Corollary 2. *Let Ψ be a sequentially continuous mapping of $S(\mathfrak{X})$ into B. If $\mathfrak{X}$ is $[B\times B, D]$-reducible (e. g. if $\overline{\overline{\mathfrak{X}}}$ is less than the first inaccesible aleph), there exists an enumerable set $P\subset\mathfrak{X}$ such that $\Psi(X)=\Psi(XP)$ for all $X\subset\mathfrak{X}$. Consequently Ψ is continuous.*

The same result holds if B is metrizable and $\mathfrak{X}$ is $\mathfrak{R}$-reducible.

This follows immediately from Theorem III where $T=\mathfrak{X}$, A_t contains only two points: the numbers 0 and 1, $a_t=0$, and $\Phi(f)=\Psi(\underset{t}{E}(f(t)=1))$. In fact the transformation

$$X\rightarrow \text{the characteristic function of } X$$

is a homeomorphism of the Hausdorff space $S(\mathfrak{X})$ onto A.

The Role of the Axiom of Induction in Elementary Arithmetic.

By

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In the usual formulations of Peano's axioms for arithmetic, the axiom of induction must be formulated as a scheme containing an infinite number of proper axioms. An axiomatization of arithmetic by means of a finite number of axioms can be achieved if one includes among the primitive notions of arithmetic *e. g.* the notion of sets or of propositional functions. In the present paper I shall discuss the question whether it is possible to obtain a finite axiomatization of arithmetic, using only those primitive notions as are admitted ordinarily in Peano's system, that is: $=$, $<$, and an arbitrary number of arithmetical functions such as $x+y$, $x\cdot y$, x^y etc.

I shall show that no finite number of proper axioms, involring only these primitive terms, suffices to prove all the particular cases of the scheme of induction. Thus, Peano's arithmetic is not finitely axiomatizable if, only, the traditional primitive notions are allowed in the axioms.

From the methodological point of view, it may be interesting to note that the non-classical models of arithmetic (the existence of which was first proved by Skolem in [1]) are the chief tools used in my proof.

This paper is self-contained and all auxiliary theorems are explicitly stated and proved. The author believes that some of them may also prove useful in further investigations of related problems.

1. The lower functional calculus. The expressions of this system (which we shall call briefly the *system LF*) are built up from the following symbols:

(1) $x, y, z, \ldots, x_1, y_1, z_1, \ldots, x_2, y_2, z_2, \ldots$ (individual variables),

(2) $\varphi,\psi,\chi,\ldots,\varphi_1,\psi_1,\chi_1,\ldots,\varphi_2,\psi_2,\chi_2,\ldots$ (functors with an arbitrary non-negative number of arguments),

(3) $=,<$ (the signs of equality and the less-than relation),

(4) propositional connectives and quantifiers.

We shall often use letters $x,y,z,\ldots$ also as names of arbitrary variables and letters $\varphi,\psi,\chi,\ldots$ as names of arbitrary functors. The use of the same symbols with two different meanings is evidently not quite correct but will allow us to simplify our formulae and is not very dangerous in itself.

Variables are also called *numerical expressions* of rank 0. If $\Gamma_1,\Gamma_2,\ldots,\Gamma_k$ are numerical expressions of ranks $\leqslant n$ and φ is a functor with k arguments, then $\varphi(\Gamma_1,\ldots,\Gamma_k)$ is called the *numerical expression* of a rank $\leqslant n+1$. Numerical expressions will always be denoted by letters $\Gamma,\Delta,E,\ldots$ with or without subscripts. A numerical expression in which the variables $x,y,z,\ldots$ occur will be denoted by symbols like $\Gamma(x,y,z,\ldots)$.

If Γ and Δ are numerical expressions, then the expressions $\Gamma=\Delta$ and $\Gamma<\Delta$ are called *matrices* of rank 0. If Φ and Ψ are matrices of ranks $\leqslant n$, then $\sim\Phi,\Phi\vee\Psi,\Phi\cdot\Psi,\Phi\to\Psi,\Phi\leftrightarrow\Psi,\prod\limits_x\Phi$ and $\sum\limits_x\Phi$ are matrices of rank $\leqslant n+1$. Arbitrary matrices will be denoted by Greek capitals $\Phi,\Psi,\Theta,\ldots$

We assume as known the distinction between the free and bound variables of a matrix. A matrix in which the variables $x,y,z,\ldots$ are free will usually be denoted by symbols like $\Phi(x,y,z,\ldots)$.

The result of substitution of a numerical expression Γ for a free variable x throughout a matrix Φ will be denoted by $S^x_\Gamma\Phi$. We omit the enumeration of conditions which must be satisfied in order that this operation can be performed. Instead of $S^x_\Gamma S^y_\Delta S^z_E\ldots\Phi(x,y,z,\ldots)$ we shall also write $\Phi(\Gamma,\Delta,E,\ldots)$. Similar symbols will also be used for substitutions throughout the numerical expressions.

Finite strings of variables will be denoted sometimes by a single German letter *e. g.* $\mathfrak{x}$, $\mathfrak{y}$, or $\mathfrak{z}$. If $\mathfrak{x}=(x_1,\ldots,x_n)$, then $\sum\limits_{\mathfrak{x}}$ stands for $\sum\limits_{x_1},\ldots,\sum\limits_{x_n}$ and $\prod\limits_{\mathfrak{x}}$ for $\prod\limits_{x_1},\ldots,\prod\limits_{x_m}$. Among all matrices we distinguish in the well known way the axioms of the propositional calculus and the axioms of the functional calculus. We admit further the following axioms:

Axioms of equality:

$$x=y\to[\Phi\to S^x_y\Phi].$$

Axioms of order:

$$\sim(x<x),\qquad (x<y)\vee(x=y)\vee(y<x),$$
$$(x<y)\to[(y<z)\to(x<z)].$$

Matrices obtainable from the axioms by means of the usual rules of proof are called *theorems of the system LF*. If Φ is a theorem of LF, we shall write $\vdash_{LF}\Phi$.

The following abbreviations will often be used in the sequel:

$x\leqslant y$ for $(x<y)\vee(x=y)$,

$x\neq y$ for $\sim(x=y)$,

$\sum\limits_{j=1}^{n}\Phi_j$ for $\Phi_1\vee\Phi_2\vee\ldots\vee\Phi_n$,

$\prod\limits_{j=1}^{n}\Phi_j$ for $\Phi_1\cdot\Phi_2\cdot\ldots\cdot\Phi_n$.

Let $\mathfrak{x}=(x_1,\ldots,x_n)$ and $\mathfrak{y}=(y_1,\ldots,y_n)$ be two strings of variables. We put for $n=1,2,\ldots$

$$\Lambda_n(\mathfrak{x},\mathfrak{y})=\sum_{m=1}^{n}[(x_m<y_m)\cdot\prod_{j=1}^{m-1}(x_j=y_j)].$$

The matrix $\Lambda_n(\mathfrak{x},\mathfrak{y})$ can be read as: $\mathfrak{x}$ preceeds $\mathfrak{y}$ in the lexicographical ordering of all n-tuples.

For any matrix $\Phi(\mathfrak{x},\mathfrak{u})$ we shall write

$$I_\Phi(\mathfrak{u})\quad\text{for}\quad\sum_{\mathfrak{x}}\Phi(\mathfrak{x},\mathfrak{u})\cdot\prod_{\mathfrak{x}}\prod_{\mathfrak{y}}[\Phi(\mathfrak{x},\mathfrak{u})\cdot\Phi(\mathfrak{y},\mathfrak{u})\to(\mathfrak{x}=\mathfrak{y})].$$

The matrix $I_\Phi(\mathfrak{u})$ can be read as: $\Phi(\mathfrak{x},\mathfrak{u})$ satisfies the condition of existence and uniqueness with respect to the first n arguments.

The following theorem is easily derivable from the axioms of LF:

$$\vdash_{LF}I_\Phi(\mathfrak{u})\cdot\Phi(\varphi_1(\mathfrak{u}),\ldots,\varphi_n(\mathfrak{u}),\mathfrak{u})\to [t_i=\varphi_i(\mathfrak{u})\leftrightarrow\sum_{t_1}\ldots\sum_{t_{i-1}}\sum_{t_{i+1}}\ldots\sum_{t_n}\Phi(t_1,\ldots,t_n,\mathfrak{u})],\qquad i=1,2,\ldots,n. \tag{5}$$

In the rest of the paper we shall consider various systems which can be described as follows:

Let $\varphi_1, \varphi_2, \ldots, \varphi_k$ be k arbitrary functors and $\Phi_1, \Phi_2, \ldots, \Phi_l$ arbitrary matrices involving only the functors $\varphi_1, \ldots, \varphi_k$. An expression is called *meaningful* in the system $S = S(\varphi_1, \ldots, \varphi_k, \Phi_1, \ldots, \Phi_l)$ if it contains no functor other than $\varphi_1, \ldots, \varphi_k$. A meaningful matrix Φ is said to be *provable* in S (in symbols: $\vdash_S \Phi$) if it can be obtained from the axioms of LF and matrices $\Phi_1, \ldots, \Phi_l$ by repeated use of the rules of proof. It is evident from these definitions that the system S is wholly determined by its functors $\varphi_1, \ldots, \varphi_k$ and specific axioms $\Phi_1, \ldots, \Phi_l$. When speaking of systems, we shall always have in mind only systems S whose structure has been described above.

2. Models. Let S be a system and let $\varphi_1, \ldots, \varphi_k$ be its functors. Every $k+2$-tuple

$$M = [I, \prec, f_1, \ldots, f_k]$$

consisting of a set, a binary relation and k functions will be called a *pseudo-model* of S if the following conditions are satisfied: (i) the set I is ordered by $\prec$; (ii) the number of arguments of f_j is equal to the number of arguments of φ_j and the arguments of f_j as well as its values run over the set I $(j = 1, 2, \ldots, k)$.

A pseudo-model is called *well ordered* if I is well ordered by the relation $\prec$. Note that the relation $\prec$ is irreflexive in every pseudo-model M.

Sometimes, it is convenient to allow a larger number of functions in M than there are functors in S. In this case M contains k functions $f_1, \ldots, f_k$ which correspond to the functors of S and an arbitrary number of wholly arbitrary functions.

Let M be a pseudo-model of S, $\Gamma(x_i, x_j, \ldots, x_h)$ a numerical expression of S, and $a_i, a_j, \ldots, a_h$ elements of I. We shall define by *induction on the rank of* Γ the element $\Gamma_M(a_i, a_j, \ldots, a_h)$, called the *value* of Γ in M for the arguments $a_i, a_j, \ldots, a_h$.

If Γ has the rank 0, then Γ is a variable, *e. g.* x_i, and we put $\Gamma_M(a_i) = a_i$. If Γ is the expression $\varphi_j(\Delta^{(1)}, \ldots, \Delta^{(s)})$ and the values $\Delta_M^{(p)}(a_i, a_j, \ldots, a_h) = b_p$ are already defined for $p = 1, 2, \ldots, s$, then we put

$$\Gamma_M(a_i, a_j, \ldots, a_h) = f_j(b_1, \ldots, b_s).$$

Now let $\Phi(x_i, x_j, \ldots, x_h)$ be a matrix of S. We shall define by *induction on the rank of* Φ the following relation: elements $a_i, a_j, \ldots, a_h$ satisfy in M the matrix Φ (in symbols $\models_M \Phi(a_i, \ldots, a_h)$).

If the rank of Φ is 0, *i. e.* if Φ has the form $\Gamma(x_i, \ldots, x_h) = \Delta(x_i, \ldots, x_h)$ or $\Gamma(x_i, \ldots, x_h) \prec \Delta(x_i, \ldots, x_h)$, then we define $\models_M \Phi(a_i, \ldots, a_h)$ as equivalent to $\Gamma_M(a_i, \ldots, a_h) = \Delta_M(a_i, \ldots, a_h)$ or $\Gamma_M(a_i, \ldots, a_h) \prec \Delta_M(a_i, \ldots, a_h)$. If the rank of Φ is $n+1$, and Φ has one of the forms: (i) $\sim\Psi$, (ii) $\Psi \cdot \Theta$, (iii) $\Psi \vee \Theta$, (iv) $\Psi \to \Theta$, (v) $\Psi \leftrightarrow \Theta$, then we define $\models_M \Phi(a_i, \ldots, a_h)$ as follows: in case (i) we define $\models_M \Phi(a_i, \ldots, a_h)$ as *equivalent with non* $\models_M \Psi(a_i, \ldots, a_h)$. If Φ has the form $\Psi \cdot \Theta$, then we define $\models_M \Phi(a_i, \ldots, a_h)$ as *equivalent with the conjunction* $\models_M \Psi(a_i, \ldots, a_h)$ *and* $\models_M \Theta(a_i, \ldots, a_h)$, and we adopt similar definitions if Φ has the forms $\Psi \vee \Theta$, $\Psi \to \Theta$, and $\Psi \leftrightarrow \Theta$. Finally, if Φ has the form $\Sigma_x \Psi(x, x_i, \ldots, x_h)$ or $\Pi_x \Psi(x, x_i, \ldots, x_h)$, then we define $\models_M \Phi(a_i, \ldots, a_h)$ as *equivalent* with the following:

$$\textit{there is an } a \in I \textit{ such that } \models_M \Psi(a, a_i, \ldots, a_h)$$

or

$$\textit{for every } a \in I, \models_M \Psi(a, a_i, \ldots, a_h).$$

A matrix $\Phi(x_i, \ldots, x_h)$ is called *true* in M if $\models_M \Phi(a_i, \ldots, a_h)$ for arbitrary $a_i, \ldots, a_h$ in I.

A pseudo-model M is called a *real model* (or simply a *model*) of S if all axioms of S are true in M.

The basic properties of the notions defined above are taken for granted. In particular, we shall use the following two well-known results:

Theorem 1. *If M is a model of S and Φ is not true in M, then non $\vdash_S \Phi$.*

Theorem 2. *If a model of S exists, then S is self consistent.*

Let $M = [I, \prec, f_1, \ldots, f_k]$ be a pseudo-model of S and let J be a subset of I such that $f_j(a_1, \ldots, a_s) \in J$ whenever $a_1, \ldots, a_s \in J$ $(j = 1, 2, \ldots, k)$. Let $\prec'$ be the relation $\prec$ restricted to J and $f_1', \ldots, f_k'$ functions $f_1, \ldots, f_k$ restricted to J. If these conditions are satisfied, we shall call the $k+2$-tuple $M' = [J, \prec', f_1', \ldots, f_k']$ a *(pseudo) sub-model* of M. The following theorem can be proved without difficulty:

Theorem 3. *If $a_i, \ldots, a_h \in J$, then $\Gamma_{M'}(a_i, \ldots, a_h) = \Gamma_M(a_i, \ldots, a_h)$ for every numerical expression Γ; if Φ contains no quantifiers, then the conditions $\models_{M'} \Phi(a_i, \ldots, a_h)$ and $\models_M \Phi(a_i, \ldots, a_h)$ are equivalent.*

In the next section we shall still need the following rather special result: Let $S = S(\varphi_1, \ldots, \varphi_k, \Phi_1, \ldots, \Phi_l)$ and $S' = S'(\varphi, \varphi_1, \ldots, \varphi_k, \Phi_1', \ldots, \Phi_m')$ and let $M = [I, \prec, f_1, \ldots, f_k]$, $M' = [I, \prec, f, f_1, \ldots, f_k]$ be two pseudo-models of S and S' (note that M' results from M by adjunction of but one function and that the sets I and relations $\lessgtr$ are identical in the two pseudo-models). Let $\Psi(x,\mathfrak{x})$ be a matrix of S and denote by $\Psi'(\mathfrak{x})$ the matrix $\Psi(\varphi(\mathfrak{x}),\mathfrak{x})$. We have then

Theorem 4. *The conditions* $\models_M \Psi(f(a_1,\ldots,a_s), a_1,\ldots,a_s)$ *and* $\models_{M'} \Psi'(a_1,\ldots,a_s)$ *are equivalent.*

In other words the theorem states that elements $f(a_1,\ldots,a_s)$, $a_1,\ldots,a_s$ satisfy in M the matrix Ψ if and only if the elements $a_1,\ldots,a_s$ satisfy in M' the matrix Ψ'. Note that although f is not a function of M, the element $f(a_1,\ldots,a_s)$ is a perfectly defined element of M.

Theorem 5. *If the functor* φ *does not occur in* $\Theta(\mathfrak{x})$, *then the conditions* $\models_M \Theta(a_1,\ldots,a_s)$ *and* $\models_{M'} \Theta(a_1,\ldots,a_s)$ *are equivalent.*

Both theorems can easily be proved by induction on the ranks of Ψ and Θ. We omit the details of these proofs.

3. Extensions of systems. A system $S'=S(\varphi_1',\ldots,\varphi_k',\Phi_1',\ldots,\Phi_l')$ is called an *extension* of a system $S=S(\varphi_1,\ldots,\varphi_m,\Phi_1,\ldots,\Phi_n)$ if (i) every φ_j $(j=1,2,\ldots,m)$ is identical with one of the functors φ_s' $(s=1,2,\ldots,k)$; (ii) every Φ_j $(j=1,2,\ldots,n)$ is provable in S'.

An extension S' is called *non-essential* if to every matrix Φ' of S' corresponds a matrix Φ of S with the same free variables such that $\vdash_{S'} \Phi' \leftrightarrow \Phi$.

Theorem 6. *In order that an extension* S' *of* S *be non-essential it is necessary and sufficient that for every* φ_s' *which is not a functor of* S, *there exists a matrix* Ψ_s *of* S *such that*

(6) $$\vdash_{S'} x=\varphi_s'(\mathfrak{x}) \leftrightarrow \Psi_s(x,\mathfrak{x}).$$

Proof. Necessity follows at once from the definitions. To show the sufficiency, we first prove the following

Lemma. *For every numerical expression* $\Gamma(\mathfrak{x})$ *of* S' *there exists a matrix* $\Theta_\Gamma(x,\mathfrak{x})$ *of* S *such that*

(7) $$\vdash_{S'} x=\Gamma(\mathfrak{x}) \leftrightarrow \Theta_\Gamma(x,\mathfrak{x}).$$

Proof of the lemma. If Γ is a variable, *e. g.* x_i, we take as $\Theta_\Gamma(x,x_i)$ the matrix $x=x_i$. Hence, the lemma is true for expressions of rank 0. Assume that the lemma holds for expressions of ranks $\leqslant n$ and that

$$\Gamma=\varphi_s'(\Delta_1,\ldots,\Delta_r),$$

where $\Delta_1,\ldots,\Delta_r$ are numerical expressions of ranks $\leqslant n$. According to the inductive assumption, there exist matrices $\Theta_j(x_j,\mathfrak{z}_j)$ such that

$$\vdash_{S'} x_j=\Delta_j(\mathfrak{z}_j) \leftrightarrow \Theta_j(x_j,\mathfrak{z}_j)$$

($\mathfrak{z}_j$ is the string of all free variables of Δ_j). Now, we take Θ_Γ equal to

$$\Sigma_{x_1}\ldots\Sigma_{x_r}[\Psi_s(x,x_1,\ldots,x_r)\cdot\prod_{j=1}^{r}\Theta_j(x_j,\mathfrak{z}_j)]$$

defined as follows: if φ_s' is a functor of S, then Ψ_s is the matrix $x=\varphi_s'(x_1,\ldots,x_r)$; otherwise, Ψ_s is the matrix, the existence of which has been assumed in (6).

Now it can easily be shown that this choice of Θ_Γ satisfies the condition (7). The Lemma is thus proved.

Proof of theorem 6. We have to show that for every matrix Φ' of S' there exists a matrix Φ of S such that $\vdash_{S'}\Phi'\leftrightarrow\Phi$.

If Φ' has one of the forms $\Gamma(\mathfrak{x})<\Delta(\mathfrak{y})$ or $\Gamma(\mathfrak{x})=\Delta(\mathfrak{y})$, we define Φ as $\sum\limits_x\sum\limits_y[\Theta_\Gamma(x,\mathfrak{x})\cdot\Theta_\Gamma(y,\mathfrak{y})\cdot x<y]$ or as $\sum\limits_x\sum\limits_y[\Theta_\Gamma(x,\mathfrak{x})\cdot\Theta_\Gamma(y,\mathfrak{y})\cdot x=y]$. If Φ' has one of the forms $\sim\Phi_1'$, $\Phi_1'\vee\Phi_2'$, $\Phi_1'\cdot\Phi_2'$, $\Phi_1'\to\Phi_2'$, $\Phi_1'\leftrightarrow\Phi_2$, $\Sigma_x\Phi_1'$, $\Pi_x\Phi_1'$ and Φ_i correspond to Φ_i' $(i=1,2)$, we define Φ as $\sim\Phi_1$, $\Phi_1\vee\Phi_2$, $\Phi_1\cdot\Phi_2$, $\Phi_1\to\Phi_2$, $\Phi_1\leftrightarrow\Phi_2$, $\Sigma_x\Phi_1$, $\Pi_x\Phi_1$. It can be proved by a straightforward induction that this choice of Φ satisfies our requirements. Theorem 6 is thus proved.

Theorem 7. *Let* Φ_1 *have the form* $\Pi_x\Psi(x)$. *Then the system* $S(\varphi_1,\ldots,\varphi_k,\Psi(x),\Phi_2,\ldots,\Phi_l)$ *is a non-essential extension of* $S(\varphi_1,\ldots,\varphi_k,\Phi_1,\ldots,\Phi_l)$.

The proof is obvious.

Theorem 8. *Let* Φ_1 *have the form* $\Sigma_x\Psi(\mathfrak{x},x)$ *and let* φ *be a functor different from all the functors* $\varphi_1,\ldots,\varphi_k$. *Further, let* Θ_1 *and* Θ_2 *be matrices*

$$\Psi(\mathfrak{x},\varphi(\mathfrak{x})),\quad \Psi(\mathfrak{x},x)\to\varphi(\mathfrak{x})\leqslant x.$$

Under these assumptions the system $S'=S(\varphi,\varphi_1,\ldots,\varphi_k,\Theta_1,\Theta_2,\Phi_2,\ldots,\Phi_k)$ *is a non-essential extension of* $S=S(\varphi_1,\ldots,\varphi_k,\Phi_1,\ldots,\Phi_l)$.

Proof. S' is an extension of S since $\vdash_{LF}\Theta_1\to\Phi_1$ and $\vdash_{S'}\Phi_1$. In order to show that the extension is non-essential it is sufficient to remark that

$$\vdash_{S'} x=\varphi(\mathfrak{x}) \leftrightarrow \Psi(\mathfrak{x},x)\cdot\Pi_y[\Psi(\mathfrak{x},y)\to y\leqslant x],$$

and to apply theorem 6.

Theorem 9. *For every system $S=S(\varphi_1,\dots,\varphi_k,\Phi_1,\dots,\Phi_l)$ there exists a non-essential extension $S'=S(\psi_1,\dots,\psi_m,\Psi_1,\dots,\Psi_n)$ such that no quantifiers occur in the axioms $\Psi_1,\dots,\Psi_m$ of S'.*

Proof. We can evidently assume that $\Phi_1,\dots,\Phi_l$ has been brought to the prenex normal form. Let t_j be the number of quantifiers occurring in Φ_j and rearrange the axioms so that $t_1=t_2=\dots=t_s>t_{s+1}\geqslant t_{s+2}\geqslant\dots\geqslant t_l$. The pair (t_1,s) will be called the *characteristic pair* of S. We say that (t',s') is *lower* than (t,s) if $t'<t$ or $t'=t$ and $s'<s$. Note that the characteristic pairs of systems constructed in theorems 7 and 8 are lower than the characteristic pair of the system S.

If the characteristic pair of S is $(0,l)$, it is sufficient to take $S'=S$. Hence, it will be sufficient to show that for every system S with a characteristic pair (t_1,s) where $t_1>0$, there exists a non-essential extension S' with a lower characteristic pair.

If Φ_1 begins with a general quantifier, we obtain the required extension using theorem 7; and if Φ_1 begins with an existential quantifier, we obtain it by theorem 8.

Theorem 9 is thus proved.

We shall still investigate the problem whether the extension obtained in theorem 9 is self-consistent.

Theorem 10. *If S has a well ordered model $M=[I,\prec,f_1,\dots,f_k]$, then so do the systems S' constructed in theorems 7 and 8; well ordered models of these systems can be obtained by adjunction of at most one function to M.*

Proof. The Theorem is trivial in case of an extension described in theorem 7. In the other case we remark that if $a_1,\dots,a_h$ are arbitrary elements of I, then $\models_M\Phi_1(a_1,\dots,a_k)$, i. e. there exists in I at least one element a such that $\models_M\Psi(a_1,\dots,a_h,a)$. Let $f(a_1,\dots,a_h)$ be the first such element a. Thus we have

$$(8)\qquad \models_M\Psi(a_1,\dots,a_h,f(a_1,\dots,a_h)),$$

$$(9)\qquad \text{if}\quad b\prec f(a_1,\dots,a_h),\quad \text{then}\quad \text{non}\models_M\Psi(a_1,\dots,a_h,b).$$

Adjoining f to the model M, we obtain a well ordered pseudo-model $M'=[I,\prec,f,f_1,\dots,f_k]$. Formulae (8) and (9) prove, according to theorem 4 and the definition of matrices Θ_1 and Θ_2 (see theorem 8), that $\models_{M'}\Theta_1(a_1,\dots,a_h)$ and $\models_{M'}\Theta_2(a_1,\dots,a_h,b)$ for arbitrary $a_1,\dots,a_h,b\in I$. Thus Θ_1 and Θ_2 are true in M'. The remaining axioms of S' are true in M' since the functor φ does not occur in them (cf. theorem 5). Hence, M' is a real model of S'.

We observe that the extension constructed in theorem 9 has been obtained from S by successive extensions of the types described in theorems 7 and 8, and we obtain from theorem 10 the following corollary:

Theorem 11. *If S has a well ordered model M, then the extension S' of S described in theorem 9 also has a well ordered model M' which arises from M by adjunction of a finite number of new functions.*

4. Majorizing functors. In this section we shall assume that S is an arbitrary system of the form $S(\varphi_1,\dots,\varphi_k,\Phi_1,\dots,\Phi_l)$ such that for at least one φ_j,

$$\vdash_S\varphi_j(x)=x.$$

Every such φ_j is called an *identity-functor*. We denote by h_i the number of arguments of φ_i $(i=1,2,\dots,k)$.

Definition. φ_m is a *majorizing functor* of S if

$$\vdash_S(x_1\leqslant x)\dots(x_{h_i}\leqslant x)\to\varphi_i(x_1,\dots,x_{h_i})\leqslant\varphi_m(x)\qquad(i=1,2,\dots,k),$$

(h_i=number of arguments of φ_i).

Theorem 12. *For every system S containing an identity functor there exists a non essential extension S' containing a majorizing functor. If the axioms of S do not contain quantifiers, then the axioms of S' enjoy the same property.*

Proof. Let $S=S(\varphi_1,\dots,\varphi_k,\Phi_1,\dots,\Phi_l)$. Let φ be a functor with one argument different from $\varphi_1,\dots,\varphi_k$ and let $\psi_1^{(i)},\dots,\psi_{h_i}^{(i)}$ $(i=1,2,\dots,k)$ be functors with 2 arguments different from each other and different from $\varphi_1,\dots,\varphi_k$. We adjoin to S the functors φ and $\psi_n^{(i)}$ $(i=1,2,\dots,k;\ n=1,2,\dots,h_i)$ and axioms $(i=1,2,\dots,k)$:

$$(10)\qquad (x_1\leqslant x)\dots(x_{h_i}\leqslant x)\to\varphi_i(x_1,\dots,x_{h_i})\leqslant\varphi(x),$$

$$(11)\qquad y<\varphi(x)\to\psi_n^{(i)}(y,x)\leqslant x\qquad(n=1,2,\dots,h_i),$$

$$(12)\qquad y<\varphi(x)\to\sum_{i=1}^{k}y<\varphi_i(\psi_1^{(i)}(y,x),\dots,\psi_{h_i}^{(i)}(y,x)),$$

$$(13)\qquad \begin{gathered}y<\varphi(x)\cdot A_h(z_1^{(1)},\dots,z_{h_k}^{(k)};\psi_1^{(1)}(y,x),\dots,\psi_{h_k}^{(k)}(y,x))\to\\ \to\prod_{i=1}^{k}[\varphi_i(z_1^{(i)},\dots,z_{h_i}^{(i)})\leqslant y],\end{gathered}$$

$$(14)\qquad \sim(y<\varphi(x))\to\psi_n^{(i)}(y,x)=\varphi(x)\qquad(n=1,2,\dots,h_i),$$

where $h=h_1+h_2+\dots+h_k$.

The intuitive content of these axioms is the following: $\varphi(x)$ is the first element which is greater than the elements $\varphi_i(x_1,\dots,x_{h_i})$ for $i=1,2,\dots,k$ and $x_1\leqslant x,\dots,x_{h_i}\leqslant x$. If $\sim(y<\varphi(x))$, then $\psi_n^{(i)}(y,x)$ is equal to $\varphi(x)$; if $y<\varphi(x)$, then the elements $\psi_1^{(1)}(y,x),\dots,\psi_{h_k}^{(k)}(y,x)$ form the first (with respect to the lexicographical ordering) h_k-tuple $(u_1^{(1)},\dots,u_{h_k}^{(k)})$ such that $\varphi_i(u_1^{(1)},\dots,u_{h_i}^{(i)})\leqslant y$ for at least one i.

We shall show that the following equivalence is provable in (S'):

$$(15)\quad \begin{aligned}&\vdash_{S'} z=\varphi(x)\leftrightarrow\prod_{i=1}^{k}\{\prod_{x_1}\dots\prod_{x_{h_i}}[(x_1\leqslant x)\dots(x_{h_i}\leqslant x)\rightarrow\varphi_i(x_1,\dots,x_{h_i})\leqslant z]\}\\ &\cdot\prod_{y}\{y<z\rightarrow\sum_{i=1}^{k}\sum_{x_1}\dots\sum_{x_{h_i}}[(x_1\leqslant x)\dots(x_{h_i}\leqslant x)\cdot y<\varphi_i(x_1,\dots,x_{h_i})]\}.\end{aligned}$$

Indeed, the implication $\rightarrow$ results immediately from axioms (10), (11) and (12). To prove the converse implication, we denote by $H_1(z)$ and $H_2(z)$ the two matrices on the right side of (15). From (11) we obtain

$$\vdash_{S'}(z<\varphi(x))\cdot H_1(z)\rightarrow\prod_{i=1}^{k}[\varphi_i(\psi_1^{(i)}(y,x),\dots,\psi_{h_i}^{(i)}(y,x))<z].$$

Combining this with axiom (12), we get by the propositional calculus

$$(16)\quad \vdash_{S'}H_1(z)\rightarrow\sim(z<\varphi(x)).$$

Writing $H_2(z)$ in the form $\prod_y H_2'(z,y)$ and using $\vdash_{LF}H_2(z)\rightarrow H_2'(z,\varphi(x))$, we obtain after some transformations

$$\begin{aligned}\vdash_{S'}H_2(z)\rightarrow\{\prod_{i=1}^{k}\prod_{x_1}\dots\prod_{x_{h_i}}[(x_1\leqslant x)\dots(x_{h_i}\leqslant x)\rightarrow\\ \rightarrow\sim(\varphi(x)<\varphi_i(x_1,\dots,x_{h_i}))]\rightarrow\sim(\varphi(x)<z)\},\end{aligned}$$

whence (by (10) and the axioms of order)

$$\vdash_{S'}H_2(z)\rightarrow\sim(\varphi(x)<z).$$

Using (16) and the axioms of order, we finally obtain formula (15). It can be shown in essentially the same way that

$$(17)\quad\begin{aligned}&\vdash_{S'}(z_1^{(1)}=\psi_1^{(1)}(y,x))\dots(z_{h_k}^{(k)}=\psi_{h_k}^{(k)}(y,x))\leftrightarrow\sim(y<x)\cdot\prod_{i=1}^{k}\prod_{n=1}^{h_i}(z_n^{(i)}=\varphi(x))\\ &\vee(y<x)\cdot\sum_{i=1}^{k}(y<\varphi_i(z_1^{(i)},\dots,z_{h_i}^{(i)}))\cdot\prod_{t_1^{(1)}}\dots\prod_{t_{h_k}^{(k)}}[A_h(t_1^{(1)},\dots,t_{h_k}^{(k)},z_1^{(1)},\dots,z_{h_k}^{(k)})\rightarrow\\ &\rightarrow\prod_{i=1}^{k}(\varphi_i(t_1^{(i)},\dots,t_{h_i}^{(i)})\leqslant y)].\end{aligned}$$

The formal derivation of this formula is a matter of routine and can be omitted here.

Denoting by $P(z_1^{(1)},\dots,z_{h_k}^{(k)},y,x)$ the matrix on the right side of (17), we get

$$\vdash_{S'}P(z_1^{(1)},\dots,z_{h_k}^{(k)},y,x)\cdot P(\bar z_1^{(1)},\dots,\bar z_{h_k}^{(k)},y,x)\rightarrow\prod_{i=1}^{k}\prod_{n=1}^{h_i}(z_n^{(i)}=\bar z_n^{(i)}),$$

$$\vdash_{S'}\sum_{z_1^{(1)}}\dots\sum_{z_{h_k}^{(k)}}P(z_1^{(1)},\dots,z_{h_k}^{(k)},y,x),$$

i. e. $\vdash_{S'}I_P(y,x)$ (cf. § 1, p. 241). By using formula (5) of § 1, we obtain from (17)

$$(18)\quad \vdash_{S'}z_n^{(i)}=\psi_n^{(i)}(y,x)\leftrightarrow\sum_{z_1^{(1)}}\dots\sum_{z_{n-1}^{(i)}}\sum_{z_{n+1}^{(i)}}\dots\sum_{z_{h_k}^{(k)}}P(z_1^{(1)},\dots,z_{h_k}^{(k)},y,x).$$

Now, according to theorem 6, formulae (15) and (18) prove that S' is a non-essential extension of S.

It remains to prove that φ is a majorizing functor of S'. Since one of the φ_j's is an identity functor, we obtain $\vdash_{S'}x\leqslant\varphi(x)$ and hence by axioms (11) and (14)

$$(19)\quad \vdash_{S'}\psi_n^{(i)}(y,x)\leqslant\varphi(x)\qquad(i=1,2,\dots,k;\ n=1,2,\dots,h_i).$$

It is easy to prove that

$$(20)\quad \vdash_{S'}x_1\leqslant x_2\rightarrow\varphi(x_1)\leqslant\varphi(x_2);$$

thus formula (19) yields

$$(21)\quad \vdash_{S'}(y\leqslant x)\cdot(x_1\leqslant x)\rightarrow\psi_n^{(i)}(y,x_1)\leqslant\varphi(x)\quad(i=1,\dots,k;\ n=1,\dots,h_i).$$

Formulae (20), (21) and (10) prove that φ is a majorizing functor of S'. Theorem 12 is thus proved.

The problem of consistency of the extension S' constructed in theorem 12 can be settled in the same way as an analogous problem dealt with in section 3:

Theorem 13. *If S has a well ordered model M, then so does the system S' obtained from S by the method described in theorem 12. The model of S' can be obtained by adjunction of $h+1$ functions to M.*

The method of proof is the same as in theorem 11.

5. Elementary arithmetic A_0. In the rest of this paper we shall be concerned with a formal system of arithmetic A_0 and with its various extensions. In this section we describe the system A_0 itself; the next one is devoted to extensions of A_0.

There are 8 functors in the system A_0:

$$\varphi_1,\ \varphi_2,\ \varphi_3,\ \varphi_4,\ \varphi_5,\ \varphi_6,\ \varphi_7,\ \varphi_8.$$

The numbers of their arguments are respectively

$$0,2,2,2,1,1,2,1.$$

The intuitive interpretation of the functors is as follows: φ_1 denotes the constant one; φ_2, φ_3 and φ_4 denote the arithmetical functions: sum, product and power; φ_5 denotes the function $p_n = n$-th prime and φ_6 a function whose value for the argument a is equal to the greatest index n such that p_n still divides a. φ_7 denotes the function $\lambda(n,a)$ such that p_n occurs in a with the exponent $\lambda(n,a)-1$. Finally φ_8 is the identity functor.

When writing down the axioms and theorems of A_0, we shall use ordinary mathematical symbols for the functors of A_0: 1 instead of φ_1, $x+y$ instead of $\varphi_2(x,y)$, $x\cdot y$ or xy instead of $\varphi_3(x,y)$, x^y instead of $\varphi_4(x,y)$, p_x or $p(x)$ instead of $\varphi_5(x)$. Further we shall write $\mu(x)$ instead of $\varphi_6(x)$, $\lambda(x,y)$ instead of $\varphi_7(x,y)$, and x instead of $\varphi_8(x)$. Of course every formula can be translated back to the "official" language of A_0 without any difficulty, but the resulting expression is usually extremely unintelligible. We shall also use the customary abbreviation $x|y$ instead of $\Sigma_z(xz=y)$ and write $P(x)$ instead of $(x\neq 1)\cdot\prod_z\{z|x\to[(z=1)\vee(z=x)]\}$.

Axioms of A_0 will be divided into 9 groups:

I. Axioms of addition:

1. $x+y=y+x$,
2. $x+(y+z)=(x+y)+z$,
3. $1\neq x+y$,
4. $x+y=z+y\to x=z$.

II. Axioms of order:

5. $x\leqslant y\leftrightarrow\sum_z(x+z=y)$,
6. $1\leqslant x$,
7. $x<y+1\leftrightarrow x\leqslant y$.

III. Axioms of multiplication.

8. $xy=yx$,
9. $x(yz)=(xy)z$,
10. $x1=x$,
11. $x(y+z)=xy+xz$.

IV. Axioms of power:

12. $x^1=x$,
13. $x^{y+z}=x^y\cdot x^z$.

V. Axioms of primes:

14. $p_1=1+1$,
15. $P(p_x)$,
16. $P(x)\cdot p_y<x\to p_{y+1}\leqslant x$,
17. $P(x)\to\sum_y(x=p_y)$,
18. $x<y\to p_x<p_y$.

VI. Axiom for the functor φ_8:

19. $\varphi_8(x)=x$.

VII. Axioms for the functor μ:

20. $\mu(1)=1$,
21. $1<x\to p(\mu(x))|x$,
22. $1<x\cdot\mu(x)<y\to\sim p_y|x$.

VIII. Axioms for the functor λ:

23. $\lambda(x,1)=1$,
24. $p_x|y\to p_x^{\lambda(x,y)}|y$,
25. $p_x|y\to\sim(p_x^{1+\lambda(x,y)}|y)$.

In the last group of axioms we shall use the abbreviation $T_2(x,y,z)$ for

$$\prod_t[t\leqslant\mu(y)\to\lambda(t,y)=\lambda(t,x)]\cdot\prod_t[t\leqslant\mu(z)\to\lambda(t,z)=\lambda(t+\mu(y)+1,x)]\cdot$$
$$\cdot\prod_t(p_t|x\to\{p_t|y\vee\sum_{t_1}[(t_1+\mu(y)=t)\cdot p_{t_1}|z]\}).$$

IX. Axioms concerning representability of integers by primes:

26. $(1<x_1)\cdot(1<x_2)\cdot\prod_t[(p_t|x_1\leftrightarrow p_t|x_2)\cdot(\lambda(t,x_1)=\lambda(t,x_2)]\to x_1=x_2$,
27. $(1<x)\cdot t<\mu(x)\to\sum_y\{(\mu(y)=t)\cdot\prod_z[z\leqslant t\to\lambda(z,y)=\lambda(z,x)]\}$,
28. $(1<y)\cdot(1<z)\to\sum_x T_2(x,y,z)$.

Axiom 26 states of course that every integer greater than 1 is uniquely representable as a product of primes. Axiom 27 states that if $x=p_1^{a_1}\ldots p_\mu^{a_\mu}$ $(\mu=\mu(x))$ and $t<\mu(x)$, then the integer $p_1^{a_1}\ldots p_t^{a_t}$ exists. Finally axiom 28 states that if $y=p_{i_1}^{a_1}\ldots p_{i_k}^{a_k}$, where $i_1<i_2<\ldots<i_k$ and $z=p_{j_1}^{b_1}\ldots p_{j_l}^{b_l}$, where $j_1<j_2<\ldots<j_l$ then there exists an integer with the development

$$p_{i_1}^{a_1}\ldots p_{i_k}^{a_k}p_{i_k+1+j_1}^{b_1}\ldots p_{i_k+1+j_l}^{b_l}.$$

The description of the system A_0 is thus complete. The system A_0 can of course be represented as $S(\varphi_1,\ldots,\varphi_8,\Phi_1,\ldots,\Phi_{28})$, where Φ_1-Φ_{28} are the axioms of A_0.

To complete this section, we shall note further theorems of A_0 and introduce a number of abbreviations.

Definition. Functors (without arguments) $1=\varphi_1$, $2=1+1$, $3=2+1,\ldots$ are called *numerals*. The n-th numeral will be denoted by $\boldsymbol{n}$.

Theorem 14. $\vdash_{A_0} x<\boldsymbol{n}+1\to x=1\vee\ldots\vee x=\boldsymbol{n}$.

Proof. We use induction with respect to n. If $n=1$, the theorem to be proved becomes $\vdash_{A_0} x<2 \leftrightarrow x=1$.

This follows immediately from axioms 3 and 5-8. If the theorem holds for an integer n it does so for the next higher integer, since by axiom 7 we have $\vdash_{A_0} x<\boldsymbol{n}+2 \leftrightarrow (x<\boldsymbol{n}+1) \vee (x=\boldsymbol{n}+1)$.

Definition. Let $\mathfrak{x}$ be an n-tuple of variables $(x_1,\ldots,x_n)$. We put

$$T_n(x,\mathfrak{x}) = \sum_{z_1} \ldots \sum_{z_{n-2}} [T_2(z_1,x_1,x_2)\cdot T_2(z_2,z_1,x_3)\cdot T_2(z_3,z_2,x_4)\ldots$$
$$\ldots T_2(z_{n-2},z_{n-3},x_{n-1})\cdot T_2(x,z_{n-1},x_n)].$$

Theorem 15. (i) $\vdash_{A_0}(1<x_1)\ldots(1<x_n)\rightarrow \sum_x T_n(x,\mathfrak{x})$;

(ii) $\vdash_{A_0}(1<x_1)\ldots(1<x_n)\rightarrow T_n(x',\mathfrak{x})\cdot T_n(x'',\mathfrak{x})\rightarrow x'=x''$.

Proof. (i) can be obtained from axiom 28 by induction on n. To prove (ii), we first investigate the case $n=2$.

From the definition of T_2 we easily obtain

$$\vdash_{A_0}(1<y)\cdot(1<z)\cdot T_2(x',y,z)\cdot T_2(x'',y,z)\rightarrow$$
$$\rightarrow \prod_t [(p_t|x' \leftrightarrow p_t|x'')\cdot(\lambda(t,x')=\lambda(t,x''))],$$

whence by axiom 26:

$$\vdash_{A_0}(1<y)\cdot(1<z)\cdot T_2(x',y,z)\cdot T_2(x'',y,z)\rightarrow x'=x''.$$

The general case now follows by an easy induction.

Matrices T_n will be particullary useful when we discuss sequences of integers and their representability in the system A_0.

A finite sequence $a_1,\ldots,a_k$ can be represented by a single integer $2^{a_1}3^{a_2}\ldots p_k^{a_k}$. If x_i represents a sequence $a_1^{(i)},\ldots,a_{k_i}^{(i)}$ and $T_n(x,\mathfrak{x})$, then x represents the sequence $a_1^{(1)},\ldots,a_{k_1}^{(1)},a_1^{(2)},\ldots,a_{k_2}^{(2)},\ldots,a_1^{(n)},\ldots,a_{k_n}^{(n)}$.

Not every integer represents a sequence but only integers in which all primes up to a certain one occur. In connection with this remark we introduce the matrix $Y(x)$ defined as follows:

$$(1<x)\cdot\prod_y [p_y|x \leftrightarrow y\leqslant\mu(x)].$$

We note without proof some theorems concerning the matrices Y and T_n:

Theorem 16. (i) $\vdash_{A_0}\prod_{i=1}^{n} Y(x_i)\cdot T_n(x,x_1,\ldots,x_n)\rightarrow Y(x)$,

(ii) $\vdash_{A_0}\prod_{i=1}^{n} Y(x_i)\cdot T_n(x,x_1,\ldots,x_n)\rightarrow \mu(x)=\mu(x_1)+\ldots+\mu(x_n)$,

(iii) $\vdash_{A_0}\prod_{i=1}^{n}\{Y(x_i)\cdot T_n(x,x_1,\ldots,x_n)\rightarrow$
$\rightarrow[\lambda(1,x)=\lambda(1,x_1)]\cdot[\lambda(\mu(x),x)=\lambda(\mu(x_n),x_n)]\}$,

(iv) $\vdash_{A_0}\prod_{i=1}^{n} Y(x_i)\cdot T_n(x,x_1,\ldots,x_n)\cdot z\leqslant\mu(x)\rightarrow$
$\rightarrow\sum_{j=1}^{n}\sum_u[(u\leqslant\mu(x_j)\cdot(z=\mu(x_1)+\ldots+\mu(x_{j-1})+u)\cdot(\lambda(z,x)=\lambda(u,x_j))]$.

The last auxiliary matrix which we shall need later is introduced by the following

Definition. If $\mathfrak{x}=(x_1,\ldots,x_n)$, we use $x=1+\max(\mathfrak{x})$ as an abbreviation of $\prod_{i=1}^{n}(x_i<x)\cdot\sum_{i=1}^{n}(x_i+1=x)$.

Theorem 17. $\vdash_{A_0} x=1+\max(\mathfrak{x})\rightarrow[x\leqslant y+1\leftrightarrow\prod_{i=1}^{n}(x_i\leqslant y)]$.

The proof proceeds by induciton on n.

6. Models and extensions of A_0. In the following theorem we exhibit a well ordered model of A_0:

Theorem 18. *A_0 has the model $M_0=[I_0,<,f_1,\ldots,f_8]$, where I_0 is the set of positive integers, $<$ is the ordinary less-than relation and f_1-f_8 have the following meanings: $f_1=1$, $f_2(m,n)=m+n$, $f_3(m,n)=m\cdot n$, $f_4(m,n)=m^n$, $f_5(m)=p_m$, $f_6(m)=$ largest n such that $p_n|m$, $f_7(m,n)=$ largest s such that $p_m^{s-1}|n$, $f_8(m)=m$.*

The proof of this theorem is obvious.

M_0 will be called the *natural model* of A_0.

The set of matrices of A_0 which are true in M_0 will be called *Peanian arithmetic*. If $\Phi_{29},\ldots,\Phi_l$ belong to Peanian arithmetic, then $S(\varphi_1,\ldots,\varphi_8,\Phi_1,\ldots,\Phi_l)$ will be called a *Peanian system*.

To give an example of Peanian system, we remark that if $\Theta(x)$ is an arbitrary matrix of A_0, then the sentence

(22) $$\Theta(1)\cdot\prod_x[\Theta(x)\rightarrow\Theta(x+1)]\rightarrow\prod_x\Theta(x)$$

belongs to Peanian arithmetic. Hence, if $\Phi_{29},\ldots,\Phi_l$ all have the form (22), then $S(\varphi_1,\ldots,\varphi_8,\Phi_1,\ldots,\Phi_l)$ is a Peanian system.

Besides Peanian systems we shall consider their extensions obtained by the methods described in theorems 9 and 12. These *extended Peanian systems*, as we shall call them, possess well ordered models $M_0'=[I_0,<,f_1,\ldots,f_k]$ which can be obtained from M_0 by adjunctions of new functions. Models of this kind will also be called *natural models of the extended systems*.

Natural models are not the only models of Peanian systems. As Skolem [1] has shown the contrary is the case:

Theorem 19. *Every (extended) Peanian system has at least one non-natural model* $M=[I,\precsim,\bar{f}_1,\ldots,\bar{f}_k]$; *the set* I_0 *of integers is an initial segment of* I *and the relation* $\precsim$ *and the functions* $\bar{f}_1,\ldots,\bar{f}_k$ *restricted to* I_0 *are identical with the relation* $<$ *and the functions* $f_1,\ldots,f_k$ *of the natural model* M_0.

Remark. It can be shown that I_0 is an initial segment of every non-normal model of a Peanian system and that the functions of the model restricted to I_0 are always identical with the functions of the natural model. For our purposes it is however sufficient to know that there exists at least one non-normal model with this property; and precisely this has been proved by Skolem.

Using the notion of normal models, we can replace unprecise remarks concerning the intuitive content of the axioms and definitions introduced in A_0 by precise theorems. For instance, we have

Theorem 20. *If* $m>1$, $n>1$, $m=2^{a_1}\ldots p_k^{a_k}$, $n=2^{b_1}\ldots p_l^{b_l}$, *then* $\models_{M_0} T_2(q,m,n)$ *is equivalent to* $q=2^{a_1}\ldots p_k^{a_k}p_{k+1}^{b_1}\ldots p_{k+l}^{b_l}$.

A similar result can be formulated for the matrix T_n as well as for other matrices defined at the end of section 6.

7. The set $D(a)$.

Let $S=S(\varphi_1,\ldots,\varphi_k,\Phi_1,\ldots,\Phi_l)$ be an arbitrary Peanian system and let $M=[I,\precsim,\bar{f}_1,\ldots,\bar{f}_k]$ be its model. We assume that $\bar{f}_j$ has m_j arguments $(j=1,2,\ldots,k)$.

Let a be an arbitrary element of I. We call a the 1-st *descendent* of a. Assume that we have already defined the i-th descendents of a $(i=1,\ldots,n)$. An element $b\in I$ will be called an $n+1$-st *descendent* of a it there is a $j\leqslant k$ such that $b=\bar{f}_j(c_1,\ldots,c_{m_j})$, where c_h is a i_h-th descendent of a $(i_h\leqslant n, h=1,2,\ldots,m_j)$ and not all i_h are less than n.

The set of all descendents of a is denoted by $D(a)$. It is clear that if $c_1,\ldots,c_{m_j}\in D(a)$, then $\bar{f}_j(c_1,\ldots,c_{m_j})\in D(a)$. Hence restricting $\precsim$ and $\bar{f}_1,\ldots,\bar{f}_k$ to $D(a)$ we obtain a pseudo-submodel $M(a)=[D(a),\precsim',\bar{f}'_1,\ldots,\bar{f}'_k]$ of M (cf. section 2, p. 243). We shall call $M(a)$ the (pseudo) model *generated* by a.

To every element c of $D(a)$ we let correspond a numerical expression $\Gamma_c(x)=\Gamma(x)$ which is said to be *attached* to c. If $c=a$, we put $\Gamma(x)=x$. If $c=\bar{f}_j(c_1,\ldots,c_{m_j})$, we put $\Gamma_c(x)=\varphi_j(\Gamma_{c_1}(x),\ldots,\Gamma_{c_{m_j}}(x))$.

In general, there will be many numerical expressions attached to one and the same descendent of a. Note that the numerical expression $\mathbf{n}$ is attached to the integer $n=1+1\ldots+1$. The following theorem is obvious:

Theorem 21. *If* $c\in D(a)$ *and* $\Gamma(x)$ *is attached to* c, *then* $\Gamma(x)$ *has exactly one free variable and* $\Gamma_M(a)=c$.

Theorem 22. *In order that* c *be an at most* n*-th descendent of* a, *it is necessary and sufficient that there exist two finite sequences*

$$a_1,\ldots,a_h,\quad n_1,\ldots,n_h \tag{23}$$

such that

$$a_j\in I,\quad n_1=1,\quad n_j\leqslant n \qquad (j=1,\ldots,h), \tag{24}$$

$$a_1=a,\quad a_h=c, \tag{25}$$

(26) *if* $1<s\leqslant h$, *then either* $a_s=a$ *and* $n_s=1$ *or there exist integers* $j\leqslant k$ *and* $t_1,\ldots,t_{m_j}<s$ *such that* $a_s=\bar{f}_j(a_{t_1},\ldots,a_{t_{m_j}})$ *and* $n_s=1+\max(n_{t_1},\ldots,n_{t_{m_j}})$.

Proof. Let us first assume that sequences (23) satisfy conditions (24)-(26); we prove then by induction on s that a_s is an n_s-th descendent of a. Hence, the condition given in the theorem is sufficient.

To prove that it is necessary, we construct the required sequences (23) for an arbitrary $c\in D(a)$. If $c=a$, then we put $h=1$, $a_1=a$, $n_1=1$. Let us assume that sequences (23) with the properties (24)-(26) exist for arbitrary i-th descendents of a $(i=1,\ldots,n)$. Let c be an $n+1$-st descendent of a, *i. e.* let $c=\bar{f}_j(c_1,\ldots,c_{m_j})$, where c_p is an i_p-th descendent of a $(i_p\leqslant n; p=1,2,\ldots,m_j)$. According to the inductive hypothesis there exist m_j sequences $a_1^{(p)},\ldots,a_{h_p}^{(p)}$, $n_1^{(p)},\ldots,n_{h_p}^{(p)}$ $(p=1,2,\ldots,m_j)$ which satisfy the conditions analogous to (24)-(26). Now, we consider sequences

$$a_1^{(1)},\ldots,a_{h_1}^{(1)},\ldots,a_1^{(p)},\ldots,a_{h_{m_j}}^{(m_j)},c;$$

$$n_1^{(1)},\ldots,n_{h_1}^{(1)},\ldots,n_1^{(p)},\ldots,n_{h_{m_j}}^{(m_j)},n+1,$$

and prove without difficulty that they satisfy conditions (24)-(26). Hence, the condition given in theorem 22 is necessary.

We shall now construct a matrix $R(x,y,z)$ the intuitive content of which is: x is an at most y-th descendent of z.

First we define the auxiliary matrix $U(x,y,z,t,s)$:

$$Y(t)\cdot Y(s)\cdot[\mu(t)=\mu(s)]\cdot[\lambda(1,t)=z]\cdot[\lambda(\mu(t),t)=x]\cdot[\lambda(1,s)=1]\cdot \prod_v[v\leqslant\mu(s)\to\lambda(v,s)\leqslant y]\cdot\prod_v\{(v\leqslant\mu(t)\to(\lambda(v,t)=z)\cdot(\lambda(v,s)=1)\vee\sum_{j=1}^{k}\sum_{s_1}\dots\sum_{s_{m_j}}(s_1<v)\dots(s_{m_j}<v)(\lambda(v,t)=\varphi_j(\lambda(s_1,t),\dots,\lambda(s_{m_j},t))\cdot[\lambda(v,s)=1+\max(\lambda(s_1,s),\dots,\lambda(s_{m_j},s))]\}.$$

The matrix $R(x,y,z)$ is now defined as

$$\sum_t\sum_s U(x,y,z,t,s).$$

The meaning of the matrices U and R is explained in the following

Theorem 23. *Let M_0' be the natural model of S and a,n,c,u,v arbitrary integers, $u=2^{a_1}\dots p_h^{a_h}$, $v=2^{n_1},\dots,p_{h'}^{n_{h'}}$. Then, $\models_{M_0'}U(c,n,a,u,v)$ holds if and only if $h=h'$ and the sequences $a_1,\dots,a_h$, $n_1,\dots,n_h$ consist of positive integers and satisfy the conditions (24)-(26); $\models_{M_0'}R(c,n,a)$ holds if and only if c is at most an n-th descendent of a.*

Proof. Let us first assume that $\models_{M_0'}U(c,n,a,u,v)$. It follows that $\models_{M_0'}Y(u)\cdot Y(v)$ and hence $a_1,\dots,a_h$ and $n_1,\dots,n_{h'}$ must be positive. Since u and v satisfy the matrix $\mu(s)=\mu(t)$, it follows that $h=h'$. Considering the next factors of the matrix U, we infer that $a_1=a$, $a_h=c$, $n_1=1$ and $n_j\leqslant n$ for $j=1,2,\dots,h$. (26) results immediately from the assumption that u and v satisfy the last factor of U. It can be proved similary that if c,n,a,u,v satisfy conditions (24)-(26), then $\models_{M_0'}U(c,n,a,u,v)$.

If $\models_{M_0'}R(c,n,a)$, then there are u,v such that $\models_{M_0'}U(c,n,a,u,v)$ and hence there exist sequences satisfying conditions (24)-(26), *i. e.* c is an at most n-th descendent of a. Conversely, if c is such a descendent of a, then there exist sequences satisfying (24)-(26) and hence there exist integers u,v such that $\models_{M_0'}U(c,n,a,u,v)$ which supplies $\models_{M_0'}R(c,n,a)$.

Theorem 23 is thus proved.

Theorem 24. $\vdash_S y_1\leqslant y_2\to[R(x,y_1,z)\to R(x,y_2,z)]$.

To prove this theorem it is sufficient to remark that the variable y occurs in R, only in the factor

$$\prod_v[v\leqslant\mu(s)\to v\leqslant y].$$

Theorem 25. $\vdash_S U(x,y+1,z,t,s)\cdot(u<\mu(t))\to(\lambda(u,s)\leqslant y)$.

This theorem is an immediate consequence of the definition of U and of theorem 17.

Theorem 26. (i) $\vdash_S U(x,1,z,2^z,2)\leftrightarrow x=z$,

(ii) $\vdash_S R(x,1,z)\leftrightarrow x=z$.

This theorem results immediately from the definition of matrices U and R; it is sufficient to remark that if we substitute in U the numeral 1 for the variable y, we can omit the last factor of U since it is automatically a theorem of S.

Theorem 27.

(i) $\vdash_S U(x,y+1,z,t,s)\cdot(q<\mu(t))\cdot(\mu(t')=q)\cdot(\mu(s')=q)\cdot\prod_u\{u\leqslant q\to[\lambda(u,t')=\lambda(u,t)]\cdot[\lambda(u,s')=\lambda(u,s)]\}\to U(\lambda(q,t'),y,z,t',s')$,

(ii) $\vdash_S R(x,y+1,z)\to R(x,y,z)\vee$

$$\vee\sum_{j=1}^{k}\sum_{x_1}\dots\sum_{x_{m_j}}\prod_{i=1}^{m_j}[R(x_i,y,z)]\cdot[x=\varphi_j(x_1,\dots,x_{m_j})].$$

Formula (i) results without any serious difficulty from the definition of the matrix U. The intuitive content of this formula is the following: if $t=2^{a_1}\dots p_k^{a_k}$ and $s=2^{n_1}\dots p_k^{n_k}$ are integers such that the sequences

$$a_1,\dots,a_k,\quad n_1,\dots,n_k\quad(n_i\leqslant n+1\text{ for }i=1,2,\dots,k)$$

satisfy the conditions (24)-(26), then the integers $t'=2^{a_1}\dots p_q^{a_q}$ and $s'=2^{n_1}\dots p_q^{n_q}$ enjoy the same property and $n_i\leqslant n$ for $i=1,2,\dots,q$.

Formula (ii) can be proved as follows. First we have

$$\vdash_S U(x,y+1,z,t,s)\cdot(\mu(t)=1)\to x=z$$

and hence by theorems 26 (ii) and 24

$$(27)\qquad \vdash_S U(x,y+1,z,t,s)\cdot(\mu(t)=1)\to R(x,y,z).$$

From the definition of the matrix U we obtain

$$(28)\qquad \vdash_S U(x,y+1,z,t,s)\cdot(\mu(t)\neq 1)\to x=z\vee\sum_{j=1}^{k}\sum_{s_1}\dots\sum_{s_{m_j}}\prod_{i=1}^{m_j}[(s_i<\mu(t))\cdot x=\varphi_j(\lambda(s_1,t),\dots,\lambda(s_{m_j},t))].$$

From formula (i) we obtain

$$\vdash_S U(x,y+1,z,t,s)\cdot u<\mu(t)\to R(\lambda(n,t),y,z),$$

and hence (28) proves that

$$\vdash_S U(x,y+1,z,t,s)\cdot(\mu(t)\neq 1)\to$$
$$\to R(x,y,z)\vee\sum_{j=1}^{k}\sum_{u_1}\dots\sum_{u_{m_j}}\{\prod_{i=1}^{m_j}[R(u_i,y,z)]\cdot(x=\varphi_j(u_1,\dots,u_{m_j}))\}.$$

This formula together with (27) yields

$$U(x,y+1,z,t,s)\to R(x,y,z)\vee\sum_{j=1}^{k}\sum_{u_1}..\sum_{u_{m_j}}\{\prod_{i=1}^{m_j}[R(u_i,y,z)]\cdot x=\varphi_j(u_1,\dots,u_{m_j})\},$$

and we obtain the desired result.

Theorem 28. *Let c be an at most n-th descendent of a and let $\Gamma(x)$ be a numerical expression attached to c. Then $\vdash_S R(\Gamma(z),\boldsymbol{n},z)$.*

Proof. We proceed by induction on n. If $n=1$, then $\Gamma(z)$ is z and the formula to be proved becomes $\vdash_S R(z,1,z)$. This results immediately from theorem 26 (ii).

Let us assume that the theorem holds for i-th descendents $(i=1,2,\dots,n)$ and let $c=f_j(c_1,\dots,c_{m_j})$ be an $n+1$-st descendent of a. Let c_i be an n_i-th descendent of a $(n_i<n)$ and let $\Gamma(x)$ be the numerical expression $\varphi_j(\Gamma_1(x),\dots,\Gamma_{m_j}(x))$, where the expressions $\Gamma_i(x)$ are attached to c_i $(i=1,2,\dots,m_j)$. According to the inductive hypothesis we have

$$\vdash_S\sum_{t_i}\sum_{s_i}U(\Gamma_i(z),\boldsymbol{n}_i,z,t_i,s_i).$$

Using theorem 15 (i), we infer that it will be sufficient to prove the following formula:

$$\vdash_S\prod_{i=1}^{m_j}U(\Gamma_i(z),\boldsymbol{n}_i,z,t_i,s_i)\cdot T_{m_j+1}(t,t_1,\dots,t_{m_j},2^{\Gamma(z)})\cdot T_{m_j+1}(s,s_1,\dots,s_{m_j},2^{n+1})\to$$
$$\to U(\Gamma(z),\boldsymbol{n},z,t,s).$$

Let H be the antecedent of this implication. From theorem 16 (i), (ii) and (iii) we obtain

$$H\to Y(t)\cdot Y(s)\cdot(\mu(t)=\mu(s))\cdot(\lambda(1,t)=z)\cdot(\lambda(\mu(t),t)=\Gamma(z))\cdot(\lambda(1,s)=1).$$

From theorem 16 (iv) we obtain further

$$H\to\prod_{v}[v\leqslant\mu(s)\to(\lambda(v,s)\leqslant\boldsymbol{n}+1)].$$

Thus it remains to prove that

$$\vdash_S H\cdot(v\leqslant\mu(t))\to\lambda(v,t)=z\cdot\lambda(v,s)=1$$
$$\vee\sum_{h=1}^{k}\sum_{s_1}\dots\sum_{s_{m_h}}\prod_{i=1}^{m_h}(s_i<v)\cdot[\lambda(v,t)=\varphi_h(\lambda(s_1,t),\dots,\lambda(s_{m_h},t))]\cdot$$
$$\cdot[\lambda(v,s)=1+\max(\lambda(s_1,s),\dots,\lambda(s_{m_h},s))].$$

We write this formula briefly thus: $\vdash_S H\cdot(v\leqslant\mu(t))\to C(v,t,s)$. To prove this formula, we first remark that

$$\vdash_S H\cdot(v=\mu(t))\to(\lambda(\mu(t),t)=\Gamma(z))\cdot(\lambda(\mu(s),s)=\boldsymbol{n}+1)$$

and

$$\vdash_S H\to\lambda(\mu(t_1)+\dots+\mu(t_i)+1,t)=\Gamma_i(z),$$
$$\vdash_S H\to\lambda(\mu(t_1)+\dots+\mu(t_i)+1,s)=\boldsymbol{n}_i.$$

Since $\vdash_S H\to\mu(t_1)+\dots+\mu(t_i)<\mu(t)$ for $i=1,2,\dots,m_j$, and $n+1=1+\max(n_1,\dots,n_{m_j})$ we infer from these formulae that

$$\vdash_S H\cdot(v=\mu(t))\to\sum_{s_1}\dots\sum_{s_{m_j}}\prod_{i=1}^{m_j}\{(s_i<v)\cdot[\lambda(v,t)=\varphi_j(\lambda(s_1,t),\dots,$$
$$\lambda(s_{m_j},t))]\cdot[\lambda(v,s)=1+\max(\lambda(s_1,s),\dots,\lambda(s_{m_h},s))]\}.$$

Since $\vdash_S H\to U(\Gamma_{i+1}(z),\boldsymbol{n}_{i+1},z,t_{i+1},s_{i+1})$, we have

(30) $$\vdash_S H\cdot(r<\mu(t_{i+1}))\to C(r,t_{i+1},s_{i+1}).$$

Now, we remark that

(31) $$\vdash_S H\cdot(v=\mu(t_1)+\dots+\mu(t_i)+r)\cdot(r<\mu(t_{i+1}))\to(\lambda(r,t_{i+1})=\lambda(v,t)),$$

and

(32) $$\vdash_S H\cdot(v=\mu(t_1)+\dots+\mu(t_i)+r)\cdot(r<\mu(t_{i+1}))\cdot u<r\to$$
$$\to(\mu(t_1)+\dots\mu(t_i)+u<v)\cdot(\lambda(\mu(t_1)+\dots+\mu(t_i)+u,t)=\lambda(u,t_{i+1})).$$

It follows from formulae (30)-(32) that

$$\vdash_S H\cdot(v=\mu(t_1)+\dots+\mu(t_i)+r)\cdot(r<\mu(t_{i+1}))\to C(v,t,s)$$

and since (by theorem 16 (iv))

$$\vdash_S H\cdot(v<\mu(t))\to\sum_{i=0}^{m_{j-1}}\sum_{r}(v=\mu(t_1)+\dots+\mu(t_i)+r)\cdot(r<\mu(t_{i+1})),$$

we obtain

$$\vdash_S H\cdot(v<\mu(t))\to C(v,t,s).$$

Combining this with (29) we obtain the desired formula

$$\vdash_S H\cdot(v\leqslant\mu(t))\to C(v,t,s).$$

In this way we have proved that $\vdash_S H\to U(\Gamma(z),\boldsymbol{n},z,t,s)$ and theorem 28 is proved.

8. Proof that the Peanian arithmetic is non-axiomatizable. We shall prove in this section the following

Theorem 29. *For every Peanian system* S (cf. § 6, p. 253) *there exists a matrix* $\Theta(x)$ *such that*

(33) $$non\vdash_S\Theta(1)\cdot\prod_x[\Theta(x)\to\Theta(x+1)]\to\prod_x\Theta(x).$$

This theorem shows that Peanian arithmetic (§ 6, p. 253) is not axiomatizable by means of any finite system of axioms.

Theorem 29 results of course from the slightly stronger

Theorem 30. *For every extended Peanian system*

$$S=S(\varphi_1,\ldots,\varphi_k,\Phi_1,\ldots,\Phi_l)$$

(§ 6, p. 253) *there exists a matrix* $\Theta(x)$ *such that* (33).

It will therefore be sufficient to prove theorem 30. We divide the proof into two parts:

I. Axioms $\Phi_1,\ldots,\Phi_l$ of S do not contain quantifiers and S has a majorizing functor, *e. g.* φ_k.

In this case we consider the matrix $R(x,y,z)$ defined in section 7 and put

$$Q(y,z)\equiv\sum_t\prod_x[R(x,y,z)\to x\leqslant t],$$
$$\Psi(y)\equiv\prod_z Q(y,z).$$

We shall prove three lemmas.

Lemma 1. $\vdash_S\Psi(1)$.

Indeed, from theorem 26 (ii), we have $\vdash_S\prod_x R(x,1,z)\to x\leqslant z]$ and hence $\vdash_S Q(1,z)$ which proves that $\vdash_S\Psi(1)$.

Lemma 2. $\vdash_S\Psi(y)\to\Psi(y+1)$.

Proof. From theorem 27 (ii) we obtain

$$\vdash_S\prod_x[R(x,y,z)\to x\leqslant t]\cdot R(x,y+1,z)\to x\leqslant t$$
$$\vee\sum_{j=1}^{k}\sum_{v_1}\cdots\sum_{v_{m_j}}[\prod_{l=1}^{m_j}(v_l\leqslant t)\cdot x=\varphi_j(v_1,\ldots,v_{m_j})].$$

φ_k being a majorizing functor of S, we obtain

$$\vdash_S\prod_{i=1}^{m_j}(v_i\leqslant t)\cdot x=\varphi_j(v_1,\ldots,v_{m_j})\to x\leqslant\varphi_k(t)$$

and the two last formulae yield

$$\vdash_S\prod_x[R(x,y,z)\to x\leqslant t]\cdot R(x,y+1,z)\to x\leqslant t\vee x\leqslant\varphi_k(t).$$

Since S contains an identity functor, we obtain further $\vdash_S x\leqslant t\to x\leqslant\varphi_k(t)$, and it follows by the previous formula

$$\vdash_S\prod_x[R(x,y,z)\to x\leqslant t]\to\prod_x[R(x,y+1,t)\to x\leqslant\varphi_k(t)].$$

Transforming this formula by means of easy logical calculations, we infer that $\vdash_S Q(y,z)\to Q(y+1,z)$, which gives $\vdash_S\Psi(y)\to\Psi(y+1)$. Lemma 2 is thus proved.

Lemma 3. *There exists a model of* S *in which the sentence* $\prod_y\Psi(y)$ *is false.*

Proof. We start with an arbitrary non-normal model $M^*=[I,\precsim^*,f_1^{\mu^*},\ldots,f_k^*]$, *e. g.* with the model constructed by Skolem (cf. theorem 19). The set I_0 of positive integers is, as we have remarked above, an initial segment of I. Let ω be an arbitrary element of I which does not belong to I_0 and consider the set $D(\omega)$ of all descendents of ω. We denote by $\precsim$ the relation $\precsim^*$ restricted to $D(\omega)$ and by $f_1,\ldots,f_k$ functions $f_1^*,\ldots,f_k^*$ restricted to $D(\omega)$. The set I_0 is again an initial segment of $D(\omega)$ since 1 (which is a second descendent of ω obtained by means of the function without arguments $f_1^*=1$) belongs to $D(\omega)$ as do $1+1$, $1+1+1,\ldots$

The set $D(\omega)$ being closed with respect to operation $f_1,\ldots,f_k$ and axioms $\Phi_1,\ldots,\Phi_l$ being matrices without bound variables, we infer from theorem 3 that

$$M=[D(\omega),\precsim,f_1,\ldots,f_k]$$

is a model of S.

First we prove that

(34) $$n\precsim\omega\quad\text{for } n=1,2,3,\ldots$$

Indeed, the axiom $x\leqslant y\vee y<x$ being true in M, one of the formulae $n\precsim\omega$, $n=\omega$, $\omega\precsim n$ must be true. The second formula is false since $\omega\,\text{non}\,\epsilon\,I_0$. If the third were true, we would obtain from

theorem 14 that $\models_M \omega=1 \vee \omega=2 \vee \ldots \vee \omega=n-1$. This is impossible since $1,2,\ldots$ are numerical expressions attached to integers $1,2,\ldots$ and hence $\models_M \omega=i$ would give $\omega=i$, *i. e.* $\omega \in I_0$ contrary to the definition of ω. Formula (34) is thus proved.

We can now show that $\prod_y \Psi(y)$ is false in M. Suppose the contrary. It follows that $\models_M \Psi(\omega)$ and hence $\models_M Q(\omega,\omega)$. It follows from the definition of Q that there exists an element $c \in D(\omega)$ such that for every $d \in D(\omega)$

(35) $\quad$ if $\models_M R(d,\omega,\omega)$, then $d \prec c$ or $d=c$.

Since $c+1 \in D(\omega)$ it is a descendent of ω. Suppose that it is an n-th one. From theorem 28 we obtain the result that if $\Gamma(z)$ is a numerical expression attached to $c+1$, then the elements $\Gamma_M(\omega)$, n, and ω satisfy the matrix $R(x,y,z)$ in M:

$$\models_M R(\Gamma_M(\omega),n,\omega).$$

Since $\Gamma_M(\omega)=c+1$ by theorem 21, we obtain

$$\models_M R(c+1,n,\omega).$$

Applying now theorem 24 and formula (34), we get

$$\models_M R(c+1,\omega,\omega).$$

Putting $d=c+1$ in (35), we obtain therefore $c+1 \prec c$ or $c+1=c$. On the other hand the axiom $x<x+1$ is true in M and hence $c \prec c+1$. Using the transitivity of $\prec$, we obtain finally $c \prec c$ which is impossible since $\prec$ is an irreflexive relation.

Lemma 3 is thus proved.

Lemmas 1, 2, 3 prove according to theorem 1 that the sentence

$$\Psi(1) \cdot \prod_y [\Psi(y) \to \Psi(y+1)] \to \prod_y \Psi(y)$$

is not provable in S. Theorem 30 is thus proved in case I.

II. S is an arbitrary (extended) Peanian system.

In this case we construct a non-essential extension S' of S containing a majorizing functor and such that no axiom of S' contains quantifiers (cf. § 4, theorem 9 and § 5, theorem 12). To S' we can apply the result obtained in case I above and obtain a matrix $\Psi'(x)$ such that

(36) $\quad$ non $\vdash_{S'} \Psi'(1) \cdot \prod_y [\Psi'(y) \to \Psi'(y+1)] \to \prod_y \Psi'(y)$.

S' being a non-essential extension of S, there exists a matrix $\Theta(x)$ of S such that $\vdash_{S'} \Psi'(x) \leftrightarrow \Theta(x)$. Formula (36) proves that

$$\text{non} \vdash_{S'} \Theta(1) \cdot \prod_y [\Theta(y) \to \Theta(y+1)] \to \prod_y \Theta(y)$$

and therefore the sentence $\Theta(1) \cdot \prod_y [\Theta(y) \to \Theta(y+1)] \to \prod_y \Theta(y)$ cannot be provable in S since all theorems of S are at the same time theorems of S'. Theorem 30 is thus proved.

***Remarks.* 1.** Our proof of theorem 30 is effective in the sense that it enables us to construct effectively a matrix $\Theta(x)$ with the property (33) once the system S is effectively given.

2. Our proof applies not only to Peanian arithmetic but to arbitrary self consistent formal systems S embracing arithmetic provided that there exist well ordered models of S. As examples of such systems S we can cite various axiomatic systems of set-theory, *e. g.* the systems of Zermelo, v. Neumann-Bernays and several others. It follows, namely, from the well-known work of Gödel [2] concerning the Generalized Continuum Hypothesis that all these systems possess well-ordered models.

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