Let φ_2 be a generalized homeomorphism of X_2 on Y_2 and let

$$\varphi(x)=\varphi_2(x) \quad \text{for} \quad x \in X_2$$

and

$$\varphi(x)=\varphi_0(x) \quad \text{for} \quad x \in \mathfrak{X}-X_2.$$

φ is a generalized homeomorphism of $\mathfrak{X}$ on $\mathcal{Y}$ and

$$[\varphi(X)]=[\varphi_2(X)]+[\varphi_0(X-X_2)]=[\varphi_0(X)]-[\varphi_0(X_2)]=[\varphi_0(X)]=h([X])$$

for every $X \in \mathfrak{B}(\mathfrak{X})$ since $\varphi_2(X)\subset Y_2 \in \boldsymbol{J}$ and $\varphi_0(X_2)\subset Y_2 \in \boldsymbol{J}$.

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On joins of spherical mappings.

By

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1. Introdution. In a recent work of G. W. Whitehead, [4], an important generalization of H. Freudenthal's Einhängung, [3], has been introduced which seems to be one of the essential intruments for the attacking of the unsolved problem of calculating the homotopy groups of spheres. For each pair of elements $\alpha \in \pi^p(S^m)$, $\beta \in \pi^q(S^n)$, a unique element $\alpha\vee\beta \in \pi^{p+q+1}(S^{m+n+1})$ is determined, which will be called the join of α and β. If $q=n$ and β is of degree $+1$, then $\alpha\vee\beta$ is the $(n+1)$-fold Einhängung of α.

The object of the present paper is to give a detailed investigation of this joining operation. Instead of considering it as an operation on the homotopy groups, we shall present it by an imbedding of the product space $(S^m)^{S^p}\times(S^n)^{S^q}$ into the space $(S^{m+n+1})^{S^{p+q+1}}$, where Y^X denotes, as usual, the space of all mappings (i. e. continuous transformations) of X into Y.

In another recent work of G. W. Whitehead, [5], it has been proved in a quite complicate way that the Einhängung of a Whitehead product, [6], is always inessential. By using our methods, we are able to prove its generalization that the join $\alpha\vee\beta$ is inessential if at least one of the elements α,β is a product.

2. The imbedding by means of joining. For the sake of briefness, we shall denote by $\{p,m\}$ the space $(S^m)^{S^p}$. In the present paragraph, we shall define an imbedding of the product space $\{p,m\}\times\{q,n\}$ into the space $\{p+q+1, m+n+1\}$ which forms the kernel of the whole investigation.

Let R^{p+1}, R^{q+1} be two euclidean spaces with coordinates systems $(x_0,x_1,\dots,x_p)$, $(y_0,y_1,\dots,y_q)$ respectively. Let

$$R^{p+q+2}=R^{p+1}\times R^{q+1},$$

then an arbitrary point $z \in R^{p+q+2}$ has coordinates

$$(x,y)=(x_0,\dots,x_p,\ y_0,\dots,y_q).$$

We distinguish neither $x \in R^{p+1}$, $(x,0) \in R^{p+q+2}$, nor $y \in R^{q+1}$, $(0,y) \in R^{p+q+2}$; and we shall use the vector notations freely. Let $S^p \subset R^{p+1}$, $S^q \subset R^{q+1}$, $S^{p+q+1} \subset R^{p+q+2}$ be the spheres defined by

$$S^p: \quad |x|^2 \equiv x_0^2+x_1^2+\dots+x_p^2=1,\ y=0;$$
$$S^q: \quad |y|^2 \equiv y_0^2+y_1^2+\dots+y_q^2=1,\ x=0;$$
$$S^{p+q+1}: \ |z|^2 \equiv |x|^2+|y|^2=1.$$

Similarly, let R^{m+1}, R^{n+1} be euclidean spaces with coordinates systems $(u_0,u_1,\dots,u_m)$, $(v_0,v_1,\dots,v_n)$ respectively; and let $R^{m+n+2}=R^{m+1}\times R^{n+1}$ with $w=(u,v)$ as coordinates. Let S^m, S^n, S^{m+n+1} be defined by $|u|^2=1$, $v=0$; $|v|^2=1$, $u=0$; $|w|^2=1$ respectively.

For an arbitrary pair of mappings $f \in \{p,m\}$, $g \in \{q,n\}$, we define the join $\varphi=f^{\vee}g \in \{p+q+1, m+n+1\}$ by taking

$$\varphi(x\cos\theta+y\sin\theta)=f(x)\cos\theta+g(y)\sin\theta$$

for each $x \in S_p$, $y \in S_q$, and $0 \leqslant \theta \leqslant \pi/2$. Thus we have defined a transformation J of the product space $\{p,m\}\times\{q,n\}$ into the space $\{p+q+1, m+n+1\}$.

Let $\varrho(f_1,f_2)$ denote the Fréchet metric of the mappings f_1,f_2. Let $f_i \in \{p,m\}$, $g_i \in \{q,n\}$, $(i=1,2)$, be arbitrary mappings, and let $\varphi_i=f_i^{\vee}g_i$, $(i=1,2)$. Then for each $z=x\cos\theta+y\sin\theta$ of S^{p+q+1}, where $x \in S^p$, $y \in S^q$, and $0 \leqslant \theta \leqslant \pi/2$, we have

$$|\varphi_1(z)-\varphi_2(z)|^2=|f_1(x)-f_2(x)|^2\cos^2\theta+|g_1(y)-g_2(y)|^2\sin^2\theta;$$

whence we deduce that

$$[\varrho(\varphi_1,\varphi_2)]^2 \leqslant [\varrho(f_1,f_2)]^2+[\varrho(g_1,e_2)]^2.$$

Hence J is continuous, i. e. a mapping.

Since $\varphi=f^{\vee}g$ coincides with f,g on S^p,S^q respectively, J is univalent. Since $\varrho(f_1,f_2)\leqslant\varrho(\varphi_1,\varphi_2)$ and $\varrho(g_1,g_2)\leqslant\varrho(\varphi_1,\varphi_2)$, the inverse J^{-1} is also continuous. Hence J is a homeomorphism and defines an imbedding of $\{p,m\}\times\{q,n\}$ into $\{p+q+1, m+n+1\}$.

3. Fibres and components of the space $\{p,m\}$. For the use of the sequel, we shall describe some special mappings as follows.

Let E^p denote the p-element in an euclidean p-space with coordinates system $\xi_1,\xi_2,\dots,\xi_p$ defined by $|\xi|^2\leqslant 1$, and S^{p-1} be the boundary sphere of E^p defined by $|\xi|^2=1$. Then by λ_p we denote the mapping of E^p onto S^p defined by

$$x_0=2|\xi|^2-1,$$
$$x_i=2\sqrt{1-|\xi|^2}\,\xi_i, \qquad (i=1,2,\dots,p).$$

It follows immediately that

$$\lambda_p(S^{p-1})=p_0=(1,0,\dots,0), \quad \lambda_p(0)=p_*=(-1,0,\dots,0),$$

and λ_p maps the interior of E^p topologically onto S^p-p_0.

Next, let E^{r+1} be the $(r+1)$-element of the euclidean $(r+1)$-space with coordinates system $t_0,t_1,\dots,t_r$ defined by $|t|^2\leqslant 1$. Let

$$\Sigma^{p+r}=(S^r\times E^p)\cup(E^{r+1}\times S^{p-1})\subset E^{r+1}\times E^p,$$

and let S^{p+r} be the $(p+r)$-sphere in the $(p+r+1)$-space with the coordinates system $t_0^*,t_1^*,\dots,t_r^*,\xi_1^*,\dots,\xi_p^*$, defined by $|t^*|^2+|\xi^*|^2=1$. By $\mu_{r,p}$ we denote the homeomorphism of S^{p+r} onto Σ^{p+r} defined by the following equations:

$$t=\frac{t^*}{|t^*|}, \quad \xi=\frac{\xi^*}{|t^*|}, \quad \text{if} \quad |t^*|\geqslant|\xi^*|,$$
$$t=\frac{t^*}{|\xi^*|}, \quad \xi=\frac{\xi^*}{|\xi^*|}, \quad \text{if} \quad |t^*|\leqslant|\xi^*|.$$

Let H_1 and H_2 be the subsets of S^{p+r} defined by $|t^*|\geqslant|\xi^*|$ and $|t^*|\leqslant|\xi^*|$ respectively. Then clearly $\mu_{r,p}$ maps H_1,H_2, and $H_1\cap H_2$ onto $S^r\times E^p$, $E^{r+1}\times S^{p-1}$, and $S^r\times S^{p-1}$ respectively.

Let τ denote the projection of $\{p,m\}$ onto S^m, defined by $\tau f=f(p_0)$ for each $f\in\{p,m\}$. By Borsuk's Fibre Theorem, [2], τ is a fibre mapping. Let $\{p,m;u\}$ denote the fibre $\tau^{-1}(u)$ for $u\in S^m$. For the remaining of this paragraph, we assume $p>0$, $m>0$. Since S^m is p-simple in the sense of S. Eilenberg, [1], each component of $\{p,m\}$ contains a unique component of the fibre $\{p,m;u\}$ and corresponds a unique element of the homotopy group $\pi^p(S^m)$. Let us denote by $\{p,m\}_\alpha$ and $\{p,m;u\}_\alpha$ the components of $\{p,m\}$ and $\{p,m;u\}$ respectively, which correspond to the element $\alpha\in\pi^p(S^m)$. From the arcwise connectedness of S^m, it follows that τ is a fibre mapping of each component $\{p,m\}_\alpha$ onto S^m, denoted by τ_α, with connected fibres

$$\tau_\alpha^{-1}(u)=\{p,m;u\}_\alpha.$$

Choose a fixed $u_* \in S^m$. Whenever there is no danger of ambiguity, we shall use the simpler notations

$$G_\alpha = \{p, m\}_\alpha, \quad F_\alpha = \tau_\alpha^{-1}(u_*).$$

Choose an arbitrary but fixed $a \in F_\alpha$ as the base point of the homotopy groups concerned. According to J. H. C. Whitehead, [7], there is a beginningless chain of homomorphisms

$$\xrightarrow{k} \pi^{r+1}(G_\alpha, F_\alpha) \xrightarrow{\partial} \pi^r(F_\alpha) \xrightarrow{j} \pi^r(G_\alpha) \xrightarrow{k} \pi^r(G_\alpha, F_\alpha) \xrightarrow{\partial} \pi^{r-1}(F_\alpha) \xrightarrow{j}$$

This chain is exact in the sense that the kernel of each homomorphism is exactly the image of the one which precedes. The homomorphism j is induced by the injection mapping $F_\alpha \to G_\alpha$ and will be called the injection homomorphism. The homomorphism k of $\pi^r(G_\alpha)$ is induced by the correspondence $\varphi \to \varphi\lambda_r$ for each mapping φ: $S^r \to G_\alpha$ with $\varphi(r_0) = a$, r_0 being the fixed point $(1, 0, \dots, 0)$ on S^r. The homomorphism ∂ of $\pi^r(G_\alpha, F_\alpha)$ is induced by the correspondence $\psi \to \psi|S^{r-1}$ for each mapping ψ: $E^r \to G_\alpha$, which will be called the boundary homomorphism.

Since τ_α is a fibre mapping of G_α onto S^m with F_α as a connected fibre, it follows that τ_α induces an isomorphism of $\pi^r(G_\alpha, F_\alpha)$ onto $\pi^r(S^m)$, which will be still denoted by τ_α and called the fibre isomorphism.

According to G. W. Whitehead, [5], there is an isomorphism I_α of $\pi^r(F_\alpha)$ onto $\pi^{p+r}(S^m)$, called the Hurewicz isomorphism. Let $\beta \in \pi^r(F_\alpha)$ be represented by φ: $S^r \times S^p \to S^m$ with

$$\varphi|r_0 \times S^p = a, \quad \varphi(S^r \times p_0) = u_*.$$

Let ψ: $S^r \times E^p \to S^m$ be defined by $\psi(t, \xi) = \varphi(t, \lambda_p \xi)$ for each $t \in S^r$, $\xi \in E^p$. Then $I_\alpha(\beta)$ is represented by the mapping φ^*: $S^{p+r} \to S^m$ defined for each $\zeta^* = (t^*, \xi^*) \in S^{p+r}$ by

$$\varphi^*(\zeta^*) = \begin{cases} \psi\mu_{r,p}(\zeta^*), & \text{if } \zeta^* \in H_1, \\ u_*, & \text{if } \zeta^* \in H_2. \end{cases}$$

4. Homomorphism and multiplications induced by J. First suppose $p > 0$, $m > 0$ and $g \in \{q, n\}$ be a fixed mapping. Then the mapping $f \to f^\vee g$ defines an imbedding J_g of the space $\{p, m\}$ into the space $\{p+q+1, m+n+1\}$. From the continuity of J_g, it follows that each component $\{p, m\}_\alpha$, $\alpha \in \pi^p(S^m)$, is mapped by J_g into some definite component $\{p+q+1, m+n+1\}_\delta$, $\delta \in \pi^{p+q+1}(S^{m+n+1})$. Hence J_g determines a transformation $\alpha \to \delta$ of $\pi^p(S^m)$ into $\pi^{p+q+1}(S^{m+n+1})$, which will still be denoted by $\delta = J_g(\alpha)$.

Theorem 4.1. *J_g: $\pi^p(S^m) \to \pi^{p+q+1}(S^{m+n+1})$ is a homomorphism, called the homomorphism by joining g.*

Proof. Let E_1^p, E_2^p be the subsets of S^p defined by the conditions $x_p \geqslant 0$, $x_p \leqslant 0$. Let α', α'' be two arbitrary elements of $\pi^p(S^m)$. Choose $p_0 = (1, 0, \dots, 0)$, $u_* \in S^m$ as the base points $\pi^p(S^m)$. There exists representatives f', f'' of α', α'' respectively such that $f'(E_1^p) = u_* = f''(E_2^p)$. Let $\varphi' = f'^\vee g$, $\varphi'' = f''^\vee g)$, then by definition we have

$$\varphi'(x\cos\theta + y\sin\theta) = f'(x)\cos\theta + g(y)\sin\theta,$$
$$\varphi''(x\cos\theta + y\sin\theta) = f''(x)\cos\theta + g(y)\sin\theta,$$

for each $x \in S^p$, $y \in S^q$, $0 \leqslant \theta \leqslant \pi/2$. Let E_1^{p+q+1}, E_2^{p+q+1} be the subsets of S^{p+q+1} defined by $x_p \geqslant 0$, $x_p \leqslant 0$; and let E_*^{p+q+1} denote the $(p+q+1)$-element defined by

$$x_p = 0, \quad x_0^2 + x_1^2 + \dots + x_{p-1}^2 + y_0^2 + y_1^2 + \dots + y_q^2 \leqslant 1.$$

Then E_*^{p+q+1}, E_1^{p+q+1}, E_2^{p+q+1} have the $(p+q)$-sphere

$$S^{p+q}\colon\ x_p = 0, \quad x_0^2 + x_1^2 + \dots + x_{p-1}^2 + y_0^2 + y_1^2 + \dots + y_q^2 = 1,$$

as their common boundary. Let π_i denote the vertical projection of E_i^{p+q+1} onto E_*^{p+q+1}. Let

$$\Phi' = \varphi'|E_1^{p+q+1}, \qquad \Psi' = \varphi'|E_2^{p+q+1},$$
$$\Phi'' = \varphi''|E_2^{p+q+1}, \qquad \Psi'' = \varphi''|E_1^{p+q+1}.$$

Then it follows that $\Psi'\pi_2^{-1} = \Psi''\pi_1^{-1}$, over E_*^{p+q+1}. Call this mapping Ψ. Then clearly

$$\Psi(z) = \Phi'(z) = \Phi''(z), \qquad (z \in S^{p+q}).$$

Let $z_* = (p_0, 0) \in S^{p+q+1}$, $w_* = (u_*, 0) \in S^{m+n+1}$. Let Ψ_t: $E_*^{p+q+1} \to S^{m+n+1}$ be a homotopy such that $\Psi_0 = \Psi$, $\Psi_1(E_*^{p+q+1}) = w_*$, and $\Psi_t(z_*) = w_*$ for each $0 \leqslant t \leqslant 1$. Since $\Psi_t|S^{p+q}$ defines a partial homotopy of both Φ' and Φ'', it has extensions

$$\Phi_t'\colon\ E_1^{p+q+1} \to S^{m+n+1}, \qquad \Phi_t''\colon\ E_2^{p+q+1} \to S^{m+n+1}$$

such that $\Phi_0' = \Phi'$, $\Phi_0'' = \Phi''$. Define homotopies φ_t', φ_t'': $S^{p+q+1} \to S^{m+n+1}$ by taking

$$\varphi_t'(z) = \begin{cases} \Phi_t'(z), & \text{if } z \in E_1^{p+q+1}, \\ \Psi_t(\pi_2(z)), & \text{if } z \in E_2^{p+q+1}; \end{cases}$$

$$\varphi_t''(z) = \begin{cases} \Psi_t(\pi_1(z)), & \text{if } z \in E_1^{p+q+1}, \\ \Phi_t''(z), & \text{if } z \in E_2^{p+q+1}. \end{cases}$$

Evidently $\varphi_0'=\varphi'$, $\varphi_0''=\varphi''$, $\varphi_1'(E_2^{p+q+1})=w_*=\varphi_2(E_1^{p+q+1})$. Define a homotopy φ_t: $S^{p+q+1}\to S^{m+n+1}$ by taking

$$\varphi_t(z)=\begin{cases}\varphi_t'(z), & \text{if} \quad z\in E_1^{p+q+1},\\ \varphi_t''(z), & \text{if} \quad z\in E_2^{p+q+1}.\end{cases}$$

Then φ_0 represents $J_g(\alpha'+\alpha'')$ and φ_1 represents $J_g(\alpha')+J_g(\alpha'')$. Hence $J_g(\alpha'+\alpha'')=J_g(\alpha')+J_g(\alpha'')$, which proves the theorem. Q. E. D.

Similarly, if $q>0$, $n>0$, and $f\in\{p,m\}$ be a fixed mapping, then the mapping $g\to f^\vee g$ defines an imbedding ${}_fJ$ of the space $\{q,n\}$ into the space $\{p+q+1, m+n+1\}$. Analogous to (4.1), we have:

Theorem 4.2. *${}_fJ$: $\pi^q(S^n)\to\pi^{p+q+1}(S^{m+n+1})$ is a homomorphism, called the homomorphism by joining to f.*

Let E denote the operation, called Einhängung by H. Freudenthal, [3]. Now let $q=n=0$ and g be the identity mapping of S^0.

Theorem 4.3. *For each $\alpha\in\pi^p(S^m)$, we have*

$$J_g(\alpha)=E(\alpha), \tag{4.31}$$
$$ {}_gJ(\alpha)=(-1)^{p+m}E(\alpha). \tag{4.32}$$

Proof. (4.31) follows directly from the definitions. To prove (4.32), let φ: $S^{p+1}\to S^{m+1}$ be a representative of $J_g(\alpha)$. Let ϱ,σ be mappings of S^{p+1} and S^{m+1} onto themselves defined by

$$\begin{aligned}\varrho:&\quad z_i^*=z_{i+1},\quad (i=0,1,\ldots,p+1),\qquad z_{p+2}^*=z_0;\\ \sigma:&\quad w_0^*=w_{m+2},\ w_i^*=w_{i+1},\ (i=1,2,\ldots,m+2).\end{aligned}$$

Then ϱ,σ are of degrees $(-1)^{p+1}$, $(-1)^{m+1}$ respectively; and ${}_g\boldsymbol{J}(\alpha)$ is represented by $\sigma\varphi\varrho$. By the definition of the homotopy groups, $\varphi\varrho$ represents $(-1)^{p+1}E(\alpha)$; by a theorem of H. Freudenthal, [3, (3.7)], $\sigma\varphi\varrho$ represents $(-1)^{p+m}E(\alpha)$, which proves (4.32). Q. E. D.

Suppose that p,q,m,n are all positive. Then for each component $\{p,m\}_\alpha$, $\alpha\in\pi^p(S^m)$, of the space $\{p,m\}$, and each component $\{q,n\}_\beta$, $\beta\in\pi^q(S^n)$, of the space $\{q,n\}$, the homeomorphism J maps the product space $\{p,m\}_\alpha\times\{q,n\}_\beta$ into a definite component $\{p+q+1, m+n+1\}_\delta$, $\delta\in\pi^{p+q+1}(S^{m+n+1})$, of the space $\{p+q+1, m+n+1\}$. We call the element δ the join of the elements α,β and use the notation $\delta=\alpha^\vee\beta$. From (4.1) and (4.2), the following theorem follows immediately.

Theorem 4.4. *The joining operation J defines a group multiplication of the homotopy groups $\pi^p(S^m)$ and $\pi^q(S^n)$ with values in the homotopy group $\pi^{p+q+1}(S^{m+n+1})$.*

Again, let us suppose $p>0$, $m>0$, and $g\in\{q,n\}$ a fixed mapping, and consider the operation J_g. Let $q_0\in S^q$ be the point $(1,0,\ldots,0)$ and $v_*=g(q_0)$. Call $z_*=(0,q_0)\in S^{p+q+1}$, $w_*=(0,v_*)\in S^{m+n+1}$. Let $\bar\tau$ be the projection of the space $\{p+q+1, m+n+1\}$ onto S^{m+n+1} defined by $\bar\tau\varphi=\varphi(z_*)$ for each $\varphi\in\{p+q+1, m+n+1\}$. Let

$$G_\alpha=\{p,m\}_\alpha,\quad F_\alpha=\{p,m;u_*\}_\alpha;$$
$$\Omega_\delta=\{p+q+1, m+n+1\}_\delta,\quad \Lambda_\delta=\{p+q+1, m+n+1; w_*\}_\delta.$$

From the definition of $f^\vee g$, it is clear that G_α is mapped by J_g into Λ_δ, where $\delta=J_g(\alpha)$. Hence, if we denote by $\bar\partial$, $\bar j$, $\bar k$ the chain of homomorphisms of the groups $\pi^r(\Lambda_\delta)$, $\pi^r(\Omega_\delta)$, and $\pi^r(\Omega_\delta,\Lambda_\delta)$, then we have the following theorem

Theorem 4.5. *The homeomorphism J_g induces, for each α of $\pi^p(S^m)$ and each $r\geqslant 1$, the homomorphisms of $\pi^r(F_\alpha)$ into $\pi^r(\Lambda_\delta)$, of $\pi^r(G_\alpha)$ into $\pi^r(\Omega_\delta)$, and of $\pi^{r+1}(G_\alpha,F_\alpha)$ into $\pi^{r+1}(\Omega_\delta,\Lambda_\delta)$, where $\delta=J_g(\alpha)$, still denoted by J_g. Further, we have*

$$J_g(\pi^r(G_\alpha,F_\alpha))=0, \tag{4.51}$$
$$J_g(\pi^r(G_\alpha))\subset\bar j(\pi^r(\Lambda_\delta)). \tag{4.52}$$

Since $J_g(G_\alpha)\subset\Lambda_\delta$, the following theorem is trivial

Theorem 4.6. *For each $\alpha\in\pi^p(S^m)$ and each $r\geqslant 1$, the homeomorphism J_g induces a homomorphism $\bar J_g$ of $\pi^r(G_\alpha)$ into $\pi^r(\Lambda_\delta)$, where $\delta=J_g(\alpha)$. Further, we have*

$$J_g=\bar j\bar J_g \quad \text{on} \quad \pi^r(G_\alpha), \tag{4.61}$$
$$J_g=\bar J_g j \quad \text{on} \quad \pi^r(F_\alpha). \tag{4.62}$$

Consider the following diagram:

$$\begin{tikzcd}
\pi^{r+1}(G_\alpha,F_\alpha) \arrow[r, "\partial"] \arrow[d, "J_g"'] & \pi^r(F_\alpha) \arrow[d, "J_g"] \\
\pi^{r+1}(\Omega_\delta,\Lambda_\delta) \arrow[r, "\bar\partial"] & \pi^r(\Lambda_\delta)
\end{tikzcd}$$

Theorem 4.7. *$J_g\partial=\bar\partial J_g$ on $\pi^{r+1}(G_\alpha,F_\alpha)$.*

Proof. Let $\gamma\in\pi^{r+1}(G_\alpha,F_\alpha)$ be represented by a mapping φ: $E^{r+1}\to G_\alpha$; then $\bar\partial J_g(\gamma)$ is represented by the mapping $J_g\varphi|S^r$, while $J_g\partial(\gamma)$ is represented by the mapping $J_g\psi$, where S^r denotes the boundary sphere of E^{r+1} and $\psi=\varphi|S^r$. Clearly $J_g\varphi|S^r=J_g\psi$, which proves the theorem. Q. E. D.

Since $J_g(\gamma)=0$ for each $\gamma\in\pi^{r+1}(G_\alpha,F_\alpha)$ by (4.51), we have

Corollary 4.8. *$J_g\partial(\pi^{r+1}(G_\alpha,F_\alpha))=0$.*

5. Relation between joining homomorphisms and Hurewicz Isomorphisms and Whitehead products. Consider the following diagram:

$$\begin{array}{ccc}
\pi^r(F_\alpha) & \xrightarrow{J_g} & \pi^r(\Lambda_\delta) \\
I_\alpha \updownarrow & & \updownarrow I_\delta \\
\pi^{p+r}(S^m) & \xrightarrow{J_g} & \pi^{p+q+r+1}(S^{m+n+1}).
\end{array}$$

The principal result of this section is:

Theorem 5.1. *$J_gI_\alpha=I_\delta J_g$ on $\pi^r(F_\alpha)$, $\delta=J_g(\alpha)$.*

Proof. Let E^{r+1} be the $(r+1)$-element, in the $(r+1)$-space with coordinates system $t_0,t_1,\ldots,t_r$, defined by $|t|^2\leqslant1$. Let S^r be the boundary sphere $|t|^2=1$ of E^{r+1}. Choose $r_0=(1,0,\ldots,0)\in S^r$, $a\in F_\alpha$ as the base points for $\pi^r(F_\alpha)$.

Let $\gamma\in\pi^r(F_\alpha)$ be an arbitrary element, it is to prove that $J_gI_\alpha(\gamma)=I_\delta J_g(\gamma)$.

Let φ: $S^r\to F_\alpha$ be a representative of γ, then $\varphi(r_0)=a$. Therefore, φ can be considered as a mapping of $S^r\times S^p$ into S^m such that

$$\varphi|r_0\times S^p=a,\quad \varphi(S^r+p_0)=u_*.$$

Let E^p denote the p-element, in the p-space with coordinates system $\xi_1,\ldots,\xi_p$, defined by $|\xi|^2\leqslant1$; and S^{p-1} be the boundary sphere of E^p, defined by $|\xi|^2=1$. Let

$$\Sigma^{p+r}=(S^r\times E^p)\cup(E^{r+1}\times S^{p-1}).$$

Let φ_0: $\Sigma^{p+r}\to S^m$ be the mapping defined by

$$\varphi_0(t,\xi)=\begin{cases}\varphi(t,\lambda_p\xi), & \text{if}\quad (t,\xi)=S^r\times E^p,\\ u_*, & \text{if}\quad (t,\xi)=E^{r+1}\times S^{p-1},\end{cases}$$

where λ_p is the mapping described in § 3.

Let S^{p+r} be the $(p+r)$-sphere, in the $(p+r+1)$-space with the coordinates system $t_0^*,t_1^*,\ldots,t_r^*$, $\xi_1^*,\ldots,\xi_p^*$, defined by $|t^*|^2+|\xi^*|^2=1$. Let φ_1: $S^{p+r}\to S^m$ be the mapping defined by $\varphi_1=\varphi_0\mu_{r,p}$, where $\mu_{r,p}$ is the homeomorphism of S^{p+r} onto Σ^{p+r} described in § 3. Then φ_1 is given by

$$\varphi_1(t^*,\xi^*)=\begin{cases}\varphi\left(\dfrac{t^*}{|t^*|},\lambda_p\left(\dfrac{\xi^*}{|t^*|}\right)\right), & \text{if}\quad |t^*|\geqslant|\xi^*|,\\ u_*, & \text{if}\quad |t^*|\leqslant|\xi^*|.\end{cases}$$

By definition, $I_\alpha(\gamma)$ is represented by φ_1.

Now let $S^{p+q+r+1}$ be the $(p+q+r+1)$-sphere, in the $(p+q+r+2)$-space with the coordinates system $t_0^*,t_1^*,\ldots,t_r^*$, $\xi_1^*,\ldots,\xi_p^*$, $\eta_0^*,\eta_1^*,\ldots,\eta_q^*$, defined by $|t^*|^2+|\xi^*|^2+|\eta^*|^2=1$. Imbed S^{p+r} and S^q in $S^{p+q+r+1}$ by identifying $(t^*,\xi^*)\in S^{p+q}$ with $(t^*,\xi^*,0)\in S^{p+q+r+1}$, and $\eta^*\in S^q$ with $(0,0,\eta^*)\in S^{p+q+r+1}$. An arbitrary point (t^*,ξ^*,η^*) of $S^{p+q+r+1}$ can be uniquely expressed in the form

$$(t^*,\xi^*,\eta^*)=(\bar t^*\cos\theta,\ \bar\xi^*\cos\theta,\ \bar\eta^*\sin\theta),$$

where $(\bar t^*,\bar\xi^*)\in S^{p+r}$, $\bar\eta^*\in S^q$, and $0\leqslant\theta\leqslant\pi/2$. Indeed, we have $\sin\theta=|\eta^*|$ and $\cos\theta=\sqrt{1-|\eta^*|^2}$.

Let S^{m+n+1} be described as in § 2. Then $J_gI_\alpha(\gamma)$ is represented by the mapping φ_2: $S^{p+q+r+1}\to S^{m+n+1}$, given by the equality

$$\varphi_2(\bar t^*\cos\theta,\ \bar\xi^*\cos\theta,\bar\eta^*\sin\theta)=\varphi_1(\bar t^*,\bar\xi^*)\cos\theta+g(\bar\eta^*)\sin\theta.$$

Hence for each $(t^*,\xi^*,\eta^*)\in S^{p+q+r+1}$, we have

$$\varphi_2(t^*,\xi^*,\eta^*)=\begin{cases}\sqrt{1-|\eta^*|^2}\,\varphi\left(\dfrac{t^*}{|t^*|},\lambda_p\left(\dfrac{\xi^*}{|t^*|}\right)\right)+|\eta^*|\,g\left(\dfrac{\eta^*}{|\eta^*|}\right), & \text{if}\quad |t^*|\geqslant|\xi^*|,\\ \sqrt{1-|\eta^*|^2}\,u_*+|\eta^*|\,g\left(\dfrac{\eta^*}{|\eta^*|}\right), & \text{if}\quad |t^*|\leqslant|\xi^*|.\end{cases}$$

Next, let S^{p+q+1} be described in § 2. Then $J_g(\gamma)$ is represented by a mapping φ_3: $S^r\times S^{p+q+1}\to S^{m+n+1}$ such that, for each pair

$$z=x\cos\theta+y\sin\theta\in S^{p+q+1},\quad t\in S^r$$

we have

$$\varphi_3(t,x\cos\theta+y\sin\theta)=\varphi(t,x)\cos\theta+g(y)\sin\theta.$$

Hence for an arbitrary $(x,y)=(x_0,\ldots,x_p,y_0,\ldots,y_q)$ of S^{p+q+1} and an arbitrary $t=(t_0,\ldots,t_r)$ of S^r, we have

$$\varphi_3(t,x,y)=|x|\,\varphi\left(t,\frac{x}{|x|}\right)+|y|\,g\left(\frac{y}{|y|}\right).$$

Let E^{p+q+1} be the $(p+q+1)$-element, in the $(p+q+1)$-space with the coordinates system $\xi_1,\dots,\xi_p,\eta_0,\eta_1,\dots,\eta_q$, defined by $|\xi|^2+|\eta|^2\leqslant 1$, and S^{p+q} be the boundary sphere of E^{p+q+1}. Let λ_{p+q+1} denote the mapping of E^{p+q+1} onto S^{p+q+1} given by

$$\begin{aligned} x_0&=2(|\xi|^2+|\eta|^2)-1, & \\ x_i&=2\sqrt{1-|\xi|^2-|\eta|^2}\,\xi_i, & (i=1,2,\dots,p),\\ y_i&=2\sqrt{1-|\xi|^2-|\eta|^2}\,\eta_i, & (i=0,1,\dots,q).\end{aligned}$$

Let

$$\Sigma^{p+q+r+1}=(S^r\times E^{p+q+1})\cup(E^{r+1}\times S^{p+q})$$

and let $w_*=(u_*,0)\in S^{m+n+1}$. Let $\varphi_4\colon\ \Sigma^{p+q+r+1}\to S^{m+n+1}$ be the mapping defined by

$$\varphi_4(t,\zeta)=\begin{cases}\varphi_3(t,\lambda_{p+q+1}\zeta), & \text{if}\quad t\in S^r,\ \zeta\in E^{p+q+1},\\ w_*, & \text{if}\quad t\in E^{r+1},\ \zeta\in S^{p+q}.\end{cases}$$

Let $S^{p+q+r+1}$ be described as above, and let K_1,K_2 denote the subsets of $S^{p+q+r+1}$ defined by the conditions

$$|t^*|^2\geqslant|\xi^*|^2+|\eta^*|^2,\qquad |t^*|^2\leqslant|\xi^*|^2+|\eta^*|^2$$

respectively. Let $\mu_{r,p+q+1}$ denote the homeomorphism of $S^{p+q+r+1}$ onto $\Sigma^{p+q+r+1}$ defined by

$$\mu_{r,p+q+1}(t^*,\xi^*,\eta^*)=\begin{cases}\left(\dfrac{t^*}{|t^*|},\dfrac{\xi^*}{|t^*|},\dfrac{\eta^*}{|t^*|}\right), & \text{if}\quad (t^*,\xi^*,\eta^*)\in K_1,\\ \left(\dfrac{t^*}{\sqrt{1-|t^*|^2}},\dfrac{\xi^*}{\sqrt{1-|t^*|^2}},\dfrac{\eta^*}{\sqrt{1-|t^*|^2}}\right), & \text{if}\quad (t^*,\xi^*,\eta^*)\in K_2.\end{cases}$$

Then $I_\delta J_g(\gamma)$ is represented by the mapping $\varphi_5=\varphi_4\mu_{r,p+q+1}$. Hence for each $(t^*,\xi^*,\eta^*)\in S^{p+q+r+1}$, we have

$$\varphi_5(t^*,\xi^*,\eta^*)=\begin{cases}\dfrac{|\xi^*|}{|t^*|}\sqrt{2|t^*|^2-1}\,\varphi\left(\dfrac{t^*}{|t^*|},\dfrac{\xi^*}{|\xi^*|}\right)+\dfrac{|\eta^*|}{|t^*|}\sqrt{2|t^*|^2-1}\,g\left(\dfrac{\eta^*}{|\eta^*|}\right),\\ \qquad\qquad\qquad\qquad\qquad\qquad\text{if}\quad |t^*|^2\geqslant|\xi^*|^2+|\eta^*|^2;\\ w_*, \qquad\qquad\qquad\qquad\qquad\text{if}\quad |t^*|^2\leqslant|\xi^*|^2+|\eta^*|^2.\end{cases}$$

Comparing φ_2 and φ_5, we conclude that they are nowhere antipodal and hence homotopic. Then it follows that $J_g I_\alpha(\gamma)=I_\delta J_g(\gamma)$. Q. E. D.

Theorem 5.2. *If $\gamma\in\pi^{p+r}(S^m)$ is a Whitehead product, then $J_g(\gamma)=0$.*

Proof. This is an immediate consequence of (4.8), (5.1), and a theorem of G. W. Whitehead, [5, (3.2)], as indicated by the following diagram:

$$\begin{tikzcd} \pi^{r+1}(G_\alpha,F_\alpha) \arrow[r, "\partial"] \arrow[d, leftrightarrow, "\tau_\alpha"] & \pi^r(F_\alpha) \arrow[r, "J_g"] \arrow[d, leftrightarrow, "I_\alpha"] & \pi^r(\Delta_\delta) \arrow[d, leftrightarrow, "I_\delta"] \\ \pi^{r+1}(S^m) \arrow[r, "\varrho_\alpha"] & \pi^{p+r}(S^m) \arrow[r, "J_g"] & \pi^{p+q+r+1}(S^{m+n+1}). \end{tikzcd}$$

If g is the identity mapping of S^0, then our theorem reduces to the Theorem (3.11) of G. W. Whitehead, [5], as a special case.

Corollary 5.3. *If one or both of the elements $\alpha\in\pi^p(S^m)$, $\beta\in\pi^q(S^n)$ are Whitehead products, then $\alpha^\vee\beta=0$.*

6. Associative law for joins. ***Theorem 6.1.*** *For arbitrary $f\in\{p,m\}$, $g\in\{q,n\}$, $h\in\{r,t\}$, we always have $f^\vee(g^\vee h)=(f^\vee g)^\vee h$.*

Proof. Let $S^{p+q+r+2}$ be the unit sphere in the space with coordinates system

$$(x,y,z)=(x_0,\dots,x_p,y_0,\dots,y_q,z_0,\dots,z_r),$$

and $S^{m+n+t+2}$ be the unit sphere in the space with coordinates system

$$(u,v,w)=(u_0,\dots,u_m,v_0,\dots,v_n,w_0,\dots,w_t).$$

Then it follows from a simple calculation that both $f^\vee(g^\vee h)$ and $(f^\vee g)^\vee h$ are identical with the mapping $\varphi\colon\ S^{p+q+r+2}\to S^{m+n+t+2}$ given by

$$\varphi(x,y,z)=|x|\,f\left(\frac{x}{|x|}\right)+|y|\,g\left(\frac{y}{|y|}\right)+|z|\,h\left(\frac{z}{|z|}\right).$$

An immediate consequence of (6.1) is:

Theorem 6.2. *The joining operation J defines an associative multiplication in the family of homotopy groups of spheres,* i. e.

$$(\alpha^\vee\beta)^\vee\gamma=\alpha^\vee(\beta^\vee\gamma)=\alpha^\vee\beta^\vee\gamma.$$

Let $E^{(n+1)}$ denote the $(n+1)$-fold Einhängung of H. Freudenthal. Then we have the

Theorem 6.3. *If g is a mapping of S^n onto itself with degree b, then for each $a \in \pi^p(S^m)$ we always have*

(6.31) $$J_g(a) = bE^{(n+1)}(a).$$

(6.32) $${}_gJ(a) = (-1)^{(n+1)(m+p)} bE^{(n+1)}(a).$$

Proof. Let g_0 denote the identity mapping of S^n. Because of (4.3), we may suppose $n > 0$. Let g_0, g represent $\beta_0, \beta \in \pi^n(S^n)$ respectively, then $\beta = b\beta_0$. Hence we have $J_g(a) = a^\vee \beta = b(a^\vee \beta_0)$. Let a be represented by f: $S^p \to S^m$, then $a^\vee \beta_0$ is represented by $f^\vee g_0 = f^\vee \psi_0^\vee \psi_1^\vee \dots ^\vee \psi_n$, where ψ_i, $(i = 0, 1, \dots, n)$, denotes the identity mapping of S^0. Hence, by (4.31), $f^\vee g_0$ represents $E^{(n+1)}(a)$, and (6.31) is proved. (6.32) can be proved by successive use of (4.32).

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Squares are normal.

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1. Introduction. Two plane sets are *finitely equivalent* if and only if they can be split respectively into sets $\alpha_1, \alpha_2, \dots, \alpha_m$ and into sets $\alpha_1', \alpha_2', \dots, \alpha_m'$ in such a way that the corresponding subdivisions are congruent. A plane set S is *paradoxical* if and only if it can be split into two sets each of which is finitely equivalent to S. A plane set which is not paradoxical is *normal*. It has been known [1]) for some time that squares and a variety of other plane sets are normal. However, all known verifications of the normality of squares so far published depend in an essential way on the axiom of choice. By making use of appropriate known devices for establishing the existence of certain linear functionals we find it is indeed possible to show, without the axiom of choice, that any bounded plane set with inner points is normal.

Nowhere in the sequel directly or indirectly do we employ the axiom of choice.

If G is a group then those members of G of the form $bab^{-1}a^{-1}$ are *commutators*; the smallest subgroup of G containing the set of all commutators is the *commutator subgroup* of G.

If by forming successive commutator subgroups of G we reach, in a finite number of steps, the subgroup consisting of the identity then G is a *solvable* group.

[1]) S. Banach and A. Tarski, *Sur la décomposition des ensembles de points en parties respectivement congruentes*, Fund. Math. **6** (1924), pp. 244-277. See also the abstract of a paper of. Z. Waraszkiewicz, *Sur l'équivalence de deux carrés*, Ann. Soc. Polon. Math. **19** (1947), p. 239 (meeting of the Society of Oct. 19, 1945).