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On Continuous Mappings on Cartesian Products.

By

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In every Hausdorff space we can distinguish two different topologies: the original *neighbourhood topology* and the *sequential topology*. The sequential topology is determined by the concept of a convergent sequence defined in the neighbourhood topology [1]). These two topologies are not, in general, equivalent (under sequential topology the space is only a Fréchet $\mathcal{L}^*$-space). The equivalence holds if the space satisfies the first axiom of countability [2]).

Let A and B be two Hausdorff spaces. By a *continuous mapping* of A into B we shall always understand a mapping Φ continuous in the neighbourhood topology, that is: for every neighbourhood V of $\Phi(a)$ there is a neighbourhood U of $a \in A$ with $\Phi(U) \subset V$. We shall say that a mapping Φ of A into B is *sequentially continuous* if it is continuous in the sequential topology of A and B, *i. e.* if $a = \lim a_n$ in A implies $\Phi(a) = \lim \Phi(a_n)$ in B.

The two above notions of continuity, corresponding to the classical definitions of Cauchy and Heine respectively, are not, in general, equivalent. Continuity always implies sequential continuity; the converse is true only under certain additional hypotheses, *e. g.* if A satisfies the first axiom of countability, in particular if A satisfies the second axiom of countability [3]) or if A is metrizable.

In this paper it will be shown that the equivalence of neighbourhood and sequential continuity holds also if the space B has the property

[1]) We write $a = \lim a_n$ if every neighbourhood of a contains all elements a_n except a finite number.

[2]) That is, for each point a there is a sequence of its neighbourhoods $\{U_n\}$ such that if U is any neighbourhood of a, then $U_n \subset U$ for an integer n.

[3]) That is, the space possesses an enumerable open basis.

(D) the diagonal D of $B\times B$ is a G_δ-set in the sequential topology (*i. e.* $B\times B-D=\sum_{n=1}^{\infty}F_n$ where the sets F_n are sequentially closed);

and if $A=\underset{t\in T}{P}A_t$ is the Cartesian product of spaces A_t satisfying the second axiom of countability, under a very general hypothesis about the cardinal $\overline{\overline{T}}$. An analogous statement will also be proved for mappings defined on some subspaces of A. For instance, if A_t are separable metric spaces and $\overline{\overline{T}}$ is less than the first inaccessible aleph [4]), then every sequentially continuous mapping of $A=\underset{t\in T}{P}A_t$ into any metric space B is continuous.

The above equivalence follows from the fact that, under these hypotheses, a sequentially continuous mapping Φ on the Cartesian product A depends only on an enumerable set of coordinates, that is $\Phi=\Phi_0\pi$, where Φ_0 is a continuous mapping on $A_0=\underset{t\in T_0}{P}A_t$ with $\overline{\overline{T}}_0\leqslant\aleph_0$, and π is the projection of A onto A_0.

The proof of this result is based on a theorem which is a generalization of Ulam's [5]) theorem on measures in abstract sets.

I. A generalization of Ulam's results.

The class of all subsets of an abstract set $\mathfrak{X}$ will be denoted by $\boldsymbol{S}(\mathfrak{X})$. The class $\boldsymbol{S}(\mathfrak{X})$ may be considered as a Hausdorff (bicompact) space [6]) where the sets

$$\underset{X}{E}(x\in X\subset\mathfrak{X}) \quad\text{and}\quad \underset{X}{E}(x \text{ non}\in X\subset\mathfrak{X}) \quad (x\in\mathfrak{X})$$

form the open subbasis [7]). Note that $X=\lim X_n$ in this topology if and only if $X=\sum_{n=1}^{\infty}\prod_{m=1}^{\infty}X_{n+m}=\prod_{n=1}^{\infty}\sum_{m=1}^{\infty}X_{n+m}$, *i. e.* if $\{X_n\}$ converges to X in the sense defined in the General Theory of Sets. In this section we shall consider only the sequential topology of $\boldsymbol{S}(\mathfrak{X})$.

Let $\boldsymbol{X}$ be a class of sets and let $\mathfrak{X}_0$ be the sum of all $X\in\boldsymbol{X}$. If $\boldsymbol{X}$ is closed in the sequential topology of $\boldsymbol{S}(\mathfrak{X}_0)$ (*i. e.* if $X_n\in\boldsymbol{X}$ implies $\lim X_n\in\boldsymbol{X}$ whenever this limit exists), the class $\boldsymbol{X}$ will be called an $\mathfrak{F}$-*class*. Analogously $\boldsymbol{X}$ will be called a $\mathfrak{G}_\delta$-*class* if it is a G_δ-set in the sequential topology of $\boldsymbol{S}(\mathfrak{X}_0)$, *i. e.* if $\boldsymbol{S}(\mathfrak{X}_0)-\boldsymbol{X}=\sum_{n=1}^{\infty}\boldsymbol{X}_n$ where $\boldsymbol{X}_n$ are $\mathfrak{F}$-classes. Note that if $\boldsymbol{X}$ is an $\mathfrak{F}$-class or a $\mathfrak{G}_\delta$-class, it is also a sequentially closed set or a sequential G_δ-set in every space $\boldsymbol{S}(\mathfrak{X})$ where $\mathfrak{X}_0\subset\mathfrak{X}$.

If $\boldsymbol{X}\subset\boldsymbol{S}(\mathfrak{X})$ and if φ is a mapping of $\mathfrak{X}$ into another set $\mathfrak{Y}$, then $\boldsymbol{X}_\varphi$ will denote the class of all $Y\subset\mathfrak{Y}$ such that $\varphi^{-1}(Y)\in\boldsymbol{X}$. For every set $\mathfrak{Z}$ the symbol $\boldsymbol{X}|\mathfrak{Z}$ will denote the class of all sets $X\in\boldsymbol{X}$ contained in $\mathfrak{Z}$. It is easy to verify that if $\boldsymbol{X}$ is an $\mathfrak{F}$-class ($\mathfrak{G}_\delta$-class), then $\boldsymbol{X}_\varphi$ and $\boldsymbol{X}|\mathfrak{Z}$ are also $\mathfrak{F}$-classes ($\mathfrak{G}_\delta$-classes).

Let $\mathfrak{P}$ be a property of classes of sets which satisfies the following conditions:

(a) Every class $\boldsymbol{X}$ with the property $\mathfrak{P}$ is simultaneously an $\mathfrak{F}$-class and a $\mathfrak{G}_\delta$-class.

(b) If $\boldsymbol{X}$ has the property $\mathfrak{P}$, then $\boldsymbol{X}_\varphi$ and $\boldsymbol{X}|\mathfrak{Z}$ also have this property.

A class with the property $\mathfrak{P}$ will be called, for brevity, a $\mathfrak{P}$-*class*. A set $\mathfrak{X}$ is said to be $\mathfrak{P}$-*reducible* if every $\mathfrak{P}$-class $\boldsymbol{X}\subset\boldsymbol{S}(\mathfrak{X})$ containing all finite subsets of $\mathfrak{X}$ also contains the set $\mathfrak{X}$.

(i) *If $\mathfrak{X}$ is $\mathfrak{P}$-reducible and $\mathfrak{X}_0\subset\mathfrak{X}$, then $\mathfrak{X}_0$ is also $\mathfrak{P}$-reducible.*

Let $\boldsymbol{X}\subset\boldsymbol{S}(\mathfrak{X}_0)$ be a $\mathfrak{P}$-class containing all finite subsets of $\mathfrak{X}_0$, and let $\varphi(x)=x$ for $x\in\mathfrak{X}_0$. Then $\boldsymbol{X}_\varphi$ is the class of all $X\subset\mathfrak{X}$ such that $X\mathfrak{X}_0\in\boldsymbol{X}$. The class $\boldsymbol{X}_\varphi$ contains all finite subsets of $\mathfrak{X}$ and is a $\mathfrak{P}$-class by (b). Hence $\mathfrak{X}\in\boldsymbol{X}_\varphi$, *i. e.* $\mathfrak{X}_0\in\boldsymbol{X}$.

(ii) *If $\mathfrak{X}$ is $\mathfrak{P}$-reducible, and if $\boldsymbol{X}$ is a $\mathfrak{P}$-class, containing all finite subsets of $\mathfrak{X}$, then $\boldsymbol{X}=\boldsymbol{S}(\mathfrak{X})$.*

Let $\mathfrak{X}_0\subset\mathfrak{X}$. Since $\mathfrak{X}_0$ is $\mathfrak{P}$-reducible by (i), we infer from (b) that $\mathfrak{X}_0\in\boldsymbol{X}|\mathfrak{X}_0$, that is $\mathfrak{X}_0\in\boldsymbol{X}$, Q. E. D.

(iii) *Let $\boldsymbol{X}$ be a $\mathfrak{P}$-class of subsets of the set $\mathfrak{X}=\sum_{s\in S}X_s$. If the set S is $\mathfrak{P}$-reducible and if every set X contained in a finite sum $X_{s_1}+\ldots+X_{s_n}$ belongs to $\boldsymbol{X}$, then $\mathfrak{X}\in\boldsymbol{X}$.*

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[4]) An aleph $\aleph_\lambda$ is said to be *inaccessible* provided that $\lambda>0$ is a limit ordinal, and that $\sum_{s\in S}\mathfrak{m}_s<\aleph_\lambda$ whenever $\overline{\overline{S}}<\aleph_\lambda$ and $\mathfrak{m}_s<\aleph_\lambda$. See A. Tarski, *Über unerreichbaren Kardinalzahlen*, Fund. Math. **30** (1938), pp. 68-89.

[5]) S. Ulam, *Zur Masstheorie in der allgemeinen Mengenlehre*, Fund. Math. **16** (1930), pp. 140-150.

[6]) The generalized discontinuous space of Cantor. See the remarks at the end of this paper.

[7]) An open *subbasis* of a topological space is a class K of open sets such that each open set is a union of finite intersections of sets in K.

Let $\{Y_s\}_{s\in S}$ be a class of disjoint sets such that $Y_s \subset X_s$ and $\mathfrak{X} = \sum_{s\in S} Y_s$. We have $Y_{s_1} + \ldots + Y_{s_n} \in \boldsymbol{X}$ for every finite sequence $s_i \in S$. Put $\varphi(x) = s$ for $x \in Y_s$. The $\mathfrak{P}$-class $\boldsymbol{X}_\varphi \subset \boldsymbol{S}(S)$ contains all finite subsets of S. Consequently $S \in \boldsymbol{X}_\varphi$, that is $\mathfrak{X} \in \boldsymbol{X}$.

(iv) *If $\mathfrak{Y}$ is $\mathfrak{P}$-reducible and $\overline{\overline{\mathfrak{Y}}} = \overline{\overline{\mathfrak{X}}}$, then $\mathfrak{X}$ is also $\mathfrak{P}$-reducible.*

Let φ be a one-one transformation of $\mathfrak{X}$ onto $\mathfrak{Y}$ and let $\boldsymbol{X} \subset \boldsymbol{S}(\mathfrak{X})$ be a $\mathfrak{P}$-class containing all finite subsets of $\mathfrak{X}$. Then $\boldsymbol{X}_\varphi \subset \boldsymbol{S}(\mathfrak{Y})$ is a $\mathfrak{P}$-class (see (b)) containing all finite subsets of $\mathfrak{Y}$. Consequently $\mathfrak{Y} \in \boldsymbol{X}_\varphi$, hence $\mathfrak{X} \in \boldsymbol{X}$.

On account of (iv) it is convenient to introduce the following definition: a cardinal $\mathfrak{m}$ is *$\mathfrak{P}$-reducible*, if a set $\mathfrak{X}$ with $\overline{\overline{\mathfrak{X}}} = \mathfrak{m}$ is $\mathfrak{P}$-reducible.

(v) *$\aleph_0$ is $\mathfrak{P}$-reducible.*

This follows from (a) and the fact that every enumerable set is the limit of a sequence of finite sets.

Theorem I[8]. *If a cardinal $\mathfrak{m}$ is $\mathfrak{P}$-reducible and $\mathfrak{n} \leqslant \mathfrak{m}$, then $\mathfrak{n}$ is also $\mathfrak{P}$-reducible. If $\mathfrak{m} = \sum_{s\in S} \mathfrak{m}_s$, if $\overline{\overline{S}}$ is $\mathfrak{P}$-reducible and if every cardinal $\mathfrak{m}_s$ is $\mathfrak{P}$-reducible, then $\mathfrak{m}$ is also $\mathfrak{P}$-reducible. If $\aleph_\mu$ is $\mathfrak{P}$-reducible, then $\aleph_{\mu+1}$ is also $\mathfrak{P}$-reducible.*

Consequently every cardinal $\mathfrak{m}$ less than the first inaccessible aleph[4]) *is $\mathfrak{P}$-reducible.*

The first remark follows from (i) and (iv). The second — from (iii) and (iv), since $\overline{\overline{X_1 + \ldots + X_n}} = \max(\overline{\overline{X_1}}, \ldots, \overline{\overline{X_n}})$ for infinite sets.

Suppose $\aleph_\mu$ is $\mathfrak{P}$-reducible. Let $\overline{\overline{\mathfrak{X}}} = \aleph_{\mu+1}$ and let $\{E_{\alpha,\beta}\}$ (where $\alpha < \omega_\mu$, $\beta < \omega_{\mu+1}$) be Ulam's [5]) decomposition of $\mathfrak{X}$, *i. e.*

(c) $E_{\alpha,\beta} E_{\alpha,\beta'} = 0$ for $\beta \neq \beta'$ and $\overline{\overline{\mathfrak{X} - \sum_{\alpha<\omega_\mu} E_{\alpha,\beta}}} \leqslant \aleph_\mu$ for each $\beta < \omega_{\mu+1}$.

Let $\{D_{\alpha,\beta}\}$ $(\alpha < \omega_\mu, \beta < \omega_{\mu+1})$ be a family of empty or one-element sets such that for every $\beta < \omega_{\mu+1}$

$$\mathfrak{X} = \sum_{\alpha<\omega_\mu} E_{\alpha,\beta} + \sum_{\alpha<\omega_\mu} D_{\alpha,\beta} \quad \text{and} \quad \Big(\sum_{\alpha<\omega_\mu} E_{\alpha,\beta}\Big)\Big(\sum_{\alpha<\omega_\mu} D_{\alpha,\beta}\Big) = 0.$$

[8]) The proof of Theorem I and the preceding lemmas is a modification of Ulam's proof of analogous theorems on measure. See Ulam's paper cited in footnote [5]) and S. Banach et C. Kuratowski, *Sur une généralisation du problème de la mesure*, Fund. Math. **14** (1929), pp. 127-131; S. Banach, *Über additive Massfunktionen in abstrakten Mengen*, Fund. Math. **15** (1930), pp. 97-101.

Let $\boldsymbol{X} \subset \boldsymbol{S}(\mathfrak{X})$ be any $\mathfrak{P}$-class containing all finite subsets of $\mathfrak{X}$. The two following cases should be considered: either there is an ordinal $\beta < \omega_{\mu+1}$ such that every subset of any finite sum

$$E_{\alpha_1,\beta} + \ldots + E_{\alpha_n,\beta} + D_{\gamma_1,\beta} + \ldots + D_{\gamma_m,\beta}$$

belongs to $\boldsymbol{X}$ or such an ordinal does not exist.

In order to prove that $\aleph_{\mu+1}$ is $\mathfrak{P}$-reducible it is sufficient to show that the second case is impossible. In fact, in the first case we have $\mathfrak{X} \in \boldsymbol{X}$ on account of (iii).

Suppose the second case holds, *i. e.* for every $\beta < \omega_{\mu+1}$ there is a set $A_\beta \operatorname{non} \in \boldsymbol{X}$ such that $A_\beta = E_\beta + D_\beta$ where

$$E_\beta \subset E_{\alpha_1(\beta),\beta} + \ldots + E_{\alpha_{n_\beta}(\beta),\beta} \quad \text{and} \quad D_\beta \subset D_{\gamma_1(\beta),\beta} + \ldots + D_{\gamma_{m_\beta}(\beta),\beta}.$$

Since $\alpha_i(\beta)$, $\gamma_j(\beta) < \omega_\mu$, there exist a non-enumerable set Γ of ordinals $< \omega_{\mu+1}$, two finite sequences $\alpha_1, \ldots, \alpha_p$; $\beta_1, \ldots, \beta_q$ of ordinals $< \omega_\mu$, and an integer n_0 such that for every $\beta \in \Gamma$

$$A_\beta = E_\beta + D_\beta, \quad E_\beta \subset E_{\alpha_1,\beta} + \ldots + E_{\alpha_p,\beta}, \quad D_\beta \subset D_{\gamma_1,\beta} + \ldots + D_{\gamma_q,\beta}$$

and $A_\beta \in \boldsymbol{X}_{n_0}$ where $\{\boldsymbol{X}_n\}$ is a sequence of $\mathfrak{F}$-classes such that $\boldsymbol{S}(\mathfrak{X}) - \boldsymbol{X} = \sum_{n=1}^{\infty} \boldsymbol{X}_n$ (see (a)). Since $\overline{\overline{D_\beta}} \leqslant q$, there is a sequence [9]) $\beta_n \in \Gamma$ $(\beta_i \neq \beta_j$ for $i \neq j)$ such that $\{D_{\beta_n}\}$ converges to a finite set $D \in \boldsymbol{X}$. The condition (c) implies $\lim E_{\beta_n} = 0$. Consequently $\lim A_{\beta_n} = D \in \boldsymbol{X}$. On the other hand, $\lim A_{\beta_n} \in \boldsymbol{X}_{n_0}$ since $\boldsymbol{X}_{n_0}$ is an $\mathfrak{F}$-class. This contradicts $\boldsymbol{X} \cdot \boldsymbol{X}_{n_0} = 0$.

Examples of properties satisfying conditions (a) and (b).

1. A class $\boldsymbol{X}$ is said *to have the property* $\mathfrak{M}$ if there is a σ-measure ν on $\boldsymbol{S}(\sum_{X\in \boldsymbol{X}} X)$ such that $\boldsymbol{X} = \mathop{E}_{X}(\nu(X) = 0)$.

Conditions (a) and (b) are clearly satisfied. In this case, Theorem I gives the well known result of Ulam [5]).

More generally:

2. Let C be a Hausdorff space and let a set $H \subset C$ be simultaneously closed and a G_δ in the sequential topology of C. A class $\boldsymbol{X}$ is said *to have the property* $[C, H]$ if there is a sequentially continuous mapping Ψ of the space $\boldsymbol{S}(\sum_{X\in \boldsymbol{X}} X)$ into C such that $\boldsymbol{X} = \mathop{E}_{X}(\Psi(X) \in H)$.

[9]) In fact, let $\{\beta_n^*\}$ be any sequence of different ordinals in Γ, and let P be the sum of all sets $D_{\beta_n^*}$. Subsets of P may be interpreted as points of the Cantor discontinuous set. Consequently $\{D_{\beta_n^*}\}$ contains a convergent subsequence $\{D_{\beta_n}\}$.

$\boldsymbol{X}$ is then an $\mathfrak{F}$-class since H is closed. Let $C-H=\sum_{n=1} H_n$ where H_n is sequentially closed, and let $\boldsymbol{X}_n$ be the class of all sets X with $\Psi(X)\epsilon H_n$. Clearly $\boldsymbol{X}_n$ is an $\mathfrak{F}$-class and $\boldsymbol{X}_1+\boldsymbol{X}_2+\dots$ is the complement of $\boldsymbol{X}$. Consequently $\boldsymbol{X}$ is a $\mathfrak{G}_\delta$-class.

Let $\mathcal{Z}$ be any set, and let Ψ_0 be the mapping Ψ restricted to subsets of the sum of all sets $X\epsilon\boldsymbol{X}|\mathcal{Z}$. Then $\boldsymbol{X}|\mathcal{Z}=\underset{X}{E}(\Psi_0(X)\epsilon H)$, that is, $\boldsymbol{X}|\mathcal{Z}$ also has property $[C,H]$.

If $\boldsymbol{X}\subset\boldsymbol{S}(\mathfrak{X})$ and if φ is a mapping of $\mathfrak{X}$ into $\mathcal{Y}$, let $\Psi'(Y)=\Psi(\varphi^{-1}(Y))$ for $Y\subset\mathcal{Y}$. Then $\boldsymbol{X}_\varphi=\underset{Y}{E}(\Psi'(Y)\epsilon H)$, that is, $\boldsymbol{X}_\varphi$ also has property $[C,H]$.

As we have proved, property $[C,H]$ satisfies conditions (a) and (b). A set $\mathfrak{X}$ is $[C,H]$-reducible if every sequentially continuous mapping of $\boldsymbol{S}(\mathfrak{X})$ into C transforming all finite sets in points of H, transforms all sets $X\subset\mathfrak{X}$ in points of H. By Theorem I, $\mathfrak{X}$ is $[C,H]$-reducible whenever $\overline{\overline{\mathfrak{X}}}$ is less than the first inaccessible aleph.

3. Let $\mathcal{R}$ be the set of all real numbers. The property $[\mathcal{R},(0)]$ (where $0=$ the number zero) will be denoted for brevity by $\mathfrak{R}$. Thus a set $\mathfrak{X}$ is $\mathfrak{R}$-reducible, if every real sequentially continuous function on $\boldsymbol{S}(\mathfrak{X})$ vanishing for all finite sets vanishes identically. If $\overline{\overline{\mathfrak{X}}}$ is less than the first inaccessible aleph, then $\mathfrak{X}$ is $\mathfrak{R}$-reducible.

If C is metrizable, and $H=\overline{H}\subset C$, then the property $[C,H]$ implies the property $\mathfrak{R}$. In fact, if $\boldsymbol{X}=(\underset{X}{E}\Psi(X)\epsilon H)$, then $\boldsymbol{X}=\underset{X}{E}(\Psi_0(X)=0)$ where $\Psi_0(X)$ is the distance between $\Psi(X)\epsilon C$ and H. Consequently, if $\mathfrak{X}$ is $\mathfrak{R}$-reducible, $\mathfrak{X}$ is also $[C,H]$-reducible for every metrizable space C and every set $H=\overline{H}\subset C$.

II. Theorems on continuity.

In this section the letter B always denotes a Hausdorff space with the property (D). The diagonal of $B\times B$ is denoted by D. $\{A_t\}_{t\epsilon T}$ is a fixed family of Hausdorff spaces satisfying the second axiom of countability. For every $t\epsilon T$, a_t denotes a fixed point in A_t.

The Cartesian product $\underset{t\epsilon T}{P}A_t$ will be denoted by A. Elements of A are functions f on T, such that $f(t)\epsilon A_t$. The class of sets $\underset{f\epsilon A}{E}(f(t_0)\epsilon U_0)$, where $t_0\epsilon T$ and U_0 is open in A_{t_0}, is an open subbasis [7]) of A. Obviously

$$\lim f_n=f \text{ in } A \text{ if and only if } \lim f_n(t)=f(t) \text{ in } A_t \text{ for every } t\epsilon T.$$

Let $S\subset T$ and $f\epsilon A$. The symbol f_S will denote an element in A defined by the equalities:

$$f_S(t)=f(t) \quad \text{for} \quad t\epsilon S \quad \text{and} \quad f_S(t)=a_t \quad \text{for} \quad t\epsilon T-S.$$

The symbol $f|S$ will denote the mapping f restricted to arguments $t\epsilon S$. Clearly $f|S\epsilon\underset{t\epsilon S}{P}A_t$.

The transformation π_S of A onto $\pi_S(A)=\underset{t\epsilon S}{P}A_t$ defined by the equality

$$\pi_S(f)=f|S$$

is called the *projection* of A onto $\pi_S(A)$. If $A_0\subset A$, then $\pi_S(A_0)$ is the set of all $f|S$ where $f\epsilon A_0$. The projection π_S is obviously continuous.

A set $A_0\subset A$ is said to be *invariant under projection* provided the conditions $f\epsilon A_0$, $S\subset T$ imply $f_S\epsilon A_0$.

$A^{(\mathfrak{m})}$ will denote the set of all $f\epsilon A$ such that $\overline{\overline{\underset{t}{E}(f(t)\neq a_t)}}\leqslant\mathfrak{m}$.

We shall use the word „enumerable" in the sense of „finite or countable".

(vi) *Let $\{T_s\}_{s\epsilon S}$, where $\overline{\overline{S}}>\aleph_0$, be a family of enumerable subsets of T such that $\overline{\overline{\underset{s\epsilon S}{E}(t\epsilon T_s)}}\leqslant\aleph_0$ for every $t\epsilon T$. Then there is a non-enumerable subset $S_0\subset S$ such that $T_s\cdot T_{s'}=0$ for $s\neq s'$, $s,s'\epsilon S_0$.*

Let U_s be the set of all $s'\epsilon S$ such that there is a finite sequence $s=s_0, s_1,\dots,s_n=s'$ with $T_{s_{i-1}}\cdot T_{s_i}\neq 0$ $(1\leqslant i\leqslant n)$. Every set $U_s\subset S$ is enumerable and for $s\neq s'$ either $U_s\cdot U_{s'}=0$ or $U_s=U_{s'}$. The set S_0 containing exactly one element from every set U_s is the required.

(vii) *Let $\{T_s\}_{s\epsilon S}$ be a family of subsets of T such that $\overline{\overline{T_s}}\leqslant k$ $(k<\aleph_0)$ and $\overline{\overline{S}}>\aleph_0$. Then there are a finite set Z and a non-enumerable set $S_0\subset S$ such that $T_s\cdot T_{s'}=Z$ for $s,s'\epsilon S_0$, $s\neq s'$.*

Let l be the greatest integer such that there are an l-element set Z and a non-enumerable set $S'\subset S$ with $Z\subset T_s$ for $s\epsilon S'$. For every $t\epsilon T$, the formula $t\epsilon T_s-Z$ holds only for an enumerable set of elements $s\epsilon S'$. Since the family $\{T_s-Z\}_{s\epsilon S}$ satisfies the assumptions of (vi), there is a non-enumerable set $S_0\subset S'$ such that $(T_s-Z)(T_{s'}-Z)=0$ for $s,s'\epsilon S_0$ $(s\neq s')$, *i. e.* $T_s\cdot T_{s'}=Z$, Q. E. D.

(viii) *Let $\{f_s\}_{s\epsilon S}$, where $\overline{\overline{S}}>\aleph_0$, be a family of elements in $A^{(k)}$, $k<\aleph_0$. Then there is a sequence $s_n\epsilon S$ $(n=1,2,\dots,s_n\neq s_{n'}$ for $n\neq n')$ such that $\lim f_{s_n}=f^0$ exists. Moreover, $f^0=(f_{s_0})_Z$ where $Z\subset T$ and $s_0\epsilon S$.*

Let $T_s = \underset{t}{E}(f_s(t) \neq a_t)$. Since the family $\{T_s\}_{s\in S}$ satisfies the hypothesis of (vii), there exist a non-enumerable set $S_0 \subset S$ and a finite set $Z \subset T$ such that

(d) $$T_s \cdot T_{s'} = Z \quad \text{for} \quad s', s \in S_0, \quad s \neq s'.$$

The space $\pi_Z(A) = \underset{t\in Z}{P} A_t$ fulfills the second axiom of countability. Thus there is a sequence $s_0, s_1, s_2, \ldots \in S_0$ such that

$$\lim f_{s_n}|Z = f_{s_0}|Z \quad \text{in} \quad \pi_Z(A),$$

that is

$$\lim f_{s_n}(t) = f_{s_0}(t) \quad \text{for} \quad t \in Z.$$

The condition (d) implies $\lim f_{s_n}(t) = a_t$ for $t \in T - Z$. Hence the sequence $\{s_n\}$ and the element $f^0 = (f_{s_0})_Z$ satisfy the thesis of (viii).

Theorem II. *Let a set $A_0 \subset A^{(\aleph_0)}$ be invariant under projection, and let Φ be a sequentially continuous mapping of A_0 into B. Then there is an enumerable set $P \subset T$ such that $\Phi(f) = \Phi(f_P)$ for $f \in A_0$, i. e. $f|P = f'|P$ implies $\Phi(f) = \Phi(f')$.*

Consequently $\Phi(f) = \Phi_0 \pi_P(f)$ where Φ_0 is a continuous mapping of $\pi_P(A_0)$ into B. Therefore Φ is continuous.

Let $A_k = A_0 \cdot A^{(k)}$ $(0 < k < \aleph_0)$ and let S_k be the class of all $s \in T$ such that there is an $f_s \in A_k$ with $\Phi(f_s^*) \neq \Phi(f_s)$, (*i. e.* $(\Phi(f_s), \Phi(f_s^*))$ non $\in D$), where, for brevity, $f_s^* = (f)_{sT-(s)}$.

We shall prove that S_k is enumerable. Suppose the contrary. Then, since $B \times B - D = \sum_{n=1}^{\infty} F_n$, where F_n is sequentially closed, there would exist an integer n_0 and a non-enumerable set $S \subset S_k$ such that

(e) $$(\Phi(f_s), \Phi(f_s^*)) \in F_{n_0} \quad \text{for} \quad s \in S.$$

By (viii) there is a sequence $s_0, s_1, s_2, \ldots \in S$ such that $s_n \neq s_{n'}$ for $n \neq n'$ and $\lim f_{s_n} = f^0 = (f_{s_0})_Z$. The class A_k being invariant under projection, we infer $f^0 \in A_k$. Since $s_n \neq s_{n'}$ for $n \neq n'$, we would also have $\lim f_{s_n}^* = f^0$. Hence

$$\lim\,(\Phi(f_{s_n}), \Phi)f_{s_n}^*)) = (\Phi(f^0), \Phi(f^0)) \in D.$$

On the other hand, the condition (e) implies that

$$\lim\,(\Phi(f_{s_n}), \Phi(f_{s_n}^*)) \in F_{n_0},$$

which is impossible.

Thus the set $P = \sum_{k=1}^{\infty} S_k$ is enumerable. If $f \in A_k$, then $\Phi(f) = \Phi(f_P)$ since

$$f_P = (\,\ldots((f_{T-(t_1)})_{T-(t_2)})\ldots)_{T-(t_r)}, \quad \text{where} \quad (t_1, \ldots, t_r) = \underset{t\in T-P}{E}(f(t) \neq a_t).$$

In fact, if t non $\in S_k$, then $\Phi(f) = \Phi(f_{T-(t)})$ for every $f \in A_k$. If $f \in A_0$, then $f = \lim f_n$, where $f_n \in A_{k_n}$. Consequently

$$f_P = \lim\,(f_n)_P \quad \text{and} \quad \Phi(f_P) = \lim \Phi((f_n)_P) = \lim \Phi(f_n) = \Phi(f),$$

which proves the first part of Theorem II.

The second part follows immediately from the first. In fact, the definition of Φ_0 is obvious. Φ_0 is sequentially continuous. Since $\pi_P(A_0)$ satisfies the second axiom of countability, Φ_0 is continuous. Φ is continuous since it is a superposition of two continuous mappings.

Theorem III. *Let Φ be a sequentially continuous mapping of A^* into B where $A^* \subset A$ is invariant under projection. If T is $[B \times B, D]$-reducible (in particular, if $\overline{T}$ is less than the first inaccessible aleph), then there is an enumerable set $P \subset T$ such that $\Phi(f) = \Phi(f_P)$ for every $f \in A^*$, i. e. $f|P = f'|P$ implies $\Phi(f) = \Phi(f')$.*

Consequently $\Phi(f) = \Phi_0 \pi_P(f)$ where Φ_0 is a continuous mapping of $\pi_P(A^*)$ into B. Therefore Φ is also continuous.

If B is metrizable, the same result holds whenever T is $\mathfrak{R}$-reducible [10]).

Let A_0 be the class of all elements f_Z where $f \in A^*$ and $\overline{\overline{Z}} \leqslant \aleph_0$, $Z \subset T$. The set A_0 being invariant under projection, on account of Theorem II there is an enumerable set $P \subset T$ such that $f_Z|P = f'_Z|P$ implies $\Phi(f_Z) = \Phi(f'_Z)$ for $f, f' \in A^*$. The set P satisfies the condition of Theorem III. In fact, let $f, f' \in A^*$, $f|P = f'|P$. Consider the mapping

$$\Psi(Z) = (\Phi(f_Z), \Phi(f'_Z)) \quad \text{for} \quad Z \subset T.$$

Ψ maps $\boldsymbol{S}(T)$ into $B \times B$ and $\Psi(Z) \in D$ if $\overline{\overline{Z}} \leqslant \aleph_0$. Consequently, $\Psi(Z) \in D$ for every $Z \subset T$, which proves $\Phi(f) = \Phi(f')$.

Corollary 1. *If $\mathfrak{m}$ is less than the first inaccessible aleph, every sequentially continuous mapping of the Cartesian product of $\mathfrak{m}$ separable metric spaces into any metric space is continuous.*

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[10]) See remarks at the end of the first part (p. 234).

Corollary 2. *Let Ψ be a sequentially continuous mapping of $S(\mathfrak{X})$ into B. If $\mathfrak{X}$ is $[B\times B, D]$-reducible (e. g. if $\overline{\overline{\mathfrak{X}}}$ is less than the first inaccesible aleph), there exists an enumerable set $P\subset\mathfrak{X}$ such that $\Psi(X)=\Psi(XP)$ for all $X\subset\mathfrak{X}$. Consequently Ψ is continuous.*

The same result holds if B is metrizable and $\mathfrak{X}$ is $\mathfrak{R}$-reducible.

This follows immediately from Theorem III where $T=\mathfrak{X}$, A_t contains only two points: the numbers 0 and 1, $a_t=0$, and $\Phi(f)=\Psi(\underset{t}{E}(f(t)=1))$. In fact the transformation

$$X\rightarrow \text{the characteristic function of } X$$

is a homeomorphism of the Hausdorff space $S(\mathfrak{X})$ onto A.

The Role of the Axiom of Induction in Elementary Arithmetic.

By

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In the usual formulations of Peano's axioms for arithmetic, the axiom of induction must be formulated as a scheme containing an infinite number of proper axioms. An axiomatization of arithmetic by means of a finite number of axioms can be achieved if one includes among the primitive notions of arithmetic *e. g.* the notion of sets or of propositional functions. In the present paper I shall discuss the question whether it is possible to obtain a finite axiomatization of arithmetic, using only those primitive notions as are admitted ordinarily in Peano's system, that is: $=$, $<$, and an arbitrary number of arithmetical functions such as $x+y$, $x\cdot y$, x^y etc.

I shall show that no finite number of proper axioms, involring only these primitive terms, suffices to prove all the particular cases of the scheme of induction. Thus, Peano's arithmetic is not finitely axiomatizable if, only, the traditional primitive notions are allowed in the axioms.

From the methodological point of view, it may be interesting to note that the non-classical models of arithmetic (the existence of which was first proved by Skolem in [1]) are the chief tools used in my proof.

This paper is self-contained and all auxiliary theorems are explicitly stated and proved. The author believes that some of them may also prove useful in further investigations of related problems.

1. The lower functional calculus. The expressions of this system (which we shall call briefly the *system LF*) are built up from the following symbols:

(1) $x, y, z, \ldots, x_1, y_1, z_1, \ldots, x_2, y_2, z_2, \ldots$ (individual variables),