element of H but $E(M+B;B,T)>0$. For each element h_j of H, let H_j be an S partitioning of h_j such that each element of H_j is of diameter less than $\frac{1}{3}E(M+B;B,T)$. There is a finite collection W_j of connected subsets of h_j such that:

(a) each pair of elements of W_j are at a positive distance from each other under $E(h_j)$,

(b) each element of H_j that is at a distance of zero from $B+T$ under $E(M+B)$ is a subset of an element of W_j,

(c) no element of W_j is at a zero distance from both B and T under $E(M+B)$,

(d) no collection having fewer elements than W_j satisfies conditions (a),(b), and (c).

By Theorem 3 of *Partitioning a Set* [2], there is an S partitioning H'_j of h_j such that each element of H'_j contains exactly one element of W_j. If H''_j denotes the collection of those elements of H'_j which are at a zero distance from B under $E(M+B)$, then the elements of $\sum H_j''$ are the elements $g_1, g_2, \dots, g_n$ while $g_0 = M - \sum_{i=1}^{n} \bar{g}_i$. Now g_0 is connected because it contains T and each of its components intersects T. Also, $\bar{g}_0$ intersects $M \cdot \bar{g}_i$ or else the condition (d) above was not satisfied by some W_j.

Theorem 2. *Suppose:*

(a) R is a connected domain with property S,

(b) $B_1, B_2, \dots, B_n$ is a collection of mutually exclusive closed subsets of the boundary of R,

(c) ϵ is a positive number such that for each point p of R there is an arc of diameter less than ϵ in $R + B_i$ from p to B_i.

Then for each positive number δ there is an S partitioning $g_0, g_{11}, g_{12}, \dots, g_{1n_1}, g_{21}, \dots, g_{nn_n}$ of R such that

(a) the diameter of g_{ij} is less than δ,

(b) $\bar{g}_0$ intersects $R \cdot \bar{g}_{ij}$,

(c) $\bar{g}_{ij}$ intersects B_i but $\bar{g}_0$ does not,

(d) for each point p of g_0 there is an arc of diameter less than $\epsilon + \delta$ from p to B_i in $g_0 + \bar{g}_{i1} + \bar{g}_{i2} + \dots + \bar{g}_{in_i}$.

Proof. Suppose that 2δ is less than the distance between two elements of $B_1, B_2, \dots, B_n$. Let:

(a) $[h_1, h_2, \dots, h_s]$ be an S δ-partitioning of R,

(b) p_i be a point of h_i,

(c) T be a dendron in R containing $\sum p_i$,

(d) a_{ij} be an arc of diameter less than ϵ in $R + B_i$ from p_j to B_i,

(e) θ be a positive number less than either δ or min $D(T + a_{jk}, B_i)$ $(i \neq j)$. If h_t has a point of B_i on its boundary, we let h_t, θ, and $\bar{h}_t \cdot B_i$ be the M, ϵ, and B of Theorem 1 and obtain a partitioning $g_0, g_1, \dots, g_n$ satisfying the conditions of that theorem. Then $g_1, g_2, \dots, g_n$ are elements of $g_{i1}, g_{i2}, \dots, g_{in_i}$. Similarly, all elements g_{ij} are obtained and g_0 is defined to be $R - R \cdot \sum\sum \bar{g}_{jk}$.

Theorem 3. *Suppose R is a connected set with property S while R_1 and R_2 are two connected subsets of R such that $E(R; R_1, R_2) > 0$. There is a continuous transformation T of R into a straight line interval I such that $T(R_1)$ and $T(R_2)$ are the ends of I while for each connected subset X of I containing an end point of I, $T^{-1}(X)$ is connected and has property S* [2]).

[2]) Using results obtained in the proof of Theorem 5, we find that T can be chosen so that each $T^{-1}(X)$ is uniformly locally connected under $E(R)$.

Proof. For convenience we suppose that R is closed [Lemma 1 of 2]. Let R_1' and R_2' be mutually exclusive closed subcontinua of R such that R_i' contains R_i and $R - (R_1' + R_2')$ has property S. Letting 3δ be the distance from R_1' to R_2' and applying Theorem 1 to each component of $R - (R_1' + R_2')$, we find that there is an S partitioning $[g_1, g_2, \dots, g_n]$ of R such that (a) g_1 contains R_1', (b) g_2 contains R_2', (c) $D(g_1, g_2) > 0$, and (d) $\bar{g}_i$ $(i = 3, 4, \dots, n)$ intersects both $\bar{g}_1$ and $\bar{g}_2$. We define $T(\bar{g}_1)$ and $T(\bar{g}_2)$ to be 0 and 1 respectively.

Each component C_j of $R - (\bar{g}_1 + \bar{g}_2)$ has property S and by Theorem 1 there is an S partitioning $g_{j0}, \bar{g}_{j1}, \dots, \bar{g}_{jn}$ of C_j such that the diameter of g_{ji} $(i = 1, 2, \dots, n)$ is less than either $1/2$ or $\frac{1}{3} D(g_1, g_2)$, $\bar{g}_{j0}$ intersects $\bar{g}_{ji}$, and $D(\bar{g}_{ji}, \bar{g}_1 + \bar{g}_2) \geqslant 0$, the inequality holding if and only if $i = 0$. We define $T(\sum \bar{g}_{j0})$ to be $1/2$.

This process is continued as follows: Suppose that $T(X_1)$ and $T(X_2)$ have been defined to be $p/2^q$ and $(p+1)/2^q$ and let C_s be a component of $R - (X_1 + X_2)$ whose closure intersects both X_1 and X_2. By Theorem 2 there is an S partitioning $h_{s0}, h_{s11}, \dots, h_{s1m_1}, h_{s21}, \dots, h_{s2m_2}$ of C_s such that:

(a) the diameter of h_{sij} is less than either $D(X_1, X_2)/3$ or $1/2^{q+1}$,

(b) $\bar{h}_{s0}$ intersects $\bar{h}_{sij}$,

(c) $\bar{h}_{sij}$ intersects X_i but $\bar{h}_{s0}$ does not,

(d) for each point r of h_{s0} there is an arc of diameter less than $1/2^q + 1/2^{q+1}$ from r to X_i in $\bar{h}_{s0} + \bar{h}_{si1} + \dots + \bar{h}_{sim_i}$.

Then $T(\sum \bar{h}_{s0}) = (2p+1)/2^{q+1}$.

The continuation of this process defines T except on a subset of R which is not dense in any open subset of R. The transformation is defined by continuity on this set. It may be shown to satisfy the conditions mentioned in the statement of Theorem 3.

Theorem 4. *A necessary and sufficient condition that a domain D with property S not be uniformly locally connected is that there be a point x of $\bar{D}$ and an arc pxq in $D + x$ such that $E(D; px - x, xq - x) > 0$.*

Theorem 5. *Suppose R is a connected and uniformly locally connected domain, R_1 and R_2 are two connected subsets of R which are at a positive distance from each other, and $B_1, B_2, \dots, B_n$ is a collection of subsets of the boundary of R. Then there is a brick partitioning $[h_1, h_2]$ of R such that h_j contains R_j and each domain containing a point $\bar{B}_i \cdot \bar{h}_j$ $(i = 1, 2, \dots, n;\ j = 1, 2)$ also contains a point of $B_i \cdot \bar{h}_j$ which is not a limit point h_k $(k \neq j)$.*

Proof. By Theorem 3 there is an uncountable collection W of S partitionings of R such that:

(a) each element of W contains exactly two elements, one of which contains R_1 and the other which contains R_2,

(b) R is a subset of the sum of these two elements plus their common boundary,

(c) for each pair of elements H, K of W there is an element h of H and element k of K such that $E(R;h,k)>0$.

Using Theorem 4, we find that if W_0' is the set of all elements $[h_1,h_2]$ of W such that not both h_1 and h_2 are uniformly locally connected, W_0' is at most countable. Also, W_i' is not uncountable if it is the set of all elements $[h_1,h_2]$ of W such that there is a point of $\bar{B}_i\cdot\bar{h}_1\cdot\bar{h}_2$ which is not a limit point of both $B_i\cdot(\bar{h}_1-\bar{h}_1\cdot\bar{h}_2)$ and $B_i\cdot(\bar{h}_2-\bar{h}_1\cdot\bar{h}_2)$. Each element of the uncountable collection $W-(W_0'+W_1'+\dots+W_n')$ satisfies the conditions of the theorem.

Each of the following three theorems follows from repeated applications of the theorem preceding it.

Theorem 6. *Suppose R is a connected and uniformly locally connected domain, W is a finite collection of subsets of R such that the distance between two elements of W is positive, and $B_1,B_2,\dots,B_n$ is a collection of subsets of the boundary of R. Then there is a brick partitioning $G=[g_1,g_2,\dots,g_m]$ of R such that,*

(a) *no element of G intersects two elements of W,*

(b) *each domain containing a point of $\bar{g}_j\cdot\bar{B}_k$ $(j=1,2,\dots,m;$ $k=1,2,\dots,n)$ also contains a point of B_k which is a limit point of g_j but of no other element of G,*

(c) *each domain containing a point of $\bar{g}_j\cdot\bar{g}_k$ $(j,k=1,2,\dots,m)$ contains a point of R which is a limit point of both g_j and g_k but of no other element of G.*

Theorem 7. *Suppose R is a connected and uniformly locally connected domain and $B_1,B_2,\dots,B_n$ is a collection of subsets of the boundary of R. Then for each positive number ϵ there is a brick ϵ-partitioning G of R satisfying conditions* (b) *and* (c) *of Theorem* 6.

Theorem 8. *If M is a set with property S, then there is a sequence $G_1,G_2,\dots$ such that G_i is a brick $(1/i)$-partitioning of M and G_{i+1} is a refinement of G_i.*

3. *Partition Chains.* The first theorem of this section is an extension of the lemma used extensively in *The Kline Sphere Characterization Problem* [1] and may be proved by the methods used there.

We say that M *disrupts* X from Y in R if there is an arc from X to Y in R but each such arc contains a point of M. An arc minus its end points is called an open arc.

Theorem 9. *Suppose the boundary of the connected domain R is the sum of the mutually exclusive sets M, N, and E, each of which is accessible from R, and U is a connected subdomain of R such that no pair of points of R disrupts U from $F(R)$ in $\bar{R}$ and no point of R disrupts U from either $M+E$ in $R+M+E$ or $N+E$ in $R+N+E$. Then there are open arcs (α) and (β) in R such that α intersects both M and N, β intersects both U and E, but α does not intersect β.*

Theorem 10. *Suppose $C=[c_1,c_2,\dots,c_n]$ is a partition chain from p to q, B is a closed subset of $\bar{C}^*$, and α is an arc from p to q in $\bar{C}^*-[F(\bar{C}^*)+B]$ such that $\alpha\cdot\bar{c}_i\cdot\bar{c}_{i+1}$ $(i=1,2,\dots,n-1)$ is a point. Then there is a partition chain $K=[k_1,k_2,\dots,k_n]$ running straight through C from p to q and such that B does not intersect $\bar{K}^*$ but each component of $\bar{c}_i-\bar{k}_i$ $(i=1,2,\dots,n)$ has a limit point in B if $c_i\neq k_i$.*

Proof. Let G be a brick partitioning of $\bar{C}^*-F(\bar{C}^*)$ such that G is a refinement of C and each element of G is of diameter less than $D(\alpha,B)/3$. Denote by k_i the maximal connected domain in c_i which contains $\alpha\cdot c_i$ and is such that $\bar{k}_i$ is the sum of the closures of elements of G which do not have a limit point in B. Then $K=[k_1,k_2,\dots,k_n]$ is the required chain.

Theorem 11. *Suppose that in addition to the hypotheses of Theorem* 9, *it is given that $[g_1,g_2]$ is a brick partitioning of R such that M does not intersect $\bar{g}_2$ and N does not intersect $\bar{g}_1$. Then there are arcs α and β satisfying the conditions of Theorem* 9 *and such that $\alpha\cdot\bar{g}_1\cdot\bar{g}_2$ is a point.*

Proof. Using Theorem 9, we find that there are open arcs (α) and (β) in R such that α intersects both M and N but does not intersect β, $\alpha\cdot\bar{g}_1\cdot\bar{g}_2$ is a point, β intersects E, and $\beta+E$ intersects both $\bar{g}_1$ and $\bar{g}_2$. (If $\bar{g}_i$ did not intersect E, in applying Theorem 9 we would let g_i be U of that theorem). Since we find from the method used in Theorem 10 that E may be replaced by a set containing $\beta+E$, there is no loss of generality in supposing that both $\bar{g}_1$ and $\bar{g}_2$ intersect E. Hence, we suppose this.

For convenience we suppose that U intersects g_2. Let (cb_1) be an open arc in g_2 from a point c of U to E. For each set A denote by $W(A)$ the set of all points p of R such that there is an open arc (β_p) from p to E in R and an open arc (α_p) from M to N in R such that $\alpha_p\cdot\bar{g}_1\cdot\bar{g}_2$ is a point and α_p does not intersect β_p+A. Assume that $W(E)$ does not intersect U and let r be the nearest point of the closure of $W(E)\cdot cb_1$ to c in the order from c to b_1 on cb_1.

Now r does not disrupt E from U in $R+E$ or else, by using the fact that r does not disrupt U from $N+E$ in $R+N+E$ or U from $M+E$ in $R+M+E$, it could be shown that there is an open arc (γ) from M to N in R such that $\gamma\cdot\bar{g}_1\cdot\bar{g}_2$ is a point and γ does not intersect rb_1 of cb_1. But then r would not be the nearest point of the closure of $W(E)\cdot cb_1$ to c on cb_1.

There is an open arc (cb_2) in $R-r$ from c to a point of E such that either (cb_2) is a subset of g_2 or $\bar{g}_1\cdot\bar{g}_2\cdot(cb_2)$ is a point while both $g_1\cdot cb_2$ and $g_2\cdot cb_2$ are connected. For convenience we suppose that $cb_1\cdot cb_2$ is an arc cx.

By using the methods of the lemma in *The Kline Sphere Characterization Problem* [1], it may be found that there are:

(a) an arc pxq such that px is a subset of xb_2, xq is a subset of xb_1, and p is a point of g_2 if $xb_2\cdot W(E)$ intersects g_2;

(b) an open arc (mn) in R from a point m of M to a point n of N such that $mn\cdot\bar{g}_1\cdot\bar{g}_2$ is a point and mn contains pxq;

(c) an open arc (pb_3) in R from p to E such that (pb_3) does not intersect mn;

(d) an open arc (qb_4) in R from q to E such that this open arc does not intersect mn;

(e) an open arc (st) in R from a point of s of (pxq) to a point r of $M+N+E$ such that $st\cdot pxq=s$ and s is not a point of $W[(pb_3)+(qb_4)]$.

For convenience suppose that p precedes q on mn in the order from m to n. Let v be the first point of

$$(M+mp)+(N+qn)+(E+pb_3+qb_4)$$

on st in the order from s to t. Now v is not a point of $E+pb_3+qb_4$ or else s is a point of $W[(pb_3)+(pb_4)]$. Also, v is not a point of $N+qn$, for assume that it is. If sv does not intersect $\bar{g}_1$, there is an arc γ from m to N in $mp+ps+sv+qn-q$ such that $\gamma\cdot\bar{g}_1\cdot\bar{g}_2$ is a point, but this contradicts the fact that s is not a point of $W\cdot[(pb_3)+(qb_4)]$. If sv intersects $\bar{g}_1$ in a point u, it can also be seen that there is an arc γ from M to N in $\bar{g}_1+qn+uv$ of sv such that γ does not intersect $xq+(qb_4)$ and $\gamma\cdot\bar{g}_1\cdot\bar{g}_2$ is a point. But x is not a point of $W(E)$. Similarly, we find that v is not a point of $M+mp$. Hence, the theorem is satisfied by showing that the assumption that U does not contain a point of $W(E)$ leads to a contradiction.

Theorem 12. *Suppose the boundary of the connected and uniformly locally connected domain R is the sum of the mutually exclusive sets M, N, and E while U is a connected subdomain of R such that no point of R disrupts U from either $M+E$ in $R+M+E$ or $N+E$ in $R+N+E$. Then either* (a) *there are open arcs* (α) *and* (β) *in R such that α intersects both M and N, β intersects both U and E, but α does not intersect β, or* (b) *there is a closed set X in $\bar{R}$ such that each component of X intersects E, $X+E$ does not disrupt M from N in $\bar{R}$, but there are a pair of points p_1, p_2 of R such that p_1+p_2 disrupts U from $F(R)$ in $\bar{R}$ and $X+E+p_i$ disrupts M from N in $\bar{R}$.*

Proof. Suppose that condition (a) is not satisfied. There is a positive number δ so small that if a subset of R disrupts U from $F(R)$ in $\bar{R}$, then this subset has a diameter of more than 3δ. By Theorem 7 there is a brick δ-partitioning G of R satisfying conditions (b) and (c) of that theorem with B_1, B_2, and B_3 replaced by M, N, and E.

Let A be a finite collection of arcs in $\bar{R}$ such that A^*+E does not disrupt M from N, each element of A intersects E, and if g is an element of G which does not intersect A^*, then each arc from g to E in $\bar{R}$ disrupts M from N in $\bar{R}-(A^*+E)$. Let α be an arc irreducible from M to N in $\bar{R}-(A^*+E)$. There is a brick partitioning H of R such that H is a refinement of G, each element of H is of diameter less than $D(\alpha,A^*)/3$, and H satisfies conditions (b) and (c) of Theorem 7 with B_1, B_2, and B_3 replaced by M, N, and E.

Let W_1 be the set of all connected sets w such that w is maximal with respect to being the common part of R and a domain whose closure is the closure of the sum of a collection of elements of G which do not intersect A^*. It may be seen that no element of W_1 has a limit point in $E+A^*$. Let W_2 be the set of all connected sets w such that w is maximal with respect to being the common part of an element of G which does intersect A^* and a domain whose closure does not intersect A^* but is the closure of the sum of a subcollection of H. There is a partition chain $C=[c_1,c_2,\ldots,c_m]$ such that the elements of C are elements of W_1+W_2, one end link of C has a point of M on its boundary and the other end link has a point of N on its boundary, and no chain with fewer elements than C has these properties. It may be noted that each element of H which is not in an element of W_1+W_2, has a limit point on A^*.

Let X be the closure of the sum of A^* and all elements of H which are not subsets of either an element of W_1 or of an element of C. Now each component of X intersects E and $R-R\cdot X=R'$ is a connected domain which contains each element of C and each element of W_1. Let Y be the sum of all subdomains y of R' such that each arc from y to E in $R+E$ disrupts M from N in $\bar{R}-(X+E)$. Denote the component of Y containing U by U'. It follows from Theorem 9 that some pair of points p_1, p_2 of R' disrupts U' from $M+N+X+E$ in $\bar{R}'$. We shall show that each of these points disrupts M from N in $\bar{R}-(X+E)$, thereby completing the proof of the theorem.

Now p_1 and p_2 are on the closures of elements of c_i and c_j $(i<j)$ of C which are not elements of W_1. These elements are not adjacent because they are of diameter less than δ. However, $j=i+2$ or else C could be replaced by a chain with fewer links. Now c_{i+1} is an element of W_1 and c_i and c_{i+2} are not.

If V denotes the collection of all points q of R' such that q separates M from N in $\bar{R}-(X+E)$, both p_1 and p_2 belong to V, for suppose that one does not. Then there is a simple closed curve J in $\bar{R}-(X+E)$ containing p_1+p_2 and two points r, s of $V+M+N$. For convenience, suppose r and s are points of V. Since $r+s$ is not the boundary of U', an arc in $R+E-(r+s)$ intersects both J and E. But then one arc of J from r to s would not intersect any element of W_1. This contradicts the fact that each arc in J from r to s contains a points of W_1 because one of the points r, s belongs to $\sum_{n=1}^{i}\bar{c}_n$ and the other belongs to $\sum_{n=i+2}^{m}\bar{c}_n$.

By using a type of argument similar to the preceding and by applying Theorem 11, we can obtain the following extension of Theorem 12.

Theorem 13. *Suppose that in addition to the hypotheses of Theorem 12, it is given that $[g_1,g_2]$ is a brick partitioning of R such that M does not intersect $\bar{g}_2$ and N does not intersect $\bar{g}_1$. Then either* (a) *there are open arcs* (α) *and* (β) *in R such that α intersects both M and N, $\alpha\cdot\bar{g}_1\cdot\bar{g}_2$ is a point, β intersects both U and E, but does not intersect α or* (b) *there are a closed set X in $\bar{R}$ such that each component of X intersects E, an arc γ in $\bar{R}-(X+E)$ from M to N such that $\gamma\cdot\bar{g}_1\cdot\bar{g}_2$ is a point, and a pair of points p_1, p_2 of R such that p_1+p_2 disrupts U from $M+N+E$ in $\bar{R}$ and $X+E+p_i$ disrupts M from N in $\bar{R}$.*

Theorem 14. *Suppose C is a partition chain from p to q such that no pair of points of $\bar{C}^*$ separates space and each element of C has a limit point on $F(\bar{C}^*)$. Then for each positive number ϵ, there is an ϵ-partition chain E from p to q such that:*

(a) *E runs straight through C,*

(b) *$\bar{E}^*$ does not intersect $F(\bar{C}^*)$,*

(c) *each element of E has a limit point on $F(\bar{E}^*)$,*

(d) *$\bar{C}^*-\bar{E}^*$ has property S and each point of it is joined to $F(\bar{C}^*)$ by an arc which does not intersect $\bar{E}^*$ but which is a subset of the closure of the sum of eleven elements of C.*

Part (d) of the above conclusion can be strengthened by using a smaller number than eleven, but eleven is used because this simplifies the proof and in the applications of Theorem 14 in the next sections, reducing the number does not improve the results.

Proof. Let G be a brick ϵ-partitioning of $\bar{C}^*-F(\bar{C}^*)$ such that G is a refinement of $C=[c_1,c_2,\ldots,c_n]$, one element of G contains p, and another contains q.

By repeated applications of Theorem 11, it may be found that there is a finite collection A of arcs such that each element of A intersects $F(\bar{C}^*)$, each element of A is in the sum of the closures of eleven elements of C, each element of G intersects an element of A, and there is an arc a from p to q in $\bar{C}^*-[A^*+F(\bar{C}^*)]$ such that $a\cdot\bar{c}_i\cdot\bar{c}_{i+1}$ $(i=1,2,\ldots,n-1)$ is a point. Before establishing the existence of such a set A, we proceed to show how the theorem can be completed if there is such a collection.

Denote the distance from a to $F(\bar{C}^*)+A^*$ by 3δ. Let be H a brick δ-partitioning of $\bar{C}^*-F(\bar{C}^*)$ which is a refinement of G. Denote by X the closure of the sum of all elements of H which have a point of $F(\bar{C}^*)+A^*$ on their boundaries. Let R be the set of all elements r such that r is a component of $g-X$ for some element g of G. From the elements of R which intersect a, form a chain E which runs straight through C and from p to q. The chain E satisfies the conditions of the theorem.

A method of verifying that A exists is now outlined. For convenience we suppose that n is of the form $6i+1$. It follows from repeated application of Theorem 11 that there is a finite collection A' of arcs such that no element of A' intersects $c_1+c_7+c_{13}+\ldots+c_n$ but each intersects $F(\bar{C}^*)$, A'^* intersects each element of G in c_r $(r=3,4,5,9,10,11,15,\ldots,n-2)$, and there is an arc from p to q in $\bar{C}^*-[F(\bar{C}^*)+A'^*]$ such that this arc intersects the common boundary of two adjacent elements of C in exactly one point. Using Theorem 10, we obtain a chain $K=[k_1,k_2,\ldots,k_n]$ such that $\bar{K}^*$ does not intersect A'^*, k_s is a subset of c_s, $k_t=c_t$ $(t=1,7,13,\ldots,n)$, $F(\bar{K}^*)$ is accessible from each element of K, and each point of $\bar{C}^*-\bar{K}^*$ belongs to an arc in $\bar{C}^*-\bar{K}^*$ which intersects $F(\bar{C}^*)$.

By using Theorem 11 again we find that there is a finite collection A'' of arcs such that no element of A'' intersects $k_4+k_{10}+k_{16}+\ldots+k_{n-3}$ but each intersects $F(\bar{K}^*)$, A''^* intersects each element of G not intersected by A'^*, and there is an arc from p to q in $\bar{K}^*-[F(\bar{K}^*)+A''^*]$ such that this arc intersects the common boundary of two adjacent elements of K in exactly one point. Then $A=A'+A''$.

Had we used Theorem 13 instead of Theorem 11 in the preceding argument, we could have obtained the following.

Theorem 15. *Suppose C is a partition chain from p to q such that no point of $\bar{C}^*$ separates space,* (a) *those elements of C which are not open arcs have a point of $F(\bar{C}^*)$ on their boundaries, and* (b) *each pair of complementary domains of $\bar{C}^*$ are separated by a pair of points each of which is either p, q, or a point of $\bar{C}^*$ which separates p from q in $\bar{C}^*$.*

Then for each positive number ϵ there is an ϵ-partition chain E from p to q satisfying conditions (a) *and* (b) *above with E substituted for C and such that* (c) *E runs straight through C,* (d) *$\bar{E}^*$ does not intersect $F(\bar{C}^*)$, and* (e) *if a complementary domain D of $\bar{E}^*$ intersects $F(\bar{C}^*)$, then each point of $D\cdot\bar{C}^*$ belongs to an arc in D which intersects $F(\bar{C}^*)$ and is a subset of the closure of the sum of eleven elements of C.*

The methods of Theorem 14 yield the following result for circular partition chains.

Theorem 16. *Suppose C is a circular partition chain such that no pair of points of $\bar{C}^*$ separates space and each element of C has a limit point on $F(\bar{C}^*)$. Then for each positive number ϵ there is a circular ϵ-partition chain E satisfying conditions* (b), (c), *and* (d) *of Theorem* 14 *and such that E is a refinement of C, each element of C contains an element of E, and if two elements of E are subsets of the same element of C, a subchain of E containing them lies in this same element of E.*

4. *Complements of Arcs and Dendrons.* The theorems of this section are applications of Theorems 14 and 15. There are several known methods [for example, see Whyburn's *Analytic Topology*, 4] of showing that each pair of points of a compact locally connected continuum belongs to an arc in this continuum. Defining an arc by means of a sequence of partition chains gives another method of doing this.

Theorem 17. *If p, q, and r are three points of the compact locally connected continuum M which is not separated by any pair of its points, then there is an arc from p to q in M such that the complement of this arc is connected, has property S and contains r.*

Proof. By Theorem 5 there is a brick partitioning $[C_1,k]$ of M such that C_1 contains $p+q$ and k contains r. There is a regular (1/2)-partition chain C_2 in C_1 from p to q and satisfying conditions like those given in Theorem 14. In C_2 there is a regular (1/4)-partition chain C_3 satisfying like conditions. Similarly, there are chains $C_4, C_5,\ldots$ The common part of $\bar{C}_1, \bar{C}_2^*, \bar{C}_3^*,\ldots$ is an arc from p to q whose complement is connected, has property S, and contains r.

Theorem 18. *If p and q are two points of the compact locally connected continuum M, there is an arc from p to q such that each complementary domain of this arc has property S and each such pair of complementary domains is separated in M by a pair of points. Furthermore, for each positive number ϵ there are only a finite number of these complementary domains of diameter more than ϵ.*

Proof. The proof is similar to that given for the previous theorem except that we apply Theorem 15 instead of Theorem 14.

Theorem 19. *Suppose W is a totally disconnected closed subset of the compact locally connected continuum M. Then there is a dendron (acyclic continuous curve) T in M containing W such that each component of $M-T$ has property S and for each positive number ϵ there are no more than a finite number of such components of diameter more than ϵ.*

Proof. Let G_1 be a brick 1/2-partitioning of M and H_1 be the collection of elements of G_1 whose closures contain points of W. By the preceding theorem, there is a finite collection A_1 of arcs in M such that for each positive number ϵ and each element α of A_1, $M-\alpha$ has no more than a finite number of components of diameter more than ϵ, each of these components has property S, $\bar{H}_1^*+A_1^*=M_1$ is connected, but if B is a proper closed subset of A_1^*, $\bar{H}_1^*+B$ is not connected.

Let G_2 be a brick 1/4-partitioning of M_1 and H_2 be the elements of G_2 whose closures intersect W. There is a finite collection A_2 of arcs in M_1 satisfying conditions like those satisfied by the collection A_1. We denote $\bar{H}_2^*+A_2^*$ by M_2. Similarly, we define $M_3, M_4, \ldots$ The dendron T is the common part of $M_1, M_2, \ldots$

It would be interesting to know the answer to the following.

Question. For each closed set W whose complement has property S, does there exist a countable collection A of arcs such that $W+A^*$ is a continuous curve each of whose complementary domains has property S?

5. *A Sphere Characterization.* J. R. Kline conjectured that a compact locally connected nondegenerate continuum was a simple surface (set topologically equivalent to the surface of a sphere) if it was separated by no pair of its points but by each simple closed curve in it. In the following theorem I use much the same methods as I used in proving this conjecture was true. However, partitionings are used here instead of open coverings.

Theorem 20. *If M is a compact locally connected nondegenerate continuum which is not separated by any pair of its points, then either M is a simple surface or there is a simple closed curve J in M such that $M-J$ is connected and has property S.*

Proof. Suppose M is not a simple surface. Zippin has shown [5] that some arc pq in M separates two points from each other in M. Without loss of generality we suppose that pq separates x from y but that no proper subarc of pq separates x from y. Let D_x, and D_y be the components of $M-pq$ containing x and y respectively [3].

[3]) By methods more complicated than those used here, I have shown that there is a simple closed curve J satisfying the conditions of this theorem, containing $p+q$, and such that one component of $J-(p+q)$ lies except for a totally disconnected set in D_x and the other component lies except for a totally disconnected set in D_y.

Let ϵ_1 and ϵ_2 be positive numbers so small that (a) no pair of subsets of M each of diameter less than ϵ_1 separates two points of M from each other if these points are further from the subsets than $D(p,q)/8$ and (b) if β is a subarc of pq containing an end point of pq, then β is of diameter less than $\epsilon_1/3$ if its end points are closer together than ϵ_2. Let G be a brick ϵ_2-partitioning of M such that p and q belong to the elements g_p and g_q of G.

There is an arc α_z $(z=x,y)$ in D_z which is irreducible from $\bar{g}_p$ to $\bar{g}_q$. Let H be a brick partitioning of M such that H is a refinement of G and no element of H is of diameter more than $D(pq,\alpha_x+\alpha_y)/3$. Denote by R the sum of all closures r of elements of H such that r either intersects pq or is a subset of $\bar{g}_p+\bar{g}_q$. Now R is the sum of three mutually exclusive connected sets R_1, U, and R_3 such that R_1 is closed and contains $\bar{g}_p$, R_3 is closed and contains $\bar{g}_q$, while U is an open subset of R whose boundary in R is a subset of $\bar{g}_p+\bar{g}_q$.

Since both R_1 and R_3 are of diameter less than ϵ_1, each component of $M-R$ that has both a boundary point on R_1 and a boundary point on R_3 is accessible from U. Let C_2 and C_4 be the components of $M-R$ containing (α_x) and (α_y) respectively, E be the sum of $\bar{U}$ and the closures of all components of $M-(R+C_2+C_4)$ such that these components have a point of U on their boundaries, C_i $(i=1,3)$ be a regular connected domain whose closure is the sum of R_i and the closures of all components of $M-R$ which have a boundary point on R_i but none on U. Then $C_1=[c_1,c_2,c_3,c_4]$ is a circular partition chain such that $M-\bar{C}_1^*$ is a connected domain which has property S and whose closure is E.

Theorem 16 gives that there is a sequence $C_1, C_2, \ldots$ of circular partition chains such that (a) C_{i+1} is a refinement of C_i and each element of C_i contains an element of C_{i+1}, (b) each element of C $(i=2,3,\ldots)$ is of diameter less than $1/2^i$, (c) $\bar{C}_{i+1}^*$ does not intersect $F(\bar{C}_i^*)$, (d) each element of C_i is accessible from the closure $M-\bar{C}_i^*$, (e) if two elements of C_{i+1} are subsets of the same element of C_i, then a subchain of C_{i+1} between these two elements lies in this same element of C_i and (f) $M-\bar{C}_i^*$ has property S while each point of $\bar{C}_i^*-\bar{C}_{i+1}^*$ is joined to the boundary of $\bar{C}_i^*$ by an arc in $M-\bar{C}_{i+1}^*$ which is a subset of the closure of the sum of eleven elements of C_i.

The common part of $\bar{C}_1^*, \bar{C}_2^*, \ldots$ is a simple closed curve J and $M-J$ is connected and has property S.

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Supplément au mémoire „Sur l'ensemble des points singuliers d'une fonction d'une variable réelle admettant les dérivées de tous les ordres“ [1]).

Par

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Une remarque publiée dans mon mémoire cité, p. 244, concernant une analyse se trouvant dans le Jahrb. Fortschr. Math. (de 1935) d'un article de M. R. P. Boas Jr. est superflue, car l'auteur de cette analyse a publié une correction dans le même volume de Jahrb. Fortschr., que je n'ai pas remarquée.

A propos d'une analyse de mon mémoire cité (Math. Reviews **10** (1949), p. 23), je tiens à remarquer que, d'après mon avis, M. R. P. Boas Jr. est le seul qui a la priorité de la démonstration du théorème de Pringsheim. M. V. Ganapathy Iyer, dans une note *Sur un problème de M. Carleman*, C. R. Acad. Paris, **199** (1934), p. 1371-1373, a démontré seulement un théorème plus faible, équivalent au lemme 1 de mon mémoire cité, p. 187.

Comme j'ai constaté l. c. p. 187, une démonstration beaucoup plus simple de ce lemme se trouve déjà dans le *Cours d'Analyse Mathématique* de É. Goursat, publié en 1917-1918. La première partie (nécessité) coïncide chez M. Ganapathy avec celle de Goursat. Dans la seconde partie de la démonstration M. Ganapathy prouve que l'inégalité

(1) $|f^{(n)}(x)| \leqslant n! \cdot M^n$ pour tout $x \in [a,b]$ et tout $n=1,2,\ldots$

implique l'inégalité

(2) $r(x) \geqslant \delta > 0$ pour tout $x \in [a,b]$

[1]) Z. Zahorski, Fund. Math. **34** (1947), pp. 183-245.