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Remarks on some topological spaces of high power.

By

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The subject of this paper [1]) is the study of those topological spaces (called *ω_μ-additive spaces*) [2]) which satisfy the following axioms:

I. For every α-sequence [3]) of sets $\{X_\xi\}$, $\alpha < \omega_\mu$,

$$\overline{\sum_{0 \leqslant \xi < \alpha} X_\xi} = \sum_{0 \leqslant \xi < \alpha} \overline{X}_\xi.$$

II. $\overline{X} = X$ for every finite set X.

III. $\overline{X} = \overline{\overline{X}}$.

If $\mu = 0$, axioms I-III coincide with the well-known axioms of Kuratowski [4]). If $\mu > 0$, axiom I is stronger than the first axiom of Kuratowski.

It will be shown that in the case $\mu > 0$ it is convenient to modify some topological notions and definitions. The idea of the modification is that the words: „an enumerable sequence", „a finite set", „an enumerable set" should be replaced by „an ω_μ-sequence", „a set of a potency $< \aleph_\mu$", „a set of the power $\aleph_\mu$" respectively. After this modification many topological theorems on separable metric spaces holds also for ω_μ-additive spaces whose power, in general, is $\geqslant \aleph_\mu$.

It is not the purpose of this paper to specify all topological theorems which can be generalized in the above-mentioned way. Only the direction of the generalization will be shown and some singularities which appear in connection with the notion of compactness and of completeness will be discussed. The final section contains an application to the theory of Boolean algebras.

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[1]) Presented at the Mathematical Congress in Wrocław on December 14, 1946.

[2]) ω_μ always denotes an initial ordinal (i. e. ω_μ is the least ordinal such that the set of all ordinals $\xi < \omega_\mu$ is of the power $\aleph_\mu$).

[3]) For brevity's sake we say „an α-sequence" instead of „a (transfinite) sequence of the type α".

[4]) Kuratowski [1], p. 20.

In the rest of this paper it is always supposed that μ is *a* fixed ordinal such that ω_μ is regular [5]). All spaces considered in this paper are ω_μ-additive. In §§ 1-4 we assume that $\mu>0$.

§ 1. The modification of topological notions. Since axioms I-III imply Kuratowski's axioms, all theorems given in Kuratowski [1] Chapter I, hold for arbitrary ω_μ-additive spaces. Certain theorems can be expressed in a stronger form. E. g. ($\mathfrak{X}$ denotes an ω_μ-additive space):

(i) *If $\alpha<\omega_\mu$, the sum (product) of any α-sequence of closed (open) subsets of $\mathfrak{X}$ is closed (open).*

In particular (since $\mu>0$),

(i') *The sum (product) of any enumerable sequence of closed (open) subsets of $\mathfrak{X}$ is closed (open).*

We obtain from (i) that

(ii) *Every subset of power $<\aleph_\mu$ is isolated and closed.*

It follows from (ii) that ω_μ-additive spaces of power $<\aleph_\mu$ are not interesting. If a set X is dense in an ω_μ-additive space of power $\geqslant\aleph_\mu$, then X is also of power $\geqslant\aleph_\mu$. This fact shows that the question whether there exists an enumerable dense subset in an ω_μ-additive space has no topological sense [6]). On the other hand, the question whether there exists a dense subset of power $\leqslant\aleph_\mu$ can be interesting from the topological point of view. Therefore the definition of the separability should be modified if we want this notion to keep its topological contents. Consequently we assume the following definition: an ω_μ-additive space is said to be *separable* provided it contains a dense subset of power $\leqslant\aleph_\mu$.

For the same reason [7]) we must modify the notion of a basis of a space in the following way: a *basis* of an ω_μ-additive space $\mathfrak{X}$ is an ω_μ-sequence $\{G_\xi\}$ of open sets such that every open subset of $\mathfrak{X}$ is the sum of a subsequence of this sequence.

Analogously we must modify the notion of an $\boldsymbol{F}_\sigma$ and a $\boldsymbol{G}_\delta$ on account of (i'). By (i) it is convenient to assume the following definition: a subset X of an ω_μ-additive space is said to be an $\boldsymbol{F}_\sigma$ (a $\boldsymbol{G}_\delta$) provided that X is the sum (product) of an ω_μ-sequence of closed (open) sets.

The examples discussed above show exactly the direction of the modification of the topological notions: the words „a finite set", „an enumerable sequence" should be systematically replaced by „a set of power $<\aleph_\mu$", „an ω_μ-sequence" etc. E. g. the class of all Borel subsets [8]) of an ω_μ-additive space $\mathfrak{X}$ should be defined as the least class $\mathfrak{B}$ such that: 1° every closed set belongs to $\mathfrak{B}$; 2° if $X\in\mathfrak{B}$, then $\mathfrak{X}-X\in\mathfrak{B}$ also; 3° the sum of any ω_μ-sequence of sets belonging to $\mathfrak{B}$ belongs also to $\mathfrak{B}$.

Some difficulties arise in connexion with the notion of the first category. In accordance with the above-accepted direction of the modification, a subset X of an ω_μ-additive space $\mathfrak{X}$ is said to be of the first category if $\mathfrak{X}$ is the sum of an ω_μ-sequence of nowhere dense sets. The analogy with the case $\mu=0$ is not complete, since the sum of an α-sequence ($\omega\leqslant\alpha<\omega_\mu$) of nowhere dense sets is not, in general, nowhere dense [9]).

§ 2. Properties of ω_μ-additive spaces.

(iii) *If $\{G_n\}$ is an ω-sequence of open subsets of an ω_μ-additive space such that $\overline{G}_{n+1}\subset G_n$ $(n=1,2,...)$, then the product $G_1G_2G_3...$ is both open and closed.*

This follows from (i') and from the equation $G_1G_2G_3...=\overline{G}_1\overline{G}_2\overline{G}_3...$

A topological space is called *0-dimensional* if, for every open set G and for every element $p\in G$, there exists a set H which is simultaneously open and closed and such that $p\in H\subset G$.

(iv) *Every regular ω_μ-additive space is 0-dimensional.*

[5]) I. e. $\sum_{\xi<\alpha}\mathfrak{m}_\xi<\aleph_\mu$ for every α-sequence ($\alpha<\omega_\mu$) of cardinals $\mathfrak{m}_\xi<\aleph_\mu$.

If a space $\mathfrak{X}$ is ω_μ-additive, where ω_μ is not regular, $\mathfrak{X}$ is also $\omega_{\mu+1}$-additive. Since $\omega_{\mu+1}$ is regular, the above assumption does not restrict the generality of our consideration.

[6]) More generally: the question whether there exists a dense subset of power $<\aleph_\mu$. In fact, such a subset exists if and only if the space is of power $<\aleph_\mu$ (see (ii)).

[7]) For if an ω_μ-additive space $\mathfrak{X}$ possesses an enumerable basis (or, more generally, a basis of power $<\aleph_\mu$) every one-point set $(x)\subset\mathfrak{X}$ is open on account of (i) since (x) is the product of all basic sets which contain the point x. Thus $\mathfrak{X}$ is isolated.

[8]) The theory of Borel sets in an ω_μ-metric space (see § 3) can be developed analogously to the case $\mu=0$.

[9]) See theorem (xiii). On the other hand, there exist ω_μ-additive spaces such that the sum of any α-sequence ($\alpha<\omega_\mu$) of nowhere dense sets is nowhere dense. See theorem (xvi).

In fact, we can easily define an ω-sequence $\{G_n\}$ such that $p \in G_{n+1} \subset \overline{G}_{n+1} \subset G_n \subset G$. The product H of all sets G_n is both open and closed by (iii), and $p \in H \subset G$.

In an analogous way we can prove that

(v) *If F_1 and F_2 are two disjoint closed subsets of an ω_μ-additive normal space, then there exists a set H such that $F_1 \subset H$, $F_2 H = 0$, and H is simultaneously open and closed.*

It follows from (iv) that the only connected subsets of an ω_μ-additive regular space are one-point sets and the empty set.

It follows from (v) that

(vi) *Every ω_μ-additive normal space with a basis possesses a basis composed of sets which are both open and closed*[10].

Analogously as in the case $\mu = 0$ we can prove that

(vii) *Every regular ω_μ-additive space with a basis is normal.*

§ 3. ω_μ-metric spaces. Let A be an ordered group[11]. We say that A is *of character* ω_μ[12] if there exists a decreasing[13] ω_μ-sequence $\{\varepsilon_\xi\}$ of positive elements of A with the property:

(*) for every positive element $\varepsilon \in A$ there exists an ordinal $\xi_0 < \omega_\mu$ such that $\varepsilon_\xi < \varepsilon$ for every $\xi > \xi_0$ $(\xi < \omega_\mu)$.

In every ordered group A of character ω_μ we can define a limit of an ω_μ-sequence $\{a_\xi\}$ of elements of A. Viz. we say that $\{a_\xi\}$ *converges* to $a \in A$ — in symbols:

$$a = \lim_{\xi < \omega_\mu} a_\xi,$$

if for every positive $\varepsilon \in A$ there exists an ordinal $\xi_0 < \omega_\mu$ such that[14] $|a - a_\xi| < \varepsilon$ for $\xi > \xi_0$ $(\xi < \omega_\mu)$.

[10]) See Kuratowski [1], p. 133.

[11]) I. e. an ordered set in which there is defined a sum of two elements such that: 1° $a + (b + c) = (a + b) + c$; 2° $a + b \leqslant b + c$ if and only if $a \leqslant c$; 3° for every a and b there exists an element c such that $a = b + c$. The symbol 0 denotes the element satisfying the equality $a + 0 = a$. An element $a \in A$ is *positive* if $a > 0$. $|a|$ denotes the greater of the elements a and $-a$.

[12]) Such groups exist. A general method for their construction follows from Hausdorff's definition of the power of ordered sets. See Hausdorff [1], p. 194. Other examples of groups with the character ω_μ are ordered algebraic fields of the character ω_μ. See Sikorski [1], pp. 73-76.

[13]) I. e. $0 < \varepsilon_\xi < \varepsilon_\eta$ for $\eta < \xi$.

[14]) E. g. if $\{\varepsilon_\xi\}$ possesses the property (*), we have $\lim_{\xi < \omega_\mu} (a + \varepsilon_\xi) = a$. Thus there are non-trivial convergent ω_μ-sequences.

Suppose now that with every pair $\{p, q\}$ of elements of a set $\mathfrak{X}$ there is associated an element $\varrho(p,q)$ of an ordered group A such that

a) $\varrho(p,p) = 0$; $\varrho(p,q) > 0$ if $p \neq q$.

b) $\varrho(p,q) \leqslant \varrho(p,r) + \varrho(q,r)$.

The function $\varrho(p,q)$ satisfying a) and b) is called *a metric of* $\mathfrak{X}$. The set $\mathfrak{X}$ with the metric ϱ is called an *ω_μ-metric space*[15].

We say that an ω_μ-sequence $\{p_\xi\}$ of elements of $\mathfrak{X}$ *converges* to a point $p \in \mathfrak{X}$ (in symbols: $p = \lim_{\xi < \omega_\mu} p_\xi$) if $\lim_{\xi < \omega_\mu} \varrho(p, p_\xi) = 0$.

The closure $\overline{X}$ of a set $X \subset \mathfrak{X}$ is the set of all limits of ω_μ-sequences whose elements belong to X.

The ω_μ-metric spaces defined above possess many properties of metric spaces. In particular:

(viii) *Every ω_μ-metric space is an ω_μ-additive normal space.*

(ix) *An ω_μ-metric space possesses a basis if and only if it is separable.*

The question arises whether Urysohn's metrization theorem can be generalized to the case $\mu > 0$. The answer is affirmative: every regular ω_μ-additive space with a basis is metrizable by the ordered algebraic field W_μ (theorem (x)). W_μ denotes here the least algebraic field containing the set P_μ of all ordinals $\xi < \omega_\mu$[16]. (The only algebraic operations in P_μ are the so-called „natural sum" and „natural product" of Hessenberg)[17].

Let $\mathfrak{D}_\mu$ denote the set of all ω_μ-sequences whose elements are the numbers 0 and 1[18]. For every two elements $p = \{a_\eta\}$ and $q = \{b_\eta\}$ of $\mathfrak{D}_\mu$ let

$$\varrho(p,q) = 0 \in W_\mu \quad \text{if} \quad p = q \quad \text{and} \quad \varrho(p,q) = \frac{1}{\eta_0} \in W_\mu \quad \text{if} \quad p \neq q;$$

η_0 denotes here the least ordinal such that $a_{\eta_0} \neq b_{\eta_0}$.

It is easy to see that $\varrho(p,q)$ fulfils the conditions a) and b). Thus $\mathfrak{D}_\mu$ is an ω_μ-metric space.

[15]) For example, every ordered group A (of the character ω_μ) with the metric $\varrho(a,b) = |a - b| \in A$ is an ω_μ-metric space.

ω_μ-metric spaces were considered by Hausdorff in [1], p. 285-286. See also Cohen and Goffman [1] and [2].

[16]) The exact definition of W_μ is given in Sikorski [1], p. 82. Every element $w \in W$ can be represented in the form $w = (\alpha - \beta)/(\gamma - \delta)$ where $\alpha, \beta, \gamma, \delta \in P_\mu$, $\gamma \neq \delta$.

[17]) See Hausdorff [2], p. 68.

[18]) $\mathfrak{D}_\mu$ is a generalization of Cantor's discontinuous set.

(x) *Every regular ω_μ-additive space $\mathfrak{X}$ with a basis is homeomorphic to a subset of $\mathfrak{D}_\mu$.*

On account of (vi) and (vii) there exists a basis $\{G_\xi\}$ of $\mathfrak{X}$ whose elements are both open and closed. The formula

$$h(x)=\{a_\eta\}\in\mathfrak{D}_\mu$$

where

$$a_\eta=0 \quad \text{if} \quad x \text{ non}\in G_\eta \quad \text{and} \quad a_\eta=1 \quad \text{if} \quad x\in G_\eta$$

defines the homeomorphism of $\mathfrak{X}$ in $\mathfrak{D}_\mu$.

The exact proof is the same as the proof of the analogous theorem for 0-dimensional spaces in the case $\mu=0$ [19]).

The problem whether the space $\mathfrak{D}_\mu$ possesses a basis is equivalent (see theorem (ix)) to the hypothesis: $2^{\aleph_\nu}\leqslant\aleph_\mu$ for $\nu<\mu$. In fact, the set $D_{\mu,\alpha}$ of all sequences $\{a_\eta\}\in\mathfrak{D}_\mu$ such that $a_\eta=0$ for $\alpha\leqslant\eta<\omega_\mu$, is an isolated subset of $\mathfrak{D}_\mu$ of the power $2^{\overline{\alpha}}$. The sum $\sum\limits_{0\leqslant\alpha<\omega_\mu} D_{\mu,\alpha}$ is dense in $\mathfrak{D}_\mu$.

§ 4. Compact and complete spaces.

An ω_μ-metric space $\mathfrak{X}$ (with $\varrho(p,q)\in A$) is called

compact, if every ω_μ-sequence $\{p_\xi\}$ of points of $\mathfrak{X}$ contains an ω_μ-subsequence $\{p_{\eta_\xi}\}$ convergent to a point $p\in\mathfrak{X}$;

complete, if every ω_μ-sequence $\{p_\xi\}$ satisfying Cauchy's condition [20]) converges to a point $p\in\mathfrak{X}$;

totally bounded, if for every positive element $\varepsilon\in A$ there exists a decomposition $\mathfrak{X}=\sum\limits_{0\leqslant\xi<\alpha} X_\xi$, where $\alpha<\omega_\mu$, and all sets X_ξ are of diameter [21]) $\leqslant\varepsilon$.

Cantor's theorems on intersections of closed sets hold also for $\mu>0$. Namely:

(xi) *If $\{F_\xi\}$ is a decreasing ω_μ-sequence of non-empty closed subsets of an ω_μ-metric compact space, then* $\prod\limits_{0\leqslant\xi<\omega_\mu} F_\xi\neq 0$ [22]).

[19]) See Kuratowski [1], p. 173-174.

[20]) I. e. for every positive $\varepsilon\in A$ there exists an ordinal $\xi_0>\omega_\mu$ such that $\varrho(p_\xi,p_\eta)<\varepsilon$ for $\xi,\eta\geqslant\xi_0$.

[21]) We say that a set $X\subset\mathfrak{X}$ is *of diameter* $\leqslant\varepsilon$ if $\varrho(p,q)\leqslant\varepsilon$ for every $p,q\in X$.

[22]) The intersection of a decreasing α-sequence (where $\alpha<\omega_\mu$ is a limit number) of closed sets can be, however, empty. Example: $\{p_\eta\}$ is an α-sequence of different points of $\mathfrak{X}$, and F_ξ is the set of all points p_μ with $\eta>\xi$.

(xii) *If $\{F_\xi\}$ is a decreasing ω_μ-sequence of non-empty closed subsets of an ω_μ-metric complete space such that the diameter of F_ξ converges [23]) to 0, then* $\prod\limits_{0\leqslant\xi<\omega_\mu} F_\xi\neq 0$.

The proof of these theorems is the same as for $\mu=0$.

In the case $\mu=0$ theorem (xiii) implies the well-known theorem of Baire that in every complete space the complement of a set of the first category is dense [24]). This implication does not hold for $\mu>0$; Baire's theorem cannot be generalized (theorem (xiii)).

Let D_μ^0 denote the set of all sequences $\{a_\eta\}\in\mathfrak{D}_\mu$ such that the equality $a_\eta=1$ holds only for a finite number of ordinals $\eta<\omega_\mu$.

(xiii) *D_μ^0 is a dense in itself, compact, ω_μ-metric space. D_μ^0 is the sum of an enumerable sequence of nowhere dense sets.*

Let $\{p_\xi\}$ be an ω_μ-sequence of elements of D_μ^0. Since $\mu>0$ and ω_μ is regular, there exists an integer n and an ω_μ-*subsequence* $p_{\lambda_\xi}=\{a_\eta^\xi\}$ such that the equality $a_\eta^\xi=1$ holds for n ordinal numbers $\eta_\xi^1<\eta_\xi^2<\ldots<\eta_\xi^n$.

Let k denote the least integer such that the ω_μ-sequence $\{\eta_\xi^k\}$ contains an increasing ω_μ-subsequence. If all sequences are bounded, let $k=n+1$. By the definition of k, there exists an increasing ω_μ-sequence $\{\tau_\xi\}$ of ordinals $<\omega_\mu$ such that

1⁰ $\eta_{\tau_\xi}^i=\eta^i$ for $1\leqslant i<k$, where η^i does not depend on ξ.

2⁰ if $k\leqslant n$, the sequence $\eta_{\tau_\xi}^k$ is increasing.

The ω_μ-sequence $\{p_{\lambda_{\tau_\xi}}\}$ converges to the element $p=\{a_\eta\}\in D_\mu$ where $a_\eta=1$ if $\eta=\eta^i$ $(1\leqslant i<k)$ and $a_\eta=0$ for all remained $\eta<\omega_\mu$.

Thus the space D_μ^0 is compact. It is clear that D_μ^0 is of potency $\aleph_\mu$.

Let now D_n denote the set of all $p=\{a_\eta\}\in D_\mu^0$ such that the equality $a_\eta=1$ holds for at most n different ordinals η. Obviously $D_\mu^0=D_1+D_2+D_3+\ldots$

It is easy to see that D_n is closed in D_μ^0. Every point $p=\{a_\eta\}\in D_n$ is the limit of an ω_μ-sequence $\{p_\xi\}\in D_\mu^0-D_n$ (i. e. D_n is nowhere dense in D_μ^0). In fact, let $\eta_1<\eta_2<\ldots<\eta_k$ be all ordinals such that

[23]) I. e. for every positive element $\varepsilon\in A$ there exists an ordinal $\xi_0<\omega_\mu$ such that F_ξ is of diameter $\leqslant\varepsilon$ for every $\xi>\xi_0$.

[24]) See Kuratowski [1], p. 320.

$a_{\eta_i}=1$ $(1\leqslant i\leqslant k,\ k\leqslant n)$ and let $p_\xi=\{a_\eta^\xi\}$, where

$$a_\eta^\xi=1 \quad \text{for} \quad \eta=\eta_i \quad (1\leqslant i\leqslant k),$$

$$a_\eta^\xi=1 \quad \text{for} \quad \eta=\eta_k+\xi+i \quad (1\leqslant i\leqslant n+1),$$

$$a_\eta^\xi=0 \quad \text{for all remaining ordinals} \quad \eta<\omega_\mu.$$

By this definition p_ξ non $\epsilon\, D_n$ and $\lim\limits_{\xi<\omega_\mu} p_\xi=p$.

Since $p_\xi\neq p$, we also infer that D_μ^0 is dense in itself. Theorem (xiv) is proved.

In the case $\mu=0$ every dense in itself, compact, metric space is of the power $2^{\aleph_0}$. One might think that, analogously, every dense in itself, compact ω_μ-metric space should be of the power $2^{\aleph_\mu}$. Theorem (xiv) shows that this hypothesis is false. I know no example of an ω_μ-metric compact space [25]) of power $>\aleph_\mu$.

(xiv) *The space $\mathfrak{D}_\mu$ is complete.*

Let $p_\xi=\{a_\eta^\xi\}$ be any ω_μ-sequence (of points of $\mathfrak{D}_\mu$) satisfying Cauchy's condition. For every $\xi<\omega_\mu$ let $a_{\xi'}$ denote an ordinal such that $|p_{\xi'}-p_{\xi''}|<1/\xi$ for $a_\xi'\leqslant\xi'<\xi''<\omega_\mu$. The sequence $\{p_\xi\}$ converges to the point $p=\{a_\eta\}$, where $a_\eta=a_\eta^{\alpha_\eta}$, q. e. d.

If $\aleph_\mu$ is not an inaccessible aleph (in the strict sense) [26]), $\mathfrak{D}_\mu$ is not compact. In fact, the space $\mathfrak{D}_\mu$ contains then an isolated subset of power $\geqslant\aleph_\mu$ (see the remarks at the end of § 3).

If $\aleph_\mu$ is an inaccessible aleph (in the strict sense), then $\mathfrak{D}_\mu$ is complete and totally bounded. The question whether $\mathfrak{D}_\mu$ is then compact, is unsolved.

The space $\mathfrak{D}_\mu$ fulfils Baire's theorem. Namely:

(xv) *Let $\{X_\xi\}$ be an ω_μ-sequence of nowhere dense subsets of $\mathfrak{D}_\mu$. For every $\alpha<\omega_\mu$ the set $\sum\limits_{0\leqslant\xi<\alpha} X_\xi$ is nowhere dense. The set $D-\sum\limits_{0\leqslant\xi<\omega_\mu} X_\xi$ is dense in $\mathfrak{D}_\mu$.*

For every β-sequence $\mathfrak{b}=\{b_\eta\}$ $(0<\beta<\omega_\mu)$ whose elements are the numbers 0 and 1 let $D(\mathfrak{b})$ denote the set of all $p=\{a_\eta\}\,\epsilon\,\mathfrak{D}_\mu$ such that $a_\eta=b_\eta$ for $0\leqslant\eta<\beta$. The set $D(\mathfrak{b})$ is both open and closed in $\mathfrak{D}_\mu$.

[25]) Other instances of compact ω_μ-metric spaces are bounded closed subsets of W_μ. Such sets are of power $\leqslant\aleph_\mu$. See Sikorski [1], p. 84 and p. 87.

[26]) See Tarski [1], p. 69.

Let G be any open non-empty subset of D_μ. It is easy to define an increasing ω_μ-sequence $\{\beta_\xi\}$ of ordinals $<\omega_\mu$ and an ω_μ-sequence $\{\mathfrak{b}^\xi\}$ such that

(a) $\mathfrak{b}^\xi$ is an β_ξ-sequence whose terms are the numbers 0 and 1;

(b) $D(\mathfrak{b}^{\xi_2})\subset D(\mathfrak{b}^{\xi_1})$ for $\xi_1<\xi_2$ (i. e. $\mathfrak{b}^{\xi_1}$ is the ξ_1-segment [27]) of $\mathfrak{b}^{\xi_2}$);

(c) $D(\mathfrak{b}^\alpha)\subset G-\sum\limits_{0\leqslant\xi<\alpha} X_\xi$ for $0\leqslant\alpha\leqslant\omega_\mu$.

By (xii) and (xiv), $\prod\limits_{0\leqslant\alpha<\omega_\mu} D(\mathfrak{b}^\alpha)\neq 0$, i. e.

(d) $G-\sum\limits_{0\leqslant\xi<\omega_\mu} X_\xi\neq 0$.

G being an arbitrary open set, we infer from (c) that $\sum\limits_{0\leqslant\xi<\alpha} X_\xi$ is nowhere dense for every $\alpha<\omega_\mu$. Analogously we infer from (d) that $\mathfrak{D}_\mu-\sum\limits_{0\leqslant\xi<\omega_\mu} X_\xi$ is dense in $\mathfrak{D}_\mu$, q. e. d.

Analogously to the case $\mu=0$ it can be proved:

(xvi) *Every compact ω_μ-metric space is complete and totally bounded.*

In the case $\mu=0$ the converse theorem is also true. This does not hold for $\mu>0$. Namely:

(xvii) *If $\mu=\nu+1$, if ω_ν is regular, and if $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ for $\alpha<\nu$, then the ω_μ-metric space $\mathfrak{D}_\mu$ contains a closed isolated and totally bounded subset D of the power $\aleph_\mu$.*

The ω_μ-metric space D is thus complete and totally bounded, but not compact.

As Specker [28]) has proved, if the conditions mentioned above are satisfied [29]), then there exists an ω_μ-sequence of sets $\{S_\xi\}$ such that:

(a) elements of S_ξ are ξ-sequences whose terms are the numbers 0 and 1;

(b) $\overline{\overline{S}}_\xi=\aleph_\nu$ $(0<\xi<\omega_\mu)$;

(c) if $0<\xi<\eta<\omega_\mu$ and $\mathfrak{b}\,\epsilon\,S_\eta$, the ξ-segment of $\mathfrak{b}$ belongs to S_ξ;

(d) there exists no ω_μ-sequence whose ξ-segment belongs to S_ξ for every $\xi<\omega_\mu$.

[27]) An α-sequence $\mathfrak{a}=\{a_\eta\}$ is the *α-segment* of a β-sequence $\mathfrak{b}=\{b_\eta\}$ $(\alpha<\beta)$ if $a_\eta=b_\eta$ for $\eta<\alpha$.

[28]) Specker [1].

[29]) This holds e. g. for $\mu=1$.

Let $p_\xi=\{a_\eta^\xi\}_{\eta<\omega_\mu}$ be a point of $\mathfrak{D}_\mu$ such that the ξ-segment of p_ξ belongs to S_ξ, and $a_\eta^\xi=0$ for $\xi\leqslant\eta<\omega_\mu$. Let D be the set of all points p_ξ $(\xi<\omega_\mu)$.

The ω_μ-sequence $\{p_\xi\}$ contains no convergent ω_μ-subsequence. In fact, suppose $\lim\limits_{\xi<\omega_\mu} p_{\tau_\xi}=p=\{a_\eta\}$ in $\mathfrak{D}_\mu$. We may assume that $\varrho(p,p_{\tau_\xi})<1/\xi$. Then the ξ-segment of p and the ξ-segment of p_{τ_ξ} are identical. Since $\tau_\xi\geqslant\xi$, the ξ-segment of p_{τ_ξ} belongs to S_ξ by (c). Thus the ξ-segment of p belongs to S_ξ for every $\xi<\omega_\mu$, which is impossible on account of (d).

Consequently the set D is closed, isolated, and of the potency $\aleph_\mu$. If $a<\omega_\mu$, we have by (c)

$$D=\sum_{\xi<a}(p_\xi)+\sum_{\mathfrak{b}\in S_a}D\cdot D(\mathfrak{b}).$$

Since $D\cdot D(\mathfrak{b})$ is of diameter $\leqslant 1/a$, the set D is totally bounded on account of (b).

§ 5. An application to the theory of Boolean algebras.

An ω_μ-additive regular space $\mathfrak{X}$ is said to be *ω_μ-bicompact* provided every open covering $\mathfrak{X}=\sum\limits_\tau G_\tau$ contains a subcovering $\mathfrak{X}=\sum\limits_{\xi<a}G_{\tau_\xi}$, where $a<\omega_\mu$.

A Boolean algebra $\boldsymbol{K}$ is called ω_μ-complete[30]) if, for every class $\boldsymbol{C}\subset\boldsymbol{K}$ with $\overline{\overline{\boldsymbol{C}}}<\aleph_\mu$, there exists the sum of all elements $A\in\boldsymbol{C}$.

The following theorem may be considered as a generalization of a well-known theorem of Stone[31]):

(xviii) *Let $\boldsymbol{K}$ be an ω_μ-complete Boolean algebra. The two following conditions are equivalent:*

(a) *$\boldsymbol{K}$ is isomorphic to the field of all both open and closed subsets of an ω_μ-bicompact ω_μ-additive space $\mathfrak{X}$;*

(b) *every ω_μ-additive proper ideal*[32]) *of $\boldsymbol{K}$ is contained in an ω_μ-additive prime ideal of $\boldsymbol{K}$.*

[30]) In accordance with this definition every Boolean algebra is ω-complete. In this section we assume $\mu\geqslant 0$.

[31]) Stone [1], p. 378.

[32]) An ideal $\boldsymbol{I}$ of $\boldsymbol{A}$ is *proper* if $\boldsymbol{I}\neq\boldsymbol{A}$; it is *$\omega_\mu$-additive* if $\sum\limits_{\xi<a}A_\xi\in\boldsymbol{I}$ for every a-sequence $A_\xi\in\boldsymbol{I}$ $(a<\omega_\mu)$.

$(a)\rightarrow(b)$. It is sufficient to prove that the field[33]) $\boldsymbol{F}$ of all both open and closed subsets of an ω_μ-additive ω_μ-bicompact space $\mathfrak{X}$ possesses the property (b). Let $\boldsymbol{I}$ be an ω_μ-additive proper ideal of $\boldsymbol{F}$, and let X_0 be the sum of all sets $X\in\boldsymbol{I}$.

Suppose $X_0=\mathfrak{X}$. The space $\mathfrak{X}$ being ω_μ-bicompact, there is an a-sequence $X_\xi\in\boldsymbol{I}$ $(a<\omega_\mu)$ such that $\mathfrak{X}=X_0=\sum\limits_{\xi<a}X_\xi$. Consequently $\mathfrak{X}\in\boldsymbol{I}$, i. e., in contradiction to our assumption, $\boldsymbol{I}$ is not proper.

Thus we infer there is a point $x_0\in\mathfrak{X}-X_0$. The class $\boldsymbol{I}_0$ of all $X\in\boldsymbol{F}$ such that $x_0\operatorname{non}\in X$ is an ω_μ-additive prime ideal, and $\boldsymbol{I}\subset\boldsymbol{I}_0$.

$(b)\rightarrow(a)$. Suppose $\boldsymbol{K}$ satisfies the condition (b). The construction of the required space $\mathfrak{X}$ is analogous to the well known construction of Stone.

For every $A\in\boldsymbol{K}$ let $h(A)$ be the class of all ω_μ-additive prime ideals $\boldsymbol{I}$ such that $A\operatorname{non}\in\boldsymbol{I}$. The class $\boldsymbol{F}$ of all sets $h(A)$, where $A\in\boldsymbol{K}$ is an ω_μ-additive fields of subsets of the set $\mathfrak{X}=h(E)$ where E is the unit element of $\boldsymbol{K}$. The mapping h is an isomorphism of $\boldsymbol{K}$ on $\boldsymbol{F}$.

Consider $\mathfrak{X}$ as a topological space with $\boldsymbol{F}$ as the class of neighbourhoods. $\boldsymbol{F}$ being ω_μ-additive, the space $\mathfrak{X}$ is also ω_μ-additive.

Suppose $\mathfrak{X}$ is not ω_μ-bicompact, i. e. there is an open covering $\mathfrak{X}=\sum\limits_\tau X_\tau$ such that

$$(*)\quad \mathfrak{X}\neq\sum_{\xi<a}X_{\tau_\xi}\ \text{for every } a\text{-sequence } \{X_{\tau_\xi}\},\ a<\omega_\mu.$$

We may assume that $X_\tau=h(A_\tau)\in\boldsymbol{F}$. The condition (*) implies that the least ω_μ-additive ideal $\boldsymbol{I}$ containing all elements $A_\tau\in\boldsymbol{K}$ is proper. By (b), $\boldsymbol{I}$ is contained in an ω_μ-additive prime ideal $\boldsymbol{I}_0$. Since $A_\tau\in\boldsymbol{I}_0$, we have $\boldsymbol{I}_0\operatorname{non}\in X_\tau=h(A_\tau)$ for every τ. This is impossible, since $\mathfrak{X}=\sum\limits_\tau X_\tau$.

Thus the space $\mathfrak{X}$ is ω_μ-bicompact. Consequently $\boldsymbol{F}$ is the field of all both open and closed subsets of $\mathfrak{X}$, q. e. d.

It is to be remarked that if an ω_μ-complete Boolean algebra $\boldsymbol{K}$ has the property (b), every quotient algebra $\boldsymbol{K}/\boldsymbol{I}$, where $\boldsymbol{I}$ is an

[33]) A field of sets $\boldsymbol{F}$ is said to be *ω_μ-additive* if $\sum\limits_{\xi<a}X_\xi\in\boldsymbol{F}$ for every a-sequence $X_\xi\in\boldsymbol{F}$ $(a<\omega_\mu)$. On account of (i), the field of all both open and closed subsets of an ω_μ-additive space $\mathfrak{X}$ is ω_μ-additive.

ω_μ-additive ideal of $\boldsymbol{K}$, also possesses this property. Consequently every such quotient algebra $\boldsymbol{K}/\boldsymbol{I}$ is isomorphic to an ω_μ-additive field of sets.

An instance of an ω_μ-complete Boolean algebra with the property (b) is the field of all both open and closed subsets of D_μ^0.

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On an Irreducible 2-dimensional Absolute Retract.

By

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In 1934 Mazurkiewicz and the author of the present paper [1]) constructed in the Euclidean 3-dimensional space E_3 an absolute retract [2]) which cannot be split into finite sum of proper subcontinua having the 1-dimensional Betti number vanishing.

The purpose of this paper is to give an example of an absolute retract P_∞ (lying also in E_3) which is a 2-dimensional Cantor-surface [3]), such that every proper 2-dimensional closed subset of it has the infinite 1-dimensional Betti number. In particular P_∞ contains no 2-dimensional proper subset being an absolute retract.

1. Irreducible cuttings. A subcompactum C of the 3-dimensional Euclidean space E_3 is said to be an *irreducible cutting* of E_3 provided that $E-C$ is not connected, but for every closed proper subset A of C the set $E-A$ is connected. Any irreducible cutting of E_3 is a 2-dimensional Cantor-surface.

It is known [4]) that irreducible cuttings of E_3 can be characterized as compacta $C \subset E_3$ such that [5])

(1) $$p^2(C) > 0,$$

(2) $$\text{if} \quad A = \bar{A} \subset C \quad \text{and} \quad A \neq C, \quad \text{then} \quad p^2(A) = 0.$$

[1]) K. Borsuk and S. Mazurkiewicz, *Sur les rétractes absolus indécomposables*, Comptes Rendus de l'Académie des Sciences **199** (Paris, 1934), p. 110-112.

[2]) A subset A of a space M is called a *retract* of M, if there exists a continuous mapping f (called a *retraction*) of M onto A such that $f(x) = x$ for every $x \in A$. A compactum A is said to be an *absolute retract* provided it is a retract of every space $M \supset A$.

[3]) A compact 2-dimensional space is called a *Cantor-surface* if it cannot be disconnected by any subset of dimension 0. See P. Urysohn, *Mémoire sur les multiplicités Cantoriennes*, Fund. Math. **7** (1925), p. 122, 123.

[4]) See P. Alexandroff and H. Hopf, *Topologie I*, Berlin, Springer 1935, p. 391.

[5]) $p^k(C)$ denotes the k-dimensional Betti number of the compactum C.