The measure $\bar{\mu}$ is isomorphic to $\bar{\nu}|\boldsymbol{Y}_0$ since there is a measure-preserving Baire transformation of $I\times I$ onto I. The measure $\bar{\nu}$ is trivially isomorphic to $\bar{\mu}|\boldsymbol{X}_0$. The measures $\bar{\mu}$ and $\bar{\nu}$, however, are not isomorphic since there is no Boolean isomorphism of $\boldsymbol{X}$ onto $\boldsymbol{Y}$.

In fact, suppose h is an isomorphism of $\boldsymbol{X}$ onto $\boldsymbol{Y}$. Since all one-point subsets of $I\times I$ belong to $\boldsymbol{X}$ and to $\boldsymbol{Y}$, there is a one-one mapping[10]) φ of $I\times I$ into $I\times I$ such that $h(X)=\varphi^{-1}(X)\in\boldsymbol{Y}_0$ for $X\in\boldsymbol{X}$. Thus φ is a Baire mapping. Consequently φ^{-1} is also a Baire mapping[11]). Let $X_0\in\boldsymbol{X}-\boldsymbol{Y}$. We have $\varphi(X_0)\in\boldsymbol{X}$ and $X_0=\varphi^{-1}(\varphi(X_0))=h(\varphi(X_0))\in\boldsymbol{Y}$ which is impossible.

The above example shows that the assumption in Theorem (T) that measures are strictly positive is essential.

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[10]) See e. g. E. Szpilrajn-Marczewski, *On the isomorphism and the equivalence of classes and sequences of sets*, Fund. Math. **32** (1939), pp. 133-148; in particular p. 137.

[11]) See e. g. C. Kuratowski, *Topologie I* (second edition), Warszawa-Wrocław 1948, p. 398, th. 3.

Państwowy Instytut Matematyczny.

Algebraic Treatment of the Functional Calculi of Heyting and Lewis[1]).

By

H. Rasiowa[2]) (Warszawa).

Every formula α of a functional calculus can be interpreted as a functional on an abstract set I with values in a suitable abstract algebra $\mathfrak{A}$. This functional will be denoted by „Φ_α" and will be called the *$(I,\mathfrak{A})$-functional* determined by α[3]).

For instance, every formula of the ordinary functional calculus can be interpreted as an $(I,\mathfrak{A})$-functional, where $\mathfrak{A}$ is a complete Boolean algebra[4]); every formula of the functional calculus of Heyting can be interpreted as an $(I,\mathfrak{B})$-functional, where $\mathfrak{B}$ is a complete Brouwerian algebra[5]), and every formula of the functional calculus of Lewis[6]) can be interpreted as an $(I,\mathfrak{C})$-functional, where $\mathfrak{C}$ is a complete closure algebra[7]).

The above interpretation is a generalization of the well-known matrix method in sentential calculi. The connection between the

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[1]) This paper was presented to the Warsaw University in candidacy for the degree of Doctor of Philosophy and accepted in May 1950. The results were announced at the Polish-Czechoslovak Mathematical Congress in Prague in September 1949. The results of this paper together with that of *„A Proof of the Completeness Theorem of Gödel"* published by the author and R. Sikorski (Fundamenta Mathematicae **37** (1950), pp. 193-200) were announced at the meeting of the Association for Symbolic Logic in December 1949 The Journal of Symbolic Logic **15** (1950), p. 79).

[2]) The author wishes to thank Professor A. Mostowski for suggestions and criticisms in connection with the writing of this Thesis.

[3]) The notion of the $(I,\mathfrak{A})$-functional and the idea of treating the functional calculi algebraically is due to Mostowski [2]. For a definition of the $(I,\mathfrak{A})$-functional Φ_α see § 4, p. 113.

[4]) See Rasiowa and Sikorski [1].

[5]) See Mostowski [2].

[6]) The system considered here is based on the system S.4 of the sentential calculus of Lewis and Langford [1], p. 501.

[7]) See § 5, p. 119.

two-valued sentential calculus and Boolean algebras is well known. An analogous connection has been established between the sentential calculus of Heyting and Brouwerian algebras, and between the sentential calculus of Lewis and closure algebras [8]). This explains the fact that algebras of these kinds appear in the discussion of the calculi mentioned above. The hypothesis, that all these algebras are complete, has to be assumed in order to make certain that all the infinite operations (corresponding to logical quantifiers), which occur in the functionals, can be performed.

In the case of the ordinary functional calculus the interpretation of formulae, as algebraic functionals, permit us to put in algebraic terms the semantic notions of satisfiability and validity [9]). Gödel's completeness theorem [10]) can then be formulated in the following equivalent form, where I_0 is the set of all positive integers and $\mathfrak{A}_0$ is the (complete) two-element Boolean algebra:

(A) *A formula α of the ordinary functional calculus is provable if and only if the $(I_0,\mathfrak{A}_0)$-functional Φ_α is identically equal to the unit element of $\mathfrak{A}_0$* [11]).

It is easily shown that, if this formula α is provable, then the $(I,\mathfrak{A})$-functional Φ_α is identically equal to the unit element of $\mathfrak{A}$, for every complete Boolean algebra $\mathfrak{A}$ and every non-void set I. Consequently, by (A), we have:

(A′) *A formula α of the ordinary functional calculus is provable if and only if for every non-void set I and for every complete Boolean algebra $\mathfrak{A}$ the $(I,\mathfrak{A})$-functional Φ_α is identically equal to the unit element of $\mathfrak{A}$.*

It will be proved that there exists a complete Brouwerian algebra $\mathfrak{B}_0$ and a complete closure algebra $\mathfrak{C}_0$ such that the following conditions are satisfied (where I_0 is the set of all positive integers):

(B) *A formula α of the functional calculus of Heyting is provable if and only if the $(I_0,\mathfrak{B}_0)$-functional Φ_α is identically equal to the zero-element of $\mathfrak{B}_0$* [12]).

[8]) See McKinsey and Tarski [3].

[9]) See Rasiowa and Sikorski [1]. For the explanation of the notions of satisfiability and validity, see Tarski [1].

[10]) See Gödel [1].

[11]) See Rasiowa and Sikorski [1].

[12]) Theorem (B) is the solution of the problem proposed by Mostowski [2], p. 207.

(C) *A formula α of the functional calculus of Lewis is provable if and only if the $(I_0,\mathfrak{C}_0)$-functional Φ_α is identically equal to the unit element of $\mathfrak{C}_0$.*

The above theorems imply [13]):

(B′) *A formula α of the functional calculus of Heyting is provable if and only if, for every non-empty set I and every complete Brouwerian algebra $\mathfrak{B}$, the $(I,\mathfrak{B})$-functional Φ_α is identically equal to the zero element of $\mathfrak{B}$* [14]).

(C′) *A formula α of the functional calculus of Lewis is provable if and only if, for every non-empty set I and every complete closure algebra $\mathfrak{C}$, the $(I,\mathfrak{C})$-functional Φ_α is identically equal to the unit element of $\mathfrak{C}$* [15]).

It is clear that theorems (B) ((B′)) and (C) ((C′)) [16]) are completeness theorems (in the same sense that theorem (A) ((A′)) is the completeness theorem for ordinary functional calculus) for the functional calculi of Heyting and Lewis, respectively.

The proof of theorems (B), (B′) and (C), (C′) is the subject of § 4 and § 5. Paragraphs 1-3 contain the description of the systems considered and some lemmas on extensions of Brouwerian and closure algebras (§ 3).

§ 1. The functional calculus of Heyting.

We shall refer to the functional calculus of Heyting as the system $\mathcal{H}$. $\mathcal{H}$ can be described briefly as follows.

The symbols of the system are: the individual variables $x_1, x_2, \ldots$, the sentential variables $a_1, a_2, \ldots$, the k-argument functional variables $F_1^k, F_2^k, \ldots$ $(k=1,2,\ldots)$, constants and parentheses.

The constants are: the conjunction sign $\wedge$, the disjunction sign $\vee$, the implication sign $\supset$, the negation sign $\sim$, the sign of the general quantifier (x_k), and the sign of the existential quantifier $(\exists x_k)$.

The *class of formulae* of the system $\mathcal{H}$ is the smallest class $\boldsymbol{H}$ which contains all sentential variables, all expressions of the form $F_i^k(x_{j_1},\ldots,x_{j_k})$ and which is closed under the following six opera-

[13]) See § 4, p. 119 and § 5, p. 125.

[14]) Theorem (B′) is the solution of the problem proposed by Mostowski [2], p. 207.

[15]) Theorems (C) and (C′) solve the question proposed to me by Mostowski.

[16]) Theorems (B) ((B′)), (C) ((C′)) are stronger than similar results obtained independently by Henkin. See Henkin [1].

tions: forming the conjunction $(\alpha\wedge\beta)$, the disjunction $(\alpha\vee\beta)$, the implication $(\alpha\supset\beta)$ from two expressions α and β, taking the negation $(\sim\alpha)$ of an expression α, and putting the existential quantifier $(\exists x_k)$ or the universal quantifier (x_k) in front of an expression α to obtain the expression $((\exists x_k)\alpha)$, or $((x_k)\alpha)$, respectively.

Among the occurrences of individual variables in a formula, we distinguish in a familiar way between *free* and *bound* occurrences. By $\alpha(x_{k_1},\dots,x_{k_n})$ we mean a formula in which at least one occurrence of each of the variables x_{k_i} $(i=1,2,\dots,n)$ is free.

We introduce the following abbreviation:

(I) $(\alpha\equiv\beta)$ for $((\alpha\supset\beta)\wedge(\beta\supset\alpha))$.

In writing formulae, we shall practice the omission of parentheses, the rule being that: (1) each of the operators $\sim$, $\wedge$, $\supset$, $\equiv$, $\vee$ binds one or two expressions less strongly than the preceding one, and (2) the quantifiers bind them more strongly than any one of the operators just listed.

If α, β, γ, are arbitrary formulae, the following formulae are called *axioms*[17]:

A. 1	$\alpha\supset\alpha\wedge\alpha$	A. 7	$\alpha\supset(\alpha\vee\beta)$
A. 2	$\alpha\wedge\beta\supset\beta\wedge\alpha$	A. 8	$(\alpha\vee\beta)\supset(\beta\vee\alpha)$
A. 3	$(\alpha\supset\beta)\supset((\alpha\wedge\gamma)\supset(\beta\wedge\gamma))$	A. 9	$(\alpha\supset\gamma)\wedge(\beta\supset\gamma)\supset((\alpha\vee\beta)\supset\gamma)$
A. 4	$(\alpha\supset\beta)\wedge(\beta\supset\gamma)\supset(\alpha\supset\gamma)$	A. 10	$\sim\alpha\supset(\alpha\supset\beta)$
A. 5	$\beta\supset(\alpha\supset\beta)$	A. 11	$(\alpha\supset\beta)\wedge(\alpha\supset\sim\beta)\supset\sim\alpha$
A. 6	$\alpha\wedge(\alpha\supset\beta)\supset\beta$	A. 12	$(x_k)\alpha\supset\alpha$

A. 13 $\alpha\supset(\exists x_k)\alpha$.

There are four *rules of inference* in the system $\mathcal{H}$:

R. 1.1 *modus ponens*: from α and $\alpha\supset\beta$ to infer β;

R. 1.2 the *rule of substitution* for individual variables[18];

R. 1.3 the *rule for* (x_k): from $\alpha\supset\beta$ to infer $\alpha\supset(x_k)\beta$ provided that no free occurrence of x_k appears in α.

R. 1.4 the *rule for* $(\exists x_k)$: from $\alpha\supset\beta$ to infer $(\exists x_k)\alpha\supset\beta$ provided that no free occurrence of x_k appears in β.

A finite sequence of formulae each of which is either an axiom, or results from one of two preceding formulae of the sequence by applying one of the rules R. 1.1-R. 1.4, is called a *formal proof* in $\mathcal{H}$.

[17]) A. 1—A. 11 are substitutions of the axioms of the sentential calculus of Heyting. See Heyting [1].

[18]) This rule is the well-known rule of substitution for individual variables in the ordinary functional calculus of the first order. See Mostowski [1], p. 53.

If α is the last formula of a formal proof then α is called a *provable* formula of $\mathcal{H}$. We write then $\vdash\alpha$.

It is easy to show that every $\alpha\in\mathcal{H}$, which is a substitution of a provable formula of the sentential calculus of Heyting[19], is also a provable one in $\mathcal{H}$. Hence, if α, β, γ, δ, are any formulae, the following are provable formulae of $\mathcal{H}$[20]:

2.2	$\vdash\alpha\wedge\beta\supset\alpha$	2.3	$\vdash(\alpha\wedge\beta)\wedge\gamma\supset\alpha\wedge(\beta\wedge\gamma)$
2.21	$\vdash\alpha\supset\alpha$	2.32	$\vdash\alpha\wedge(\beta\wedge\gamma)\supset(\alpha\wedge\beta)\wedge\gamma$
2.22	$\vdash\alpha\wedge\beta\supset\beta$	3.2	$\vdash(\alpha\vee\beta)\vee\gamma\supset\alpha\vee(\beta\vee\gamma)$
2.23	$\vdash(\alpha\supset\beta)\wedge(\gamma\supset\delta)\supset(\alpha\wedge\gamma\supset\beta\wedge\delta)$	3.21	$\vdash\alpha\vee(\beta\vee\gamma)\supset(\alpha\vee\beta)\vee\gamma$
2.24	$\vdash(\alpha\supset\beta)\wedge(\alpha\supset\gamma)\equiv\alpha\supset\beta\wedge\gamma$	3.22	$\vdash\alpha\vee\alpha\supset\alpha$
2.26	$\vdash\beta\supset(\alpha\supset\alpha\wedge\beta)$	3.3	$\vdash(\alpha\supset\beta)\wedge(\gamma\supset\delta)\supset((\alpha\vee\gamma)\supset(\beta\vee\delta))$
2.27	$\vdash\alpha\supset(\beta\supset\gamma)\equiv\alpha\wedge\beta\supset\gamma$	3.6	$\vdash(\alpha\vee\beta)\supset((\alpha\supset\beta)\supset\beta)$
2.271	$\vdash\alpha\supset(\beta\supset\gamma)\equiv\beta\supset(\alpha\supset\gamma)$	4.1	$\vdash\sim\alpha\supset(\alpha\supset\beta)$
2.29	$\vdash(\alpha\supset\beta)\supset((\beta\supset\gamma)\supset(\alpha\supset\gamma))$	4.2	$\vdash\alpha\supset\beta\supset(\sim\beta\supset\sim\alpha)$
2.291	$\vdash(\beta\supset\gamma)\supset((\alpha\supset\beta)\supset(\alpha\supset\gamma))$	4.21	$\vdash(\alpha\supset\sim\beta)\supset(\beta\supset\sim\alpha)$

4.3 $\vdash\alpha\supset\sim\sim\alpha$.

The following formulae are also provable[21] in $\mathcal{H}$:

T. 1	$\vdash\alpha\equiv\alpha$	[2.21, 2.26, I]
T. 2	$\vdash\alpha\wedge\beta\equiv\beta\wedge\alpha$	[A. 2, 2.26, I]
T. 3	$\vdash\alpha\vee\beta\equiv\beta\vee\alpha$	[A. 8, 2.26, I]
T. 4	$\vdash\alpha\wedge\alpha\equiv\alpha$	[2.2, A. 1, 2.26, I]
T. 5	$\vdash(\alpha\vee\alpha)\equiv\alpha$	[3.22, A. 7, 2.26, I]
T. 6	$\vdash\alpha\wedge(\beta\wedge\gamma)\equiv(\alpha\wedge\beta)\wedge\gamma$	[2.32, 2.3, 2.26, I]
T. 7	$\vdash(\alpha\vee(\beta\vee\gamma))\equiv((\alpha\vee\beta)\vee\gamma)$	[3.21, 3.2, 2.26, I]
T. 8	$\vdash\alpha\wedge\beta\supset(\alpha\equiv\beta)$	

[19]) By the sentential calculus of Heyting we understand the system based on the following axioms (see Heyting [1]):
2.1 $a_1\supset a_1\wedge a_1$, 2.11 $a_1\wedge a_2\supset a_2\wedge a_1$, 2.12 $(a_1\supset a_2)\supset(a_1\wedge a_3\supset a_2\wedge a_3)$, 2.13 $(a_1\supset a_2)\wedge(a_2\supset a_3)\supset(a_1\supset a_3)$, 2.14 $a_2\supset(a_1\supset a_2)$, 2.15 $a_1\wedge(a_1\supset a_2)\supset a_2$, 3.1 $a_1\supset(a_1\vee a_2)$, 3.11 $(a_1\vee a_2)\supset(a_2\vee a_1)$, 3.12 $(a_1\supset a_3)\wedge(a_2\supset a_3)\supset((a_1\vee a_2)\supset a_3)$, 4.1 $\sim a_1\supset(a_1\supset a_2)$, 4.2 $(a_1\supset a_2)\wedge(a_1\supset\sim a_2)\supset\sim a_1$.

There are two rules of inference: R. 1.1 and the rule of substitution for sentential variables.

[20]) The numbers 2.2-4.3 of these formulae refer to the numbers (see Heyting [1]) of the provable formulae of Heyting's sentential calculus, from which they are obtained by substitution.

[21]) In the description of the formal proofs of T. 1-T. 17 we do not mention applications of the rule R. 1.1.

Proof. 8.1 $\vdash(\alpha\wedge\beta)\wedge\alpha\supset\beta$ [2.2, 2.22, 2.29]

8.2 $\vdash\alpha\wedge\beta\supset(\alpha\supset\beta)$ [8.1, 2.27, I, 2.22]

8.3 $\vdash\beta\wedge\alpha\supset(\alpha\supset\beta)$ [A. 2, 8.2, 2.29]

8.4 $\vdash\alpha\wedge\beta\supset(\beta\supset\alpha)$ [8.3]

8.5 $\vdash\alpha\wedge\beta\supset(\alpha\supset\beta)\wedge(\beta\supset\alpha)$ [8.2, 8.4, 2.26, 2.24, I, 2.2]

8.6 $\vdash\alpha\wedge\beta\supset(\alpha\equiv\beta)$ [8.5, I]

T. 9 $\vdash\alpha\wedge(\alpha\vee\beta)\equiv\alpha$

Proof. 9.1 $\vdash\alpha\supset\alpha\wedge(\alpha\vee\beta)$ [2.21, A. 7, 2.26, 2.24, I, 2.2]

9.2 $\vdash\alpha\wedge(\alpha\vee\beta)\equiv\alpha$ [9.1, 2.2, 2.26, I]

T. 10 $\vdash(\alpha\vee\alpha\wedge\beta)\equiv\alpha$

Proof. 10.1 $\vdash(\alpha\vee\alpha\wedge\beta)\supset\alpha$ [2.21, 2.2, 2.26, A. 9]

10.2 $\vdash(\alpha\vee\alpha\wedge\beta)\equiv\alpha$ [10.1, A. 7, 2.26, I]

T. 11 $\vdash\beta\wedge\sim(\alpha\supset\alpha)\equiv\sim(\alpha\supset\alpha)$

Proof. 11.1 $\vdash(\alpha\supset\alpha)\supset(\sim(\alpha\supset\alpha)\supset\beta)$ [A. 10, 2.271, I, 2.2]

11.2 $\vdash\sim(\alpha\supset\alpha)\supset\beta$ [11.1, 2.21]

11.3 $\vdash\sim(\alpha\supset\alpha)\supset\beta\sim\wedge(\alpha\supset\alpha)$ [11.2, 2.21, 2.26, 2.24, I, 2.2]

11.4 $\vdash\beta\wedge\sim(\alpha\supset\alpha)\equiv\sim(\alpha\supset\alpha)$ [2.22, 11.3, 2.26, I]

T. 12 $\vdash\sim\beta\equiv\beta\supset\sim(\alpha\supset\alpha)$

Proof. 12.1 $\vdash(\alpha\supset\alpha)\vee\sim\beta$ [A. 7, 2.21]

12.2 $\vdash((\alpha\supset\alpha)\supset\sim\beta)\supset\sim\beta$ [12.1, 3.6]

12.3 $\vdash(\beta\supset\sim(\alpha\supset\alpha))\supset((\alpha\supset\alpha)\supset\sim\beta)$ [4.21]

12.4 $\vdash(\beta\supset\sim(\alpha\supset\alpha))\supset\sim\beta$ [12.3, 2.271, 2.21]

12.5 $\vdash\sim\beta\equiv\beta\supset\sim(\alpha\supset\alpha)$ [4.1, 12.4, 2.26, I]

T. 13 $\vdash(x_k)(\alpha\wedge\beta)\equiv(x_k)\alpha\wedge(x_k)\beta$ [22])

T. 14 $\vdash(x_k)(\alpha\supset\beta)\supset((x_k)\alpha\supset(x_k)\beta)$

T. 15 $\vdash(x_k)(\alpha\supset\beta)\supset((\exists x_k)\alpha\supset(\exists x)\beta)$

T. 16 $\vdash\alpha\equiv(x_k)\alpha$ } provided that there is no free occurrence

T. 17 $\vdash\alpha\equiv(\exists x_k)\alpha$ } of x_k in α.

Lemma 1.1. *If* $\vdash\alpha\wedge(\beta\wedge\gamma)\equiv\beta\wedge\gamma$, *then* $\vdash(\beta\supset\alpha)\wedge\gamma\equiv\gamma$ (*for arbitrary* $\alpha,\beta,\gamma\in\boldsymbol{H}$).

Proof. Suppose $\vdash\alpha\wedge(\beta\wedge\gamma)\equiv\beta\wedge\gamma$, then

(1) $\vdash\beta\wedge\gamma\supset\alpha\wedge(\beta\wedge\gamma)$ [I, 2.22]

(2) $\vdash\gamma\wedge\beta\supset\alpha$ [A. 2, (1), 2.2, 2.29]

(3) $\vdash\gamma\supset(\beta\supset\alpha)$ [(2), 2.27, I, 2.22]

(4) $\vdash\gamma\supset(\beta\supset\alpha)\wedge\gamma$ [(3), 2.21, 2.26, 2.24, I, 2.2]

(5) $\vdash(\beta\supset\alpha)\wedge\gamma\equiv\gamma$ [2.22, (4), 2.26, I]

[22]) The formal proofs of T. 13-T. 17 are omitted; they coincide with the formal proofs of T. 13-T. 17 in the ordinary functional calculus.

Lemma 1.2. *If* $\vdash(\beta\supset\alpha)\wedge\gamma\equiv\gamma$, *then* $\vdash\alpha\wedge(\beta\wedge\gamma)\equiv\beta\wedge\gamma$ (*for arbitrary* $\alpha,\beta,\gamma\in\boldsymbol{H}$).

Proof. Suppose $\vdash(\beta\supset\alpha)\wedge\gamma\equiv\gamma$, then

(1) $\vdash\gamma\supset(\beta\supset\alpha)\wedge\gamma$ [I, 2.22]

(2) $\vdash\beta\wedge\gamma\supset(\beta\wedge(\beta\supset\alpha))\wedge\gamma$ [(1), A. 3, A. 2, 2.29, 2.32, 2.29]

(3) $\vdash\beta\wedge(\beta\supset\alpha)\supset\alpha\wedge\beta$ [A. 6, 2.22, 2.26, 2.24, I, 2.2]

(4) $\vdash\beta\wedge\gamma\supset\alpha\wedge(\beta\wedge\gamma)$ [(2), (3), A. 3, 2.29, 2.3, 2.29]

(5) $\vdash\alpha\wedge(\beta\wedge\gamma)\equiv\beta\wedge\gamma$ [2.22, (4), 2.26, I]

Lemma 1.3. *If* $\vdash\alpha$ *and* $\vdash\beta$, *then* $\vdash\alpha\equiv\beta$ (*for arbitrary* $\alpha,\beta\in\boldsymbol{H}$).

Proof. [2.26, T. 8].

Lemma 1.4. *If* $\vdash\alpha$, *then* $\vdash(x_k)\alpha$ (*for arbitrary* $\alpha\in\boldsymbol{H}$).

Lemma 1.5. *If* k, i, *are arbitrary positive integers such that neither of* $(\exists x_i)$ *and* (x_i), *nor* x_i *itself occurs in* $\alpha(x_k)\in\boldsymbol{H}$, *then*

$$\vdash(x_k)\,\alpha(x_k)\equiv(x_i)\,\alpha(x_i)\quad\textit{and}\quad\vdash(\exists x_k)\,\alpha(x_k)\equiv(\exists x_i)\,\alpha(x_i),$$

where $\alpha(x_i)$ *arises from* $\alpha(x_k)$ *by the substitution the variable* x_i *for the variable* x_k.

The proofs of Lemmas 1.4 and 1.5 are obvious.

§ 2. Functional calculus of Lewis.

We shall refer to the functional calculus of Lewis as the system $\mathcal{L}$. $\mathcal{L}$ can be described briefly as follows:

The symbols of the system are: the individual variables $x_1, x_2, \ldots$, the sentential variables $\alpha_1, \alpha_2, \ldots$, the k-argument functional variables $F_1^k, F_2^k, \ldots$ $(k=1,2,\ldots)$, constants and parentheses.

The constants are: the negation sign $\sim$, the conjunction sign $\wedge$, the sign of possibility $\Diamond$, the sign of the general quantifier (x_k) and the sign of the existential quantifier $(\exists x_k)$.

The *class of formulae* of the system $\mathcal{L}$ is the smallest class L which contains all sentential variables, all expressions of the form $F_i^k(x_{j_1},\ldots,x_{j_k})$ and which is closed under the following five operations: forming the conjunction $(\alpha\wedge\beta)$ of two expressions α and β, taking the negation $(\sim\alpha)$ or the possibility $(\Diamond\alpha)$ of an expression α, and putting the existential quantifier $(\exists x_k)$ or the universal quantifier (x_k) in front of an expression α to obtain the expressions $((\exists x_k)\alpha)$ or $((x_k)\alpha)$ respectively.

The meaning of *bound* and *free* occurrences of individual variables remains the same as in the system $\mathcal{H}$. The same applies to the notation $\alpha(x_{k_1},\ldots,x_{k_n})$.

We introduce the following abbreviations [23]:

I. $(\alpha\vee\beta)$ for $(\sim((\sim\alpha)\wedge(\sim\beta)))$,
II. $(\alpha\supset\beta)$ for $(\sim(\alpha\wedge(\sim\beta)))$,
III. $(\alpha\equiv\beta)$ for $((\alpha\supset\beta)\wedge(\beta\supset\alpha))$,
IV. $(\alpha<\beta)$ for $(\sim(\Diamond(\alpha\wedge(\sim\beta))))$,
V. $(\alpha=\beta)$ for $((\alpha<\beta)\wedge(\beta<\alpha))$.

In writing formulae, we shall practice the omission of parantheses, the rule being that: (1) each of the operators $\wedge$, $\vee$, $\supset$, $<$, $\equiv$, $=$ binds two expressions less strongly than the previous one; (2) each of the operators $\sim$ and $\Diamond$ binds an expression more strongly than any one of the two-argument operators; (3) the quantifiers bind them more strongly than any one of the operators just listed.

If α,β,γ are arbitrary formulae, the following formulae are called *axioms* [24]:

B*. 1 $\vdash\alpha\wedge\beta<\beta\wedge\alpha$,
B*. 2 $\vdash\alpha\wedge\beta<\alpha$,
B*. 3 $\vdash\alpha<\alpha\wedge\alpha$,
B*. 4 $\vdash(\alpha\wedge\beta)\wedge\gamma<\alpha\wedge(\beta\wedge\gamma)$,
B*. 5 $\vdash\alpha<\sim\sim\alpha$,
B*. 6 $\vdash(\alpha<\beta)\wedge(\beta<\gamma)<(\alpha<\gamma)$,
B*. 7 $\vdash\alpha\wedge(\alpha<\beta)<\beta$,
B*. 8 $\vdash\Diamond(\alpha\wedge\beta)<\Diamond\alpha$,
C*. 10.1 $\vdash\Diamond\Diamond\alpha=\Diamond\alpha$,
D. 1 $\vdash(x_k)\alpha<\alpha$,
D. 2 $\vdash\alpha<(\exists x_k)\alpha$.

There are six *rules of inference* in the system $\mathcal{L}$:

R. 2.1 *modus ponens*: from α and $\alpha<\beta$ to infer β;

R. 2.2 the *rule of adjunction*: from α and β to infer $\alpha\wedge\beta$;

R. 2.3 the *rule of replacement*: if β occurs as a part of $\alpha(\beta_1)$, then from $\beta_1=\beta_2$ we infer $\alpha(\beta_2)$, where $\alpha(\beta_2)$ is the formula obtained from $\alpha(\beta_1)$ by substitution of β_2 for β_1;

R. 2.4 the *rule of substitution* for individual variables;

R. 2.5 the *rule for* (x_k): from $\alpha<\beta$ we infer $\alpha<(x_k)\beta$ provided that no free occurrence of x_k appears in α;

R. 2.6 the *rule for* $(\exists x_k)$: from $\alpha<\beta$ we infer $(\exists x_k)\alpha<\beta$ provided that no free occurrence of x_k appears in β.

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[23]) See the definitions 11.01 (p. 123), 14.01, 14.02 (p. 136), 11.02 and 11.03 (p. 124) of Lewis and Langford [1].

[24]) The axioms B*.1-B*.8, C*.10.1 are substitutions of the axioms of the system S.4 of the sentential calculus of Lewis. See Lewis and Langford [1], pp. 493 and 497.

The notions of a formal proof and of a provable formula in the system $\mathcal{L}$ are analogous to those of a formal proof and a provable formula in $\mathcal{H}$ [25]).

It is easy to show that every formula of the system $\mathcal{L}$ which is a substitution of a provable formula of the system S. 4 of the sentential calculus of Lewis [26]), is also provable in $\mathcal{L}$.

Let $\alpha,\beta,\gamma,\delta$, be arbitrary formulae of $\mathcal{L}$. It follows from the above remark that the following formulae are provable in $\mathcal{L}$:

T*. 1	$\vdash\alpha=\alpha$	[12.11]
T*. 2	$\vdash\alpha\wedge\beta<\beta$	[12.17]
T*. 3	$\vdash\alpha=\sim\sim\alpha$	[12.3]
T*. 4	$\vdash\alpha<\beta=\sim\beta<\sim\alpha$	[12.44]
T*. 5	$\vdash\alpha\vee\beta=\beta\vee\alpha$	[13.11]
T*. 6	$\vdash\alpha<\beta\vee\alpha$	[13.21]
T*. 7	$\vdash(\alpha\vee\beta)\vee\gamma=\alpha\vee(\beta\vee\gamma)$	[13.41]
T*. 8	$\vdash\alpha\vee\sim\alpha$	[13.5]
T*. 9	$\vdash(\alpha<\beta)<(\alpha\supset\beta)$	[14.1]
T*. 10	$\vdash\sim(\sim\alpha\vee\sim\beta)=\alpha\wedge\beta$	[14.21]
T*. 11	$\vdash\alpha<\Diamond\alpha$	[18.4]
T*. 12	$\vdash\alpha<\beta=\sim\Diamond\sim(\alpha\supset\beta)$	[18.7]
T*. 13	$\vdash\sim\Diamond\sim(\alpha\vee\sim\alpha)$	[18.81]
T*. 14	$\vdash\alpha=\alpha\wedge\beta\vee\alpha\wedge\sim\beta$	[18.92]
T*. 15	$\vdash\alpha\wedge\sim\alpha\vee\beta=\beta$	[19.58]
T*. 16	$\vdash(\alpha<\gamma)\wedge(\beta<\gamma)<((\alpha\vee\beta)<\gamma)$	[19.65]
T*. 17	$\vdash\sim\Diamond\sim\alpha<(\beta<\alpha)$	[19.75]
T*. 18	$\vdash\sim\Diamond\sim\alpha\wedge\sim\Diamond\sim\beta=\sim\Diamond\sim(\alpha\wedge\beta)$	[19.81]
T*. 19	$\vdash\Diamond(\alpha\vee\beta)=\Diamond\alpha\vee\Diamond\beta$	[19.82]
T*. 20	$\vdash\sim\Diamond\sim\alpha\wedge\sim\Diamond\sim\beta<(\alpha=\beta)$	[19.84]
T*. 21	$\vdash\sim\Diamond\sim\alpha=\sim\Diamond\sim\sim\Diamond\sim\alpha$	[C. 10]
T*. 22	$\vdash\Diamond(\alpha\wedge\sim\alpha)=\alpha\wedge\sim\alpha$	
T*. 23	$\vdash(\alpha<\beta)<(\Diamond\alpha<\Diamond\beta)$.	

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[25]) See p. 102 and 103.

[26]) By the system S.4 of the sentential calculus of Lewis we understand the system based on the following axioms:

B.1 $a_1\wedge a_2<a_2\wedge a_1$, B.2 $a_1\wedge a_2<a_1$, B.3 $a_1<a_1\wedge a_1$, B.4 $(a_1\wedge a_2)\wedge a_3<a_1\wedge(a_2\wedge a_3)$, B.5 $a_1<\sim\sim a_1$, B.6 $(a_1<a_2)\wedge(a_2<a_3)<(a_1<a_3)$, B.7 $a_1\wedge(a_1<a_2)<a_2$, B.8 $\Diamond(a_1\wedge a_2)<\Diamond a_1$, C. 10 $\Diamond\Diamond a_1=\Diamond a_1$.

There are four rules of inference: R. 2.1, R. 2.2, R. 2.3 and the *rule of substitution* for sentential variables. For information regarding the system S.4 the reader is referred to Lewis and Langford [1].

The numbers in brackets refer to the numbers (see Lewis and Langford [1]) of the formulae of S. 4, from which the formulae T*.1-T*.21 are obtained by substitution. T*. 22 is a substitution of a provable formula the formal proof of which is given by McKinsey [8], p. 126. T*. 23 is a substitution of the axiom L. 12 in McKinsey and Tarski [3], p. 2.

The following formulae are also provable [27]) in $\mathcal{L}$.

T*. 24 $\vdash \sim(\sim\alpha \vee \sim\beta) \vee \sim(\sim\alpha \vee \beta) = \alpha$

Proof. 24.1 $\vdash \alpha = \sim(\sim\alpha \vee \sim\beta) \vee \sim(\sim\alpha \vee \sim\sim\beta)$ [T*. 14, T*. 10, R. 2.3]

24.2 $\vdash \sim(\sim\alpha \vee \sim\beta) \vee \sim(\sim\alpha \vee \beta) = \alpha$ [R. 2.3, 24.1, T*. 3]

T*. 25 $\vdash \alpha \vee \Diamond\alpha = \Diamond\alpha$

Proof. 25.1 $\vdash \Diamond\alpha < \Diamond\alpha$ [T*. 1, V, B*. 2]

25.2 $\vdash \alpha \vee \Diamond\alpha < \Diamond\alpha$ [T*. 11, 25.1, R. 2.2, T*. 16]

25.3 $\vdash \alpha \vee \Diamond\alpha = \Diamond\alpha$ [25.2, T*. 6, R. 2.2, V]

T*. 26 $\vdash \beta \vee (\alpha \vee \sim\alpha) = \alpha \vee \sim\alpha$

Proof. 26.1 $\vdash \beta < \alpha \vee \sim\alpha$ [T*. 17, T*. 13]

26.2 $\vdash \alpha \vee \sim\alpha < \alpha \vee \sim\alpha$ [T*. 1, V, B*. 2]

26.3 $\vdash \beta \vee (\alpha \vee \sim\alpha) < \alpha \vee \sim\alpha$ [26.1, 26.2, R. 2.2, T*. 16]

26.4 $\vdash \beta \vee (\alpha \vee \sim\alpha) = \alpha \vee \sim\alpha$ [26.3, T*. 6, R. 2.2, V]

T*. 27 $\vdash (\alpha = \beta) = \sim\Diamond\sim(\alpha \equiv \beta)$

Proof. 27.1 $\vdash (\alpha < \beta) \wedge (\beta < \alpha) = \sim\Diamond\sim(\alpha \supset \beta) \wedge \sim\Diamond\sim(\beta \supset \alpha)$ [T*. 1, T*. 12, R. 2.3]

27.2 $\vdash (\alpha < \beta) \wedge (\beta < \alpha) = \sim\Diamond\sim((\alpha \supset \beta) \wedge (\beta \supset \alpha))$ [27.1, T*. 18, R. 2.3]

27.3 $\vdash (\alpha = \beta) = \sim\Diamond\sim(\alpha \equiv \beta)$ [27.2, III, 27.3, V]

T*. 28 $\vdash (\alpha < \beta) < (\sim\Diamond\sim\alpha < \sim\Diamond\sim\beta)$ [T*. 23, T*. 4, R. 2.3]

T*. 29 $\vdash \alpha = (x_k)\alpha$ provided that there is no free occurrence of x_k in α [D. 1, T*. 1, V, B*. 2, R. 2.5, R. 2.2, V]

T*. 30 $\vdash \alpha = (\exists x_k)\alpha$ provided that there is no free occurrence of x_k in α [D. 2, T*. 1, V, B*. 2, R. 2.6, R. 2.2, V]

[27]) In T*. 24-T*. 30 we do not mention applications of the rule R. 2.1.

Lemma 2.1 [28]). *If* $\vdash \alpha$, *then* $\vdash \sim\Diamond\sim\alpha$ *(for arbitrary* $\alpha \in \boldsymbol{L}$*)*.

This lemma can be established by a simple induction on the length of the formal proof of α, making use of T*. 12, T*. 27, T*. 21, T*. 28 and T*. 18.

Lemma 2.2. *Given any positive integers* k, i, *if neither of* $(\exists x_i)$ *and* (x_i), *nor* x_i *itself occurs in* $\alpha(x_k) \in \boldsymbol{L}$, *then* $\vdash (x_k)\alpha(x_k) = (x_i)\alpha(x_i)$ *and* $\vdash (\exists x_k)\alpha(x_k) = (\exists x_i)\alpha(x_i)$, *where* $\alpha(x_i)$ *arises from* $\alpha(x_k)$ *by the substitution the variable* x_i *for the variable* x_k.

The proof of Lemma 2.2 is obvious.

§ 3. Extensions of closure algebras and of Brouwerian algebras.

The purpose of this section is to prove that, for every Brouwerian algebra $\mathfrak{B}$, there exists a complete Brouwerian algebra $\mathfrak{B}_e$ such that: 1° $\mathfrak{B}_e$ is an extension of $\mathfrak{B}$, 2° $\mathfrak{B}_e$ preserves all (finite and infinite) sums and products of $\mathfrak{B}$. This result follows from an analogous statement on extensions of closure algebras which is a consequence of some theorems given by Tarski and McKinsey and by MacNeille.

Definition 3.1. By an *abstract algebra* we mean an ordered class $\mathfrak{A} = \langle K, o_1, \ldots, o_n \rangle$, where K is an arbitrary non-empty set and $o_1, \ldots, o_n$ are arbitrary operations on (finitely many) elements of K. We assume that K is closed under these operations.

Definition 3.2. By a *subalgebra* of an abstract algebra $\mathfrak{A} = \langle K, o_1, \ldots, o_n \rangle$ we mean an algebra $\mathfrak{A}_s = \langle K_s, o_1, \ldots, o_n \rangle$, where K_s is a subset of K.

Definition 3.3. An algebra $\mathfrak{A}_e$ is called an *extension* of an algebra $\mathfrak{A}$, if $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}_e$.

Definition 3.4. An algebra $\mathfrak{A} = \langle K, +, \cdot \rangle$ is called a *lattice* [29]) if for every $x, y, z \in K$ the following axioms are satisfied: (1) $x \cdot x = x$, (2) $x + x = x$, (3) $x \cdot y = y \cdot x$, (4) $x + y = y + x$, (5) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, (6) $x + (y + z) = (x + y) + z$, (7) $x \cdot (x + y) = x$, (8) $x + x \cdot y = x$. If $x + y = y$, we write $x \leqslant y$. The *zero* element and the *unit* element of $\mathfrak{A}$, whenever they exist, will be denoted by „0" and „1" respectively; by definition: $0 \leqslant x$ and $x \leqslant 1$ for every $x \in K$.

[28]) The similar theorem for formulae of the system S. 4 of the sentential calculus of Lewis is proved by McKinsey and Tarski [3], p. 5. This theorem was taken by Gödel as a primitive rule of inference in the formalization of the system S. 4. See Gödel [2].

[29]) See Birkhoff [1], p. 18.

Definition 3.5. Let $\mathfrak{A}=\langle K,+,\cdot\rangle$ be a lattice and let $x_i \in K$ for every $i \in I$. The element $x \in K$ is said to be the *product (sum)* of all x_i in $\mathfrak{A}$, in symbols $x=\prod_{i\in I} x_i$ $(x=\sum_{i\in I} x_i)$, provided that 1^0 $x \leqslant x_i$ $(x_i \leqslant x)$ for every x_i, where $i \in I$, 2^0 if $y \leqslant x_i$ $(x_i \leqslant y)$ for every x_i, where $i \in I$, then $y \leqslant x$ $(x \leqslant y)$. Let $\mathfrak{A}_e$ be an extension of $\mathfrak{A}$. By saying that $\mathfrak{A}_e$ *preserves* all sums and products of $\mathfrak{A}$, we shall understand that, if $a, b, x_i \in K$ and $a=\prod_{i\in I} x_i$, $b=\sum_{i\in I} x_i$ in $\mathfrak{A}$, then also $a=\prod_{i\in I} x_i$ and $b=\sum_{i\in I} x_i$ in $\mathfrak{A}_e$.

Definition 3.6. By a *Boolean algebra* we shall mean every algebra $\mathfrak{A}=\langle K, +, \cdot, -\rangle$, where K consists of at least two different elements and for all $x,y,z \in K$ the following axioms are satisfied [30]: (1) $x+y=y+x$, (2) $(x+y)+z=x+(y+z)$, (3) $-(-x+-y)+-(-x+y)=x$, (4) $x\cdot y=-(-x+-y)$. It is known that, if $\mathfrak{A}$ is a Boolean algebra, then $\langle K, +, \cdot\rangle$ is a lattice.

Definition 3.7. An algebra $\mathfrak{C}=\langle K, +, \cdot, -, C\rangle$ is said to be a *closure algebra* [31], if $\langle K+, \cdot, -\rangle$ is a Boolean algebra and, for every $x,y \in K$, the following axioms are satisfied: (1) $x \leqslant Cx$, (2) $CCx=Cx$, (3) $C(x+y)=Cx+Cy$, (4) $C0=0$.

Definition 3.8. An element x of a closure algebra is said to be *closed*, if $x=Cx$.

Definition 3.9. An algebra $\mathfrak{B}=\langle K, +, \cdot, \dot{-}, \urcorner\rangle$ is called a *Brouwerian algebra*, if 1^0 $\langle K, +, \cdot\rangle$ is a lattice with 1, 2^0 for all $x,y,z \in K$, the formulae $x \dot{-} y \leqslant z$ and $x \leqslant y+z$ are equivalent, 3^0 $\urcorner x = 1 \dot{-} x$ for each $x \in K$.

Definition 3.10. A lattice (Boolean, closure, Brouwerian algebra) is said to be *complete*, if, for every subset of elements of $\mathfrak{A}$, there exist the sum and the product.

Definition 3.11. Let $\mathfrak{C}=\langle K, +, \cdot, -, C\rangle$ be a closure algebra. Let K^* be the set of all closed elements of K. Then, by the *algebra of closed elements of* $\mathfrak{C}$ [32], we mean the algebra $\mathfrak{C}^*=\langle K^*, +, \cdot, \dot{-}, \urcorner\rangle$, where $x \dot{-} y = C(x\cdot -y)$ and $\urcorner x = C(-x)$.

It is known [33] that $\mathfrak{C}^*$ is a Brouwerian algebra.

[30]) See Huntington [1].
[31]) See McKinsey and Tarski [2], p. 146.
[32]) See McKinsey and Tarski [2], p. 130.
[33]) See ibidem, p. 130.

Lemma 3.12. *Let $\mathfrak{A}=\langle K, +, \cdot, -\rangle$ be a complete Boolean algebra, let C be a unary operation defined over a subalgebra $\mathfrak{A}_s=\langle K_s, +, \cdot, -\rangle$ of $\mathfrak{A}$ in such a way, that $\mathfrak{A}_{sc}=\langle K_s, +, \cdot, -, C\rangle$ is a closure algebra. For every $x \in K$, let $C_e x$ be the product (in $\mathfrak{A}$) of all closed elements $y \in K_s$ such that $x \leqslant y$ and $Cy=y$. (Since $\mathfrak{A}$ is complete, this product always exists). Then $\mathfrak{A}_c=\langle K, +, \cdot, -, C_e\rangle$ is a closure algebra and $\mathfrak{A}_{sc}$ is the subalgebra of $\mathfrak{A}_c$.*

This lemma follows immediately from a more general lemma given by McKinsey and Tarski [34].

Theorem 3.13. *Let $\mathfrak{C}=\langle K, +, \cdot, -, C\rangle$ be a closure algebra. Then there exists a complete closure algebra $\mathfrak{C}_e$ such that 1^0 $\mathfrak{C}_e$ is an extension of $\mathfrak{C}$, 2^0 $\mathfrak{C}_e$ preserves [35] all sums and products of $\mathfrak{C}$.*

Proof. Let $\mathfrak{A}_e=\langle K_e, +, \cdot, -\rangle$ be the minimal extension [36] (in the sense determined by MacNeille) of the Boolean algebra $\mathfrak{A}=\langle K, +, \cdot, -\rangle$. It is known [37] that $\mathfrak{A}_e$ preserves all sums and products of $\mathfrak{A}$. Let C_e (be the operation defined over the algebra $\mathfrak{A}_e$ in the same way as in Lemma 3.12. It follows, from Lemma 3.12, that $\mathfrak{C}_e=\langle K_e, +, \cdot, -, C_e\rangle$ is a closure algebra and $\mathfrak{C}$ is a subalgebra of $\mathfrak{C}_e$, q. e. d.

The closure algebra $\mathfrak{C}_e$, obtained in this way from $\mathfrak{C}$, will be called a *minimal closure extension* [38] of $\mathfrak{C}$.

Theorem 3.14. *For every Brouwerian algebra $\mathfrak{B}$ there exists a closure algebra $\mathfrak{C}$ such that $\mathfrak{B}=\mathfrak{C}^*$* (where $\mathfrak{C}^*$ is the algebra of closed elements of $\mathfrak{C}$).

This theorem is proved by McKinsey and Tarski [39].

Lemma 3.15. *Let $\mathfrak{C}^*$ be the algebra of closed elements of a closure algebra $\mathfrak{C}=\langle K, +, \cdot, -, C\rangle$. Let $b \in K$ and $x_i=Cx_i \in K$ for $i \in I$. Then the conditions $b=\prod_{i\in I} x_i$ in $\mathfrak{C}$ and $b=\prod_{i\in I} x_i$ in $\mathfrak{C}^*$ are equivalent.*

Proof. Suppose $b=\prod_{i\in I} x_i$ in $\mathfrak{C}$, i. e., b is the largest element of K such that $b \leqslant x_i = Cx_i$ for $i \in I$. Consequently, $Cb \leqslant x_i$. Therefore $b=Cb$ and b is the largest closed element such that $b \leqslant x_i$ for $i \in I$. Hence, $b=\prod_{i\in I} x_i$ in $\mathfrak{C}^*$.

[34]) See McKinsey and Tarski [1], p. 148.
[35]) See Definition 3.5.
[36]) See MacNeille [1], p. 437.
[37]) This was prooved by MacNeille [1].
[38]) See Sikorski [1], p. 174.
[39]) See McKinsey and Tarski [2], p. 130.

Conversely, if $b = Cb = \prod_{i \in I} x_i$ in $\mathfrak{C}^*$ and $a \in K$ satisfies $b \leqslant a \leqslant x_i$ for $i \in I$, then $b \leqslant Ca \leqslant x_i$ for $i \in I$. Hence, by the definition of the product in $\mathfrak{C}^*$, $Ca \leqslant b$. Therefore $b = Ca$ and $b = a$. Hence, b is the largest element of $\mathfrak{C}$ such that $b \leqslant x_i$ for every $i \in I$, i. e. $b = \prod_{i \in I} x_i$ in $\mathfrak{C}$, q. e. d.

Lemma 3.16. *Let $\mathfrak{C}$ be a complete closure algebra. Then, the algebra $\mathfrak{C}^*$ of closed elements of $\mathfrak{C}$ is also complete. Moreover, if $a = \sum_{i \in I} x_i$ in $\mathfrak{C}$, where $x_i = Cx_i$ for $i \in I$, then $Ca = \sum_{i \in I} x_i$ in $\mathfrak{C}^*$.*

Proof. Let us start with the second part of 3.16. If $b = Cb \geqslant x_i$ for every $i \in I$, then $b \geqslant \sum_{i \in I} x_i = a$.

Hence, $b \geqslant Ca \geqslant x_i$ for $i \in I$ which proves that $Ca = \sum_{i \in I} x_i$ in $\mathfrak{C}^*$.

The first part of 3.16 follows from the second one and from 3.15.

Lemma 3.17. *Let $\mathfrak{C} = \langle K, +, \cdot, -, C \rangle$ be a closure algebra, and let $\mathfrak{C}_e = \langle K_e, +, \cdot, -, C_e \rangle$ be the minimal closure extension of $\mathfrak{C}$. Let $\mathfrak{C}^* = \langle K^*, +, \cdot, \dot{-}, \urcorner \rangle$ and $\mathfrak{C}_e^* = \langle K_e^*, +, \cdot, \dot{-}_e, \urcorner_e \rangle$ be the algebras of closed elements of $\mathfrak{C}$ and $\mathfrak{C}_e$ respectively. Then, $\mathfrak{C}_e^*$ is an extension of $\mathfrak{C}^*$, and $\mathfrak{C}_e^*$ preserves all sums and products of the algebra $\mathfrak{C}^*$.*

Proof. The first statement is obvious. Suppose $a = \sum_{i \in I} x_i$ in $\mathfrak{C}^*$, where $a, x_i \in K$. Let $c = \sum_{i \in I} x_i$ in $\mathfrak{C}_e^*$. As a result of 3.16, $c = C_e d$, where $d = \sum_{i \in I} x_i$ in $\mathfrak{C}_e$. Since $a = \sum_{i \in I} x_i$ in $\mathfrak{C}^*$, we obtain by the definition of the operation C_e (Lemma 3.12) $C_e d = a$. Hence, $a = c$. Now, suppose $b = \prod_{i \in I} x_i$ in $\mathfrak{C}^*$, where $b, x_i \in K^*$. It follows, from 3.15, $b = \prod_{i \in I} x_i$ in $\mathfrak{C}$. Since $\mathfrak{C}_e$ is the minimal closure extension of $\mathfrak{C}$ [40]), we have $b = \prod_{i \in I} x_i$ in $\mathfrak{C}_e$. Hence, by 3.15, $b = \prod_{i \in I} x_i$ in $\mathfrak{C}_e^*$, q. e. d.

Theorem 3.18. *Let $\mathfrak{B}$ be a Brouwerian algebra. Then, there exists a complete Brouwerian algebra $\mathfrak{B}_e$ such that 1⁰ $\mathfrak{B}_e$ is an extension of $\mathfrak{B}$ and 2⁰ $\mathfrak{B}_e^*$ preserves all sums and products of $\mathfrak{B}$.*

Proof. It follows, from Lemma 3.14, that there is a closure algebra $\mathfrak{C}$ such that $\mathfrak{B} = \mathfrak{C}^*$, where $\mathfrak{C}^*$ is the algebra of closed elements of $\mathfrak{C}$. Let $\mathfrak{C}_e$ be the minimal closure extension of $\mathfrak{C}$ and let $\mathfrak{C}_e^*$ be the algebra of closed elements of $\mathfrak{C}_e$. Then, by Lemmas 3.16 and 3.17, $\mathfrak{C}_e^*$ is complete and preserves all sums and products of $\mathfrak{C}^* = \mathfrak{B}$, q. e. d. [41]).

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[40]) See Lemma 3.13.

[41]) See Lemma 3.13.

§ 4. Completeness of the functional calculus of Heyting.

The purpose of this section is to establish the completeness of the system $\mathcal{H}$, in the sense mentioned in the introduction [42]).

In this section, let $\alpha, \beta, \gamma, \delta$ always be arbitrary formulae of $\mathcal{H}$ and let I_0 be the set of all positive integers.

Definition 4.1. Let $\mathcal{F}^k(I, \mathfrak{A})$ be the set of all k-argument ($k = 1, 2, \ldots$) functions the arguments of which run over a non-empty abstract set I and the values belong to an abstract algebra $\mathfrak{A}$. A function $\Phi = \Phi(x_{i_1}, \ldots, x_{i_n}, a_{j_1}, \ldots, a_{j_m}, F_{p_1}^{k_1}, \ldots, F_{p_r}^{k_r})$ is an $(I, \mathfrak{A})$-functional [43]), if its values belong to $\mathfrak{A}$, and if it has

1⁰ n arguments $x_{i_1}, \ldots, x_{i_n}$ running over I,

2⁰ m arguments $a_{j_1} \ldots a_{j_m}$ running over the set of all elements of $\mathfrak{A}$,

3⁰ r arguments $F_{p_1}^{k_1} \ldots, F_{p_r}^{k_r}$ running over $\mathcal{F}^{k_1}(I, \mathfrak{A}), \ldots, \mathcal{F}^{k_r}(I, \mathfrak{A})$ respectively.

Let $\mathfrak{B}$ be a complete Brouwerian algebra and let I be a non-empty abstract set. Then, each formula

$$\alpha = \alpha(x_{i_1}, \ldots, x_{i_n}, a_{j_1}, \ldots, a_{j_m}, F_{p_1}^{k_1}, \ldots, F_{p_r}^{k_r})$$

of $\mathcal{H}$ with n individual variables, m sentential variables, and r functional variables may be interpreted as an $(I, \mathfrak{A})$-functional

$$\Phi_\alpha = \Phi_\alpha(x_{i_1}, \ldots, x_{i_n}, a_{j_1}, \ldots, a_{j_m}, F_{p_1}^{k_1}, \ldots, F_{p_r}^{k_r})$$

by considering

(1⁰) the individual variables of α to be variables running over I,

(2⁰) the sentential variables of α to be variables running over the set of all elements of the algebra $\mathfrak{B}$,

(3⁰) the functional variables with k arguments to be variables running over $\mathcal{F}^k(I, \mathfrak{B})$,

(4⁰) the logical operations $\vee, \wedge, \sim, (x_k), (\exists x_k)$ to be the operations $\cdot, +, \urcorner, \sum_{x_k \in I}, \prod_{x_k \in I}$ of the algebra $\mathfrak{B}$, respectively. The logical operations $\supset$ will be interpreted as the operation $\dot{-}$ (of the algebra $\mathfrak{B}$) with the converse order of arguments [44]).

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[42]) See p. 101.

[43]) See Mostowski [2], p. 204.

[44]) That is $\Phi_{\alpha \supset \beta} = \Phi_\beta \dot{-} \Phi_\alpha$.

The following theorem was proved by Mostowski[45]):

(*) *If* $\vdash \alpha$, *then the* $(I,\mathfrak{B})$*-functional* Φ_α *is identically equal to the zero element of* $\mathfrak{B}$[46]) *for every complete Brouwerian algebra* $\mathfrak{B}$ *and for every non-void set* I.

We prove now the fundamental theorem of this section:

Theorem 4.2. *There exists a complete Brouwerian algebra* $\mathfrak{B}_0$ *such that, for every* α, *if* α *is not provable then* $(I_0,\mathfrak{B}_0)$*-functional* Φ_α *is not identically equal to* 0.

The proof of 4.2 is based on the following lemmas. First of all, we introduce the binary relation $\cong$ defined by

$$\alpha\cong\beta \quad \text{if and only if} \quad \vdash \alpha\equiv\beta.$$

Lemma 4.3. *The relation* $\cong$ *is a congruence relation, i. e. the following nine conditions are satisfied:*

(1) $\alpha\cong\alpha$,
(2) *if* $\alpha\cong\beta$, *then* $\beta\cong\alpha$,
(3) *if* $\alpha\cong\beta$ *and* $\beta\cong\gamma$, *then* $\alpha\cong\gamma$,
(4) *if* $\alpha\cong\beta$, *then* $\sim\alpha\cong\sim\beta$,
(5) *if* $\alpha\cong\beta$, *then* $(x_k)\alpha\cong(x_k)\beta$,
(6) *if* $\alpha\cong\beta$, *then* $(\exists x_k)\,\alpha\cong(\exists x_k)\,\beta$,
(7) *if* $\alpha\cong\beta$ *and* $\gamma\cong\delta$, *then* $\alpha\wedge\gamma\cong\beta\wedge\delta$,
(8) *if* $\alpha\cong\beta$ *and* $\gamma\cong\delta$, *then* $\alpha\vee\gamma\cong\beta\vee\delta$,
(9) *if* $\alpha\cong\beta$ *and* $\gamma\cong\delta$, *then* $\alpha\supset\gamma\cong\beta\supset\delta$.

Proof. (1) and (2) follow from T.1 and A. 2, I (§ 1), respectively. (3) follows from 2.2, 2.22, 2.26, A. 4, R. 1.1 (§ 1). The proof of (4) is based on 2.2, 2.22, 4.2, 2.26 and R. 1.2 (§ 1). (5) and (6) are proved by using 2.2, 2.22, Lemma 1.4, T. 13 (in the case of (5)), T. 14 (in the case of (6)), 2.26, and R. 1.2 (§ 1). (7) and (8) follow from 2.2, 2.26, 2.23 (in the case of (7)), 3.3 (in the case of (8)). The proof of (9) is based on 2.2, 2.22, 2.29, 2.291, 2.26, A. 4 and R. 1.1 (§ 1).

For every $\alpha\in\boldsymbol{H}$[47]), let $[\alpha]$ be the set of all $\beta\in\boldsymbol{H}$ such that $\alpha\cong\beta$. Obviously $[\alpha]=[\beta]$ if and only if $\alpha\cong\beta$. In view of Lemma 4.3 the following definition may be introduced:

[45]) See Mostowski [2], p. 205.

[46]) By saying that $(I,\mathfrak{B})$-functional Φ_α *is identically equal to the zero element of* $\mathfrak{B}$, we mean that this functional assumes the value 0 for every choice of arguments.

[47]) See § 1, p. 101.

Definition 4.4. By a *Lindenbaum algebra*[48]) for the system $\boldsymbol{H}$ we mean the algebra $\mathfrak{B}_h=\langle K_h,+,\cdot,\dot{-},\urcorner\rangle$ defined as follows: 1) K_h is the set of all classes $[\alpha]$ such that $\alpha\in\boldsymbol{H}$, 2) For every $[\alpha],[\beta]\in K_h$, we put

$$[\alpha]+[\beta]=[\alpha\wedge\beta],\quad [\alpha]\cdot[\beta]=[\alpha\vee\beta],$$
$$[\alpha]\dot{-}[\beta]=[\beta\supset\alpha],\quad \urcorner[\alpha]=[\sim\alpha].$$

Lemma 4.5. (i) $\mathfrak{B}_h$ *is a Brouwerian algebra*, (ii) $[\sim(\alpha\supset\alpha)]=1$, (iii) $[\alpha\supset\alpha]=0$.

Proof. It is easy to see that $\mathfrak{B}_h$ is a lattice. In fact, this follows from Definition 3.4 and the following equations:

$$[\alpha]\cdot[\alpha]=[\alpha] \quad [\text{by T. 5}],\qquad [\alpha]+[\alpha]=[\alpha]\quad [\text{by T. 4}],$$
$$[\alpha]\cdot[\beta]=[\beta]\cdot[\alpha] \quad [\text{by T. 3}],\qquad [\alpha]+[\beta]=[\beta]+[\alpha]\quad [\text{by T. 2}],$$
$$[\alpha]\cdot([\beta]\cdot[\gamma])=([\alpha]\cdot[\beta])\cdot[\gamma] \quad [\text{by T. 7}],$$
$$[\alpha]+([\beta]+[\gamma])=([\alpha]+[\beta])+[\gamma] \quad [\text{by T. 6}],$$
$$[\alpha]\cdot([\alpha]+[\beta])=[\alpha] \quad [\text{by T. 10}],$$
$$[\alpha]+([\alpha]\cdot[\beta])=[\alpha] \quad [\text{by T. 9}].$$

Then, we prove that

$$[\alpha]\dot{-}[\beta]\leqslant\gamma \quad \text{if and only if} \quad [\alpha]\leqslant[\beta]+[\gamma].$$

Suppose $[\alpha]\dot{-}[\beta]\leqslant\gamma$. Then, by Definition 3.4, $([\alpha]\dot{-}[\beta])+[\gamma]=[\gamma]$, or $[(\beta\supset\alpha)\wedge\gamma]=[\gamma]$. Hence, by Lemma 1.2, $\vdash\alpha\wedge(\beta\wedge\gamma)\equiv\beta\wedge\gamma$, so that $[\alpha\wedge(\beta\wedge\gamma)]=[\beta\wedge\gamma]$. Consequently $[\alpha]+([\beta]+[\gamma])=[\beta]+[\gamma]$, or $[\alpha]\leqslant[\beta]+[\gamma]$.

The proof of the converse implication can be carried through in a similar way, by using Lemma 1.1.

Remark (ii) follows from T. 11 and Definition 3.4. In view of (ii) and Definition 3.9, we put for every $[\beta]\in K_h$

$$\urcorner[\beta]=1\dot{-}[\beta].$$

Hence, by T. 12, we obtain $\urcorner[\beta]=[\sim\beta]$. In this way we have established (i).

To prove (iii), we notice that $0=\urcorner 1$. Therefore $0=\urcorner[\sim(\beta\supset\beta)]=$ $=[\sim\sim(\beta\supset\beta)]$. But, on account of 2.21 and 4.3 (§ 1), the formula $\sim\sim(\beta\supset\beta)$ is provable. Hence, by Lemma 1.3 and 2.21 (§ 1), we obtain $\vdash\sim\sim(\beta\supset\beta)\equiv(\alpha\supset\alpha)$, or $[\alpha\supset\alpha]=0$, q. e. d.

[48]) To construct this algebra we use the unpublished method of Lindenbaum. This method was applied by McKinsey [1], Rieger [1], [2], Henkin [1].

Lemma 4.6. *The class of all provable formulae is the zero element of* $\mathfrak{B}_h$.

Proof. In fact, if $\vdash\beta$, then $\vdash\beta\equiv\alpha\supset\alpha$ (by Lemma 1.3 and 2.21 (§ 1)). Consequently, $[\beta]=[\alpha\supset\alpha]=0$. Suppose β is not provable. Then, $\beta\equiv\alpha\supset\alpha$ is not provable, so that $[\beta]\neq[\alpha\supset\alpha]=0$, q. e. d.

Lemma 4.7. $\vdash\alpha\supset\beta$ *if and only if* $[\beta]\leqslant[\alpha]$.

Proof. This follows from Lemma 4.6 and from the fact that the conditions $[\beta]\dot{-}[\alpha]=0$ and $[\beta]\leqslant[\alpha]$ are equivalent.

Definition 4.8. Let α be a formula with the property that neither of $(\exists x_l)$ and (x_l), nor x_l itself occurs in α. We say that the operation i/l is performed on α if in α 1⁰ all quantifiers (x_i) and $(\exists x_i)$ are replaced by the quantifiers (x_l) and $(\exists x_l)$, respectively, and 2⁰ each bound occurrence of x_i is replaced by x_l.

Let α_i^l be the formula which is obtained from α by applying the operation i/l.

The following lemma follows easily from Lemmas 1.5 and 4.3 ((4)-(9)).

Lemma 4.9. $\vdash\alpha\cong\alpha_i^l$, *i. e.*, $[\alpha]=[\alpha_i^l]$.

Definition 4.10. By $\alpha\begin{pmatrix}x_{p_1}, \ldots, x_{p_n}, x_{k_{n+1}}, \ldots, x_{k_m}\\ x_{k_1}, \ldots, x_{k_n}\end{pmatrix}$ we shall mean the formula obtained from $\alpha(x_{k_1},\ldots,x_{k_m})$ in the following way:

1⁰ we perform on $\alpha(x_{k_1},\ldots,x_{k_m})$ the operations p_i/l_i $(i=1,2,\ldots,n)$, where $l_1,\ldots,l_n$ is a fixed sequence with $l_j\neq p_i$ for $j=1,2,\ldots,n$, $l_j\neq l_i$ for $j\neq i$, $l_j\neq k_r$ for $j=1,2,\ldots,n$ and $r=1,\ldots,m$ [49]).

2⁰ by using the rule R. 1.2, we substitute the variables $x_{p_1},\ldots,x_{p_n}$ for the variables $x_{k_1},\ldots,x_{k_n}$, respectively.

Formula $\alpha\begin{pmatrix}x_{p_1}, \ldots, x_{p_n}, x_{k_{n+1}}, \ldots, x_{k_m}\\ x_{k_1}, \ldots, x_{k_n}\end{pmatrix}$ defined in this way is not uniquely determined, but the element $\left[\alpha\begin{pmatrix}x_{p_1}, \ldots, x_{p_n}, x_{k_{n+1}}, \ldots, x_{k_m}\\ x_{k_1}, \ldots, x_{k_n}\end{pmatrix}\right]$ of $\mathfrak{B}_h$ is uniquely determined, since it does not depend on the choice of the integers $l_1,\ldots,l_n$, by Lemma 4.9.

[49]) Since no bound occurrence of x_{p_i} appears in α, the order, in which the operations p_i/l_i are performed, makes no difference.

Lemma 4.11.

$$(1^*)\qquad \prod_{p_i\in I_0}\left[\alpha\begin{pmatrix}x_{p_1}, x_{p_2}, \ldots, x_{p_n}, x_{k_{n+1}}, \ldots, x_{k_m}\\ x_{k_1}, x_{k_2}, \ldots, x_{k_n}\end{pmatrix}\right]= \left[(\exists x_{k_1})\,\alpha\begin{pmatrix}x_{k_1}, x_{p_2}, \ldots, x_{p_n}, x_{k_{n+1}}, \ldots, x_{k_m}\\ x_{k_2}, \ldots, x_{k_n}\end{pmatrix}\right],$$

$$(2^*)\qquad \sum_{p_i\in I_0}\left[\alpha\begin{pmatrix}x_{p_1}, x_{p_2}, \ldots, x_{p_n}, x_{k_{n+1}}, \ldots, x_{k_m}\\ x_{k_1}, x_{k_2}, \ldots, x_{k_n}\end{pmatrix}\right]= \left[(x_{k_1})\,\alpha\begin{pmatrix}x_{k_1}, x_{p_2}, \ldots, x_{p_n}, x_{k_{n+1}}, \ldots, x_{k_m}\\ x_{k_2}, \ldots, x_{k_n}\end{pmatrix}\right].$$

Proof. We shall prove this lemma for the case $n=1$. The proof in the general case is analogous to this one. For brevity, we shall mention only those variables which are essential to our proof.

In order to prove

$$(1)\qquad \prod_{p\in I_0}\left[\alpha\begin{pmatrix}x_p\\ x_k\end{pmatrix}\right]=[(\exists x_k)\,\alpha(x_k)],$$

it is sufficient to show that

(a) $[(\exists x_k)\,\alpha(x_k)]\leqslant\left[\alpha\begin{pmatrix}x_p\\ x_k\end{pmatrix}\right]$ for every $p\in I_0$,

(b) if $[\beta]\leqslant\left[\alpha\begin{pmatrix}x_p\\ x_k\end{pmatrix}\right]$ for every $p\in I_0$, then $[\beta]\leqslant[(\exists x_k)\,\alpha(x_k)]$.

(a) follows from A. 13 and Lemmas 4.7, 4.9. To prove (b) suppose $[\beta]\leqslant\left[\alpha\begin{pmatrix}x_p\\ x_k\end{pmatrix}\right]$ for every $p\in I_0$. Hence, by Lemma 4.7 $\vdash\alpha\begin{pmatrix}x_p\\ x_k\end{pmatrix}\supset\beta$ for $p\in I_0$. Let $p\in I_0$ be a positive integer such that neither (x_p) nor $(\exists x_p)$ occurs in $\alpha(x_k)$ and that x_p itself occurs neither in β nor in $\alpha(x_k)$.

Therefore

$$\vdash(\exists x_p)\,\alpha\begin{pmatrix}x_p\\ x_k\end{pmatrix}\supset\beta\quad [\text{R. } 1.4]$$

and $\vdash(\exists x_p)\,\alpha\begin{pmatrix}x_p\\ x_k\end{pmatrix}\equiv(\exists x_k)\,\alpha(x_k)$ [Lemma 1.5].

Hence, by Lemma 4.7 we obtain $[\beta]\leqslant\left[(\exists x_p)\,\alpha\begin{pmatrix}x_p\\ x_k\end{pmatrix}\right]=[(\exists x_k)\,\alpha(x_k)]$.

In order to prove

$$(2)\qquad \sum_{p\in I_0}\left[\alpha\begin{pmatrix}x_p\\ x_k\end{pmatrix}\right]=[(x_k)\,\alpha(x_k)],$$

it is sufficient to show that

(c) $$\left[\alpha\binom{x_p}{x_k}\right]\leqslant[(x_k)\alpha(x_k)] \quad \text{for every } p\in I_0,$$

(d) if $\left[\alpha\binom{x_p}{x_k}\right]\leqslant\beta$ for every $p\in I_0$, then $[(x_k)\alpha(x_k)]\leqslant[\beta]$.

From A. 12 and Lemmas 4.9 and 4.7 follows (c). To prove (d) suppose $\left[\alpha\binom{x_p}{x_k}\right]\leqslant[\beta]$ for $p\in I_0$. Hence, by Lemma 4.7, $\vdash\beta\supset\alpha\binom{x_p}{x_k}$ for $p\in I_0$. Let $p\in I_0$ be such that neither (x_p) nor $(\exists x_p)$ occurs in $\alpha(x_k)$, and that x_p itself occurs neither in β nor in $\alpha(x_k)$. Consequently,

$$\vdash\beta\supset(x_p)\alpha\binom{x_p}{x_k} \quad [\text{R. } 1.3)]$$

and $\vdash(x_p)\alpha\binom{x_p}{x_k}\equiv(x_k)\alpha(x_k)$ [Lemma 1.5].

Hence, by Lemma 4.7

$$[(x_k)\alpha(x_k)]\leqslant[\beta], \quad \text{q. e. d.}$$

As a result of Theorem 3.18, it follows that there exists a complete Brouwerian algebra $\mathfrak{B}_0=\langle K_0,+,\cdot,\dot{-},\urcorner\rangle$ which is an extension of $\mathfrak{B}_h$ and preserves all sums and products [50]) of $\mathfrak{B}_h$.

Let $\varphi_p^k\in\mathcal{F}^k(I_0,\mathfrak{B}_0)$ $(k,p=1,2,\ldots)$ be the k-argument function defined by

$$\varphi_p^k(i_1,\ldots,i_k)=[F_p^k(x_{i_1},\ldots,x_{i_k})]\in K_h\subset K_0$$

for every sequence $(i_1,\ldots,i_k)$ of k positive integers.

Consider a formula

$$\alpha=\alpha(x_{i_1},\ldots,x_{i_n},a_{j_1},\ldots,a_{j_m},F_{p_1}^{k_1},\ldots,F_{p_r}^{k_r}).$$

We shall conceive this formula as the $(I_0,\mathfrak{B}_0)$-functional

$$\Phi=\Phi_\alpha(x_{i_1},\ldots,x_{i_n},a_{j_1},\ldots,a_{j_m},F_{p_1}^{k_1},\ldots,F_{p_r}^{k_r})$$

in the sense determined in the beginning of this section. Let $\Phi_\alpha^0\binom{l_1}{x_{i_1}},\ldots,\binom{l_n}{x_{i_n}}$ be the value of Φ_α, for the following values of its arguments:

$$x_{i_1}=l_1,\ldots,x_{i_n}=l_n \quad (\text{where } l_1,\ldots,l_n\in I_0),$$

$$a_{j_1}=[a_{j_1}],\ldots,a_{j_m}=[a_{j_m}], \quad F_{p_1}^{k_1}=\varphi_{p_1}^{k_1},\ldots,F_{p_r}^{k_r}=\varphi_{p_r}^{k_r}.$$

[50]) See Definition 3.5.

Lemma 4.12. *For every* $\alpha=\alpha(x_{i_1},\ldots,x_{i_n},a_{j_1},\ldots,a_{j_m},F_{p_1}^{k_1},\ldots,F_{p_r}^{k_r})$ *we have*

$$\Phi_\alpha^0\left(\frac{l_1}{x_{i_1}},\ldots,\frac{l_q}{x_{i_q}},\frac{i_{q+1}}{x_{i_{q+1}}},\ldots,\frac{i_n}{x_{i_n}}\right)=\left[\alpha\left(\frac{x_{l_1}}{x_{i_1}},\ldots,\frac{x_{l_q}}{x_{i_q}},x_{i_{q+1}},\ldots,x_{i_n}\right)\right].$$

This lemma may be established by induction on the length of α, making use of Lemma 4.11 and the fact that $\mathfrak{B}_0$ preserves all sums and products of $\mathfrak{B}_h$. The easy proof of this lemma is omitted.

To prove Theorem 4.2, let us suppose a formula

$$\alpha=\alpha(x_{i_1},\ldots,x_{i_n},a_{j_1},\ldots,a_{j_m},F_{p_1}^{k_1},\ldots,F_{p_r}^{k_r})$$

to be not provable. Consequently, by Lemmas 4.12 and 4.6,

$$\Phi_\alpha^0\left(\frac{i_1}{x_{i_1}},\ldots,\frac{i_n}{x_{i_n}}\right)=[\alpha(x_{i_1},\ldots,x_{i_n})]=[\alpha]\neq 0.$$

Hence, the $(I_0,\mathfrak{B}_0)$-functional Φ_α is not identically equal to zero. Thus Theorem 4.2 is proved.

Theorems (B) and (B′) [51]) are immediate consequences of Theorem 4.2 and Mostowski's Theorem (*) [52]).

Theorems (B) and (B′) can be considered as generalizations of the similar theorems of McKinsey and Tarski [53]) for the sentential calculus of Heyting.

§ 5. Completeness of the functional calculus of Lewis.

The purpose of this section is to establish the completeness of the system $\mathcal{L}$, in the sense mentioned in the introduction [54]).

In this section let $\alpha,\beta,\gamma,\delta$ always be arbitrary formulae of $\mathcal{L}$; let I_0 and I be the set of all positive integers and a non-void abstract set, respectively; and let $\mathfrak{C}$ be a complete closure algebra.

Every formula α will be interpreted as an $(I,\mathfrak{C})$-functional [55]) Φ_α by considering 1° the individual variables of α to be variables running over I, 2° the sentential variables of α to be variables running over the set of all elements of the algebra $\mathfrak{C}$, 3° the functional variables with k arguments to be variables running over $\mathcal{F}^k(I,\mathfrak{C})$, 4° the logical operations $\sim$, $\Diamond$, $\wedge$, (x_k), $(\exists x_k)$ to be the operations $-$, C, $\cdot$, $\prod_{x_k\in I}$, $\sum_{x_k\in I}$, of the algebra $\mathfrak{C}$, respectively.

[51]) See pp. 100-101.
[52]) See p. 114.
[53]) See McKinsey and Tarski [3].
[54]) See p. 101.
[55]) See § 4, p. 113.

Theorem 5.1. *If α is provable, then the $(I, \mathfrak{C})$-functional Φ_α is identically equal to the unit element of $\mathfrak{C}$, for every complete closure algebra $\mathfrak{C}$ and every non-void set I. (We write then $\Phi_\alpha=1$).*

Proof. We prove this theorem by induction on the length of the formal proof of α.

If α arises by substitution from one of the axioms of the system S. 4 of the sentential calculus of Lewis, then, as shown by McKinsey and Tarski [56]), $\Phi_\alpha=1$.

If α has the form $(x_k)\beta<\beta$ or $\sim\Diamond[(x_k)\beta\wedge\sim\beta]$, then $\Phi_\alpha=-C[\prod_{x_k\in I}\Phi_\beta\cdot-\Phi_\beta]=-C(0)=-0=1$.

If α has the form $\beta<(\exists x_k)\beta$ or $\sim\Diamond[\beta\wedge\sim(\exists x_k)\beta]$, then $\Phi_\alpha=-C[\Phi_\beta\cdot-\sum_{x_k\in I}\Phi_\beta]=-C(0)=-0=1$.

Therefore Theorem 5.1 is true if α is an axiom of the system $\mathcal{L}$.

Now, let β and γ be formulae such that $\Phi_\beta=1$ and $\Phi_\gamma=1$. We shall show that the use of each of the rules of inference gives a formula α such that $\Phi_\alpha=1$.

Rule R. 2.1. In this case, γ is of the form $\beta<\alpha$, $\Phi_\gamma=1$ and $\Phi_\beta=1$. Then $\Phi_\gamma=-C[\Phi_\beta\cdot-\Phi_\alpha]=1$. Hence $C[\Phi_\beta\cdot-\Phi_\alpha]=0$. Consequently $\Phi_\beta\cdot-\Phi_\alpha=0$ or $1\cdot-\Phi_\alpha=0$. Thus we obtain $\Phi_\alpha=1$.

Rule R. 2.2. In this case, α is of the form $\beta\wedge\gamma$, $\Phi_\beta=1$ and $\Phi_\gamma=1$. Then $\Phi_\alpha=\Phi_\beta\cdot\Phi_\gamma=1$.

Rule R. 2.3. In this case, γ is of the form $\beta_1=\beta_2$, $\Phi_\beta=1$ and $\Phi_\gamma=\Phi_{\beta_1=\beta_2}=1$. Then $-C[\Phi_{\beta_1}\cdot-\Phi_{\beta_2}]\cdot-C[\Phi_{\beta_2}\cdot-\Phi_{\beta_1}]=1$. Consequently, $\Phi_{\beta_1}\cdot-\Phi_{\beta_2}=0$ and $\Phi_{\beta_2}\cdot-\Phi_{\beta_1}=0$, so that $\Phi_{\beta_1}=\Phi_{\beta_2}$. Therefore, if α is the formula which arises from β by replacing β_1 by β_2, then $\Phi_\alpha=\Phi_\beta=1$.

Rule R. 2.4. If α arises from β by substitution and $\Phi_\beta=1$, then obviously $\Phi_\alpha=1$.

Rule R. 2.5. In this case, β is of the form $\delta_1<\delta_2$ and $\Phi_\beta=1$. Then, $C[\Phi_{\delta_1}\cdot-\Phi_{\delta_2}]=1$, or $\Phi_{\delta_1}\cdot-\Phi_{\delta_2}=0$, so that $\Phi_{\delta_1}\leqslant\Phi_{\delta_2}$. Suppose no free occurrence of x_k appears in δ_1. Then, $\Phi_{\delta_1}\leqslant\prod_{x_k\in I}\Phi_{\delta_2}$ so that $\Phi_{\delta_1}\cdot-\prod_{x_k\in I}\Phi_{\delta_2}=0$. Consequently, $-C[\Phi_{\delta_1}\cdot-\prod_{x_k\in I}\Phi_{\delta_2}]=1$. Finally

$$\Phi_{\delta_1<(x_k)\delta_2}=\Phi_\alpha=1.$$

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[56]) See McKinsey and Tarski [3].

Rule R. 2.6. In this case β is of the form $\delta_1<\delta_2$, α is of the form $(\exists x_k)\delta_1<\delta_2$ (where no free occurrence of x_k appears in δ_2) and $\Phi_\beta=1$. The proof that $\Phi_\alpha=1$ is similar to that used in the case of rule R. 2.5.

In this way Theorem 5.1 is proved.

The following is the fundamental theorem of this section:

Theorem 5.2. *There exists a complete closure algebra $\mathfrak{C}_0$, such that, for every α, if α is not provable, then $(I_0, \mathfrak{C}_0)$-functional Φ_α is not identically equal to the unit element of $\mathfrak{C}_0$.*

The proof of this theorem is similar to that of Theorem 4.2.

By saying that $\alpha\cong\beta$, we shall mean that $\vdash\alpha=\beta$.

Lemma 5.3. *The relation $\cong$ is a congruence relation in the sense of modern algebra, i. e., the following conditions are satisfied:*

(1) $\alpha\cong\alpha$,
(2) *if* $\alpha\cong\beta$, *then* $\beta\cong\alpha$,
(3) *if* $\alpha\cong\beta$ *and* $\beta\cong\gamma$, *then* $\alpha\cong\gamma$,
(4) *if* $\alpha\cong\beta$, *then* $\sim\alpha\cong\sim\beta$,
(5) *if* $\alpha\cong\beta$, *then* $(x_k)\alpha\cong(x_k)\beta$,
(6) *if* $\alpha\cong\beta$, *then* $(\exists x_k)\alpha\cong(\exists x_k)\beta$,
(7) *if* $\alpha\cong\beta$, *then* $\Diamond\alpha\cong\Diamond\beta$,
(8) *if* $\alpha\cong\beta$ *and* $\gamma\cong\delta$, *then* $\alpha\wedge\gamma\cong\beta\wedge\delta$.

Proof. The proofs of (1)-(4), (7), (8) are the same as in McKinsey [1], p. 123. To show (5) and (6) suppose $\alpha\cong\beta$. Then $\vdash\alpha=\beta$. By $\vdash\alpha=\alpha$ [T*. 1], we obtain $\vdash(x_k)\alpha=(x_k)\alpha$ and $\vdash(\exists x_k)\alpha=(\exists x_k)\alpha$. The use of rule R. 2.3 gives $\vdash(x_k)\alpha=(x_k)\beta$ and $\vdash(\exists x_k)\alpha=(\exists x_k)\beta$ or $(x_k)\alpha\cong(x_k)\beta$ and $(\exists x_k)\alpha\cong(\exists x_k)\beta$, q. e. d.

For every $\alpha\in\boldsymbol{L}$, let $[\alpha]$ be the set of all $\beta\in\boldsymbol{L}$ such that $\alpha\cong\beta$. Obviously, $[\alpha]=[\beta]$ if and only if $\alpha\cong\beta$.

In view of Lemma 5.3 the following definition may be introduced.

Definition 5.4. By a *Lindenbaum algebra* for the system $\mathcal{L}$ we shall mean the algebra $\mathfrak{C}_l=\langle K_l, +, \cdot, -, C\rangle$ defined as follows: 1) K_l is the set of all classes $[\alpha]$ such that $\alpha\in\boldsymbol{L}$. 2) For every $[\alpha], [\beta]\in K_l$, we put

$$-[\alpha]=[\sim\alpha], \qquad C[\alpha]=[\Diamond\alpha],$$
$$[\alpha]\cdot[\beta]=[\alpha\wedge\beta], \qquad [\alpha]+[\beta]=[\alpha\vee\beta].$$

Lemma 5.5. (i) $\mathfrak{C}_l$ *is a closure algebra,* (ii) $[\alpha\wedge\sim\alpha]=0$.

Proof. It is easily shown that, for any $[\alpha],[\beta],[\gamma]\in K_l$, the following conditions are satisfied:

$$[\alpha]+[\beta]=[\beta]+[\alpha] \qquad [\mathrm{T}^*.\ 5],$$
$$([\alpha]+[\beta])+[\gamma]=[\alpha]+([\beta]+[\gamma]) \qquad [\mathrm{T}^*.\ 7],$$
$$-(-[\alpha]+-[\beta])+-(-[\alpha]+[\beta])=[\alpha] \qquad [\mathrm{T}^*.\ 24],$$
$$[\alpha]\cdot[\beta]=-(-[\alpha]+-[\beta]) \qquad [\mathrm{T}^*.\ 10],$$
$$[\alpha\wedge\sim\alpha]\neq[\alpha\vee\sim\alpha].$$

Hence, $\mathfrak{C}_l$ is a Boolean algebra.

To prove (ii), we notice that $\vdash\alpha\wedge\sim\alpha\vee\beta=\beta$ [T*. 15].

Therefore, $[\alpha\wedge\sim\alpha]+[\beta]=[\beta]$. Consequently, $[\alpha\wedge\sim\alpha]\leqslant[\beta]$ for every $[\beta]\in K_l$. Hence, $[\alpha\wedge\sim\alpha]=0$. The class K_l is closed under the operation C. Moreover, it is easily seen that for every $[\alpha]$, $[\beta]\in K_l$ the following conditions are satisfied:

(1) $[\alpha]\leqslant C[\alpha]$ [T*. 25],

(2) $CC[\alpha]=C[\alpha]$ [C* 10.1],

(3) $C[\alpha]+C[\beta]=C([\alpha]+[\beta])$ [T*. 19],

(4) $C0=0$ [(ii), T*. 22].

Since $\mathfrak{C}_l$ is a Boolean algebra, we infer from (1)-(4) that $\mathfrak{C}_l$ is a closure algebra, q. e. d.

Lemma 5.6. (i) $[\alpha\vee\sim\alpha]=1$, (2) *The class of all provable formulae is the unit element of* $\mathfrak{C}_l$.

Proof. To prove (i), we notice that $\vdash\beta\vee(\alpha\vee\sim\alpha)=\alpha\vee\sim\alpha$ [T*. 26]. Hence, $[\beta]\leqslant[\alpha\vee\sim\alpha]$ for every $[\beta]\in K_l$, so that $[\alpha\vee\sim\alpha]=1$. To show (2), suppose $\vdash\beta$. By T*. 8 and Lemma 2.1 we have $\vdash\sim\Diamond\sim\beta$ and $\vdash\sim\Diamond\sim(\alpha\vee\sim\alpha)$. The use of R. 2.2 gives $\vdash\sim\Diamond\sim\beta\wedge\sim\Diamond\sim(\alpha\vee\sim\alpha)$. On account of T*. 20 we infer that $\vdash\beta=\alpha\vee\sim\alpha$. Therefore, if $\vdash\beta$, then $[\beta]=[\alpha\vee\sim\alpha]$. Conversely, if β is not provable, then $\beta=\alpha\vee\sim\alpha$ is not provable so that $[\beta]\neq[\alpha\vee\sim\alpha]$, which proves (2).

Lemma 5.7. $\vdash\alpha<\beta$ *if and only if* $[\alpha]\leqslant[\beta]$.

Proof. In fact, $\vdash\alpha<\beta$ if and only if $\vdash\alpha\supset\beta$ [T*. 9, Lemma 2.1, T*. 12]. Since the conditions $\vdash\alpha\supset\beta$ and $[\sim\alpha\vee\beta]=1$ are equivalent [Lemma 5.6], we infer that $\vdash\alpha<\beta$ if and only if $[\alpha]\leqslant[\beta]$.

Lemma 5.8. $\alpha\cong\alpha_i^l$, *i. e.,* $[\alpha]=[\alpha_i^l]$, *where* α_i^l *is the formula obtained from* α *by applying the operation* i/l *described in Definition* 4.8.

This lemma follows easily from Lemmas 2.2 and 5.3.

Let the meaning of $\alpha\left(\binom{x_{p_1}}{x_{k_1}},\dots,\binom{x_{p_n}}{x_{k_n}},x_{k_{n+1}},\dots,x_{k_m}\right)\in L$ remain the same as in Definition 4.10 [57]). This formula is not uniquely determined, but the element $\left[\alpha\left(\binom{x_{p_1}}{x_{k_1}},\dots,\binom{x_{p_n}}{x_{k_n}},x_{k_{n+1}},\dots,x_{k_m}\right)\right]$ of $\mathfrak{C}_l$ is uniquely determined on account of Lemma 5.8.

Lemma 5.9.

(1*) $$\prod_{p_1\in I_0}\left[\alpha\left(\binom{x_{p_1}}{x_{k_1}},\binom{x_{p_2}}{x_{k_2}},\dots,\binom{x_{p_n}}{x_{k_n}},x_{k_{n+1}},\dots,x_{k_m}\right)\right]= \left[(x_{k_1})\,\alpha\left(x_{k_1},\binom{x_{p_2}}{x_{k_2}},\dots,\binom{x_{p_n}}{x_{k_n}},x_{k_{n+1}},\dots,x_{k_m}\right)\right],$$

(2*) $$\sum_{p_1\in I_0}\left[\alpha\left(\binom{x_{p_1}}{x_{k_1}},\binom{x_{p_2}}{x_{k_2}},\dots,\binom{x_{p_n}}{x_{k_n}},x_{k_{n+1}},\dots,x_{k_m}\right)\right]= \left[(\exists x_{k_1})\,\alpha\left(x_{k_1},\binom{x_{p_2}}{x_{k_2}},\dots,\binom{x_{p_n}}{x_{k_n}},x_{k_{n+1}},\dots,x_{k_m}\right)\right].$$

Proof. We shall prove this lemma for the case $n=1$. The proof in the general case is analogous to this one. For brevity, we shall mention only those variables, which are essential for our proof.

In order to prove

(1) $$\prod_{p\in I_0}\left[\alpha\binom{x_p}{x_k}\right]=[(x_k)\,\alpha(x_k)],$$

it is sufficient to show that:

(a) $[(x_k)\,\alpha(x_k)]\leqslant\left[\alpha\binom{x_p}{x_k}\right]$ for every $p\in I_0$,

(b) if $[\beta]\leqslant\left[\alpha\binom{x_p}{x_k}\right]$ for every $p\in I_0$ then $[\beta]\leqslant[(x_k)\,\alpha(x_k)]$.

(a) follows from D. 1 (§ 2) and Lemmas 5.7, 5.8. To prove (b) suppose $[\beta]\leqslant\left[\alpha\binom{x_p}{x_k}\right]$ for every $p\in I_0$. Hence, by Lemma 5.7 $\vdash\beta<\alpha\binom{x_p}{x_k}$ for every $p\in I_0$. Let $p\in I_0$ be a positive integer such that neither (x_p) nor $(\exists x_p)$ occurs in $\alpha(x_k)$ and x_p itself occurs neither in β nor in $\alpha(x_k)$. Then: $\vdash\beta<(x_p)\,\alpha\binom{x_p}{x_k}$ [R. 2.5] and $\vdash(x_p)\,\alpha\binom{x_p}{x_k}=(x_k)\,\alpha(x_k)$ [Lemma 2.2].

[57]) See p. 116.

Hence, by Lemma 5.7 $[\beta] \leqslant \left[(x_p)\,\alpha\binom{x_p}{x_k}\right] = [(x_k)\,\alpha(x_k)]$.

To prove

(2) $$\sum_{p\in I_0}\left[\alpha\binom{x_p}{x_k}\right] = [(\exists x_k)\,\alpha(x_k)],$$

it is sufficient to show

(c) $$\left[\alpha\binom{x_p}{x_k}\right] \leqslant [(\exists x_k)\,\alpha(x_k)] \quad \text{for every} \quad p \in I_0,$$

(d) if $\left[\alpha\binom{x_p}{x_k}\right] \leqslant [\beta]$ for every $p \in I_0$, then $[(\exists x_k)\,\alpha(x_k)] \leqslant [\beta]$.

(c) follows from D. 2 and Lemmas 5.7 and 5.8. To prove (d) suppose $\left[\alpha\binom{x_p}{x_k}\right] \leqslant [\beta]$ for each $p \in I_0$. Hence, by Lemma 5.7 $\vdash \alpha\binom{x_p}{x_k} < \beta$ for each $p \in I_0$. Let $p \in I_0$ be a positive integer such that neither (x_p) nor $(\exists x_p)$ occurs in $\alpha(x_k)$, and x_p itself occurs neither in β nor in $\alpha(x_k)$. Consequently, $\vdash (\exists x_p)\,\alpha\binom{x_p}{x_k} < \beta$ [R. 2.6] and $\vdash (\exists x_p)\,\alpha\binom{x_p}{x_k} = (\exists x_k)\,\alpha(x_k)$ by Lemma 2.2. As a result of Lemma 5.7, we obtain

$$[(\exists x_k)\,\alpha(x_k)] = \left[(\exists x_p)\,\alpha\binom{x_p}{x_k}\right] \leqslant \beta, \quad \text{q. e. d.}$$

It follows from Theorem 3.13 that there exists a complete closure algebra $\mathfrak{C}_0 = \langle K_0, +, \cdot, -, C_0\rangle$ which is an extension of $\mathfrak{C}_l$ and preserves all sums and products (finite and infinite) of $\mathfrak{C}_l$.

Let $\varphi_p^k \in \mathcal{F}^k(I_0, \mathfrak{C}_0)$ $(k,p = 1,2,\ldots)$ be the k-argument function defined as follows:

$$\varphi_p^k(i_1,\ldots,i_k) = [F_p^k(x_{i_1},\ldots,x_{i_k})] \in K_l \subset K_0$$

for every sequence $(i_1,\ldots,i_k)$ of k positive integers. Consider a formula

$$\alpha = \alpha(x_{i_1},\ldots,x_{i_n}, a_{j_1},\ldots,a_{j_m}, F_{p_1}^{k_1},\ldots,F_{p_r}^{k_r}).$$

We shall consider this formula to be the $(I_0, \mathfrak{C}_0)$-functional $\Phi_\alpha = \Phi_\alpha(x_{i_1},\ldots,x_{i_n}, a_{j_1},\ldots,a_{j_m}, F_{p_1}^{k_1},\ldots,F_{p_r}^{k_r})$ in the sense determined in the beginning of this section. Let $\Phi_\alpha^0\left(\genfrac{}{}{0pt}{}{l_1}{x_{i_1}},\ldots,\genfrac{}{}{0pt}{}{l_n}{x_{i_n}}\right)$ be the value of Φ_α for the following values of its arguments $x_{i_1} = l_1,\ldots,x_{i_n} = l_n$ (where $l_1,\ldots,l_n \in I_0$), $a_{j_1} = [a_{j_1}],\ldots,a_{j_m} = [a_{j_m}]$, $F_{p_1}^{k_1} = \varphi_{p_1}^{k_1},\ldots,F_{p_r}^{k_r} = \varphi_{p_r}^{k_r}$.

Lemma 5.10. *For every* $\alpha = \alpha(x_{i_1},\ldots,x_{i_n}, a_{j_1},\ldots,a_{j_m}, F_{p_1}^{k_1},\ldots,F_{p_r}^{k_r})$, *we have*

$$\Phi_\alpha^0\left(\genfrac{}{}{0pt}{}{l_1}{x_{i_1}},\ldots,\genfrac{}{}{0pt}{}{l_q}{x_{i_q}},\genfrac{}{}{0pt}{}{i_{q+1}}{x_{i_{q+1}}},\ldots,\genfrac{}{}{0pt}{}{i_n}{x_{i_n}}\right) = \left[\alpha\left(\genfrac{}{}{0pt}{}{x_{l_1}}{x_{i_1}},\ldots,\genfrac{}{}{0pt}{}{x_{l_q}}{x_{i_q}}, x_{i_{q+1}},\ldots,x_{i_n}\right)\right].$$

This lemma may be established by induction on the length of α, making use of Lemma 5.9 and of the fact that $\mathfrak{C}_0$ preserves all sums and products of $\mathfrak{C}_l$.

To prove Theorem 5.2 let us suppose a formula

$$\alpha = \alpha(x_{i_1},\ldots,x_{i_n}, a_{j_1},\ldots,a_{j_m}, F_{p}^{k_1},\ldots,F_{p_r}^{k_r})$$

to be not provable. Consequently, by Lemmas 5.10 and 5.6 (2)

$$\Phi_\alpha^0\left(\genfrac{}{}{0pt}{}{i_1}{x_{i_1}},\ldots,\genfrac{}{}{0pt}{}{i_n}{x_{i_n}}\right) = [\alpha(x_{i_1},\ldots,x_{i_n})] = [\alpha] \neq 1.$$

Hence, the $(I_0, \mathfrak{C}_0)$-functional Φ_α is not identically equal to unit element of $\mathfrak{C}_0$. Thus Theorem 5.2 is established.

Theorems (C) and (C′) [58] are immediate consequences of Theorems 5.1 and 5.2.

Theorems (C) and (C′) can be considered as generalizations of the similar theorems of McKinsey and Tarski [59] for the sentential calculus of Lewis.

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[58]) See p. 101.

[59]) See McKinsey and Tarski [3].

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A Characterization of L Spaces[1]).

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1. Introduction. Kakutani [2] has characterized a space of integrable functions as a Banach lattice satisfying the following three conditions:

(1) there exists a unit element $e > \theta$ such that $x > \theta$ implies $e \wedge x > \theta$;

(2) $x \geqslant \theta$, $y \geqslant \theta$ imply $\|x+y\| = \|x\| + \|y\|$;

(3) $x \wedge y = \theta$ implies $\|x-y\| = \|x+y\|$.

The set of points, Ω, over which the L space is defined can be assumed to have measure 1. Kakutani [3] has also given a similar type of characterization for Banach lattices of functions continuous over a bicompact Hausdorff space. More recently, Clarkson [1] has characterized a Banach space of continuous functions in terms of the shape of the unit sphere. In this characterization an order relation is introduced by means of a certain type of cone used in the construction of the unit sphere, and under this ordering the space in shown to be an M space and hence equivalent to a space of continuous functions. In this paper spaces of integrable functions will be characterized by the shape of their unit spheres, making use of methods similar to those of Clarkson. The Borel field of measurable subsets of the space Ω will be shown to correspond to the family of maximal convex subsets of the unit sphere in a manner similar to the role played by this family in the case of a space of continuous functions as investigated by Eilenberg [4]. The case in which the measure is completely atomic is of particular interest and will be treated in more detail.

[1]) Presented to the American Mathematical Society, September 9, 1948.