described for the system H_f, the only change being that to the conditions (i), (ii), (iii) and (iv) we add a condition (v) which ψ' must satisfy:

(v) $\psi'(A(A_1,\dots,A_n)) = +(\psi'(A_1),\dots,\psi'(A_n))$ for all wffs $A_1,\dots,A_n$.

From this description it can be seen that the presence of symbols from the propositional calculus (other than „$\supset$") is *irrelevant* for the characterization of quantifiers which we have given. It is this fact which provides the justification for our having abstracted from these symbols and worked with the system of basic implication.

The Tychonoff Product Theorem Implies the Axiom of Choice.

By

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Recently S. Kakutani has conjectured that the axiom of choice is a consequence of that theorem of Tychonoff [1]) which states that the Cartesian product of compact topological spaces is compact. It is the purpose of this note to show that this conjecture is correct.

1. *Definitions.* We first review the pertinent definitions. A topological space is a set X, together with a family $\mathcal{J}$ of subsets (called *open* subsets), the family $\mathcal{J}$ having as members the void set, X, all finite [2]) intersections and arbitrary unions of members of $\mathcal{J}$. If we adjoin the requirement that complements of finite sets be open, the topological space is a Kuratowski closure space [3]). In the proof which we give the topological spaces constructed are closure spaces.

The space is compact (=bicompact) if each covering of X by members of $\mathcal{J}$ has a finite subcovering. (In particular, the void set A with the topology $\{A\}$, is compact). If, for each member a of a set A, X_a is a set, the product $\mathbf{P}_{a\in A}X_a$ is the set of all functions x on A for which, for each $a \in A$, $x_a \in X_a$. If each X_a has a topology we let $\mathcal{S}$ be the family of all subsets of the Cartesian product which, for some set U open in some X_a, are the set of all x with $x_a \in U$. The product is then topologized by calling a set open if it is the union of finite intersections of members or $\mathcal{S}$.

[1]) Mathematische Annalen, vol. **111** (1935), pp. 762-766.

[2]) In the absence of the axiom of choice it is necessary to define „finite". We agree that a set is finite if it may be ordered so that every non-void subset has both a first and a last element in the ordering. Then the axiom of choice for finite families of sets can be proved. See A. Tarski, Fund. Math. **6** (1924), pp. 49-95, for a full discussion of this and related questions.

[3]) See C. Kuratowski, *Topologie I*, Monogr. Mat. **3** (1933), p. 15.

2. ***Proof of the theorem.*** We now demonstrate the following statement of the axiom of choice:

If for each $a \in A$, X_a is a non-void set, then the Cartesian product $\mathbf{P}_{a \in A} X_a$ is non-void.

We begin by adjoining a single point, say Δ, to each of the sets X_a: Let $Y_a = X_a \cup \{\Delta\}$. We assign a topology for Y_a by defining the void set and complements of finite sets to be open. It is clear that Y_a, with this topology, is compact.

For each $a \in A$, let Z_a be that subset of $\mathbf{P}_{a \in A} Y_a$ consisting of all points whose a-th coordinate lies in X_a. Surely Z_a is closed in $\mathbf{P}_{a \in A} Y_a$ since X_a is closed in Y_a. Moreover, for any finite subset B of A the intersection $\bigcap_{a \in B} Z_a$ is non-void, for, since each X_a is non-void we may by the finite axiom of choice choose $x_a \in X_a$ for $a \in B$, and set $x_a = \Delta$ for $a \in A - B$. Consequently the family of all sets of the form Z_a, for some $a \in A$, is a family of closed subsets of $\mathbf{P}_{a \in A} Y_a$, with the property that the intersection of any finite subfamily is non-void. Hence, since by the Tychonoff Theorem $\mathbf{P}_{a \in A} Y_a$ is compact, the intersection $\bigcap_{a \in A} Z_a$ is non-void. But this intersection is precisely $\mathbf{P}_{a \in A} X_a$, and the axiom of choice is proved.

3. ***Remarks.*** It is of some interest to note, in the various proofs of Tychonoff's theorem, the precise lemmas which require the axiom of choice. In each of the proofs which have been published the axiom of choice is used in the proof of two distinct subsidiary propositions. In what is probably the most illuminating proof [4]), that of J. W. Alexander, these results are:

i) Let $\mathcal{S}$ be the family of subsets of a Cartesian product of compact spaces as defined in Section 1. Then every covering of the product by members of $\mathcal{S}$ has a finite subcovering.

ii) Let $\mathfrak{R}$ be any family of sets with the property: any subfamily which covers the union $\bigcup_{A \in \mathfrak{R}} A$ has itself a finite subfamily which also covers. Then the family $\mathfrak{Q}$ of all finite intersections of members of $\mathfrak{R}$ enjoys the same property.

Proposition i) implies the axiom of choice. Indeed, the above proof uses only i). However, I am unable to discover whether ii) does or does not imply the choice axiom.

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[4]) Proceedings of the National Academy of Sciences **25** (1939), pp. 296-298.

A Paradoxical Theorem.

By

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In this paper the following theorem is proved: B being an uncountable closed subset of the set C of all countable ordinal numbers, let $f(x)$ be a single-valued transformation of B onto $A \subset C$ having the property that $f(x) < x$ for all $x \in B$. Then there exists a countable ordinal number $\alpha \in A$ and an uncountable subset $B^* \subset B$ such that $f(x) = \alpha$ for all $x \in B^*$.

This theorem is used in the first instance to prove Theorem 2, which in a special case gives this paradoxical result: We take away one element s_1 from the given infinite countable set A_0, we add a new infinite countable set A_1 to the remainder, from the set $\bigcup_{\lambda<2} A_\lambda - \bigcup_{\lambda<2} s_\lambda$ we take away one element s_2, add a new infinite countable set A_2 and we continue in this way so that from the set $\bigcup_{\lambda<\alpha} A_\lambda - \bigcup_{\lambda<\alpha} s_\lambda$ (unless it is empty) we take away one element s_α and then we add a new infinite countable set A_α. Then there exists a countable ordinal number ϑ such that the set of all given and added elements is the same as the set of elements taken away i. e. $\bigcup_{\lambda<\vartheta} A_\lambda = \bigcup_{\lambda<\vartheta} s_\lambda$.

In the second instance the theorem mentioned above is used to prove Theorems 3 and 4, in which necessary and sufficient conditions are given for ordered continuum with the Souslin property (i. e. every disjoint system of intervals is countable) to be a linear set. One of these conditions is the existence of a rational dyadic partition of the ordered continuum with the Souslin property (Theorem 3) and the second condition is the existence of a closed dyadic partition (Theorem 4).

Theorem 1. *Let*

(1) $$\beta_0 < \beta_1 < \ldots < \beta_\lambda < \ldots$$

be an increasing sequence of ordinal numbers $\beta_\lambda < \Omega$ such that

(2) $$\lim \beta_\nu = \beta_{\lim \nu}$$