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Weak compactness in Banach spaces

by

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The author was murdered by the Germans during the second world war. The present work was received by the editor in 1941 and has been prepared for print by A. Alexiewicz.

In his monography [4][1]) Banach calls only little attention to the notion of weak compactness. In recent years, however, the importance of this concept for Functional Analysis has been emphasized in several papers, to mention only its role in the theory of stochastic processes. The present paper [2]) deals with a detailed study of the weakly compact subsets of Banach spaces, especially of the separable ones. When considering the compact subsets in Banach spaces, I follow the way indicated by Gelfand [10].

The contents of this paper consist of two parts. In the first part (§ 1 - § 7), necessary and sufficient conditions for weak compactness of subsets of separable Banach spaces are given and the connectedness of the weak convergence with the weak compactness are studied. In the second part (§ 8 - § 10) we consider the general forms of weak compactness in some concrete Banach spaces and we apply the results to the study of weakly completely continuous operations.

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[1]) The numbers in brackets refer to the bibliography at the end of the present paper (p. 93-94).

[2]) the principal results of which were announced without proofs in the note [15].

Terminology and notation.

$\underset{x}{E}\{w(x)\}$ denotes the set of the elements x satisfying the condition $w(x)$.

$\bar{R}$ denotes the closure of the set R in any topological space.

$\|x\|$ denotes the norm of the element x of a Banach space, E^* — the space conjugate to the Banach space E.

$|Q|$ denotes the Lebesgue measure of the set Q.

If $f(x)$ is a function, $f(\cdot)$ will denote this function considered as an element of some class of functions.

$\boldsymbol{C}$ denotes the space of continuous functions $x=x(t)$ in an interval $[a,b]$, with the norm $\|x\|=\max\limits_{a\leqslant t\leqslant b} x(t)$.

$\boldsymbol{c}$ denotes the space of the sequences $x=(x^{(1)},x^{(2)},\dots)$ convergent to $x^{(\infty)}$, with the norm $\|x\|=\sup\limits_{n=1,2,\dots}|x^{(n)}|$.

$\boldsymbol{m}$ denotes the space of bounded sequences $x=(x^{(1)},x^{(2)},\dots)$ with the norm $\|x\|=\sup\limits_{n=1,2,\dots}|x^{(n)}|$.

$\boldsymbol{M}$ denotes the space of the functions $y=y(t)$ essentially bounded in $[a,b]$, with the norm $\|y\|=\operatorname{ess\,sup}\limits_{a\leqslant t\leqslant b}|y(t)|=\inf\limits_{|Q|=b-a}\sup\limits_{t\in Q}|y(t)|$.

$\boldsymbol{V_0}$ denotes the space of functions $y=y(t)$ of bounded variation in $[a,b]$ and vanishing for $t=a$, with the norm $\|y\|=\operatorname{var}\limits_{a\leqslant t\leqslant b} y(t)$.

$\boldsymbol{l}$ denotes the space of sequences $x=(x^{(1)},x^{(2)},\dots)$ for which $\sum\limits_{n=1}^{\infty}|x^{(n)}|<\infty$, the norm of the element x being, by definition, the sum of the last series.

$\boldsymbol{L}$ denotes the space of the functions $y=y(t)$ for which $\int\limits_a^b|y(t)|dt<\infty$, the norm of the element y being, by definition, the value of this integral.

$\boldsymbol{V_l}$ denotes the space of the functions $y=y(t)$ of bounded variation in $[a,b]$, continuous at the left, and vanishing for $t=a$.

In the sequel the term *function* refers to real-valued functions.

§ 1. A sequence $f_1(x),f_2(x),\dots$ of real-valued functions defined in a quite arbitrary set G is said (Arzelà [1]) to be *quasi-uniformly convergent in G to $f(x)$* if it converges to $f(x)$ and if, given any $\varepsilon>0$ and n_0, there exists an index $n_1\geqslant n_0$ such that

$$\min_{n_0\leqslant n\leqslant n_1}|f_n(x)-f(x)|<\varepsilon \qquad \text{for every } x\in G.$$

Arzelà [1] has proved that a sequence of continuous functions defined on an interval $[a,b]=G$ converges to a continuous limit $f(x)$ if and only if it converges to $f(x)$ quasi-uniformly in G.

We can prove without altering the proof of Arzelà's theorem its following generalization:

1.1. Lemma. *Let $f_1(x),f_2(x),\dots$ be a sequence of functions defined on an arbitrary set G, and convergent quasi-uniformly on G to $f(x)$. Suppose further that $x_0,x_1,x_2,\dots$ is a sequence of elements of G such that*

$$\lim_{m\to\infty} f_n(x_m)=f_n(x_0)$$

for $n=1,2,\dots$ Then there exists $\lim\limits_{m\to\infty} f(x_m)$ and

$$\lim_{m\to\infty} f(x_m)=f(x_0).$$

We use also another notion of convergence intermediate between the uniform and the quasi-uniform convergence, introduced by Fichtenholz and Kantorovitch [7].

A sequence $f_1(x),f_2(x),\dots$ of functions defined in an abstract set G is said to be *almost uniformly convergent to $f(x)$ in G*, if it converges quasi-uniformly to $f(x)$ in G, together with any partial sequence.

In order to see that the almost uniform convergence is stronger than the quasi-uniform one, consider, given a sequence $f_1(x),f_2(x),\dots$ convergent not quasi-uniformly to $f_0(x)$, the new sequence $f_1(x),f_0(x),f_2(x),f_0(x),\dots$; in order to see that the almost uniform convergence is weaker than the uniform one, consider any sequence of continuous functions defined on an interval, convergent non-uniformly to a continuous limit.

1.2. Lemma. *A necessary and sufficient condition for a sequence $f_1(x),f_2(x),\dots$ of functions defined in an abstract set G to be almost uniformly convergent to 0 is that $x_i\in G$ imply*

$$\lim_{n\to\infty}\varlimsup_{i\to\infty}|f_n(x_i)|=0. \tag{1}$$

Proof. Suppose, that (1) does not hold. Then there exist elements $x_i \epsilon G$ and a sequence $n_k \to \infty$ such that

$$\overline{\lim_{i\to\infty}} |f_{n_k}(x_i)| \geqslant \delta > 0 \qquad \text{for } k=1,2,\ldots;$$

hence there exists for each k an index i_k such that $i \geqslant i_k$ implies

$$|f_{n_k}(x_i)| \geqslant \delta/2.$$

Choose now freely two integers k_0 and $k_1 > k_0$, and put

$$i > \max_{k_0 \leqslant k \leqslant k_1} i_k;$$

thus

$$\min_{k_0 \leqslant k \leqslant k_1} |f_{n_k}(x_i)| \geqslant \delta/2,$$

which inequality shows that the sequence $f_{n_1}(x), f_{n_2}(x), \ldots$ does not converge quasi-uniformly to 0. The condition is thus necessary.

Suppose now the condition (1) satisfied and the sequence $f_1(x), f_2(x), \ldots$ not almost uniformly convergent to 0. Thus there exists a $\delta > 0$, an integer k_0, and elements $x_i \epsilon G$ such that

$$\min_{k_0 \leqslant k \leqslant k_1} |f_{n_k}(x_i)| > \delta \qquad \text{for } i=1,2,\ldots,$$

and this implies

$$\overline{\lim_{i\to\infty}} |f_{n_k}(x_i)| \geqslant \delta$$

contrarily to (1). The condition is thus sufficient.

1.3. Corollary. *$f_1(x), f_2(x), \ldots$ being a sequence of functions defined on an abstract set G, convergent almost uniformly to 0 in G, suppose that $\lim_{m\to\infty} f_n(x_m) = \xi_n$ exists for $n=1,2,\ldots$ Then*

$$\lim_{n\to\infty} \xi_n = 0.$$

§ 2. A subset R of a topological space P ([1], p. 37) will be called *compact* if any infinite part of P contains a sequence convergent to some element of P.

By a familiar argument of General Topology we can easily prove the following

2.1. Lemma. *Let the sequence $U_1, U_2, \ldots$ of open sets cover[3]) the set $\bar{R}$ and let the set $\bar{R}$ be compact. Then there exists a finite sequence $U_1, U_2, \ldots, U_m$ which covers the set R.*

Let E be any Banach space. Consider the following neighbourhood system in E: given an $x_0 \epsilon E$, an $\varepsilon > 0$, and n elements $f_1, f_2, \ldots, f_n$ of the space E^* (with arbitrary n), every set

$$\underset{x}{E}\{|f_1(x-x_0)| < \varepsilon, \ldots, f_n(x-x_0)| < \varepsilon\}$$

constitutes a neighbourhood of x_0. The topology generated by this neighbourhood system is called the *weak topology* of E. With this topology E is a locally convex linear topological space (Wehausen [17]).

Any set compact in this topology will be said to be *weakly compact*.

It is easy to see that a sequence $x_1, x_2, \ldots$ of elements is convergent to x_0 in the weak topology, if and only if for any functional $f \epsilon E^*$

$$\lim_{n\to\infty} f(x_n) = f(x_0). \tag{2}$$

Any sequence satisfying the condition (2) for each $f \epsilon E^*$ is termed *weakly convergent* to x_0.

Hence a set $R \subset E$ is weakly compact, if and only if any infinite part of R contains a sequence weakly convergent to an element of E.

§ 3. Now a sufficient condition for weak compactness in the space C will be given, which will be proved later on (p. 79) to be also necessary.

Let R be any set of functions $x(t)$ defined in an interval $[a,b]$.

We can define the equicontinuity as follows: the set R is said to be *equicontinuous*, if $a_n, b_n \epsilon [a,b]$, $b_n - a_n \to 0$ implies

$$x(b_n) - x(a_n) \to 0, \tag{3}$$

uniformly for all $x(t) \epsilon R$.

Using the language of Functional Analysis we can express this fact otherwise.

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[3]) The sequence $U_1, U_2, \ldots$ *covers* the set Q, if $Q \subset U_1 + U_2 + \ldots$

Considering the functions $x(t)$ as elements of the space $\boldsymbol{C}$, one will see that the expressions $x(b_n)-x(a_n)$ are linear functionals on the space $\boldsymbol{C}$:

$$x(b_n)-x(a_n)=f_n(x). \tag{4}$$

Thus the set $R\subset \boldsymbol{C}$ is equicontinuous, if and only if $b_n-a_n\to 0$ implies that the sequence of functionals (4) converges to 0 uniformly in R.

We now introduce an analogous concept. A set $R\subset \boldsymbol{C}$ will be said to be *quasi-equicontinuous* if $b_n-a_n\to 0$ implies the quasi-uniform convergence in R of the sequence of functionals (4) [4]).

It is well known (Banach [4], p. 134) that the weak convergence in the space $\boldsymbol{C}$ has the following meaning: the sequence $x_1, x_2, \ldots$ converges weakly, if the functions $x_n(t)$ are equibounded in $[a,b]$ and converge at any point to a continuous function.

3.1. Theorem. *If the functions of a set $R\subset \boldsymbol{C}$ are equibounded and quasi-equicontinuous in $[a,b]$, then the set R is weakly compact.*

Proof. Let $x_1, x_2, \ldots$ be a sequence of elements of R. By the diagonal method we can extract a subsequence

$$x_{k_n}=x_n^*$$

such that $x_n^*(t)$ converges for every rational t to $x_0(t)$; we shall prove that $x_0(t)$ is uniformly continuous in the set W of the rational numbers of the interval $[a,b]$. Let $t_n', t_n''\in W$, $(t_n'-t_n'')\to 0)$, and put

$$f_n(x)=x(t_n')-x(t_n'').$$

The sequence $f_1(x), f_2(x), \ldots$ converges to 0 quasi-uniformly in R; moreover

$$\lim_{m\to\infty} f_n(x_m^*)=x_0(t_n')-x_0(t_n'').$$

By Corollary 1.2

$$\lim_{n\to\infty}[x_0(t_n')-x_0(t_n'')]=0.$$

[4]) The quasi-equicontinuity may be defined directly as follows: a set R of continuous functions in $[a,b]$ is *quasi-equicontinuous* if from any system $\mathfrak{X}$ of intervals containing intervals of $[a,b]$ of arbitrary small length it is possible, given any $\varepsilon>0$, to choose a finite subsystem $[\alpha_i,\beta_i]\in\mathfrak{X}$, $i=1,2,\ldots,m$, such that $\min\limits_{i=1,2,\ldots,m}|x(\beta_i)-x(\alpha_i)|<\varepsilon$ for any $x\in R$.

The function $x_0(t)$ being uniformly continuous in W, there exists a continuous function $x_0^*(t)$ such that $x_0(t)=x_0^*(t)$ in W. To prove that x_n^* converges weakly to x_0^*, it suffices to prove that the sequence $x_1^*(t), x_2^*(t), \ldots$ converges everywhere to $x_0^*(t)$. Given an arbitrary $t\in[a,b]$, let $t_n\in W$, $t_n\to t_0$, and write

$$f_n^*(x)=x(t_n), \qquad f^*(x)=x(t).$$

It is obvious that the sequence $f_1^*(x), f_2^*(x), \ldots$ converges to $f^*(x)$ quasi-uniformly in the set S composed of the functions

$$x_0^*, x_1^*, x_2^*, \ldots$$

and that

$$\lim_{m\to\infty} f_n^*(x_m^*)=f_n^*(x_0^*) \qquad \text{for } n=1,2,\ldots$$

Hence by Lemma 1.1

$$\lim_{m\to\infty} f^*(x_m^*)=f^*(x_0^*), \quad \text{i.e.} \quad x_0^*(t_m)\to x_0^*(t).$$

§ 4. We now can prove the first criterion of weak compactness in separable spaces.

4.1. Theorem. *A subset R of a separable Banach space is weakly compact, if and only if any sequence of functionals convergent to 0 on the whole of E, converges quasi-uniformly on R* [5]).

Proof. Let R be weakly compact, and consider any sequence of functionals $f_1(x), f_2(x), \ldots$ convergent everywhere to 0. Let $\varepsilon>0$ and n_0 be arbitrary, and consider the sets

$$U_n=\mathop{E}_{x}\{|f_n(x)|<\varepsilon\} \qquad n=n_0, n_0+1, \ldots;$$

since the sets U_n are open in the weak topology and cover the set E, by Lemna 2.1 there exists a n_1 for which

$$R\subset\sum_{n=n_0}^{n_1} U_n,$$

i.e. $f_n(x)\to 0$ quasi-uniformly on R. Thus the condition is necessary.

Let now the set R satisfy the condition of the theorem. We prove first that the set R is bounded. In the contrary case there

[5]) When this paper was finished, Mr. Šmulian informed me that he had found a modification of Gelfand's and my criterion valid in non-separable spaces.

would exist a linear functional $f(x)$ for which $\sup_{x \in R}|f(x)|=+\infty$ (Banach [4], p. 80), hence there exists a sequence $x_1, x_2, \ldots$ such that $x_n \epsilon R$, $|f(x_n)|<|f(x_{n+1})|$, $|f(x_n)|\to\infty$. Put $f_n(x)=f(x)/f(x_n)$; obviously $f_n(x)\to 0$ in E, not quasi-uniformly however in R.

In fact, choose $n_1>n_0$ arbitrarily; then

$$f_n(x_m)=f(x_m)/f(x_n)>1$$

for $n_0\leqslant n\leqslant n_1<m$, and this shows that the sequence $f_n(x)$ does not converge quasi-uniformly.

The space E being separable, it is equivalent to a subspace of the space $\boldsymbol{C}$. Denote by $U(x,t)$ the element $y(t)$ of $\boldsymbol{C}$ which corresponds in this equivalence to the element x. Gelfand has shown ([10], p. 266) that there exists for each $t\epsilon[a,b]$ a linear functional f_t such that

$$U(x,t)=f_t(x), \tag{5}$$

moreover, $t_n\to t_0$ implies $f_{t_n}(x)\to f_{t_0}(x)$ for any x.

The set H into which the operation (5) maps the set R being evidently bounded, it is sufficient to prove that the set H is compact in $\boldsymbol{C}$ i. e. quasi-equicontinuous in $[a,b]$. Let $(a_n-b_n)\to 0$; the sequence of functionals $g_n(x)=(f_{b_n}-f_{a_n})(x)$ converges to 0 in E, hence by hypothesis $g_n(x)\to 0$ quasi-uniformly in R; thus by Theorem 3.1 the set R is compact in $\boldsymbol{C}$. The condition is thus sufficient.

Remark. The condition of Theorem 4.1 is necessary for non-separable spaces too. In that Theorem we can replace the words "quasi-uniformly" by "almost uniformly".

In virtue of Lemma 1.2 we can reformulate Theorem 4.1, as follows:

4.2. Theorem. *A subset R of a separable Banach space is weakly compact, if and only if*

$$\lim_{n\to\infty}\overline{\lim_{i\to\infty}}|f_n(x_i)|=0$$

for each sequence $f_1, f_2, \ldots$ of elements of the space E^*, convergent to 0 in the whole of E, and for arbitrary $x_i \epsilon R$.

4.3 Theorem. *A subset H of the space $\boldsymbol{C}$ is weakly compact, if and only if it is bounded and quasi-equicontinuous* [6]).

Proof. The sufficiency of the condition having been proved in 3.1 we prove only its necessity. The set H must be equibounded, for in the contrary case there would exist elements $x_n \epsilon H$ such that $\|x_n\|\to\infty$, and this sequence cannot contain any weakly convergent sequence. To prove that the set H is quasi-equicontinuous, let $(b_n-a_n)\to 0$ and put $g_n(x)=x(b_n)-x(a_n)$ for $x\epsilon\boldsymbol{C}$; these linear functionals converge to 0 in $\boldsymbol{C}$; hence by Theorem 4.1 this sequence converges to 0 quasi-uniformly in H. Thus the set H is quasi-equicontinuous.

In a quite analogous way we can prove the following

4.4. Theorem. *A subset H of the space $\boldsymbol{c}$ is weakly compact, if and only if it is bounded and the sequence $x^{(n)}$ converges to $x^{(\infty)}$ quasi-uniformly on H.*

§ 5. Some new considerations must forego the proof of the second criterion of weak compactness.

I shall say that the sequence $f_1(x), f_2(x), \ldots$ of real-valued functions defined on an abstract set G *μ-converges to 0 on the set G*, if, given any $\varepsilon>0$, we can choose non-negative numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$\sum_{i=1}^{n}\lambda_i=1, \qquad \sup_{x\epsilon G}\Big|\sum_{i=1}^{n}\lambda_i f_i(x)\Big|<\varepsilon.$$

In other words, μ-convergence to 0 means that there exists a sequence of weighed means of the functions $f_i(x)$ convergent to 0 uniformly on G.

5.1. Lemma. *Let R be a subset of a Banach space E and suppose that every sequence of elements of E convergent to 0 in the whole of E, μ-converges to 0 on R. Then the set R is bounded.*

[6]) The reader will observe that the quasi-uniform convergence plays the same role for the weak compactness in the space $\boldsymbol{C}$ as the uniform one for the (ordinary) compactness. The later has been characterized by Arzelà [1]; the same author has also introduced [2] the concept of the quasi uniform convergence, and only the insufficient development of Functional Analysis in Arzelà's time accounts for the fact that he did not find the criterion of weak compactness in the space $\boldsymbol{C}$.

Proof. Suppose the contrary. Then there exist (Banach [4], p. 80), elements $x_n \epsilon R$ and a linear functional $f(x)$ for which

$$|f(x_n)| < |f(x_{n+1})|, \qquad |f(x_n)| \to \infty.$$

Put $f_n(x) = f(x)/f(x_n)$; we have $f_n(x) \to 0$ in E.
Let $\lambda_i \geqslant 0$, and

$$\sum_{i=1}^{n} \lambda_i = 1;$$

then

$$\sum_{i=1}^{n} \lambda_i f_i(x_n) = \sum_{i=1}^{n} \lambda_i \frac{f(x_n)}{f(x_i)} \geqslant \sum_{i=1}^{n} \lambda_i = 1,$$

contrarily to the hypothesis.

Denote by $\boldsymbol{B}_G$ the Banach space composed of the bounded functions $f = f(x)$ defined on an abstract set G, with the norm

$$\|f\| = \sup_{x \epsilon G} |f(x)|.$$

5.2. Theorem. *Let $f_1(x), f_2(x), \ldots$ be a sequence of elements of the space $\boldsymbol{B}_G$; the following statements are equivalent:*

(I) *the sequence is equibounded and converges to* 0 *almost uniformly on* G,

(II) *it converges weakly to* 0,

(III) *it* μ*-converges to* 0 *together with any partial sequence.*

Proof. By Lemma 1.2 the almost uniform convergency to 0 on G is equivalent to the condition

$$(\gamma) \qquad \lim_{n\to\infty} \overline{\lim_{i\to\infty}} |f_n(x_i)| = 0,$$

thus by a theorem of Banach ([4], p. 219) together with the equiboundedness it is equivalent to (II); by a theorem of Mazur ([12], p. 81) (II) it is equivalent to (III).

5.3. Theorem. *A subset R of a separable Banach space E is weakly compact, if and only if any sequence $f_n(x)$ of elements of the space E^*, convergent to* 0 *on the whole of E, μ-converges to* 0 *on R.*

Proof. This follows from Theorem 4.1, Lemmas 5.1 and 5.2, and from any linear functional being bounded in any weakly compact set.

Remark. The condition of Theorem 5.3 is necessary for non-separable spaces too.

Theorem 5.3 permits us to establish some properties of weakly compact sets. E.g. *the convex span of any weakly compact subset of a separable Banach space is weakly compact* [7]).

§ 6. We now investigate more precisely the notion of quasi-equicontinuity.

6.1. Lemma. *Let H be a quasi-equicontinuous set of functions in $[a,b]$. Given any $\alpha, \beta \epsilon [a,b]$, there exists a constant $M(\alpha,\beta)$ such that*

$$\sup_{x \epsilon H} |x(\alpha) - x(\beta)| < M(\alpha,\beta).$$

Proof. Suppose this is not the case. There exist $\alpha_1, \beta_1 \epsilon [a,b]$ and a sequence $x_n^{(1)} \epsilon H$ such that

$$\lim_{n\to\infty} |x_n^{(1)}(\alpha_1) - x_n^{(1)}(\beta_1)| = \infty.$$

This implies that

$$\overline{\lim_{n\to\infty}} \left| x_n^{(1)}(\alpha_1) - x_n^{(1)}\left(\frac{\alpha_1+\beta_1}{2}\right)\right| + \overline{\lim_{n\to\infty}} \left| x_n^{(1)}\left(\frac{\alpha_1+\beta_1}{2}\right) - x_n^{(1)}(\beta_1)\right| = \infty;$$

hence there exist α_2, β_2 and a sequence $x_n^{(2)}$ extracted from $x_n^{(1)}$ such that $\alpha_2 - \beta_2 = (\alpha_1 - \beta_1)/2$, and

$$\lim_{n\to\infty} |x_n^{(2)}(\alpha_2) - x_n^{(2)}(\beta_2)| = \infty.$$

Continuing this process we obtain functions $x_n^{(k)} \epsilon H$ such that each of the sequences $x_1^{(k)}, x_2^{(k)}, \ldots$ contains the following, and intervals $[\alpha_k, \beta_k] \subset [a,b]$, such that $\alpha_k - \beta_k \to 0$ and

$$\lim_{n\to\infty} |x_n^{(k)}(\alpha_k) - x_n^{(k)}(\beta_k)| = \infty.$$

Write $x_n(t) = x_n^n(t)$; we have

$$\lim_{n\to\infty} |x_n(\alpha_k) - x_n(\beta_k)| = \infty \qquad \text{for } k = 1, 2, \ldots$$

[7]) This theorem has been extended to non-separable spaces by Krein and Šmulian ([11], p. 581).

The set H being quasi-equicontinuous, we have

$$[x(a_k)-x(\beta_k)]\to 0$$

quasi-uniformly on H; hence there exists a k_1 such that for each $x\in H$

$$\min_{1\leqslant k\leqslant k_1}|x(a_k)-x(\beta_k)|<1.$$

It follows that there exists a k_0 and a sequence of indices n_i, such that

$$|x_{n_i}(a_{k_0})-x_{n_i}(\beta_{k_0})|<1 \qquad \text{for } i=1,2,\ldots,$$

which is impossible.

Concerning the Lemma proved above, let us remember that for the equicontinuous sets of functions a stronger proposition holds: it is possible to choose a constant M independent of the intervals $[a,\beta]$ such that

$$\sup_{x\in H}|x(a)-x(\beta)|<M.$$

A set $H\subset \boldsymbol{C}$ will be said to be *pointwise compact* if any sequence of functions of H contains a subsequence convergent everywhere to a continuous limit.

Arguing similarly as in § 4 and § 5, using the topology of Tychonoff [16] instead of the weak topology, and applying Lemma 6.1 we can prove the following

6.2. Theorem. *A set $H\subset \boldsymbol{C}$ is pointwise compact if and only if the functions of this set are bounded at any point and quasi-equicontinuous.*

Taking into account the fact that for continuous functions defined in an interval the quasi-uniform convergence is equivalent to the almost-uniform one, and using Lemma 6.1 we can easily prove the following

6.3. Theorem. *The following statements are equivalent:*

(A) *the set $H\subset \boldsymbol{C}$ is quasi-equicontinuous and the difference $|x(\beta)-x(a)|$ is uniformly bounded for all $a,\beta\in[a,b]$ and $x\in H$,*

(B) $(a_n-\beta_n)\to 0$ *implies μ-convergence to* 0 *on H of the sequence of functionals $x(a_n)-x(\beta_n)$.*

§ 7. In the space E^*, conjugate to a Banach space E, the weak convergence of the sequence $f_1,f_2,\ldots$ to f_0 means that $F(f_n)\to F(f_0)$ for every $F\in E^{**}$. A weaker notion of convergence is also introduced: the sequence $f_1,f_2,\ldots$ is *(*)-weakly convergent* to f_0 if $f_n(x)\to f_0(x)$ for every $x\in E$.

7.1. Theorem. *A sequence $f_1,f_2,\ldots$ of elements of the space E^* converges weakly to* 0, *if and only if $f_n(x)\to 0$ almost uniformly on the unit sphere of the space E.*

Proof. Put $G=\underset{x}{E}\{\|x\|\leqslant 1\}$; given an element $f\in E^*$, denote by $\hat f$ the functional $f(x)$ with its domain restricted to the set G. The correspondence $f\longleftrightarrow\hat f$ establishes an equivalence (Banach [4], p. 180) between the spaces E^* and $\boldsymbol{B}_G$. The convergence $f_n(x)\to 0$ in G implies boundedness of the sequence of the norms. Thus, the theorem results from the invariance of the weak convergence under equivalent transformations, and from Theorem 5.2.

The theorem proved above together with the foregoing one permits us to establish some already known facts. E.g. *the weak compactness of the unit sphere in E is a sufficient condition for the coincidence of the weak convergence* [8]) *with the (*)-weak convergence in E^*. In the case of E being separable this condition is also necessary* (Gantmacher and Šmulian [9]).

Further we can formulate a general principle of characterization of weakly compact sets: a subset of a separable Banach space is weakly compact if and only if each (*)-weakly convergent sequence of elements of E^* behaves on E as weakly convergent.

§ 8. We shall now characterize the weak convergence in some spaces.

8.1. The space $\boldsymbol{m}$.

Denoting by N the set of positive integers, we see that the space $\boldsymbol{m}$ is identical with the space $\boldsymbol{B}_N$. Thus 5.2 gives the following

8.1.1. Theorem. *A sequence $y_m=(y_m^{(1)},y_m^{(2)},\ldots)$ converges weakly to* 0, *if and only if $\sup_{n,m=1,2,\ldots}|y_m^{(n)}|<\infty$ and $\lim_{m\to\infty}y_m^{(n)}=0$ almost uniformly in the set N.*

[8]) Note that in this paper the weak convergence implies the existence of the weak limit (this is not required by some authors; see e. g. Gelfand [10])

8.2. The space $\boldsymbol{M}$.

8.2.1. Theorem. *A sequence $y_n = y_n(t) \epsilon \boldsymbol{M}$ converges weakly to 0, if and only if there exists a set $Q \subset [a,b]$ and a constant K such that $|Q| = b-a$, $\operatorname*{ess\,sup}_{a \leqslant t \leqslant b} |y_n(t)| < K$, and $y_n(t) \to 0$ almost uniformly on Q.*

Proof. For each $y = y_t(t) \epsilon \boldsymbol{M}$ denote by Q_y the set of the points at which $y(t)$ is equal to the derivative of its integral. We prove that $Q \subset Q_y$, $|Q - Q_y| = 0$ implies

$$\operatorname*{ess\,sup}_{a \leqslant t \leqslant b} |y(t)| = \sup_{t \epsilon Q} |y(t)|. \tag{6}$$

The inequality $\operatorname*{ess\,sup}_{a \leqslant t \leqslant b} |y(t)| < \sup_{t \epsilon Q} |y(t)|$ being obvious, we prove only the opposite inequality. Suppose that there exists an $y(t)$ for which $\operatorname*{ess\,sup}_{a \leqslant t \leqslant b} |y(t)| < a < \sup_{t \epsilon Q} |y(t)|$; then there exists a $t_0 \epsilon Q$ for which $y(t_0) > a$ or $y(t_0) < -a$; suppose the first possibility. Thus there is a $\delta > 0$ such that $\int_{t_0}^{t_0+\delta} y(t)\,dt > a\delta$; on the other hand $\operatorname*{ess\,sup}_{a \leqslant t \leqslant b} |y(t)| < a$ implies $\int_{t_0}^{t_0+\delta} y(t)\,dt < a\delta$.

Put $Q_0 = \prod_{n=1}^{\infty} Q_{y_n}$ and denote by M_0 the linear span of the elements $y_1, y_2, \ldots$ From (1) we infer that

$$\|y\| = \sup_{t \epsilon Q_0} |y(t)| \qquad \text{for each } y \epsilon M_0;$$

this shows that the linear subset $M_0 \subset \boldsymbol{M}$ is equivalent to a part of the space $\boldsymbol{B}_{Q_0}$. By the theorem on the extension of linear functionals (Banach [4], p. 55) the weak convergence of the sequence $y_1, y_2, \ldots$ in $\boldsymbol{M}$ is equivalent to the weak convergence of this sequence as elements of the space $\boldsymbol{B}_{Q_0}$. It suffices now to apply Theorem 5.2.

8.3. The space $\boldsymbol{V}_0$.

We denote by $\operatorname*{var^+}_{a \leqslant t \leqslant b} y(t)$ and $\operatorname*{var^-}_{a \leqslant t \leqslant b} y(t)$ respectively the positive and the negative variation of $y(t)$.

8.3.1. Theorem. *A sequence $y_1, y_2, \ldots$ of elements of the space $\boldsymbol{V}_0$ converges weakly to 0, if and only if the sequence $\|y_1\|, \|y_2\|, \ldots$ is bounded and the relation*

$$\lim_{n \to \infty} \sum_{i=1}^{m} [y_n(b_i) - y_n(a_i)] = 0$$

holds almost uniformly in the set $\mathfrak{S}$ of all finite systems of non-overlapping intervals $(a_1, b_1), \ldots, (a_m, b_m)$.

Proof. The set $\mathfrak{S}$ consists of all finite systems $\sigma = (e_1, e_2, \ldots, e_m)$ of non-overlapping intervals. Write for $y \epsilon V_0$ and $e = (\alpha, \beta)$,

$$y(e) = y(\beta) - y(\alpha).$$

Now, we define an operation $U(y, \sigma)$ which makes correspond to each $y \epsilon \boldsymbol{V}_0$ an element $f(\sigma)$ of the space $\boldsymbol{B}_{\mathfrak{S}}$. Write

$$U(y, \sigma) = \sum_{i=1}^{n} y(e_i) \qquad \text{for } \sigma = (e_1, e_2, \ldots, e_m). \tag{7}$$

The operation $U(y) = U(y, \sigma)$ is obviously linear. Since

$$\|U(y, \cdot)\| = \sup_{\sigma \epsilon \mathfrak{S}} \Big|\sum_{i=1}^{m} y(e_i)\Big| = \max\big[\operatorname*{var^+}_{a \leqslant t \leqslant b} y(t), \operatorname*{var^-}_{a \leqslant t \leqslant b} y(t)\big],$$

we get

$$\|U(y, \cdot)\| \leqslant \|y\|, \quad 2\|U(y, \cdot)\| \geqslant \|y\|,$$

$$\tfrac{1}{2}\|y\| \leqslant \|U(y, \cdot)\| \leqslant \|y\|. \tag{8}$$

Denote by $\boldsymbol{B}^0_{\mathfrak{S}}$ the range of the operation $U(y)$. The formula (8) shows that the operation (7) establishes an isomorphical mapping of $\boldsymbol{V}_0$ into $\boldsymbol{B}^0_{\mathfrak{S}}$. The weak convergence being invariant under isomorphical mappings, it suffices to apply Theorem 5.2.

§ 9. An operation from a Banach space E to a Banach space E_1 is said to be *weakly completely continuous,* if it maps each bounded set of the space E into a weakly compact set of the space E_1.

We prove a theorem analogous to a Gelfand's characterization ([10], p. 269) of completely continuous operations.

9.1. Theorem. *A necessary and, if E_1 is separable, also sufficient condition for a linear operation $y=U(x)$ to be weakly completely continuous is that the conjugate operation $f=U^*(g)$ maps any sequence of elements of E_1^* (*)-weakly convergent to 0 into a sequence of elements of E weakly convergent to 0.*

Proof. The condition is necessary [9]). In fact, let the operation $U(x)$ be weakly completely continuous and let a sequence $g_1, g_2, \ldots$ of elements of E_1^* converge (*)-weakly to 0. Put $f_n=U^*(g_n)$, then

$$f_n(x)=g_n(U(x)), \tag{9}$$

and because of the weak compactness of the set

$$\underset{y}{E}\{y=U(x), \|x\|\leqslant 1\}$$

we get by Theorem 4.1 $f_n(x)\to 0$ almost uniformly on the unit sphere of E, and this implies, by Theorem 7.1, the weak convergence to 0 of this sequence.

In order to prove that the condition is sufficient, suppose the operation $U(x)$ satisfies this condition and E_1 is separable. Suppose the sequence $g_1, g_2, \ldots$ of elements of E_1^* converges (*)-weakly to 0. Writing $f_n=U^*(g_n)$ we have $f_n(x)\to 0$ almost uniformly on the unit sphere in E; hence formula (9) shows that $g_n(y)\to 0$ almost uniformly on the set $\underset{y}{E}\{y=U(x), \|x\|\leqslant 1\}=Z$. By Theorem 4.1 the set Z is weakly compact.

9.2. Theorem. *Any linear operation $U(x)$ from the space $\boldsymbol{c}$ to a weakly complete Banach space E is completely continuous.*

Proof. Let us say that a sequence $x_1, x_2, \ldots$ of elements of E converges *weakly in itself*, if $\lim\limits_{n\to\infty} f(x_n)$ exists for every $f\in E^*$. In the space $\boldsymbol{c}$ every bounded sequence contains a sequence weakly convergent in itself. Hence the operation $U(x)$ maps any bounded set $P\subset \boldsymbol{c}$ in a set P_1 which has the property that any sequence of its elements contains a subsequence weakly convergent in itself. Because of the weak completeness of the space E the set P_1 is weakly compact. Thus we have shown that the operation $U(x)$ is weakly completely continuous.

[9]) proved independently by Vera Gantmacher ([8], p. 302).

Now, the range of $U(x)$ being separable, we can suppose without loss of generality that the space E is so. By Theorem 9.1 the conjugate operation $f=U^*(g)$ maps any (*)-weakly convergent sequence of elements of the space E^* in a weakly convergent one. But in the space $\boldsymbol{c}^*$ the weak and the strong convergences coincide (Banach [4], p. 67 and 137). It suffices now to apply a theorem of Gelfand ([10], p. 269) [10]).

§ 10. We presently establish general forms of weakly completely continuous operations in some concrete Banach spaces.

The general form of a linear operation $y=U(x)$ from a Banach space E to the space $\boldsymbol{C}$ is (cf. § 4)

$$U(x)=f_t(x), \tag{10}$$

where $f_t\in E^*$ and $t_n\to t_0$ implies the (*)-weak convergence of the sequence $\{f_{t_n}\}$ to f_{t_0}. Moreover

$$\|U\|=\sup_{a\leqslant t\leqslant b}\|f_t\|.$$

10.1. Theorem. *The operation* (10) *is weakly completely continuous, if and only if the function f_t is weakly continuous* [11]).

Proof. Suppose the operation (10) to be weakly completely continuous, and put

$$R=\underset{y}{E}\{y=U(x), \|x\|\leqslant 1\}; \tag{11}$$

this set is weakly compact. Let $t_n\to t_0$; then $(f_{t_n}-f_{t_0})(x)\to 0$ for each $x\in \boldsymbol{C}$; hence by Theorem 4.1 this sequence converges almost uniformly on R, and Theorem 7.1 implies that f_{t_n} converges weakly to f_{t_0}. Thus the condition is necessary.

Suppose now the condition satisfied. We prove that the set (11) is weakly compact. This set is obviously bounded in $\boldsymbol{C}$. Let $(b_n-a_n)\to 0$, and let F be an arbitrary element of E^{**}. By hypothesis $F(f_t)$ is continuous in the interval $[a,b]$, and this implies

[10]) In its formulation it must be added that the considered space is separable. This theorem can be proved directly by using the general form of linear functionals in the considered space and a theorem of Orlicz ([14], p. 246). The result of Gelfand ([10], p. 274) is weaker.

[11]) The function f_t is *weakly continuous* if, given any $F\in E^{**}$, the function $F^*(f_t)$ is continuous.

$F(f_{b_n}-f_{a_n})\to 0$, i.e. $f_{b_n}-f_{a_n}$ converges weakly to 0. By Theorem 7.1 $(f_{b_n}-f_{a_n})(x)\to 0$ almost uniformly on the unit sphere in E. We apply finally Theorem 4.1.

Now, we illustrate Theorem 10.1 by some examples. Using the general form of linear functionals (Banach [4], p. 59-72) and the results of § 8 we get the following

10.1.2. Theorem. *The general form of the weakly completely continuous linear operations from* $\boldsymbol{l}$ *to* $\boldsymbol{C}$ *is*

$$U(x)=U(x,t)=\sum_{n=1}^{\infty}x^{(n)}y^{(n)}(t),$$

where the sequence $y^{(n)}(t)$ *is equibounded and quasi-equicontinuous in* $[a,b]$; *moreover*

$$\|U\|=\sup_{\substack{a\leqslant t\leqslant b\\ n=1,2,\ldots}}|y^{(n)}(t)|.$$

10.1.3. Theorem. *The general form of the weakly completely continuous linear operations from* $\boldsymbol{L}$ *to* $\boldsymbol{C}$ *is*

$$U(x)=U(x,t)=\int_a^b K(s,t)x(s)\,ds,$$

where

1^0 $K(\cdot,t)\,\epsilon\,\boldsymbol{M}$ *for every* t,

2^0 $t_n\to t_0$ *implies* $K(s,t_n)\to K(s,t_0)$ *almost uniformly on a set* Q *(depending upon the* t_n*'s) such that* $|Q|=b-a$,

3^0 $\sup\limits_{a\leqslant t\leqslant b}\ \operatorname*{ess\,sup}\limits_{a\leqslant s\leqslant b}|K(s,t)|<\infty$.

10.1.4. Theorem. *The general form of the weakly completely continuous linear operations from* $\boldsymbol{C}$ *to* $\boldsymbol{C}$ *is*

$$U(x)=U(x,t)=\int_a^b x(s)\,d_sK(s,t),$$

where

(a) $K(\cdot,t)\,\epsilon\,\boldsymbol{V}_l$ *for any* t,

(b) $K(s,\cdot)\,\epsilon\,\boldsymbol{C}$ *for any* s,

(c) $\sup\limits_{a\leqslant t\leqslant b}\ \operatorname*{var}\limits_{a\leqslant s\leqslant b}K(s,t)<\infty$,

(d) *the set of the functions* $\sum_{i=1}^{m}[K(\beta_i,t)-K(\alpha_i,t)]$ *is equibounded and quasi-equicontinuous in* $\mathfrak{S}$ (cf. § 8).

Moreover, denoting by C_1 *the closed linear span of the range of the operation* $U(x)$, *we have* $K(s,\cdot)\,\epsilon\,C_1$ *for each* s.

Proof. Only the last part of Theorem requires a proof. Since $K(a,\cdot)=U(0)$ and $K(b,\cdot)=U(1)$, it suffices to consider the case $a<s<b$. Put

$$z_s(\vartheta)=\begin{cases}1 & \text{for}\quad a\leqslant\vartheta\leqslant s\\ 0 & \text{for}\quad s<\vartheta\leqslant b,\end{cases}$$

and let $z_{ns}(\vartheta)$ be a non-decreasing sequence of continuous functions convergent to $z_s(\vartheta)$. The sequence

$$z_{1_s},z_{2_s},\ldots \tag{12}$$

being bounded, the sequence $U(z_{ns})$ is weakly compact; moreover, since the sequence (12) is weakly convergent in itself, $U(z_{ns})$ converges weakly to an element $y_s\,\epsilon\,C_1$. We have then for every $t\,\epsilon\,[a,b]$

$$y_s(t)=\lim_{n\to\infty}\int_a^b z_{ns}(\vartheta)\,d_\vartheta K(\vartheta,t)=\lim_{n\to\infty}\int_a^s z_{ns}(\vartheta)\,d_\vartheta K(\vartheta,t),$$

and the continuity (in ϑ) at the left of the function $K(\vartheta,t)$ implies for $\vartheta=s$

$$y_s(t)=K(s,t).$$

10.2. Now, we investigate the linear completely continuous operations from the space $\boldsymbol{C}$ to an arbitrary Banach space E.

Let us recall first some concepts and results on abstract functions of bounded variation.

Let y_t be an abstract function from an interval $[a,b]$ to the space E. We denote by $V(y_t)$ the set of the elements

$$\sum_{i=1}^{n}[y_{\beta_i}-y_{\alpha_i}],$$

where (α_i,β_i) is any finite system of non-overlapping intervals.

If the set $V(y_t)$ is bounded and $f\,\epsilon\,E^*$, then $f(y_t)$ is of bounded variation; the function y_t is then called *of bounded variation.*

Let $x(t)$ be any real-valued function, continuous in $[a,b]$; for any function y_t of bounded variation, consider the Stieltjes sums

$$\sum_{i=1}^{m}x(\tau_i)[y_{t_i}-y_{t_{i-1}}]. \tag{13}$$

Dunford ([6], p. 312) has shown that these sums converge (in the sense of Moore and Smith [13]) to an element $y \in E$. This element is denoted by $\int_a^b x(t)\,dy_t$ and termed the *Stieltjes integral of* $x(t)$. It has the following property:

10.2.1. *If f is any linear functional in E, then*

$$f\Big(\int_a^b x(t)\,dy_t\Big)=\int_a^b x(t)\,df(y_t).$$

Gelfand ([10], p. 280-282) has proved that the general form of the linear operations from the space C to an arbitrary weakly complete Banach space E is

$$U(x)=\int_a^b x(t)\,dy_t, \tag{14}$$

where the set $V(y_t)$ is bounded. Moreover, if we assume that the set $V(y_t)$ is compact, (14) gives the general form of the completely continuous linear operations from C to an arbitrary Banach space.

10.2.2. Lemma. *The general form of the weakly completely continuous linear operations from the space C to an arbitrary separable*[12] *Banach space E is* (14), *where*

(i⁰) $V(y_t)$ *is weakly compact,*

(ii⁰) $y_a=0$,

(iii⁰) y_t *is weakly continuous at the left for $a<t<b$.*

Proof. The space E is equivalent to a subspace C_1 of the space C (Banach [4], p. 185). Let

$$W(x,t)=f_t(x) \tag{15}$$

be the equivalent mapping of C into C_1 (cf. § 4). Now let $U(x)$ be any linear operation from C to C_1. The operation $W(x)$ being isometrical, the operation $y=U(x)$ is weakly completely continuous if and only if the operation $z=fU(x)$ is so. By Theorem 10.1.4.

$$f_t U(x)=\int_a^b x(s)\,d_s K(s,t), \tag{16}$$

where $K(s,t)$ satisfies the conditions (a)-(d), and $K(s,\cdot)\in C_1$ for any s. Hence there exists for any s an element $y_s\in E$ which is mapped into $K(s,\cdot)$ by the operation (15):

$$K(s,t)=f_t(y_s). \tag{17}$$

The condition (d) implies that the set of functions

$$\sum_{i=1}^{m}[f_t(y_{\beta_i})-f_t(y_{\alpha_i})]=f_t\Big(\sum_{i=1}^{m}[y_{\beta_i}-y_{\alpha_i}]\Big)$$

is weakly compact. The mapping (15) establishing an equivalence between C and C_1, we see that the set $V(f_t)$ is also weakly compact. Lemma 10.2.1 and the formulae (16) and (17) imply

$$f_t U(x)=\int_a^b x(t)\,d_s f_t(y_s)=f_t\int_a^b x(t)\,dy_s.$$

The mapping (15) being one-one, we get the formula (14) with y_t satisfying the condition (i⁰). The conditions (ii⁰) and (iii⁰) result from the formula (17) and from the condition (a). Thus the necessity is established.

It is easy to see that, conversely, the conditions (i⁰), (ii⁰) and (iii⁰) imply for the kernel $K(s,t)$ the conditions (a)-(d).

Lemma 10.2.2 shows the usefullness of functions satisfying the condition (i⁰). The following theorem throws some more light on those functions:

10.2.3. Theorem. *If for an abstract function y_t the set $V(y_t)$ is weakly compact, then for each $t\in(a,b)$ there exist the weak limits y_{t+0}, y_{t-0}, and at the ends of the interval there exist y_{a+0}, y_{b-0}.*

Moreover, if the space E is separable, then $y_{t+0}=y_{t-0}$, except a denumerable set.

[12] The lemma is true also for non-separable spaces E: then (i⁰) is to be read "$V(y_t)$ *is separable and weakly compact*". But this formulation does not present any generalization since the range of any operation of considered type is separable.

Proof. The weak compactness of the set $V(y_t)$ implies the same for the range of the function y_t. For any $f \epsilon E^*$ the function $f(y_t)$ is of bounded variation; hence for any $t_0 \epsilon [a,b]$ there exists

$$\lim_{t\to t_0+0} f(y_t).$$

Thus $t_n \to t_0$, where $t_n > t_0$, implies the weak convergence in itself of the sequence $\{y_{t_n}\}$, and the weak compactness of this sequence implies the existence of the weak limit y_{t_0+0}, which obviously does not depend upon the particular choice of the sequence t_n. We prove the existence of y_{t_0-0} by changing the variable t into $-t$.

Let now the space E be separable. Then there exists a sequence $f_1, f_2, \dots$ of elements of the space E^* (*)-weakly dense in E^*, i.e. dense in the sense of the (*)-weak convergence (Banach [4], p. 124). For each n, the set D_n of the points of discontinuity of the function $f_n(y_t)$ is denumerable. We prove that

$$t_0 \epsilon (a,b) - \sum_{n=1}^{\infty} D_n \quad \text{implies} \quad y_{t_0+0} = y_{t_0-0}.$$

In fact, let $f \epsilon E^*$ and let f_{n_k} converge (*)-weakly to f. Since by Theorem 4.1 $f_{n_k}(y_t) \to f(y_t)$ quasi-uniformly in $[a,b]$, and since the functions $f_{n_k}(y_t)$ are continuous at t_0, the function $f(y_t)$ is also continuous at t_0.

Theorem 10.2.3 shows that the conditions (ii^0) and (iii^0) are superfluous in Lemma 10.2.2. In fact, write

$$y_t^* = \begin{cases} 0 & \text{for} \quad t=a, \\ y_{t+0} - y_a & \text{for} \quad a<t<b, \\ y_b - y_a & \text{for} \quad t=b. \end{cases}$$

According to a theorem of Krein and Šmulian ([11], p. 581) and to Theorem 10.2.3 the function y_t^* satisfies the conditions (i^0) - (iii^0). It is obvious that

$$\int_a^b x(t)\, dy_t^* = \int_a^b x(t)\, dy_t \qquad \text{for every } x(t) \epsilon C.$$

Hence we obtain the following

10.2.4. Theorem. *The general form of the weakly continuous linear operations from the space* C *to a separable Banach space* E *is*

$$U(x) = \int_a^b x(t)\, dy_t,$$

where, for the function y_t, *the set* $V(y_t)$ *is weakly compact.*

We give an application of the above theorem. Using the conditions for the weak convergence in the space L (Banach, [4], p. 136), the Dunford's criterion ([5], p. 643) of weak compactness in the space L and the remarks at the beginning of 10.2 we have:

10.2.5. Theorem. *The general form of the weakly completely continuous linear operations from the space* C *to the space* L *is*

$$U(x) = \frac{d}{dt} \int_a^b x(s)\, d_s K(s,t),$$

where

(*) $K(s,\cdot)$ *is an absolutely continuous function for every* s,

(**) *the set of functions* $\sum_{i=1}^{n} [K(\beta_i,t) - K(\alpha_i,t)]$, *where the system* (α_i,β_i) *runs over the set* $\mathfrak{S}$ (cf. § 8), *is equi-absolutely continuous* [13]).

Tokhovo, 1939 - Leningrad, 1940.

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[4] S. Banach, *Théorie des opérations linéares*, Monografie Matematyczne, Warszawa 1932.

[13]) A family H of functions is *equi-absolutely continuous* if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for any system $[a_1,b_1],[a_2,b_2],\dots,[a_n,b_n]$ of non-overlapping intervals

$$\sum_{i=1}^{n} (b_i - a_i) < \delta \qquad \text{implies} \qquad \sup_{x \epsilon H} \sum_{i=1}^{n} |x(b_i) - x(a_i)| < \varepsilon.$$

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Sur les solutions de l'équation linéaire du type elliptique, discontinues sur la frontière du domaine de leur existence

par

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Introduction.

1. L'une des généralisations du problème classique de Dirichlet pour l'équation de Laplace dans un domaine ouvert borné consiste à admettre que la solution cherchée peut devenir discontinue en certains points de la frontière de ce domaine. On doit à Zaremba [1]) un théorème important, concernant l'unicité de la solution de ce problème. D'après son théorème, si la fonction $u(x,y)$ de deux variables indépendantes, harmonique dans D est continue et s'annule sur la frontière $\mathrm{Fr}(D)$ de ce domaine sauf aux points $P_1, P_2, \dots, P_\mu$ appartenant à $\mathrm{Fr}(D)$, et si elle satisfait en outre à la condition

$$\lim_{r_\nu \to 0} \frac{u(x,y)}{\log r_\nu} = 0 \quad \text{pour} \quad \nu = 1, 2, \dots, \mu \quad \text{et} \quad (x,y) \,\epsilon\, D + \mathrm{Fr}(D),$$

r_ν désignant la distance du point (x,y) au point P_ν, elle est identiquement nulle dans D. Dans le cas de trois variables indépendantes, on doit remplacer la dernière condition par

$$\lim_{\overline{PP_\nu} \to 0} u(P) \cdot r_\nu = 0 \qquad \text{pour } \nu = 1, 2, \dots, \mu,$$

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