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On Labil and Stabil Points.

By

K. Borsuk and J. W. Jaworowski (Warszawa).

1. The concept of the (homotopically) labil point is due to H. Hopf and E. Pannwitz[1]). Its definition can be formulated as follows:

Definition 1. A point a of a space M is *homotopically labil* whenever for every neighbourhood U of a there exists a continuous mapping $f(x,t)$ which is defined in the Cartesian product $M\times I$ of M and of the interval $I:0\leqslant t\leqslant 1$ and which satisfies the following conditions:

(1) $f(x,t)\in M$ for every $(x,t)\in M\times I$,

(2) $f(x,0)=x$ for every $x\in M$,

(3) $f(x,t)=x$ for every $(x,t)\in (M-U)\times I$,

(4) $f(x,t)\in U$ for every $(x,t)\in U\times I$,

(5) $f(x,1)\neq a$ for every $x\in M$.

A point $a\in M$ will be called *homotopically stabil*[2]) if it is not homotopically labil.

Remark. If a is a homotopically labil point of a space M and b a point of another space N and if there exists a homeomorphic mapping h of a neighbourhood U_0 of a in M onto a neighbourhood V_0 of b in N such that $h(a)=b$, then b is a homotopically

[1]) H. Hopf and E. Pannwitz, *Über stetige Deformationen von Komplexen in sich*, Math. Ann. **108** (1933), pp. 433-465. See also P. Alexandroff and H. Hopf, *Topologie I*, Berlin 1935, p. 523. In the present paper we slightly modify the terminology. Namely we shall refer to the points called by H. Hopf and E. Pannwitz *labil, as homotopically labil*. The term "labil" will be used here in the other sense.

[2]) H. Hopf and E. Pannwitz use the term "locally stabil point".

labil point of N. In fact, let V be a neighbourhood of b in N and $U \subset U_0$ an open neighbourhood of a in M so small that $h(\overline{U}) \subset V$. If $f(x,t)$ denotes the mapping satisfying the conditions (1)-(5), then putting

$$f'(y,t) = hf(h^{-1}(y),t) \quad \text{for every} \quad (y,t) \in h(U) \times I,$$
$$f'(y,t) = y \quad \text{for every} \quad (y,t) \in (N - h(U)) \times I,$$

we obtain a mapping f' which satisfies, for the space N, for the point b and for the neighbourhood V, conditions analogous to (1)-(5). Thus we see that the *property of being a homotopically labil point is a local one.*

Let us remark that for metric spaces definition 1 is equivalent to the following one:

Definition 1'. A point a of a metric space M is *homotopically labil* if for every $\varepsilon > 0$ there exists a continuous mapping g of $M \times I$ into M satisfying the following conditions:

(6) $\quad g(x,0) = x \quad$ for every $\quad x \in M$,

(7) $\quad \varrho(x, g(x,t)) < \varepsilon \quad$ for every $\quad (x,t) \in M \times I$,

(8) $\quad g(x,1) \neq a \quad$ for every $\quad x \in M$.

It is evident that a point $a \in M$ homotopically labil in the sense of the definition 1 is also homotopically labil in the sense of the definition 1'. Now let us suppose that a is homotopically labil in the sense of definition 1'. If U is a neighbourhood of a then for an $\varepsilon > 0$ the inequality $\varrho(a,x) < 3\varepsilon$ implies $x \in U$. Let $g(x,t)$ denote a continuous mapping satisfying the conditions (6), (7), (8). Putting

$$f(x,t) = g(x,t) \quad \text{if} \quad \varrho(x,a) \leqslant \varepsilon, \quad 0 \leqslant t \leqslant 1,$$
$$f(x,t) = g\left[x, t\left(2 - \frac{\varrho(x,a)}{\varepsilon}\right)\right] \quad \text{if} \quad \varepsilon \leqslant \varrho(x,a) \leqslant 2\varepsilon, \quad 0 \leqslant t \leqslant 1,$$
$$f(x,t) = x \quad \text{if} \quad \varrho(x,a) \geqslant 2\varepsilon, \quad 0 \leqslant t \leqslant 1,$$

we obtain a continuous mapping f satisfying conditions (1)-(5).

2. Besides the concept of homotopically labil points we introduce for metric spaces another concept, namely the concept of labil points by the following

Definition 2. A point a of a metric space M is *labil* whenever for every $\varepsilon > 0$ there exists a continuous mapping f of M into itself such that

(9) $\quad \varrho(x, f(x)) < \varepsilon \quad$ for every $\quad x \in M$,

(10) $\quad f(x) \neq a \quad$ for every $\quad x \in M$.

Points which are not labil are said to be *stabil points* of M.

Example. *Every point of order* 1 [3]) *is labil.*

In fact if a is a point of order 1 of a metric space M, then for every $\varepsilon > 0$ there exists a neighbourhood U of a open in M and such that the diameter of U is $< \varepsilon$ and the set $\overline{U} \cdot (M - U)$ contains only one point b. Putting $f(x) = b$ for every $x \in U$ and $f(x) = x$ for every $x \in M - U$ we obtain a continuous mapping f satisfying conditions (9) and (10).

Let us remark that a point of order 1 can be homotopically stabil. For instance, putting $M = \sum_{n=0}^{\infty} M_n$ where M_0 contains only one point $a = (0,0)$ of the Euclidean plane and M_n denotes, for $n = 1, 2, \ldots$, the circle given by the equation

$$\left(x - \frac{3}{2^n}\right)^2 + y^2 = \frac{1}{4^n}$$

we easily see that the point a is homotopically stabil in M though it is of order 1.

Evidently every point homotopically labil is necessarily labil, since if $g(x,t)$ is a mapping satisfying conditions (6), (7) and (8), then putting $f(x) = g(x,1)$ we obtain a mapping satisfying conditions (9) and (10). The continuum just considered $M = \sum_{n=0}^{\infty} M_n$ shows that the inverse is not true. Moreover, there exist spaces in which every point is labil, but no one is homotopically labil. For instance, the locally connected curve by Sierpiński [4]), universal for plane curves, has such properties.

Remark. Let us observe that *the property of being a labil point is not local.* In fact, consider in the Euclidean plane the following sets:

[3]) A point $a \in M$ is said to be *of order* n if n is the smallest integer such that for every positive number ε there exists an open neighbourhood of a in M of diameter $< \varepsilon$ whose boundary (with respect to M) contains at most n points. See P. Urysohn, C. R. de l'Ac. Paris **175** (1922), p. 481.

[4]) W. Sierpiński, C. R. de l'Ac. Paris **162** (1916), p. 629. Also C. Kuratowski, *Topologie II*, Monografie Matematyczne XXI, Warszawa-Wrocław 1950, p. 202.

$$M_0 = \underset{(x,y)}{E}[x^2+y^2=1],$$

$$M_n = \underset{(x,y)}{E}\left[x=\left(1-\frac{1}{n}\right)\cos\vartheta,\ y=\left(1-\frac{1}{n}\right)\sin\vartheta;\ |\vartheta|\leqslant\pi-\frac{1}{n}\right];\quad n=1,2,\ldots$$

$$M=\sum_{n=0}^{\infty}M_n,\qquad N=M\cdot\underset{(x,y)}{E}[x\geqslant 0].$$

We can easily verify that M and N are compact spaces, locally homeomorphic in the point $(1,0)$. But $(1,0)$ is stabil in M and labil in N.

3. Let us show that for ANR (absolute neighbourhood retracts)[5]) the concept of labil point is the same as the concept of homotopically labil point. By **2** it suffices to prove that every labil point a of a space M being an ANR is also homotopically labil.

We can assume that M is a subset of the Hilbert-cube Q_ω. Let r be a retraction of a neighbourhood V of M in Q_ω into M. The uniform continuity of r yields for every positive ε a positive ε' such that $y\in V$ and $\varrho(x,r(y))<\varepsilon$ whenever $x,y\in Q_\omega$, $x\in M$ and $\varrho(x,y)<\varepsilon'$.

Let a be a labil point of M. There exists a continuous transformation f' of M into itself such that $\varrho(x,f'(x))<\varepsilon'$ and $f'(x)\neq a$ for every $x\in M$. It follows that the segment $\overline{xf'(x)}$ lies in V. For every $x\in M$ and $t\in I$ denote by $\varphi(x,t)$ the point dividing the segment $\overline{xf'(x)}$ in the ratio $t/(1-t)$ and put

$$f(x,t)=r\varphi(x,t)\quad\text{for every}\quad x\in M\quad\text{and}\quad t\in I.$$

Thus we obtain a continuous mapping of the Cartesian product $M\times I$ into M. Since, for every $x\in M$, $\varphi(x,0)=x$ and $r(x)=x$ the mapping f satisfies condition (6). For every $(x,t)\in M\times I$, the point $\varphi(x,t)$ lies on the segment $\overline{xf'(x)}$. Hence

$$\varrho(\varphi(x,t),x)\leqslant\varrho(x,f'(x))<\varepsilon'.$$

It follows that the distance between x and $f(x,t)=r\varphi(x,t)$ is $<\varepsilon$, *i. e.* condition (7) is satisfied. Finally $\varphi(x,1)=f'(x)\in M$ for every $x\in M$, which implies $f(x,1)=r\varphi(x,1)=f'(x)\neq a$. Hence condition (8) is also satisfied.

[5]) A subset A of a space B is called a *retract* of B if there exists a continuous mapping r (called a *retraction*) of B onto A such that $r(x)=x$ for every $x\in A$. A compactum A is said to be an *absolute neighbourhood retract* whenever a topological image A^* of A in any space X is necessarily a retract of some neighbourhood of A^* in X. In particular every (finite) polytope is an absolute neighbourhood retract. See K. Borsuk, *Über eine Klasse von lokal zusammenhängenden Räumen*, Fund. Math. **19** (1932), p. 227.

4. The concepts of the labil, stabil, homotopically labil and homotopically stabil point constitute a useful tool in the study of Cartesian products. We shall say that a topological property ω of a point a of a space M is *invariant under Cartesian multiplication* if for every two spaces A and B and for arbitrary points $a\in A$ and $b\in B$ whenever a and b have the property ω then the point $(a,b)\in A\times B$ has also the property ω. A topological property ω will be said to be *invariant under Cartesian division* if for every two spaces A and B whenever the point $(a,b)\in A\times B$ has the property ω, then also both of the points $a\in A$ and $b\in B$ have the property ω.

The properties invariant under Cartesian division are important for the study of decompositions of a space into Cartesian products. In particular all invariants of the retractions are evidently invariant under Cartesian division. But the invariants of the retractions are not sufficient to determine the topological structure of all Cartesian factors of a space. Consequently, for the theory of the decomposition into Cartesian products it is important to investigate also topological properties which are not invariant under retraction, but are invariant under Cartesian division. Let us show that the stability and also homotopical stability both belong to such properties.

Theorem. *The stability of a point and also the homotopical stability are invariant under Cartesian division, but are not invariant under retraction.*

Proof. First let us prove that if the point $(a,b)\in A\times B$ is stabil (respectively homotopically stabil) in $A\times B$ then the point $a\in A$ is stabil (respectively homotopically stabil) in A. For were a labil in A, then there would exist for every $\varepsilon>0$ a continuous mapping f of A into itself such that

$$\varrho(x,f(x))<\varepsilon\quad\text{and}\quad f(x)\neq a\quad\text{for every}\quad x\in A.$$

Setting

$$g(x,y)=(f(x),y)\quad\text{for every}\quad (x,y)\in A\times B$$

we would obtain a continuous mapping of $A\times B$ into itself satisfying the conditions:

$$\varrho((x,y),g(x,y))=\varrho(x,f(x))<\varepsilon\quad\text{and}\quad g(x,y)\neq(a,b)$$

for every $(x,y)\in A\times B$.

But this is impossible since (a,b) is stabil in $A\times B$.

Were a homotopically labil in A then there would exist for every $\varepsilon > 0$ a continuous mapping $g(x,t)$ of $A \times I$ satisfying conditions (6), (7) and (8). Putting

$$f((x,y),t) = (g(x,t),y) \quad \text{for every} \quad (x,y) \in A \times B \quad \text{and} \quad t \in I,$$

we would obtain a continuous mapping of $A \times B \times I$ into $A \times B$ satisfying the conditions:

$$f((x,y),0) = (x,y),$$
$$\varrho((x,y),f((x,y),t)) = \varrho((x,y),(g(x,t),y)) = \varrho(x,g(x,t)) < \varepsilon,$$
$$f((x,y),1) = (g(x,1),y) \neq (a,b)$$

for every $(x,y,t) \in A \times B \times I$. Hence the point (a,b) would be homotopically labil in $A \times B$.

It remains to show that both the stability and homotopical stability of points are not invariant under retraction. It suffices to observe that a simple arc L lying on the circle S is a retract of S, and that every point of S is stabil (consequently also homotopically stabil) in S, but the end points of L are homotopically labil (consequently also labil) in L.

Problem. Is the stability (respectively homotopical stability) of points invariant under Cartesian multiplication?

5. Now we shall investigate the properties of labil and stabil points using the concepts of combinatorial topology. It is sufficient for our purposes to use only homological notions in the sense modulo 2 [6]).

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[6]) By an *m-dimensional ε-simplex* of the compactum A we understand a set composed by $m+1$ points of A with diameter $<\varepsilon$. A set of a finite number of m-dimensional ε-simplexes of A will be called *m-dimensional ε-chain modulo 2 of A*. The sum of two m-dimensional ε-chains mod 2: $\varkappa_1$ and $\varkappa_2$ of A is defined as the ε-chain $\varkappa_1 \dotplus \varkappa_2$ composed by all ε-simplexes belonging to just one of the chains $\varkappa_1$ and $\varkappa_2$. The boundary $\partial\Delta$ of an m-dimensional simplex Δ is the chain composed by all $(m-1)$-dimensional faces of Δ. In the case for which the simplex Δ is 0-dimensional we understand by $\partial\Delta$ the number 1 considered as the rest modulo 2. Under the boundary of an m-dimensional ε-chain $\varkappa$ mod 2 we understand the $(m-1)$-dimensional ε-chain $\partial\varkappa$ defined as the sum of boundaries of all ε-simplexes of $\varkappa$. An ε-chain is called an *ε-cycle* if its boundary vanishes. Two ε-cycles γ_1, γ_2 are said to be *ε-homologous in A* if there exists in A an ε-chain $\varkappa$ such that $\partial\varkappa = \gamma_1 + \gamma_2$. Then we write $\gamma_1 \underset{\varepsilon}{\sim} \gamma_2$ in A.

A sequence $\varkappa = \{\varkappa_i\}$ is called an *m-dimensional true chain in A* (mod 2) if there exists a sequence of positive numbers $\{\varepsilon_i\}$ convergent to zero and such

Definition 3. Let A_0 be a closed subset of a compact space A and let a be a point of $A - A_0$. The point a will be called *linked in A with the set A_0 in the dimension m* provided there exists in A_0 an $(m-1)$-dimensional true cycle $\gamma = \{\gamma_i\}$ homologous to zero in A but not homologous to zero in any compact subset of $A - (a)$. The point a will be called *linked in A with the set A_0* whenever it is linked in A with A_0 in some dimension m.

Theorem. *Let A_0 be a closed subset of a compact space A. If a point $a \in A - A_0$ is linked in A with A_0, then a is homotopically stabil in A.*

Proof. Let $\gamma = \{\gamma_i\}$ be a true cycle in A_0 homologous to zero in A but not homologous to zero in any compact subset of $A - (a)$.

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that $\varkappa_i$ is an m-dimensional ε_i-chain mod 2 in A for $i = 1,2,\ldots$ The m-dimensional true chains in A constitute an Abelian group with the addition defined as follows

$$\{\varkappa_i'\} + \{\varkappa_i''\} = \{\varkappa_i' + \varkappa_i''\}.$$

The zero of this group is the true chain $\{\varkappa_i\}$ with $\varkappa_i = 0$ for $i = 1,2,\ldots$ If $\varkappa = \{\varkappa_i\}$ is an m-dimensional true chain in A then putting

$$\partial\varkappa = \{\partial\varkappa_i\}$$

we obtain a true $(m-1)$-dimensional chain $\partial\varkappa$ called the *boundary of $\varkappa$*. If $\partial\varkappa = 0$, then $\varkappa$ is called a *true cycle in A*. Two true cycles $\gamma = \{\gamma_i\}$ and $\gamma' = \{\gamma_i'\}$ are called *homologous in A* whenever there exists in A a true chain $\varkappa$ such that $\partial\varkappa = \gamma + \gamma'$. Then we write $\gamma \sim \gamma'$ in A.

A true cycle $\gamma = \{\gamma_i\}$ in A is called *convergent* if the cycles $\gamma_{i+1} + \gamma_i$, $i = 1,2,\ldots$ constitute a true cycle homologous to zero in A.

If f is a mapping of the compactum A into another compactum B then to every m-dimensional chain $\varkappa$ of A there corresponds the chain $\varkappa_f$ composed by all m-dimensional simplexes which are by f images of the simplexes constituting $\varkappa$. It is clear that the operation f is permutable with the addition of chains and also with the operator of the boundary ∂. In particular f maps every cycle in A onto a cycle in B.

If the mapping f is continuous then every true chain $\varkappa = \{\varkappa_i\}$ in A is carried by f onto a true chain $\varkappa_f = \{\varkappa_{if}\}$ in B and also every true cycle in A (respectively every true cycle convergent in A) $\gamma = \{\gamma_i\}$ onto a true (respectively convergent) cycle $\gamma_f = \{\gamma_{if}\}$ in B. Two true cycles γ' and γ'' homologous in A are necessarily carried by f onto two true cycles γ_f' and γ_f'' homologous in B.

We can easily see that if γ is a true cycle in a closed subset A of the Hilbert-cube Q_ω and γ is not homologous to zero in A then there exists a positive number ε such that for every continuous mapping f of A onto a set $f(A) \subset Q_\omega$ and satisfying the condition $\varrho(x,f(x)) < \varepsilon$ for every $x \in A$, the true cycle γ_f is not homologous to zero in $f(A)$.

If a were homotopically labil in A then there would exist a continuous mapping $f(x,t)$ of $A\times I$ into A such that

(11) $$f(x,0)=x \quad \text{for every} \quad x\in A,$$

(12) $$f(x,1)\neq a \quad \text{for every} \quad x\in A,$$

(13) $$\varrho(x,f(x,t))<\varrho(a,A_0) \quad \text{for every} \quad (x,t)\in A\times I.$$

It follows by (13) that $f(A_0\times I)$ is a compact subset of $A-(a)$. By (11) the mapping $f(x,1)$ is then homotopic to the identical transformation. Hence $f(x,1)$ maps the true cycle γ onto a true cycle γ' homologous to γ in the set $f(A_0\times I)\subset A-(a)$. But the true cycle γ' is homologous to zero in the compact set $f(A,1)$ contained, by (12), in $A-(a)$. Hence the true cycle γ would be homologous to zero in the compact set

$$f(A_0\times I)+f(A,1)\subset A-(a),$$

contrary to the assumption that γ is not homologous to zero in any compact subset of $A-(a)$.

Remark. The statement of the last theorem that a is homotopically stabil cannot be replaced by the stronger statement that a is stabil. In fact, consider in the Euclidean plane the set $M=\sum_{n=0}^{\infty}M_n$, where M_0 denotes the segment with the end points $(0,0)$ and $(1,0)$ and M_n denotes, for $n=1,2,\ldots$, the segment with the end points $(0,0)$ and $(1,1/n)$. Then all interior points of the segment M_0 are labil, but every one of them is linked (in the dimension 1) with the set composed of the two end-points of M_0.

A more remarkable example is given by the known indecomposable continuum K built up of all half-circles lying in the half-plane $y\geqslant 0$ and having the point $(1/2,0)$ as center with the end-points belonging to Cantor's set, and of all half-circles lying in the half-plane $y\leqslant 0$ and having as centers the points of the form $(5/2\cdot 3^n,0)$, $n=0,1,\ldots$, and as end-points $(x,0)$ belonging to Cantor's set and satisfying the inequality $2/3^n\leqslant x\leqslant 1/3^{n-1}$ [7]). It is easy to observe that every point a of K is labil though a is linked in K (in the dimension 1) with every set composed by two points $a_1,a_2\in K-(a)$ belonging to different composants of K [8]).

[7]) See C. Kuratowski, *Théorie des continus irréductibles entre deux points I*, Fund. Math. **3** (1921), p. 209.

[8]) By a *composant* of an indecomposable continuum C is meant every maximal proper subset C_0 of C such that every two points of C_0 are contained in a subcontinuum of C_0. See Z. Janiszewski and C. Kuratowski, *Sur les continus indécomposables*, Fund. Math. **1** (1920), p. 218.

6. *Theorem.* *Let A be a subcompactum of an n-dimensional space M and let a be a point of A linked in A in the dimension n with a compactum $A_0\subset A-(a)$. Then a is homotopically stabil in M.*

Proof. Suppose on the contrary that the point a is homotopically labil in M. The set $U=M-A_0$ constitutes a neighbourhood of a. Hence there exists a continuous mapping $f(x,t)$ of $M\times I$ in M satisfying conditions (1)-(5). By hypothesis there exists in A_0 a true $(n-1)$-dimensional cycle $\gamma=\{\gamma_i\}$ homologous to zero in A, but not homologous to zero in any compact subset of $A-(a)$. Hence there exists in A a true n-dimensional chain $\varkappa=\{\varkappa_i\}$ such that $\partial\varkappa=\gamma$. The function $\varphi(x)=f(x,1)$ maps the true chain $\varkappa$ onto a true chain $\varkappa_\varphi=\{\varkappa_{i\varphi}\}$ lying in the compactum $\varphi(A)\subset M-(a)$. By (3) we have $\partial\varkappa_\varphi=\gamma_\varphi=\gamma$. It follows that the n-dimensional chains

$$\gamma_i^*=\varkappa_i+\varkappa_{i\varphi} \qquad (i=1,2,\ldots)$$

are cycles. Moreover we infer, by the homotopy of the mapping $\varphi(x)=f(x,1)$ with the identical mapping $f(x,0)$, that the n-dimensional true cycle $\gamma^*=\{\gamma_i^*\}$ is homologous to zero in M.

We assert that the true cycle γ^* is not homologous to zero in the set $A+\varphi(A)$. In fact, let us choose a positive number

$$\alpha<\tfrac{1}{3}\varrho(a,\varphi(A)).$$

We can admit that the diameter of every simplex belonging to $\varkappa_i,\gamma_i,\varkappa_{i\varphi}$ or to γ_i^* is $<\alpha$. Putting

$$U_1=\underset{p\in M}{E}[\varrho(p,a)<\alpha]; \quad U_2=\underset{p\in M}{E}[\varrho(p,a)<2\alpha],$$

let us denote by $\varkappa_i'$ the chain composed by all simplexes of $\varkappa_i$ containing at least one vertex belonging to U_1. Putting

(14) $$\varkappa_i''=\varkappa_i+\varkappa_i'$$

we have

$$\gamma_i^*=\varkappa_i'+\varkappa_i''+\varkappa_{i\varphi}.$$

It follows that $\partial\varkappa_i'+\partial\varkappa_i''+\partial\varkappa_{i\varphi}=0$, *i. e.*

$$\partial\varkappa_i'=\partial(\varkappa_i''+\varkappa_{i\varphi}).$$

Hence $\partial\varkappa_i'$ is a cycle lying in $\overline{U}_2-U_1$ and homologous to zero in $A\cdot\overline{U}_2$ as well in $\varphi(A)+(A-U_1)$. Moreover, we infer by (14) that

$$\partial\varkappa_i'=\gamma_i+\partial\varkappa_i''.$$

Hence the true cycle $\{\partial\varkappa_i'\}$ is homologous in $A-U_1$ to the true cycle $\{\gamma_i\}$. It follows that the true cycle $\{\partial\varkappa_i'\}$ is not homologous to zero in any compact subset of $A-(a)$. In particular $\{\partial\varkappa_i'\}$ is not homologous to zero in the set

$$A\cdot(\overline{U}_2-U_1)=A\cdot\overline{U}_2\cdot[\varphi(A)+(A-U_1)].$$

We infer by the known Phragmen-Brouwers theorem [9]) that the n-dimensional true cycle

$$\{\gamma_i^*\}=\{\varkappa_i'+\varkappa_i''+\varkappa_{i\varphi}\}$$

is not homologous to zero in the set $A\cdot\overline{U}_2+\varphi(A)+(A-U_1)=A+\varphi(A)$. But $\{\gamma_i^*\}$ is homologous to zero in M. It follows [10]) that $\dim M>n$ and this contradicts our hypothesis.

Corollary 1. *If M is an n-dimensional space and Q an n-dimensional cube $\subset M$, then every point a of the interior of Q is homotopically stabil in M.*

In particular if M is a locally-connected curve then for every point $a\in M$ of order >1 there exists in M a simple arc L containing L in its interior [11]). Hence we can formulate the following.

Corollary 2. *If M is a locally-connected curve, then every point $a\in M$ of order >1 is homotopically stabil in M.*

It has already been observed (example of **2**) that every point of order 1 is labil in M, but not necessarily homotopically labil, even if M is a locally-connected curve. But the following corollary holds:

Corollary 3. *In a local contractible curve the labil points are the same as the points of order 1.*

Since every local contractible curve is an ANR, we infer by **3** and by corollary 2 that every point a of a local contractible curve M of order >1 is stabil in M. On the other hand, as has already been shown, every point of order 1 is labil in M.

[9]) See P. Alexandroff, *Dimensionstheorie. Ein Beitrag zur Geometrie der abgeschlossenen Mengen*, Math. Ann. **106** (1932), p. 178. Also K. Borsuk, *Über sphäroidale und H-sphäroidale Räume*, Recueil Math. Moscou **1** (43), (1936), p. 643.

[10]) See P. Alexandroff, l. c., p. 194-195.

[11]) For instance see C. Kuratowski, *Topologie II*, Warszawa-Wrocław 1950, p. 242.

7. The problem of the invariance of the stability and of the homotopical stability (posed in **4**) under Cartesian multiplication remains unsolved in the general case. But in some cases the homotopical stability of a point of a Cartesian product can be proved by the homological means based upon the following

Theorem. *Let A_0 be a closed subset of a compactum A and let B_0 be a closed subset of a compactum B. If a point $a\in A-A_0$ is linked in A with A_0 in the dimension k and a point $b\in B-B_0$ is linked in B with B_0 in the dimension l, then the point $(a,b)\in A\times B-(A_0\times B+A\times B_0)$ is linked in $A\times B$ with $A_0\times B+A\times B_0$ in the dimension $k+l$.*

First we prove the two following lemmas:

Lemma 1. *In the m-dimensional Cartesian space R_m are given: a k-dimensional chain $\varkappa$ and a cycle α' of dimension $m-k$ such that the geometrical realization $|\alpha'|$ of α' is disjoint to the geometrical realization $|\alpha|$ of the cycle $\alpha=\partial\varkappa$. In the n-dimensional Cartesian space R_m are given: an l-dimensional chain λ and a cycle β' of dimension $n-l$ such that the geometrical realization $|\beta'|$ of β' is disjoint to the geometrical realization $|\beta|$ of the cycle $\beta=\partial\lambda$. Then the $(k+l-1)$-dimensional cycle*

$$\gamma=\partial(\varkappa\times\lambda)=\varkappa\times\beta+\alpha\times\lambda \ [12]),$$

and the $[(m+n)-(k+l)]$-dimensional cycle

$$\delta=\alpha'\times\beta'$$

lying in the $(m+n)$-dimensional Cartesian space R_{m+n} have disjoint realizations $|\gamma|$ and $|\delta|$, and their linking coefficient $\mathfrak{y}$ [13]) is given by the formula

$$\mathfrak{y}(\gamma,\delta)=\mathfrak{y}(\alpha,\alpha')\cdot\mathfrak{y}(\beta,\beta').$$

[12]) $\varkappa\times\lambda$ denotes the Cartesian product of the chains $\varkappa$ and λ. See P. Alexandroff and H. Hopf, *Topologie I*, Berlin 1935, p. 302, and S. Lefschetz, *Algebraic Topology*, New York 1942, p. 138. Also K. Borsuk, *On the Decomposition of Manifolds into Products of Curves and Surfaces*, Fund. Math. **33** (1945), p. 280.

[13]) Concerning the concepts of the linking coefficient $\mathfrak{y}$ and of the intersection index $\mathfrak{X}$ see for instance L. Pontrjagin, *The General Topological Theorem of Duality for Closed Sets*, Annals of Math. **35** (1934), p. 907. In our case only chains modulo 2 are used. Hence the values of $\mathfrak{X}$ and $\mathfrak{y}$ are rests modulo 2.

Proof. Evidently, the geometrical realization of the Cartesian product of two chains (modulo 2) is equal to the Cartesian product of their geometrical realizations. It follows that

$$|\gamma|\cdot|\delta|=(|\varkappa|\times|\beta|)\cdot(|\alpha'|\times|\beta'|)+(|\alpha|\times|\lambda|)\cdot(|\alpha'|\times|\beta'|)=$$
$$=(|\varkappa|\cdot|\alpha'|)\times(|\beta|\cdot|\beta'|)+(|\alpha|\cdot|\alpha'|)\times(|\lambda|\times|\beta'|)=0.$$

By computing the linking coefficient of the cycles γ and δ we can replace $\varkappa$ and λ by chains obtained from them by a sufficiently small displacement of their vertices not belonging to $|\alpha|$ respectively to $|\beta|$. This enables us to assume that $\varkappa$ and α' are in general position in R_m[14]) and also that λ and β' are in a general position in R_n. Then $\varkappa\times\lambda$ and $\alpha'\times\beta'$ are in a general position in R_{m+n}. Let

$$\varkappa=\sum_i c_i\varkappa_i,\quad \lambda=\sum_j d_j\lambda_j,\quad \alpha'=\sum_\mu a_\mu\alpha'_\mu,\quad \beta'=\sum_\nu b_\nu\beta'_\nu.$$

Then

$$\mathfrak{y}(\gamma,\delta)=\mathfrak{y}(\partial(\varkappa\times\lambda),\alpha'\times\beta')=\mathfrak{y}(\partial(\sum_{i,j}c_i\cdot d_j\cdot\varkappa_i\times\lambda_j),\sum_{\mu,\nu}a_\mu\cdot b_\nu\cdot\alpha'_\mu\times\beta'_\nu)=$$
$$=\sum_{i,j,\mu,\nu}a_\mu\cdot b_\nu\cdot c_i\cdot d_j\cdot\mathfrak{X}(\varkappa_i\times\lambda,\alpha'_\mu\times\beta'_\nu)=\sum_{i,j,\mu,\nu}a_\mu\cdot b_\nu\cdot c_i\cdot d_j\cdot\mathfrak{X}(\varkappa_i,\alpha'_\mu)\cdot\mathfrak{X}(\lambda_j,\beta'_\nu)=$$
$$=[\sum_{i,\mu}c_i\cdot a_\mu\cdot\mathfrak{X}(\varkappa_i,\alpha'_\mu)]\cdot[\sum_{j,\nu}d_j\cdot b_\nu\cdot\mathfrak{X}(\lambda_j,\beta'_\nu)]=$$
$$=\mathfrak{y}(\partial\varkappa,\alpha')\cdot\mathfrak{y}(\partial\lambda,\beta')=\mathfrak{y}(\alpha,\alpha')\cdot\mathfrak{y}(\beta,\beta').$$

Lemma 2. *Let E_0 be a closed subset of a compactum E lying in the m-dimensional Cartesian space R_m and let $\varkappa=\{\varkappa_i\}$ be a true k-dimensional chain of E such that $\alpha=\{\alpha_i\}=\partial\varkappa$ is a convergent $(k-1)$-dimensional cycle of E_0 not homologous to zero in E_0. Moreover let F_0 be a closed subset of a compactum F lying in the n-dimensional Cartesian space R_n and let $\lambda=\{\lambda_i\}$ be a true l-dimensional chain of F such that $\beta=\{\beta_i\}=\partial\lambda$ is a convergent $(l-1)$-dimensional cycle of F_0 not homologous to zero in E_0. Then the $(k+l-1)$-dimensional true cycle*

$$\gamma=\partial(\varkappa\times\lambda)=\varkappa\times\beta+\alpha\times\lambda$$

lying in the set

$$H=E_0\times F+E\times F_0$$

is not homologous to zero in H.

[14]) *i. e.* if $|a_0a_1...a_k|$ is a simplex of $\varkappa$ and $|b_0b_1...b_p|$ is a simplex of α, then either $|a_0a_1...a_k|$ and $|b_0b_1...b_p|$ are disjoint or every system composed of $m+1$ of points $a_0a_1...a_k\,b_0b_1...b_p$ is linearly independent.

Proof. By the known theorem of Pontrjagin[15]) there exists in R_m-E_0 an $(m-k)$-dimensional cycle α' such that

$$\mathfrak{y}(\alpha,\alpha')=1.$$

Similarly there exists in R_n-F_0 an $(n-l)$-dimensional cycle β' such that

$$\mathfrak{y}(\beta,\beta')=1.$$

It follows that there exists an i_0 such that for every $i>i_0$ it is

$$\mathfrak{y}(\alpha_i,\alpha')=\mathfrak{y}(\beta_i,\beta')=1.$$

Applying lemma 1 we infer that for every $i>i_0$ it is

$$\mathfrak{y}(\partial(\varkappa_i\times\lambda_i),\alpha'\times\beta')=\mathfrak{y}(\alpha_i,\alpha')\cdot\mathfrak{y}(\beta_i,\beta')=1.$$

But the $[(m+n)-(k+l)]$-dimensional cycle $\alpha'\times\beta'$ lies in the set

$$R_m\times R_n-E_0\times F-E\times F_0=R_m\times R_n-H.$$

Hence the true cycle $\gamma=\partial(\varkappa\times\lambda)$ is not homologous to zero in H.

Proof of the theorem. We may assume that A and B are subsets of the Hilbert-cube Q_ω. Then

$$A\times B\subset Q_\omega\times Q_\omega.$$

Let $\alpha=\{\alpha_i\}$ be a $(k-1)$-dimensional true cycle in A_0 homologous to zero in A but not homologous to zero in any set of the form $A-U$, where U is a neighbourhood of a in A. Similarly let $\beta=\{\beta_i\}$ be an $(l-1)$-dimensional true cycle in B_0 homologous to zero in B but not homologous to zero in any set of the form $B-V$, where V is a neighbourhood of b in B. Let $\varkappa=\{\varkappa_i\}$ be a true chain in A such that $\partial\varkappa=\alpha$ and let $\lambda=\{\lambda_i\}$ be a true chain in B such that $\partial\lambda=\beta$. To prove our theorem we have only to show that the true cycle $\partial(\varkappa\times\lambda)$ is not homologous to zero in any set of the form $A\times B-W$, where W is a neighbourhood of (a,b) in $A\times B$.

Otherwise there exists a neighbourhood U of a in A and a neighbourhood V of b in B such that $\partial(\varkappa\times\lambda)$ is homologous to zero in $A\times B-U\times V$. Since α is not homologous to zero in $A-U$ there exist an $\varepsilon>0$ and a subsequence $\{\alpha_{i_\nu}\}$ of $\{\alpha_i\}$ such that α_{i_ν} is not ε-homologous to zero in A for every $\nu=1,2,...$ Hence we can assume that already the sequence $\{\varkappa_i\}$ satisfies the condition

(15) $\alpha_i=\partial\varkappa_i$ is not ε-homologous to zero in $A-U$.

[15]) L. Pontrjagin, l. c., p. 912.

Similarly we can assume that there exists an $\eta>0$ such that

(16) $\beta_l=\partial\lambda_l$ is not η-homologous to zero in $B-V$.

By a theorem of Alexandroff[16]) a convergent cycle may be chosen from every true cycle modulo 2 of a compactum. It follows that if we replace the true chains $\{\varkappa_l\}$ and $\{\lambda_l\}$ by suitably chosen subsequences we can assume that the true cycles $\partial\varkappa=\alpha$, $\partial\lambda=\beta$ and $\partial(\varkappa\times\lambda)=\varkappa\times\beta+\alpha\times\lambda$ are convergent. Moreover, by (15) the true cycle α is not homologous to zero in $A-U$, and by (16) the true cycle β is not homologous to zero in $B-V$.

Let ν be a natural number. Putting

$$r_\nu(x)=r_\nu(x_1,x_2,\ldots,x_\nu,x_{\nu+1},\ldots)=(x_1,x_2,\ldots,x_\nu,0,\ldots)$$

for every point $x=(x_1,x_2,\ldots,x_\nu,x_{\nu+1},\ldots)\in Q_\omega$ we obtain a continuous mapping r_ν of Q_ω onto the ν-dimensional Euclidean cube $Q_\nu=r_\nu(Q_\omega)$. Evidently, for every $\varepsilon_0>0$, there exists an index ν_{ε_0} such that

(17) $\varrho(r_\nu(x),x)<\varepsilon_0$ for every $x\in Q_\omega$ and every $\nu\geqslant\nu_{\varepsilon_0}$.

But the convergent cycles α and β are not homologous to zero in $A-U$, and $B-V$, respectively. By (17) there exists a natural ν such that $r=r_\nu$ maps α onto a convergent cycle $\alpha_r=\{\alpha_{lr}\}$ not homologous to zero in $r(A-U)$ and β onto a convergent cycle $\beta_r=\{\beta_{lr}\}$ not homologous to zero in $r(B-V)$. Moreover r maps the true chain $\varkappa=\{\varkappa_l\}$ onto a true chain $\varkappa_r=\{\varkappa_{lr}\}$ lying in $r(A)$ and the true chain $\lambda=\{\lambda_l\}$ onto a true chain $\lambda_r=\{\lambda_{lr}\}$ lying in $r(B)$.

Furthermore we have

$$\alpha_r=\partial\varkappa_r \quad\text{and}\quad \beta_r=\partial\lambda_r$$

and consequently

$$\partial(\varkappa_r\times\lambda_r)=\alpha_r\times\lambda_r+\varkappa_r\times\beta_r.$$

By lemma 2 the convergent cycle $\partial(\varkappa_r\times\lambda_r)$ is not homologous to zero in the set

$$r(A-U)\times r(B)+r(A)\times r(B-V).$$

Putting

$$s(x,y)=(r(x),r(y)) \quad\text{for every}\quad (x,y)\in Q_\omega\times Q_\omega$$

[16]) P. Alexandroff, *Dimensionstheorie. Ein Beitrag zur Geometrie der abgeschlossenen Mengen*, Math. Ann. **106** (1932), p. 180.

we obtain a continuous transformation s mapping the convergent cycle $\partial(\varkappa\times\lambda)$ onto the convergent cycle

$$[\partial(\varkappa\times\lambda)]_s=\partial(\varkappa_r\times\lambda_r)=\alpha_r\times\lambda_r+\varkappa_r\times\beta_r$$

not homologous to zero in the set

$$s(A\times B-U\times V)=r(A-U)\times r(B)+r(A)\times r(B-V).$$

But this contradicts the hypothesis that the convergent cycle $\partial(\varkappa\times\lambda)$ is homologous to zero in the set $A\times B-U\times V$. This completes the proof of the theorem.

8. A compact set $A\neq 0$ will be called *n-cyclic* if there exists in A a true n-dimensional cycle γ not homologous in A to any true cycle lying in a closed proper subset of A.

Example. The n-dimensional sphere S_n is n-cyclic. More generally, every compact subset of the Euclidean $(n+1)$-dimensional space E_{n+1} which is an irreducible cutting of E_{n+1} is n-cyclic.

Lemma. *Let γ be a true n-dimensional cycle lying in a compact subsets A of an n-dimensional space M and not homologous to zero in A and let $f(x,t)$ be a continuous mapping of $A\times I$ into M such that $f(x,0)=x$ for every $x\in A$. Then $A\cdot f(A,1)\neq 0$.*

Proof. Let $f(x,1)$ map the true cycle $\gamma=\{\gamma_i\}$ on the true cycle $\gamma^*=\{\gamma_i^*\}$. Evidently γ is homologous to γ^* in M. Hence the true n-dimensional cycle $\gamma+\gamma^*$ is homologous to zero in M.

Suppose that $A\cdot f(A,1)=0$. We assert that $\gamma+\gamma^*$ is not homologous to zero in $A+f(A,1)$. For otherwise there would exist in $A+f(A,1)$ a true chain $\varkappa=\{\varkappa_i\}$ such that $\partial\varkappa=\gamma+\gamma^*$. Cancelling a finite number of elements in the sequence $\{\varkappa_i\}$ we can assume that all vertices of every simplex of $\varkappa_i$ belong to only one of the sets A and $f(A,1)$. Thus $\varkappa_i$ is decomposed into the sum of two disjoint chains: $\varkappa_i'$ lying in A and $\varkappa_i''$ lying in $f(A,1)$. But $\partial\varkappa_i=\gamma_i+\gamma_i^*$, where γ_i lies in A and γ_i^* — in $f(A,1)$. Hence $\gamma_i=\partial\varkappa_i'$ for almost all i, *i. e.* $\gamma\sim 0$ in A, contrary to the hypothesis.

Thus the supposition that $A\cdot f(A,1)=0$ implies that the n-dimensional true cycle $\gamma+\gamma^*$ is not homologous to zero in $A+f(A)$, but homologous to zero in M. But this is impossible[10]), since $\dim M=n$.

Theorem. *If A is an n-cyclic subset of an n-dimensional space M and $f(x,t)$ is a continuous mapping of $A\times I$ into M such that $f(x,0)=x$, for every $x\in A$, then $A\subset f(A,1)$.*

Proof. Let $\gamma=\{\gamma_i\}$ be a true n-dimensional cycle in A not homologous in A to any true cycle lying in a closed proper subset of A. Suppose that there exists a point $a\in A-f(A,1)$. Then there exists a positive number ε such that $\varrho(a,f(A,1))>2\varepsilon$. Let us put

$$U_\varepsilon=\underset{p}{E}[p\in A;\ \varrho(p,a)<\varepsilon],$$
$$U_{2\varepsilon}=\underset{p}{E}[p\in A;\ \varrho(p,a)<2\varepsilon].$$

We can assume that all simplexes of γ_i have diameter $<\varepsilon$. Let $\varkappa_i$ denote the chain composed by all simplexes of γ_i containing at least one vertex belonging to U_ε and let $\lambda_i=\gamma_i+\varkappa_i$. Then $\varkappa=\{\varkappa_i\}$ is a true chain lying in $\overline{U}_{2\varepsilon}$ and $\lambda=\{\lambda_i\}$ is a true chain lying in $A-U_\varepsilon$. Let us show that the true cycle $\partial\varkappa=\partial\lambda$ lying in $\overline{U}_{2\varepsilon}-U_\varepsilon$ is not homologous to zero in $\overline{U}_{2\varepsilon}-U_\varepsilon$. For, otherwise, there would exist in $\overline{U}_{2\varepsilon}-U_\varepsilon$ a true chain $\{\mu_i\}$ such that $\partial\{\mu_i\}=\partial\{\varkappa_i\}$. The function $f(x,1)$ maps the true n-dimensional cycle $\{\varkappa_i+\mu_i\}$ lying in $\overline{U}_{2\varepsilon}$ onto a true cycle lying in the set $f(A,1)$ disjoint to $\overline{U}_{2\varepsilon}$. We infer by Lemma 1 that $\{\varkappa_i+\mu_i\}\sim 0$ in $\overline{U}_{2\varepsilon}\subset A$. Hence

$$\{\gamma_i\}\sim\{\gamma_i+\varkappa_i+\mu_i\}=\{\lambda_i+\mu_i\}\quad\text{in } A.$$

But $\{\lambda_i+\mu_i\}$ lies in the compactum $A-U_\varepsilon$. This contradicts the hypothesis that γ is not homologous in A to any true cycle lying in a closed proper subset of A.

Thus it is shown that the true cycle $\partial\varkappa=\partial\lambda$ is not homologous to zero in $\overline{U}_{2\varepsilon}-U_\varepsilon$. Let γ^* denote the true cycle being the image under $f(x,1)$ of the true cycle γ. Then $\partial\varkappa=\partial(\lambda+\gamma^*)$ is homologous to zero in $\overline{U}_{2\varepsilon}$ and also homologous to zero in $A+f(A,1)-U_\varepsilon$. It follows, by the known Phragmen-Brouwer theorem[9]), that the true cycle $\varkappa+\lambda+\gamma^*=\gamma+\gamma^*$ is not homologous to zero in $\overline{U}_{2\varepsilon}+(A-U_\varepsilon)+f(A,1)=A+f(A,1)$.

But this is impossible, because $\gamma+\gamma^*$ is homologous to zero in M and $\dim M=n$.

Corollary 1. *Let a be a point of an n-dimensional space M such that for every $\varepsilon>0$ there exists in M an n-cyclic subset $A_\varepsilon\subset M$ such that $\varrho(a,A_\varepsilon)<\varepsilon$. Then for every continuous mapping $f(x,t)$ of $M\times I$ into M such that $f(x,0)=x$, for every $x\in A$, the point a belongs to the closure $\overline{f(M,1)}$ of the set $f(M,1)$.*

Proof. For suppose that $a\,\bar{\in}\,\overline{f(M,1)}$. Then the number $\varepsilon=\varrho(a,f(M,1))$ is positive and the set A_ε satisfies the condition

$$A_\varepsilon-f(A_\varepsilon,1)\supset A_\varepsilon-f(M,1)\neq 0,$$

which contradicts the theorem.

Corollary 2. *Suppose that the n-dimensional space M is locally compact in the point $a\in M$ and that it satisfies the hypothesis of the corollary 1. Then the point a is homotopically stabil in M.*

Proof. Let U be a neighbourhood of a in M such that $\overline{U}$ is compact. If a is homotopically labil, then there exists a mapping $f(x,t)$ satisfying the conditions (1)-(5). Then $f(M,1)=f(\overline{U},1)+(M-U)$ and consequently $a\in\overline{f(\overline{U},1)}=f(\overline{U},1)\subset f(M,1)$, contrary to condition (5).

Corollary 3. *In an n-dimensional space M every point belonging to an n-cyclic subset of M is homotopically stabil in M.*

Corollary 4. *Let a be a homotopically stabil point of a locally-connected curve A and let b be an inner point of an n-dimensional cube Q. Then the point (a,b) is homotopically stabil in $A\times Q$.*

If there exists a dendrite which is a neighbourhood of a in A then by the remark of **1**, by **3** and Corollary 3 of **6** a is of order >1. Hence there exists a simple arc $L\subset A$ containing a in its interior. Then (a,b) lies in the interior of the $(n+1)$-dimensional cube $L\times Q$. By Corollary 1 of **6** the point (a,b) is homotopically stabil in the $(n+1)$-dimensional space $A\times Q$.

If, however, there exists no dendrite which is a neighbourhood of a in A, then for every $\varepsilon>0$ there exists a simple closed curve $A_\varepsilon\subset A$ such that $\varrho(a,A_\varepsilon)<\varepsilon$. Since — by a remark in **1** — homotopical stability constitutes a local property, we may replace Q by an n-dimensional sphere S and prove that the point (a,b) is homotopically stabil in $A\times S$. But $A_\varepsilon\times S$ is an $(n+1)$-dimensional manifold, hence also an $(n+1)$-cyclic subset of the $(n+1)$-dimensional space $A\times S$. Since $\varrho((a,b),A_\varepsilon\times S)<\varepsilon$, we infer by Corollary **3** that the point (a,b) is homotopically stabil in $A\times S$, hence also homotopically stabil in $A\times Q$.