Boolean algebra. But if we desire to have a non-distributive (though, of course, modular) example of the kind just mentioned, then it is sufficient to take the (simplest) five-element modular and non-distributive lattice instead of the above four-element Boolean algebra, and to perform further 4 symmetric adjunctions. Unlike these almost trivial answers to the first two questions of the problem Nr 75 of (L), an answer to the third question seems to be more difficult. This is closely related to problem Nr 56 of (L) of finding a non-Desarguesian plane projective geometry which admits orthocomplements. I could not find any satisfactory answer to this third question.

Measures in Almost Independent Fields.

By

Edward Marczewski (Wrocław).

Introduction. This paper deals with the existence of common extensions of measures defined in given fields of sets to the smallest field containing all the given fields. This problem taken in such a general way might have more than one solution. But we propose a restrictive condition viz., that the extension in question be multiplicative [1]), i. e. that all given fields be stochastically independent with respect to it. Then, as it is easy to prove, if the required extension exists, it is unique (Lemma 3).

Section 1 contains all the definitions and a few examples.

Section 2 contains the complete solution of the problem in the case of the finite additivity of fields and measures: the almost-independence is a necessary and sufficient condition for the existence of the multiplicative extension (Theorem I) [2]).

In Section 3 the same problem is considered for the denumerable additivity. Banach [2] has proved that the σ-independence is here a sufficient condition but it is easy to see that it is not a necessary condition [3]). On the other hand Helson [1] has established that, even in the case of two fields, the almost-independence (which is in this case identical with the almost-σ-independence) is not a sufficient condition. An analogous necessary and sufficient condition is not known so far. In this paper only a sufficient condition is given (Theorem III and IV), formulated thanks to some ideas of Kakutani [1]: it is namely the almost-σ-independence under the additional condition (obviously a very restrictive one)

[1]) Stochastic extension in the terminology of Helson [4] and Sikorski [10].

[2]) I proved this theorem and I presented it to the Polish Mathematical Society, Warsaw Section, in spring 1939; cf. Marczewski [8], p. 127, Théorème III, and Banach [1], pp. 159-160.

[3]) Marczewski [8], p. 130.

that each of the given fields — except at most one — contains a finite number of sets [4]).

It is interesting to note that Theorems I-IV continue to hold for Boolean algebras, whereas Banach's theorem, as recently proved by Sikorski [11], does not.

Theorem IV can be applied to problems of extensions of Lebesgue measure. Invariant (or, more exactly, invariant with respect to the congruence) σ-extensions of Lebesgue measure are well-known (see e. g. Marczewski [6]). The existence of a non-separable σ-extension of this measure has been proved by Kakutani [5]. In 1935 I proposed the problem of the existence of σ-extension of Lebesgue measure which is simultaneously invariant and non-separable ([6], p. 558). In 1950, using my Theorem IV and some ideas of Banach [1], Sierpiński [9] and Kakutani [5] I proved, with the aid of the continuum hypothesis, that such a σ-extension exists [5]); this proof is not given here, because quite recently a stronger result has been published by Kakutani, Kodaira, and Oxtoby [6]).

The following related problem of Sierpiński is not yet solved ([6], p. 558): Does there exist for each invariant σ-extension of Lebesgue measure its proper invariant σ-extension?

1. Fields and their independence. Measures and their extensions. By *field* of subsets of a fixed set X we understand any class $\boldsymbol{K}$ of subsets of X which is additive (i. e. such that if $E_1, E_2 \in \boldsymbol{K}$, then $E_1+E_2 \in \boldsymbol{K}$) and complementative (i. e. such that if $E \in \boldsymbol{K}$, then $X-E \in \boldsymbol{K}$). Any σ-additive field $\boldsymbol{K}$ (i. e. such that if $E_j \in \boldsymbol{K}$ for $j=1,2,\dots$, then $E_1+E_2+\ldots \in \boldsymbol{K}$) is called a *$\sigma$-field*. The smallest field and the smallest σ-field containing a class $\boldsymbol{Q}$ of subsets of X will be denoted by $\boldsymbol{Q}_0$ and $\boldsymbol{Q}_\beta$, respectively. A trivial but important example of a σ-field is each four-element field $\boldsymbol{K}=(X,0,E,X-E)=(E)_0=(E)_\beta$, where $E\subset X$ and $0\neq E\neq X$.

[4]) At first I proved Theorem IV, weaker than Theorem III, and I published it without proof in [8], p. 128. The stronger formulation was suggested to me by A. Götz and R. Sikorski. Theorems III and IV′ generalize a result of [7] (II théorème fondamentale, p. 25).

[5]) This result was communicated to the Polish Mathematical Society, Wrocław Section, on the 12-th of May, 1950.

[6]) K. Kodaira and S. Kakutani, *A non-separable translation invariant extension of the Lebesgue measure space*, Annals of Mathematics **52** (1950), pp. 574-579; S. Kakutani and J. C. Oxtoby, *Construction of a non-separable invariant extension of the Lebesgue measure space*, ibidem, pp. 580-590.

For each set $E\subset X$ we put $E^0=X-E$ and $E^1=E$. We denote by $A\dot{-}B$ the symmetric difference of A and B.

Lemma 1. *For each finite sequence $E_1, E_2, \dots, E_n$ of sets belonging to a field $\boldsymbol{K}$, there exists a sequence $P_1, P_2, \dots, P_m$ of disjoint sets belonging to $\boldsymbol{K}$, such that each set E_k $(k=1,2,\dots,n)$ is the sum of some sets P_l.*

In fact, all *atoms* of $E_1, E_2, \dots, E$ (i. e. the sets of the form $E_1^{i_1}E_2^{i_2}\dots E_n^{i_n}$, where $i_j=0,1$) form the required sequence $\{P_k\}$.

Lemma 2. *If $\boldsymbol{K}_1, \boldsymbol{K}_2, \dots, \boldsymbol{K}_n$ is a finite sequence of fields of subsets of X, then each set E belonging to $\boldsymbol{K}=(\boldsymbol{K}_1+\boldsymbol{K}_2+\ldots+\boldsymbol{K}_n)_0$ has the form*

$$(*)\qquad E=\sum_{j=1}^{m}\prod_{i=1}^{n}E_{ij}, \quad \text{where } E_{ij}\in\boldsymbol{K}_i \text{ for } i=1,2,\dots,n.$$

In fact, the class of all sets of the form (*) is a field containing all the fields $\boldsymbol{K}_1, \boldsymbol{K}_2, \dots, \boldsymbol{K}_n$.

Lemma 2′. *We can suppose that the sets $\prod_{i=1}^{n}E_{ij}$ in the formula (*) are disjoint.*

To prove this we apply Lemma 1 to each sequence $E_{i1}, E_{i2}, \dots, E_{in}$, $(i=1,2,\dots,m)$ in the formula (*) and we transform the obtained expression so as to give it the usual "polynomial" form.

A non-negative set function μ defined in a field $\boldsymbol{M}$ of subsets of X is called a *measure*, if $\mu(X)=1$ and if it is additive (i. e. if $\mu(E_1+E_2)=\mu(E_1)+\mu(E_2)$ whenever $E_1E_2=0$). A measure μ is called a *σ-measure* if it is defined in a σ-field and if it is σ-additive (i. e. if $\mu(E_1+E_2+\ldots)=\mu(E_1)+\mu(E_2)+\ldots$ whenever the sets E_j are disjoint). By an *extension* [or *σ-extension*] of a measure μ defined in a field $\boldsymbol{M}$ we understand any measure [σ-measure] ν defined on a field containing $\boldsymbol{M}$ and such that $\nu(E)=\mu(E)$ for $E\in\boldsymbol{M}$.

In the sequel we consider families $\{E_t\}$, $\{\boldsymbol{M}_t\}$, $\{\mu_t\}$ of sets, of fields, and of measures. The index t runs over an arbitrary set T.

The sets belonging to a family $\{E_t\}$ (where $E_t\subset X$) are called *independent* [*σ-independent*], if for each finite sequence [finite or infinite sequence] of different indices $t_n\in T$ and for each sequence $\{i_n\}$ of numbers 0 and 1 we have

$$E_{t_1}^{i_1}\cdot E_{t_2}^{i_2}\cdot\ldots\neq 0.$$

Analogically, the fields $\boldsymbol{M}_t$ (of subsets of X) are called *independent* [*σ-independent*] if for each finite sequence [infinite sequence] of different indices $t_n \in T$ and for each sequence of sets $E_n \in \boldsymbol{M}_{t_n}$, such that $E_n \neq 0$, we have

$$E_1 \cdot E_2 \cdot \ldots \neq 0.$$

Let μ_t be a measure in $\boldsymbol{M}_t$. By substituting the condition $\mu_{t_n}(E_n) \neq 0$ for $E_n \neq 0$ in the above definition we obtain the notions of *almost-independence* and *almost-σ-independence* of fields with respect to the measures μ_t.

Obviously if T is finite, the independence coincides with the σ-independence and the almost-independence with the almost-σ-independence.

If μ is a measure in the field $\boldsymbol{M}$ and $\{\boldsymbol{Q}_t\}$ is a family of subclasses of $\boldsymbol{M}$, then $\boldsymbol{Q}_t$ are called *stochastically independent* (with respect to μ), if

$$\mu(E_1 \cdot E_2 \cdot \ldots \cdot E_n) = \mu(E_1) \cdot \mu(E_2) \cdot \ldots \cdot \mu(E_n)$$

for any $E_j \in \boldsymbol{Q}_{t_j}$, where $t_i \neq t_j$ for $i \neq j$.

If $\boldsymbol{M}_t$ are subfields of the field $\boldsymbol{N}$ and μ_t are measures in $\boldsymbol{M}_t$, then a common extension ν of μ_t to $\boldsymbol{N}$ is called *multiplicative*, if $\boldsymbol{M}_t$ are stochastically independent with respect to ν.

Lemma 2′ implies the following

Lemma 3. *The multiplicative extension μ of measures μ_t defined in the fields $\boldsymbol{M}_t$ to the smallest field containing $\boldsymbol{M}_t$ is unique.*

In fact, if E is represented as a disjoint sum (*), then

$$\mu(E) = \sum_{j=1}^{m} \mu\Big(\prod_{i=1}^{n} E_{ij}\Big) = \sum_{j=1}^{m} \prod_{i=1}^{n} \mu_{t_i}(E_{ij}).$$

Examples. 1. Let $\{E_t\}$ be any family of subsets of X. Putting $\boldsymbol{M}_t = (E_t)_0$ we obtain a family of four-element fields. It is obvious that the fields $\boldsymbol{M}_t$ are independent [σ-independent] if and only if the sets E_t are independent [σ-independent]. Consequently the known examples of independent or σ-independent sets [7]) give directly analogous examples of independence of fields of sets.

2. Let $\boldsymbol{L}$ denote the σ-field of all Lebesgue-measurable subsets of the unit interval I, and $\boldsymbol{N}$ the four-element field $\boldsymbol{N} = (Z)_0$, where

[7]) Cf. e. g. Marczewski [2], pp. 16-17 and 21-22.

the Lebesgue exterior measure of Z and $I-Z$ is 1. Let us define a measure ν in $\boldsymbol{N}$ by putting $\nu(Z) = \nu(I-Z) = \frac{1}{2}$. The σ-fields $\boldsymbol{L}$ and $\boldsymbol{N}$ are obviously not independent but they are almost independent with respect to the Lebesgue measure and the measure ν.

3. Let I denote the unit interval and $X = I^2$ the unit square. Let $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ denote the class of all sets of the form $L \times I$ and $I \times L$, respectively, where L is a Lebesgue-measurable subset of I. The σ-fields $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are independent and stochastically independent with respect to the Lebesgue plane measure μ. Let us denote by μ_1 and μ_2, the measure μ restricted to the fields $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$. The set functions μ_1 and μ_2 are σ-measures, and μ is their multiplicative σ-extension.

2. Multiplicative extension of measures.

Theorem I. *Let $\{\mu_t\}$ be a family of measures defined respectively in the fields $\boldsymbol{M}_t$ of subsets of a set X. There exists a multiplicative extension of the measures μ_t if and only if the fields $\boldsymbol{M}_t$ are almost independent with respect to μ_t.*

Let us put $\boldsymbol{M} = (\sum_t \boldsymbol{M}_t)_0$.

Lemma 4. *If $\boldsymbol{M}_t$ are fields of subsets of X, then*

$$\Big(\sum_t \boldsymbol{M}_t\Big)_0 = \sum_{(t_1 t_2 \ldots t_n)} (\boldsymbol{M}_{t_1} + \boldsymbol{M}_{t_2} + \ldots + \boldsymbol{M}_{t_n})_0,$$

where $(t_1, t_2, \ldots, t_n)$ runs over the set of all finite sequences of elements of T.

To prove this it is sufficient to show that the right-hand side $\boldsymbol{R}$ of the formula is a field. Obviously the class $\boldsymbol{R}$ is complementative. If

$$A \in (\boldsymbol{M}_{u_1} + \boldsymbol{M}_{u_2} + \ldots + \boldsymbol{M}_{u_p})_0 \quad \text{and} \quad B \in (\boldsymbol{M}_{v_1} + \boldsymbol{M}_{v} + \ldots + \boldsymbol{M}_{v_q})_0,$$

then A and B also belong to the field

$$\boldsymbol{Q} = (\boldsymbol{M}_{u_1} + \ldots + \boldsymbol{M}_{u_p} + \boldsymbol{M}_{v_1} + \ldots + \boldsymbol{M}_{v_q})_0,$$

and consequently $A + B \in \boldsymbol{Q} \subset \boldsymbol{R}$. Thus the class $\boldsymbol{R}$ is additive, q. e. d.

Lemma 5. *Let $\mu_1, \mu_2, \ldots, \mu_{n+1}$ be measures in the fields $\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots, \boldsymbol{M}_{n+1}$, of subsets of X. If these fields are almost independent with respect to the considered measures, and if the measure μ is a common extension of $\mu_1, \mu_2, \ldots, \mu_n$ to the field $\boldsymbol{M} = (\boldsymbol{M}_1 + \boldsymbol{M}_2 + \ldots + \boldsymbol{M}_n)_0$, then $\boldsymbol{M}$ and $\boldsymbol{M}_{n+1}$ are almost independent with respect to μ and μ_{n+1}.*

Let $E \in \boldsymbol{M}$, $\mu(E)>0$, $Z \in \boldsymbol{M}_{n+1}$, $\mu_{n+1}(Z)>0$.

By Lemma 2

$$E=\sum_{j=1}^{s} E_{1j}\cdot E_{2j}\cdot\ldots\cdot E_{nj} \quad \text{where} \quad E_{ij} \in \boldsymbol{M}_i.$$

Since $\mu(E)>0$, there exists j_0 such that $\mu(E_{1j_0}\cdot E_{2j_0}\cdot\ldots\cdot E_{nj_0})>0$, whence a fortiori

$$\mu_i(E_{ij_0})=\mu(E_{ij_0})>0 \qquad \text{for } i=1,2,\ldots,n.$$

It follows from the almost-independence of $\boldsymbol{M}_1,\boldsymbol{M}_2,\ldots,\boldsymbol{M}_{n+1}$ that

$$EZ \subset E_{1j_0}\cdot E_{2j_0}\cdot\ldots\cdot E_{nj_0}\cdot Z \neq 0, \qquad \text{q. e. d.}$$

Lemma 6. *Let μ and ν be two measures defined respectively in two almost independent (with respect to μ and ν) fields $\boldsymbol{M}$ and $\boldsymbol{N}$ of subsets of X. Let*

$$A_1,A_2,\ldots,A_r,\ A_1^*,A_2^*,\ldots,A_{r^*}^* \tag{1}$$

$$B_1,B_2,\ldots,B_r,\ B_1^*,B_2^*,\ldots,B_{r^*}^* \tag{2}$$

be two finite sequences of sets belonging to $\boldsymbol{M}$ and $\boldsymbol{N}$ respectively. Then, if the sets A_kB_k are disjoint $(k=1,2,\ldots,r)$, if $A_k^*B_k^*$ are disjoint $(k=1,2,\ldots,r^*)$, and if

$$\sum_{k=1}^{r} A_kB_k=\sum_{k=1}^{r^*} A_k^*B_k^*, \tag{3}$$

then

$$\sum_{k=1}^{r} \mu(A_i)\nu(B_i)=\sum_{k=1}^{r^*} \mu(A_i^*)\nu(B_j^*). \tag{4}$$

By Lemma 1 there exists two finite sequences of disjoint sets: $\{G_i\}$ belonging to $\boldsymbol{M}$ and $\{H_i\}$ belonging to $\boldsymbol{N}$, such that each set (1) is the sum of some sets G_i and each set (2) is the sum of some sets H_i:

$$A_k=\sum_{i\in P_k} G_i,\quad B_k=\sum_{i\in Q_k} H_i,\quad A_k^*=\sum_{i\in P_k^*} G_i,\quad B_k^*=\sum_{i\in Q_k^*} H_i. \tag{5}$$

We put

$$R=\sum_{k=1}^{r}(P_k\times Q_k),\quad R^*=\sum_{k=1}^{r^*}(P_k^*\times Q_k^*).$$

Thus, it follows from (5) that

$$\sum_{k=1}^{r} A_kB_k=\sum_{(i,j)\in R} G_iH_j,\quad \sum_{k=1}^{r^*} A_k^*B_k^*=\sum_{(i,j)\in R^*} G_iH_j,$$

whence we obtain, in virtue of (3):

$$\sum_{(i,j)\in R} G_iH_j=\sum_{(i,j)\in R^*} G_iH_j. \tag{6}$$

Consequently, since the sets G_iH_j and G_kH_l are disjoint whenever $(i,j)\neq(k,l)$, we see that if $(i,j)\in R-R^*$, then $G_iH_j=0$. Further, since $\boldsymbol{M}$ and $\boldsymbol{N}$ are almost independent, $\mu(G_i)=0$ or $\nu(H_j)=0$. Therefore

$$\sum_{(i,j)\in R} \mu(G_i)\nu(H_j)=\sum_{(i,j)\in R^*} \mu(G_i)\nu(H_j). \tag{7}$$

Since the terms of the sums (5) are disjoint, we have

$$\mu(A_k)=\sum_{i\in P_k}\mu(G_i) \qquad \nu(B_k)=\sum_{i\in Q_k}\nu(H_i)$$

and

$$\sum_{k=1}^{r}\mu(A_k)\nu(B_k)=\sum_{k=1}^{r}\Big[\sum_{i\in P_k}\sum_{j\in Q_k}\mu(G_i)\nu(H_j)\Big]. \tag{8}$$

If a pair of indices (i,j) recurs in the last sum, i. e. if for some i,j,k,l (where $k\neq l$) we have $i\in P_kP_l$ and $j\in Q_kQ_l$, then by (5)

$$G_iH_j\subset A_kB_k \quad \text{and} \quad G_iH_j\subset A_lB_l.$$

Since A_kB_k and A_lB_l are disjoint, $G_iH_j=0$, whence by the almost-independence of $\boldsymbol{M}$ and $\boldsymbol{N}$, we have $\mu(G_i)\nu(H_j)=0$.

Consequently, in the right-hand side of (8) only those products $\mu(G_i)\nu(H_j)$ recur which are equal to zero; therefore we may write

$$\sum_{k=1}^{r}\mu(A_k)\nu(B_k)=\sum_{(i,j)\in R}\mu(G_i)\nu(H_j) \tag{9}$$

and analogically

$$\sum_{k=1}^{r^*}\mu(A_k^*)\nu(B_k^*)=\sum_{(i,j)\in R^*}\mu(G_i)\nu(H_j). \tag{9*}$$

The formulae (7), (9), and (9*), give (4), q. e. d.

Proof of Theorem I. 1^0 First we shall deduce the existence of a multiplicative extension of μ_t from the almost-independence in the case of a finite number of fields and measures. By Lemma 5 the proof reduces to the case of two measures: μ_1 and μ_2, defined on two fields: $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$. By Lemma 2′ each set $E\in\boldsymbol{M}=(\boldsymbol{M}_1+\boldsymbol{M}_2)_0$ is of the form

$$E=\sum_{k=1}^{n}A_kB_k, \quad \text{where} \quad (A_iB_i)(A_jB_j)=0 \quad \text{for} \quad i\neq j.$$

We put

$$\mu(E)=\sum_{k=1}^{n}\mu_1(A_k)\mu_2(B_k);$$

it follows from Lemma 6 that this number does not depend on the choice of A_k and B_k.

It follows directly from this definition that μ is a multiplicative extension of μ_1 and μ_2 to $\boldsymbol{M}$.

Now we pass to the case of an infinite set T of indices. Let us put for each finite sequence $U=(u_1,u_2,\dots,u_n)$ of elements of T

$$\boldsymbol{M}_U=(\boldsymbol{M}_{u_1}+\boldsymbol{M}_{u_2}+\dots+\boldsymbol{M}_{u_n})_0.$$

Since U is finite, there exists a multiplicative extension μ_U of all μ_{u_j} $(j=1,2,\dots,n)$ to $\boldsymbol{M}_U$.

We shall prove that all the measures so defined are compatible, i. e. that if $E\in\boldsymbol{M}_U\cdot\boldsymbol{M}_V$ (where U and V are finite sequences of elements of T), then

(10) $$\mu_U(E)=\mu_V(E).$$

Indeed, $E\in\boldsymbol{M}_{U,V}$, and it follows from the uniqueness of the multiplicative extension (Lemma 3) that $\mu_{U,V}$ coincides with μ_U in $\boldsymbol{M}_U$ and with μ_V in $\boldsymbol{M}_V$, whence the identity (10).

Consequently we may put for each $E\in\boldsymbol{M}_U$

(11) $$\mu(E)=\mu_U(E).$$

It follows from Lemma 4 that the function μ is defined in $\boldsymbol{M}$. By (11) we have $\mu(0)=0$ and $\mu(X)=1$.

In order to prove the additivity of μ, let us remark that for each $A,B\in\boldsymbol{M}$ there exists by Lemma 4 two finite sequences U and V of elements of T, such that $A\in\boldsymbol{M}_U$ and $B\in\boldsymbol{M}_V$, and consequently $A\in\boldsymbol{M}_{U,V}$ and $B\in\boldsymbol{M}_{U,V}$. Since $\mu(A)=\mu_U(A)=\mu_{U,V}(A)$ and $\mu(B)=\mu_V(B)=\mu_{U,V}(B)$, we have

$$\mu(A+B)=\mu_{U,V}(A+B)=\mu_{U,V}(A)+\mu_{U,V}(B)=\mu(A)+\mu(B)$$

whenever A and B are disjoint.

It follows easily from the definition (11) that μ is a common extension μ of μ_t $(t\in T)$ and that the fields $\boldsymbol{M}_t$ are stochastically independent with respect to μ.

2° We shall prove that the existence of a multiplicative extension μ of μ_t implies the almost-independence of $\boldsymbol{M}_t$. In fact, if $E_j\in\boldsymbol{M}_{t_j}$, $\mu_{t_j}(E_j)>0$, and $t_i\neq t_j$ for $i\neq j$ $(i,j=1,2,\dots,n)$, then

$$\mu(E_1\cdot E_2\cdot\ldots\cdot E_n)=\mu(E_1)\cdot\mu(E_2)\cdot\ldots\cdot\mu(E_n)=\mu_{t_1}(E_1)\cdot\mu_{t_2}(E_2)\cdot\ldots\cdot\mu_{t_n}(E_n)\neq 0.$$

Theorem I is thus proved.

Theorem II. *Let $\{\boldsymbol{M}_t\}$ denote a family of fields of subsets of a set X. Then the following statements are equivalent:*

(a) *The fields $\boldsymbol{M}_t$ are independent.*

(b) *There exists a multiplicative extension of each family of measures μ_t defined respectively in $\boldsymbol{M}_t$.*

(c) *There exists a common extension of each family of measures μ_t defined respectively in $\boldsymbol{M}_t$.*

Proof. Theorem I implies the implication (a)$\to$(b). The implication (b)$\to$(c) is trivial. Finally, in order to prove (c)$\to$(a), let us consider a finite sequence of non void sets $E_j\in\boldsymbol{M}_{t_j}$ $(j=1,2,\dots,n)$, where t_j is a sequence of different indices. Let $\{p_t\}$ be a family of points of X such that $p_{t_j}\in E_j$ for $j=1,2,\dots,n$. We define in each field $\boldsymbol{M}_t$ a measure μ_t by putting $\mu_t(E)=1$ or 0 according as p_t belongs to E or not. Obviously $\mu_t(E_j)=1$ and consequently $\mu(E_j)=1$, (for $j=1,2,\dots,n$), whence $E_1\cdot E_2\cdot\ldots\cdot E_n\neq 0$, q. e. d.

3. Multiplicative σ-extensions of σ-measures.

We prove now

Theorem III. *Let μ_t (where t runs over a set T containing an element t_0) be a family of σ-measures defined respectively on σ-fields $\boldsymbol{M}_t$ of subsets of X. Let us suppose all $\boldsymbol{M}_t$ except $\boldsymbol{M}_{t_0}$ are finite. If $\boldsymbol{M}_t$ are almost-σ-independent with respect to μ_t, then there exists a multiplicative σ-extension ν of all μ_t.*

We denote by $A_t^{(i)}$, where $i=1,2,\dots,K_t$, the sequence of all non void atoms of the field $\boldsymbol{M}_t$ $(t_0\neq t\in T)$.

Let us put $\boldsymbol{M}=(\sum_{t\in T}\boldsymbol{M}_t)_0$ and $\boldsymbol{N}=(\sum_{t\in T}\boldsymbol{M}_t)_\beta$; we have then $\boldsymbol{N}=\boldsymbol{M}_\beta$.

Lemma 6. *If $E\in\boldsymbol{M}$, then*

$$E=\sum_i A_{t_1}^{(i_1)}\cdot A_{t_2}^{(i_2)}\cdot\ldots\cdot A_{t_n}^{(i_n)}B_{i_1i_2\dots i_n}$$

where $B_{i_1i_2\dots i_n}\in\boldsymbol{M}_{t_0}$ and where $i=(i_1,i_2,\dots,i_n)$ runs over all sequences consisting of n numbers i_j such that $i_j\leqslant K_{t_j}$ [8]*).*

[8]) Cf. Kakutani [5], p. 117.

This is an easy consequence of Lemmas 4 and 2.

Using an analogous notation we have

Lemma 7. *If* $E_1 \supset E_2 \in \boldsymbol{M}$ *and*

$$(*) \qquad E_1 = \sum_i A_{t_1}^{(i_1)} \cdot A_{t_2}^{(i_2)} \cdot \ldots \cdot A_{t_{n_1}}^{(i_{n_1})} B_{i_1 i_2 \ldots i_{n_1}}^{(1)},$$

then E_2 *may be represented in the form*

$$E_2 = \sum_i A_{t_1}^{(i_1)} \cdot A_{t_2}^{(i_2)} \cdot \ldots \cdot A_{t_{n_2}}^{(i_{n_2})} B_{i_1 i_2 \ldots i_{n_2}}^{(2)},$$

where $n_1 \leqslant n_2$ *and* $B_{i_1 i_2 \ldots i_{n_1}}^{(1)} \supset B_{i_1 i_2 \ldots i_{n_2}}^{(1)}$.

By Lemma 6

$$E_2 = \sum_i A_{u_1}^{(j_1)} \cdot A_{u_2}^{(j_2)} \cdot \ldots \cdot A_{u_n}^{(j_n)} B_{j_1 j_2 \ldots j_n}, \qquad B_{j_1 j_2 \ldots j_n} \in \boldsymbol{M}_{t_0}.$$

Now we form a sequence $t_1, t_2, \ldots, t_{n_2}$ (where $n_1 \leqslant n_2$) containing all terms of the sequence $u_1, u_2, \ldots, u_n$. Consequently there is for each $k = 1, 2, \ldots, n$ an l_k such that $1 \leqslant l_k \leqslant n_2$ and $t_{l_k} = u_k$. Putting

$$B_{i_1, i_2, \ldots, i_{n_2}}^{*} = B_{i_{l_1}, i_{l_2}, \ldots, i_{l_n}}$$

we obtain easily

$$E_2 = \sum_i A_{t_1}^{(i_1)} \cdot A_{t_2}^{(i_2)} \cdot \ldots \cdot A_{t_{n_2}}^{(i_{n_2})} B_{i_1, i_2, \ldots, i_{n_2}}^{*}.$$

Since

$$A_{t_1}^{(i_1)} \cdot A_{t_2}^{(i_2)} \cdot \ldots \cdot A_{t_{n_2}}^{(i_{n_2})} B_{i_1, i_2, \ldots, i_{n_2}}^{*} \subset E_1$$

and since the set $A_{t_1}^{(i_1)} \cdot A_{t_2}^{(i_2)} \cdot \ldots \cdot A_{t_{n_1}}^{(i_{n_1})}$ is disjoint with $A_{t_1}^{(j_1)} \cdot A_{t_2}^{(j_2)} \cdot \ldots \cdot A_{t_{n_1}}^{(j_{n_1})}$ whenever $(i_1, i_2, \ldots, i_{n_1}) \neq (j_1, j_2, \ldots, j_{n_1})$, we have

$$A_{t_1}^{(i_1)} \cdot A_{t_2}^{(i_2)} \cdot \ldots \cdot A_{t_{n_2}}^{(i_{n_2})} B_{i_1 i_2 \ldots i_{n_2}}^{*} \subset A_{t_1}^{(i_1)} \cdot A_{t_2}^{(i_2)} \cdot \ldots \cdot A_{t_{n_1}}^{(i_{n_1})} \cdot B_{i_1 i_2 \ldots i_{n_1}}^{(1)}.$$

Thus, putting

$$B_{i_1 i_2 \ldots i_{n_2}}^{(2)} = B_{i_1 i_2 \ldots i_{n_2}}^{*} \cdot B_{i_1 i_2 \ldots i_{n_1}}^{(1)}$$

we obtain

$$A_{t_1}^{(i_1)} \cdot A_{t_2}^{(i_2)} \cdot \ldots \cdot A_{t_{n_2}}^{(i_{n_2})} B_{i_1 i_2 \ldots i_{n_2}}^{(2)} = A_{t_1}^{(i_1)} \cdot A_{t_2}^{(i_2)} \cdot \ldots \cdot A_{t_{n_2}}^{(i_{n_2})} \cdot B_{i_1 i_2 \ldots i_{n_2}}^{*},$$

which implies the required formula (*).

Proof of Theorem III. In view of Theorem I there is a multiplicative extension μ of μ_t to $\boldsymbol{M}$.

On account of the well-known theorem on the σ-extension of a measure, in order to prove Theorem III it is sufficient to prove that for each sequence $\{E_n\}$ such that

$$E_1 \supset E_2 \supset \ldots \qquad E_k \in \boldsymbol{M}, \quad \mu(E_k) \geqslant \delta > 0$$

we have

$$(1) \qquad E_1 E_2 \ldots \neq 0.$$

It follows from Lemma 7 that

$$(2) \qquad E_k = \sum_i A_{t_1}^{(i_1)} \cdot A_{t_2}^{(i_2)} \cdot \ldots \cdot A_{t_{n_k}}^{(i_{n_k})} B_{i_1 i_2 \ldots i_{n_k}}^{(k)},$$

where

$$B_{i_1 i_2 \ldots i_{n_k}}^{(k)} \in \boldsymbol{M}_{t_0}, \quad n_k \leqslant n_{k+1}, \quad B_{i_1 i_2 \ldots i_{n_k}}^{(k)} \supset B_{i_1 i_2 \ldots i_{n_{k+1}}}^{(k+1)}.$$

Since the terms in the sum (2) are disjoint and since the fields $\boldsymbol{M}_t$ are stochastically independent with respect to μ, we have

$$(3) \qquad \begin{aligned} \mu(E_k) &= \sum_i \mu(A_{t_1}^{(i_1)} \cdot A_{t_2}^{(i_2)} \cdot \ldots \cdot A_{t_{n_k}}^{(i_{n_k})}) \cdot \mu(B_{i_1 i_2 \ldots i_{n_k}}^{(k)}) = \\ &= \sum_i \mu(A_{t_1}^{(i_1)}) \cdot \mu(A_{t_2}^{(i_2)}) \cdot \ldots \cdot \mu(A_{t_{n_k}}^{(i_{n_k})}) \cdot \mu(B_{i_1 i_2 \ldots i_{n_k}}^{(k)}). \end{aligned}$$

Obviously

$$\sum_i \mu(A_{t_1}^{(i_1)} \cdot A_{t_2}^{(i_2)} \cdot \ldots \cdot A_{t_{n_k}}^{(i_{n_k})}) = 1,$$

and consequently we may apply the following arithmetical proposition which is easy to prove:

(A) If

$$c = \sum_{j=1}^{n} a_j b_j, \quad a_j \geqslant 0, \quad b_j \geqslant 0, \quad \sum_{j=1}^{n} a_j = 1,$$

then there is j_0 such that $c \leqslant b_{j_0}$ and $a_{j_0} \neq 0$.

Since $\mu(E_k) \geqslant \delta$, it follows from (3) and (A) that for each natural k there exists a sequence $(i_1, i_2, \ldots, i_{n_k})$ of numbers $i_j \leqslant K_{t_j}$ satisfying the following condition:

$$(4) \qquad \mu(A_{t_1}^{(i_1)}) > 0, \quad \mu(A_{t_2}^{(i_2)}) > 0 \quad, \ldots, \quad \mu(A_{t_{n_k}}^{(i_{n_k})}) > 0, \quad \mu(B_{i_1 i_2 \ldots i_{n_k}}^{(n_k)}) \geqslant \delta.$$

Obviously if $(i_1, i_2, \ldots, i_{n_{k+1}})$ satisfies (4), then $(i_1, i_2, \ldots, i_{n_k})$ satisfies (4) too.

Consequently it is easy to define by induction an infinite sequence $i_1, i_2, \ldots$ such that $(i_1, i_2, \ldots, i_{n_k})$ satisfies (4) for $k=1,2,\ldots$

Let us put

$$B = B^{(1)}_{i_1 i_2 \ldots i_{n_1}} \cdot B^{(2)}_{i_1 i_2 \ldots i_{n_2}} \cdot \ldots$$

Since μ_{t_0} is a σ-measure in the σ-field $\boldsymbol{M}_{t_0}$, we have $B \epsilon \boldsymbol{M}_{t_0}$ and $\mu(B) \geqslant \delta$. Consequently, the almost-independence of $\boldsymbol{M}_t$ implies

$$E_1 \cdot E_2 \ldots \supset B \cdot A^{i_1}_{t_1} \cdot A^{i_2}_{t_2} \cdot \ldots \neq 0.$$

The relation (1) is thus proved.

Theorem IV. *Let μ be a σ-measure in a σ-field $\boldsymbol{M}$ of subsets of X, and φ a real set function defined on a family $\boldsymbol{F}$ of subsets of X, such that always $0 \leqslant \varphi(E) \leqslant 1$. If for each sequence of sets $A_n \epsilon \boldsymbol{F}$, each set $B \epsilon \boldsymbol{M}$ such that $\mu(B) > 0$, and each sequence $\{i_n\}$ of numbers 0 and 1 we have*

$$B \cdot A_1^{i_1} \cdot A_2^{i_2} \cdot \ldots \neq 0,$$

then there is a σ-measure ν in $\boldsymbol{N} = (\boldsymbol{M} + \boldsymbol{F})_\beta$ which is an extension of μ and φ, such that

$$\mu(A_1 \cdot A_2 \cdot \ldots \cdot A_n \cdot B) = \mu(A_1) \cdot \mu(A_2) \cdot \ldots \cdot \mu(A_n) \cdot \mu(B)$$

for each $A_j \epsilon \boldsymbol{F}$ and each $B \epsilon \boldsymbol{M}$.

In order to prove this theorem it is sufficient to consider for each set belonging to the family $\boldsymbol{E} = \{A_t\}$ the four-element field $(X, 0, A_t, X - A_t)$ and the measure μ_t:

$$\mu_t(X) = 1, \quad \mu_t(0) = 0, \quad \mu_t(A_t) = \varphi(A_t), \quad \mu_t(X - A_t) = 1 - \varphi(A_t)$$

and to apply Theorem III.

We do not know a necessary and sufficient condition for the existence of multiplicative σ-extensions, analogous to that contained in Theorem I (cf. Introduction). Instead we may complete the theorem of Banach to the following one, analogous to Theorem II:

Let $\{\boldsymbol{M}_t\}$ denote a family of σ-fields of subsets of X. Then the following statements are equivalent:

(α) *The fields $\boldsymbol{M}_t$ are σ-independent.*

(β) *There exists a multiplicative σ-extension of each family μ_t of σ-measures defined respectively in $\boldsymbol{M}_t$.*

(γ) *There exists a common σ-extension of each family μ_t of σ-measures defined respectively in $\boldsymbol{M}_t$.*

References.

[1] S. Banach, *Sur les transformations biunivoques*, Fund. Math. **19** (1932), pp. 10-16.

[2] — *On measures in independent fields* (edited by S. Hartman), Studia Math. **10** (1948), pp. 159-181.

[3] P. R. Halmos, *Measure theory*, New York 1950.

[4] H. Helson, *Remark on measures in almost independent fields*, Studia Math. **10** (1948), pp. 182-183.

[5] S. Kakutani, *Construction of a non separable extension of the Lebesgue measure space*, Proc. Imp. Acad. Tokyo **20** (1944), pp. 115-119.

[6] E. Marczewski (Szpilrajn), *Sur l'extension de la mesure lebesguienne*, Fund. Math. **25** (1935), pp. 551-558.

[7] — *Ensembles indépendants et leurs applications à la théorie de la mesure*, Fund. Math. **35** (1948), pp. 13-28.

[8] — *Indépendance d'ensembles et prolongement de mesures* (Résultats et problèmes), Coll. Math. **1** (1947-48), pp. 122-132.

[9] W. Sierpiński, *Sur les translations des ensembles linéaires*, Fund. Math. **19** (1932), pp. 22-28.

[10] R. Sikorski, *Independent fields and Cartesian products*, Studia Math. **11** (1950), pp. 171-184.

[11] — *On measures in Cartesian products of Boolean algebras*, Coll. Math. **2** (1950), pp. 124-129.

[12] A. Tarski, *Ideale in vollständigen Mengenkörpern I*, Fund. Math. **32** (1939), pp. 45-63.