Now, we can answer the [7]) question whether there exists an ordered continuum C of power $2^{\aleph_0}$, where $\mathfrak{m}(C)\leqslant\aleph_1$, without containing points with character c_{00}. The answer to this question is negative. As a matter of fact, suppose on the contrary that C is such an ordered continuum. Let $\mathfrak{P}$ be a partition of C. Since C does not contain any point with character c_{00}, every symbol $I_{i_0 i_1 \dots i_n \dots (n<\omega)}$, where $i_n=0$ and $i_{n'}=1$ for infinitely many naturals n and n', denotes an interval of C belonging to $\mathfrak{P}$. The cardinal number of all intervals like these is $2^{\aleph_0}$. According to lemma 3 every two intervals of this sort have at most one point in common. As $\mathfrak{m}(C)\leqslant\aleph_1$ we have $2^{\aleph_0}=\aleph_1$ and consequently $2^{\aleph_0}<2^{\aleph_1}$. Thus the above supposition contradicts Theorem 5.

Corollary. *Every ordered continuum C with power $2^{\aleph_0}$ and with $\mathfrak{m}(C)=\aleph_1$ contains a subset of power $2^{\aleph_0}$ which is dense in C and whose points have character c_{00}.*

Proof. Every [2]) interval $J\subset C$ has the power $2^{\aleph_0}$. Since $\mathfrak{m}(J)\leqslant\aleph_1$ there is a point in J — as we have just shown — with character c_{00}. Consequently, the subset $A_{00}\subset C$ of all points with character c_{00} is dense in C and the power of A_{00} is $\geqslant\aleph_1$ and $\leqslant 2^{\aleph_0}$ at the same time. Therefore if the power of A_{00} were $<2^{\aleph_0}$ then we should have $\aleph_1<2^{\aleph_0}$. Now, let $\mathfrak{P}$ be a partition of C. The cardinal number of the system of all intervals of $\mathfrak{P}$ of order ω and of all points of C of the same order ω is $2^{\aleph_0}$. According to Lemma 3 and because $\mathfrak{m}(C)=\aleph_1$ the system of all intervals of $\mathfrak{P}$ of order ω has the power $\leqslant\aleph_1$. Therefore the cardinal number of all points in C of order ω is $2^{\aleph_0}$. The power of A_{00} is $\geqslant 2^{\aleph_0}$, every point of order ω belonging to the set A_{00}. Thus we should get a contradiction.

Remarks. I do not know whether or not there exists an ordered continuum of power $2^{\aleph_0}$ with $\mathfrak{m}(C)=\aleph_1$ such that the power of $C-A_{00}$ is $2^{\aleph_0}$.

From Theorem 5 it follows that the more general question whether there exists an ordered continuum of power $2^{\aleph_0}$ without points of character c_{00} is equivalent to the question whether $2^{\aleph_0}=2^{\aleph_1}$.

[7]) See J. Novák, l. c. ad [6]), p. 79.

Characterization of Types of Order Satisfying $\alpha_0+\alpha_1=\alpha_1+\alpha_0$.

By

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Introduction. The question of what types of order α_0 and α_1 are commutative with respect to their addition is one of several seemingly simple questions in the theory of ordered sets. A few cases of commutative types of orders have been known for a long time:

(a) $\alpha_0=\delta m$, $\alpha_1=\delta n$, with some type of order δ, m and n being natural numbers,

(b) $\alpha_0=\alpha_1\omega+\delta+\alpha_1\omega^*$ or
$\alpha_1=\alpha_0\omega+\delta+\alpha_0\omega^*$ with some type of order δ.

A. Tarski has communicated to the author that in the middle thirties he proved that (a) and (b) represent all the commutative types if one of them is assumed to be either enumerable or dispersed. He also made a conjecture that these two cases exhaust all possible commutative types, but a counter-example was constructed by A. Lindenbaum. (None of these results have been published).

In the present paper we shall give a complete characterization of all commutative pairs α_0, α_1 of types of order. The characterization is obtained by using the theory of partitions of ordered sets, and also by introducing the new notion of semi-similarity between ordered sets. We are thus able to attack the problem in a much more general form.

We consider a class of types of order $\{\alpha_i\}$, $i\in I$, where I is an arbitrary set of indices. In I we introduce two order relations which make of I two different ordered sets I' and I''. The problem is then one of characterizing all the classes $\{\alpha_i\}$ satisfying the "generalized commutativity" equation

$$\sum_{i\in I'}\alpha_i=\sum_{i\in I''}\alpha_i.$$

A particular case of this general problem is one where I' and I'' are of the same type of order; this case corresponds to a permutation in the set I. Clearly, the problem of two commutative types is the simplest one of this kind.

The specific results obtained in this paper concern the general problem in the case where the types of order of I' and I'' are both real (*i. e.* similar to a set of real numbers). In this case the general problem is reduced to the same problem considered only for real types of order α_i.

In the problem of two commutative types of order where the set I is composed of only two elements, $i=0,1$, this reduction leads to a complete characterization given in the following theorem.

Theorem I. *In order that $\alpha_0+\alpha_1=\alpha_1+\alpha_0$ it is necessary and sufficient that either α_0, α_1 be in case* (b) *or that they be representable in the form*

$$\alpha_0=\gamma'_0+\sum_{0<\xi<\xi_0}\gamma'(\zeta)+\gamma'_1, \qquad \alpha_1=\gamma'_0+\sum_{\xi_0<\xi<1}\gamma'(\zeta)+\gamma'_1, \tag{1}$$

where ε_0 is real, $0<\varepsilon_0<1$ and γ'_0, γ'_1, and $\gamma'(\xi)$ are types of order, $\gamma'(\xi)$ being defined in the interval $0<\xi<1$ and satisfying the conditions:
1^0 $\gamma'(\xi_1)=\gamma'(\xi_2)$ if $\xi_2-\xi_1=m+m_0\xi_0$ for some integers m and m_0;
2^0 $\gamma'(\xi)=\gamma'_1+\gamma'_0$ for all $\xi=m+m_0\xi_0$ with any integers m and m_0.

Case (a) above of commutative types is obtained from the general form (1) when ξ_0 is a rational number. The representation is not unique. Some commutative types belonging to (b) may be represented in form (1). Also, some types in the case (a) may be represented in form (1) with an irrational ξ_0.

These considerations may be easily generalized to types of partial order if we define their sum in a similar manner to that of types of order. The formulation of the general problem will then involve a consideration of *ordered partitions* of partially ordered sets. The consideration of partially ordered partitions of partially ordered sets leads to an extension of the general problem which seems to present some additional difficulties.

1. Partitions. We shall use the notation $\cup$ and $\cap$ for union and intersection of sets respectively. The intersection of a finite number of sets will also be denoted by $A_1A_2...A_n$. The symbol $\sum_{i\epsilon I} A_i$ will be reserved for the ordered union of ordered mutually disjoint

sets A_i where i runs through the ordered set I. The order relation will be denoted by $<$ in the strict sense, or $\leqslant$ in the large sense; if necessary we shall also use the symbols $\prec$ and $\preceq$.

For two subsets A' and A'' of an ordered set A we shall write $A'<A''$ or $A''>A'$, if for every $x'\epsilon A'$ and every $x''\epsilon A''$ we have $x'<x''$. We shall also write $x<A'$ or $A'>x$, if $x<x'$ for all $x'\epsilon A'$.

As usual, a type of order will be called *enumerable* if it is the type of an enumerable set. We shall call a type of order *real* if it is the type of a set of real numbers with their natural order relation.

A subset B of an ordered set A is called *dense* in A if for any two elements a' and a'' in A with $a'<a''$ there exists an element $b\epsilon B$ such that $a'\leqslant b\leqslant a''$ [1]). Using this definition of density we can condense some well known theorems characterizing the real types of order in the following way: *for an ordered set A to be of real type it is necessary and sufficient that it contain a dense enumerable subset.*

If A is of real type it presents an at most enumerable number of jumps. We can then adjoin to any dense enumerable subset of A all the endpoints of jumps and obtain an enumerable dense subset B of A which contains for any $a'<a''$ in A either both elements a' and a'' or an element b satisfying $a'<b<a''$.

As usual, by an interval P of an ordered set A we mean a non-empty subset of A such that with every two elements a and b in P, every element x of A with $a<x<b$ belongs to P. An interval containing more than one element will be called a *proper* interval.

A class of mutually disjoint intervals A_i, $i\epsilon I$, is called a partition of the ordered set A if $A=\bigcup_{i\epsilon I} A_i$. We shall denote this partition by $\{A_i\}_I$ or $\{A_i\}$. The order of A induces an order relation in the set of indices I: $i'<i''$ if and only if every element of $A_{i'}$ precedes every element of $A_{i''}$. Thus I becomes an ordered set and we can write $A=\sum_{i\epsilon I} A_i$.

A partition $\{A'_j\}_J$ is equivalent to the partition $\{A_i\}_I$ if there is a one-to-one correspondence between I and J such that for corresponding indices i and j we have $A_i=A'_j$. The correspondence between I and J is then a similarity. Ordinarily we shall not distinguish between equivalent partitions.

[1]) This definition of dense subsets is slightly more general than the usual one; for example, in a finite ordered set of n elements there exist dense subsets of $[n/2]$ elements.

The type of order of I is called the type of the partition. An enumerable partition is of real type. For any partition of real type we may assume that the set of indices is a set of real numbers.

If $\{A_i\}_I$ is a partition of A and if α and α_i are the types of order of A and A_i we can write

$$\alpha = \sum_{i \in I} \alpha_i. \tag{1.1}$$

A partition $\{A'_j\}_J$ is a subdivision of the partition $\{A_i\}_I$ if every A'_j is contained in some A_i. Such a subdivision induces a partition $\{J_i\}_I$ of the ordered set J: an element $j \in J$ belongs to J_i if $A'_j \subset A_i$. The subdivision induces also a partition of every A_i:

$$A_i = \sum_{j \in J_i} A'_j.$$

If $\{A'_j\}_J$ is a subdivision of $\{A_i\}_I$ we will call the first a *finer* partition than the second and the second a *smaller* partition than the first. In a class of partitions of A there may exist the smallest one — of which every partition of the class is a subdivision, or the finest one — which is a subdivision of any partition of the class.

It is clear that a necessary and sufficient condition in order that a subdivision of a partition $\{A_i\}_I$ be enumerable is that the last partition be enumerable and that the induced partition of every interval A_i also be enumerable. For real types we have the following statement.

Theorem 1.1. *In order that a subdivision $\{A'_j\}_J$ of the partition $\{A_i\}_I$ be of real type it is necessary and sufficient that 1^0 $\{A_i\}_I$ be of real type, 2^0 an at most enumerable set of intervals A_i be subdivided, and 3^0 in the subdivided intervals A_i the induced partitions be of real type.*

Proof. *Necessity*: 1^0 If $\{A'_j\}_J$ is of real type, there exists in it a dense enumerable subset of intervals $A'_{j'}$. If we consider all the intervals A_i containing one of these $A'_{j'}$ they will form, clearly, an enumerable and dense subset in the partition $\{A_i\}_I$. Hence $\{A_i\}_I$ is of real type. 2^0 If there were a non-enumerable set of subdivided intervals A_i each of these would have to contain at least one interval $A'_{j'}$ of an enumerable dense subset in $\{A'_j\}_J$. This is clearly impossible. 3^0 Since the intervals of the induced subdivision of an interval A_i form a subset of the partition $\{A'_j\}_J$ they must also form an ordered set of real type.

Sufficiency. Denote by $A_{i'}$ and $A_{i''}$ all the non-subdivided and subdivided intervals of $\{A_i\}_I$ respectively. In each $A_{i''}$ we consider the induced partition in which there exists an enumerable dense sequence $A'_{j_n(i'')}$. This sequence will be chosen so as to include the first and last intervals of the partition of $A_{i''}$ if they exist. In the set of all intervals $A_{i'}$ — which is of real type — we choose a dense sequence $A_{i'_n}$. All the intervals $A_{i'_n}$ and $A'_{j_n(i'')}$ form an enumerable subset of the partition $\{A'_j\}_J$. This subset is dense in $\{A'_j\}_J$. In fact, if we have two intervals $A'_{j'} < A'_{j''}$ we may consider the intervals $A_{i_1} \supset A'_{j'}$ and $A_{i_2} \supset A'_{j''}$. If $A_{i_1} = A'_{j'}$ and $A_{i_2} = A'_{j''}$ then i_1 and i_2 are among the indices i' and there will be some $A_{i'_n}$ such that $A_{i_1} \leqslant A_{i'_n} \leqslant A_{i_2}$. If $A_{i_1} \neq A'_{j'}$ and $i_1 \neq i_2$, then $A_{i_1} < A_{i_2}$, and i_1 is among the indices i'', and there will exist some $A'_{j_n(i_1)}$ such that $A'_{j'} \leqslant A'_{j_n(i_1)} < A_{i_2}$, and hence $A'_{j'} \leqslant A'_{j_n(i_1)} < A'_{j''}$. If $A_{i_1} \neq A'_{j'}$ and $i_1 = i_2$, then $A'_{j'}$ and $A'_{j''}$ belong to the induced partition of A_{i_1} and there will be some $A'_{j_n(i_1)}$ with $A'_{j'} \leqslant A'_{j_n(i_1)} \leqslant A'_{j''}$.

The density property can be proved similarly if $A_{i_2} \neq A_{j''}$.

If we have a sequence of partitions $\{A^n_i\}_{I^n}$ of the set A, all non-empty intersections

$$\cap A^n_{i^n}, \quad i^n \in I^n, \tag{1.2}$$

form a partition of A which is a subdivision of each partition $\{A^n_i\}$. As set of indices of this partition we take the subset I of the Cartesian product $\prod_n I^n$ of all systems $\{i^n\}$ for which the intersection (1.2) is non-empty. The order relation induced in I by this partition coincides with the lexicographic order relation in the Cartesian product [2]). The partition composed of the intervals (1.2) will be called the intersection of the partitions $\{A^n_i\}_{I^n}$ and denoted by

$$\bigcap_n \{A^n_i\}_{I^n}. \tag{1.3}$$

Theorem 1.2. *The intersection of an at most enumerable sequence of partitions of real type is of real type.*

Proof. In each I^n consider a dense enumerable subset J^n. We may and shall suppose that J^n contains the endpoints of every

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[2]) This means that $\{i_n\} < \{i'_n\}$ if and only if for the first n for which $i_n \neq i'_n$, we have $i_n < i'_n$.

jump in I^n. For each $j^n \in J^n$ and for each couple $j^m \in J^m$, $j^n \in J^n$, $m<n$, such that $A^{(m)}_{j^m} A^{(n)}_{j^n} \neq 0$, the intervals (1.2) contained in the intervals $A^{(n)}_{j^n}$ or $A^{(m)}_{j^m} A^{(n)}_{j^n}$ form a partition of these intervals. We denote by P_{j^n} or $P_{j^m j^n}$ respectively, the first interval of this partition if it exists, or a chosen interval of this partition if there is no first interval. All the intervals P_{j^n} and $P_{j^m j^n}$ for $j^m \in J^m$, $j^n \in J^n$, $m,n=1,2,\ldots$ form an enumerable set of intervals of the partition (1.3). We shall prove that this set is dense among all the intervals of this partition.

In fact, consider two intervals (1.2) corresponding to sequences $\{i_1^n\}$ and $\{i_2^n\}$. Suppose further that

$$i_1^l = i_2^l \quad \text{for} \quad l \leqslant k-1, \quad i_1^k < i_2^k.$$

If (i_1^k, i_2^k) is not a jump in I^k, there is a $j^k \in J^k$ with $i_1^k < j^k < i_2^k$. Consequently $A^{(k)}_{i_1^k} < A^{(k)}_{j^k} < A^{(k)}_{i_2^k}$ and hence

$$\bigcap_n A^{(n)}_{i^n} < P_{j^k} < \bigcap_n A^{(n)}_{i_2^n}.$$

If (i_1^k, i_2^k) form a jump in I^k, then $i_2^k = j^k \in J^k$. If P_{j^k} is the first interval of the partition of $A^{(k)}_{j^k}$, the inequalities $A^{(k)}_{i_1^k} < A^{(k)}_{j^k} = A^{(k)}_{i_2^k}$ imply $\bigcap_n A^{(n)}_{i_1^n} < P_{j^k} \leqslant \bigcap_n A^{(n)}_{i_2^n}$. If there is no first interval in the partition of $A^{(k)}_{j^k}$, this partition contains an interval $\bigcap_n A^{(n)}_{i_3^n}$ preceding $\bigcap_n A^{(n)}_{i_2^n}$. In view of $A^{(k)}_{i_1^k} < A^{(k)}_{j^k}$ this gives

$$\bigcap_n A^{(n)}_{i_1^n} < \bigcap_n A^{(n)}_{i_3^n} < \bigcap_n A^{(n)}_{i_2^n}$$

and consequently $i_3^l = i_1^l = i_2^l$ for $l \leqslant k-1$, $i_3^k = i_2^k = j^k$. Further, there must exist an $m>k$ so that $i_3^l = i_2^l$ for $l \leqslant m-1$ and $i_3^m < i_2^m$. It follows that for some $j^m \in J^m$, $i_3^m \leqslant j^m < i_2^m$. The inequalities $A^{(m)}_{i_3^m} \leqslant A^{(m)}_{j^m} < A^{(m)}_{i_2^m}$, $A^{(k)}_{j^k} A^{(m)}_{i_3^m} \neq 0$ and $A^{(k)}_{j^k} A^{(m)}_{i_2^m} \neq 0$ prove that $A^{(k)}_{j^k} A^{(m)}_{j^m} \neq 0$. Finally, the inequalities $A^{(k)}_{i_1^k} < A^{(k)}_{j^k} A^{(m)}_{j^m} < A^{(m)}_{i_2^m}$ prove that

$$\bigcap_n A^{(n)}_{i_1^n} < P_{j^k j^m} < \bigcap_n A^{(n)}_{i_2^2}.$$

2. Semi-similarities. Consider two ordered sets A and B and a one-to-one correspondence S of A onto B, $S(A)=B$. The correspondence S will be called a ***semi-similarity*** relative to a partition $\{A_i\}_I$ of the domain A of S if S, restricted to each A_i, forms a similarity transforming A_i onto an interval $S(A_i)$ of B. It is then clear that $\{S(A_i)\}_I$ is a partition of B, and that S^{-1} is a semi-similarity relative to this partition. Every correspondence S is a semi-similarity relative to the partition formed by all elements of A (considered as intervals). S is a similarity if and only if it is a semi-similarity relative to the partition composed of one interval (which is $=A$). If S is a semi-similarity relative to the partition $\{A_i\}_I$ it is also a semi-similarity relative to any subdivision of this partition.

For each correspondence S there exists the smallest partition relative to which S is a semi-similarity. The smallest partition can be obtained by considering for each $a \in A$, all intervals $A' \subset A$ such that $a \in A'$, and S is a similarity on A' transforming it onto an interval of B. It is easily seen that the union of all these intervals A' is still an interval with the same property; it is the largest of all intervals A'. We denote this largest interval by A_a. For two elements a' and a'' of A, $A_{a'}$ and $A_{a''}$ are either equal or disjoint. The intervals A_a form the smallest partition relative to which S is a semi-similarity.

Let S be a semi-similarity of A onto B relative to the partition $\{A_i\}_I$ and let T be a semi-similarity of B onto C relative to the partition $\{B_j\}_J$. The correspondence TS is then a semi-similarity of A onto C relative to the partition which has for intervals all the non-empty intersections $A_i S^{-1}(B_j)$. This partition is obtained from the intersection of the two partitions $\{S(A_i)\}_I$ and $\{B_j\}_J$ by the transformation S^{-1}.

If S is a semi-similarity of A onto B relative to the partition $\{A_i\}_I$ then the two partitions, $\{A_i\}_I$ of A and $\{S(A_i)\}_I$ of B, induce two order relations, $<$ and $\prec$, in the set of indices I. The system $[I, <, \prec]$ of the set I and its two order relations represents the type of the semi-similarity S relative to the partition $\{A_i\}_I$. If we refer the intervals of this partition to another set of indices I', this set is then in one-to-one correspondence with I. The two partitions of A and B also induce in I' two order relations, $<$ and $\prec$, and the correspondence between I and I' is a similarity with respect to the first order relation as well as to the second. The system $[I', <, \prec]$ is then also a representation — equivalent to $[I, <, \prec]$ —

of the type of S. If the partition $\{A'_j\}_J$ is a subdivision of the partition $\{A_i\}_I$, the resulting partition $\{J_i\}_I$ which is a partition of J with respect to the order relation $<$ induces, as such, the order relation $<$ in I. It is also a partition of J with respect to $\prec$ and as such induces the order relation $\prec$ in I. If $\{A_i\}_J$ is the smallest partition of A relative to which S is a semi-similarity, the corresponding type of S has the characteristic property that there exists no subset $I' \subset I$ which is a proper interval for both order relations, and in which the two order relations coincide.

If the set of indices is enumerable, the semi-similarity S relative to $\{A_i\}_I$ will be called of *enumerable type*. If the types of order of the set of indices in both order relations are real, the semi-similarity S will be called of *real type*.

Let S and T be semi-similarities of A onto B and of B onto C relative to the partitions $\{A_i\}_I$ and $\{B_j\}_J$. We know already that TS is a semi-similarity of A onto C relative to the partition composed of all non-empty sets $A_i S^{-1}(B_j)$. It is clear that if S and T are of enumerable type the same is true for TS. We have a similar result for real types:

Theorem 2.1. *If S and T are of real type, TS is also of real type.*

Proof. We must prove that the partition of A composed of non-empty sets $A_i S^{-1}(B_j)$ and that the partition of C composed of corresponding sets $TS(A_i S^{-1}(B_j)) = TS(A_i)T(B_j)$ are of real type. The intervals of both of these partitions are obtained from those of the intersection $\{B'_j\}_{J'}$ of the two partitions of B, $\{S(A_i)\}_I$ and $\{B_j\}_J$, the first by the transformation S^{-1} and the second by T. The intersection $\{B'_j\}_{J'}$ is a subdivision of each of the partitions $\{S(A_i)\}_I$ and $\{B_j\}_J$ and by Theorem 1.2 it is of real type.

We prove first that the partition of A, $\{S^{-1}(B'_j)\}_{J'}$, is of real type. By Theorem 1.1 (necessity) only an enumerable number of the intervals $S(A_i)$ are subdivided in $\{B'_j\}_{J'}$; those which are have induced partitions of real type. It follows that in the partition $\{S^{-1}(B'_j)\}_{J'}$ which is a subdivision of $\{A_i\}_I$, only an enumerable number of intervals A_i are subdivided; those which are come from the intervals $S(A_i)$ subdivided in $\{B'_j\}_{J'}$. Since S^{-1} is a similarity on the intervals $S(A_i)$ the induced partition on the subdivided intervals A_i is of the same type as on the corresponding $S(A_i)$, *i. e.* it is real. By the sufficiency in Theorem 1.1, it follows that $\{S^{-1}(B'_j)\}_{J'}$ is of real type.

The proof that the partition $\{T(B'_j)\}_{J'}$ is of real type proceeds similarly, S and S^{-1} being replaced by T^{-1} and T, and the roles of the partitions $\{S(A_i)\}_I$ and $\{B_j\}_J$ being interchanged.

We shall be particularly concerned with ***auto-semi-similarities***, *i. e.* with semi-similarities of an ordered set A onto itself.

Let S be an auto-semi-similarity (a.-s.-s.) of A relative to the partition $\{A_i\}_I$. By an easy induction we obtain from our previous remarks and theorems the following statements:

Theorem 2.2. *The correspondence S^m, $m=1,2,3,\ldots$, is an a.-s.-s. relative to the partition composed of all non-empty sets*

$$\bigcap_{k=-m+1}^{0} S^k(A_{i_k}), \quad i_k \in I. \tag{2.1}$$

The correspondence S^{-m}, $m=1,2,3,\ldots$, is an a.-s.-s. relative to the partition composed of all non-empty intersections

$$\bigcap_{k=1}^{m} S^k(A_{i_k}), \quad i_k \in I. \tag{2.1$'$}$$

Theorem 2.3. *If the a.-s.-s. S is of enumerable or of real type, then, for any integer m, the same is true of S^m.*

A partition $\{A'_j\}_J$ relative to which S is an a.-s.-s. will be called an ***invariant*** partition if the partition $\{S(A'_j)\}_J$ is equivalent to $\{A'_j\}_J$. This means that each interval $S(A'_j)$ is equal to some interval $A'_{j'}$. Generally the order relations induced in the set J by the partitions $\{A'_j\}_J$ and $\{S(A'_j)\}_J$ will be different, and may be of different types. If $\{A'_j\}_J$ is an invariant partition for S, then it is also an invariant partition for each S^m, $m=0, \pm 1, \pm 2, \ldots$ For each one-to-one correspondence S of A onto A there exists an invariant partition which is that one all of whose intervals are improper. However, for a given a.-s.-s. there will exist in general a smaller invariant partition. We have the following theorem.

Theorem 2.4. *Let S be an a.-s.-s. of A relative to the partition $\{A_i\}_I$. The class of all non-empty intersections*

$$\bigcap_{k=-\infty}^{+\infty} S^k(A_{i_k}), \quad i_k \in I, \tag{2.2}$$

forms an invariant partition for S.

Proof. We first prove that each non-empty intersection of type (2.2) is an interval. In fact, if (2.2) is not empty, each of the finite intersections

$$(2.3) \qquad \bigcap_{k=-m}^{m} S^k(A_{i_k}), \quad m=1,2,3,\dots$$

is non-empty. Applying the transformation S^{-m} to the intersection, we obtain the set

$$(2.4) \qquad \bigcap_{k=-2m}^{0} S^k(A_{i_{k+m}}).$$

This is an interval of the partition of Theorem 2.2 with m replaced by $2m+1$; S^{2m+1} is an a.-s.-s. relative to this partition which is clearly a subdivision of the partition corresponding to S^m. By applying the transformation S^m to (2.4) we obtain an interval; hence (2.3) is an interval. The set (2.2) being a non-empty intersection of the intervals (2.3) it must also be an interval.

If we take the union of the intervals (2.2) for all admissible systems $\{i_k\}$, $-\infty<k<+\infty$, (*i. e.* those systems for which (2.2) $\neq 0$) we obtain, clearly, the same result as when we take the union of all intersections (2.2), empty or non-empty. This gives

$$\bigcup_{\{i_k\}} \bigcap_{k=-\infty}^{+\infty} S^k(A_{i_k}) = \bigcap_{k=-\infty}^{+\infty} \bigcup_{i_k\in I} S^k(A_{i_k}) = \bigcap_{k=-\infty}^{\infty} S^k\Big(\bigcup_{i_k\in I} A_{i_k}\Big) = \bigcap_{k=-\infty}^{\infty} S^k(A) = A.$$

Hence (2.2) is a partition of A. To prove that it is an invariant partition for S, consider a non-empty set of form (2.2) corresponding to a system $\{i_k\}$. We have then

$$0 \neq S\Big(\bigcap_{k=-\infty}^{\infty} S^k(A_{i_k})\Big) = \bigcap_{k=-\infty}^{\infty} S^{k+1}(A_{i_k}) = \bigcap_{k=-\infty}^{\infty} S^k(A_{i_{k-1}})$$

which means that the interval corresponding to $\{i_k\}$ is transformed by S onto the interval corresponding to $\{i_{k-1}\}$.

Theorem 2.5. *If S is of real type relative to $\{A_i\}_I$ then it is also of real type relative to the invariant partition of Theorem* 2.4.

Proof. The invariant partition of Theorem 2.4 is clearly the intersection of the partitions corresponding to S^m, $m=\pm 1, \pm 2, \dots$, as given in Theorem 2.2. From Theorem 2.3 it follows that all these partitions are of real type. Hence by Theorem 1.2 the invariant partition is of real type which proves the theorem.

3. General problem. Our general problem concerning types of order may be stated as follows: Consider a set of indices I and two order relations $<$ and $\prec$ in this set. The set I ordered by these relations will be denoted by $I^{<}$ and $I^{\prec}$. We ascribe to each $i\in I$ a type of order a_i and the problem is one of finding necessary and sufficient conditions for a_i so that the following equation holds:

$$(3.1) \qquad \sum_{i\in I^{<}} a_i = \sum_{i\in I^{\prec}} a_i.$$

A system $\{a_i\}$, $i\in I$, satisfying (3.1) will be called a *solution* of (3.1) *of type* $[I, <, \prec]$. This will be an *essential* solution if all the a_i's are $\neq 0$. It is still an open question for which types $[I, <, \prec]$ there exists an essential solution (3.1). A trivial solution with all $a_i=0$ always exists. A solution $\{a_i\}$ with some a_i's vanishing will be called *parly trivial*. If, for such a solution, I' denotes the set of all i's such that $a_i\neq 0$, then the system $\{a_i\}$, $i\in I'$ forms an essential solution of type $[I', <, \prec]$.

From now on we shall be concerned only with essential solutions of type $[I, <, \prec]$.

Suppose that we have an essential solution of (3.1) of type $[I, <, \prec]$. Let a be the type of order equal to the two members of (3.1) and let A be an ordered set of type a. Since $a=\sum_{i\in I^{<}} a_i = \sum_{i\in I^{\prec}} a_i$, there exist two partitions $\{A_i\}_I$ and $\{A'_i\}_I$ of A such that A_i and A'_i have the type of order a_i and the order relations induced in I by these two partitions are precisely $<$ and $\prec$. A_i and A'_i being of the same type, we can choose a similarity S_i transforming A_i onto A'_i. The transformation S of A onto A which, on each A_i, coincides with S_i is clearly a one-to-one correspondence which is an a.-s.-s. relative to the partition $\{A_i\}_I$. We see, therefore, that each system $\{a_i\}$ solving equation (3.1) leads to an a.-s.-s. of type $[I, <, \prec]$. Conversely, if we have an a.-s.-s. of any ordered set A of type $[I, <, \prec]$ relative to a partition $\{A_i\}$, then the types of order a_i of the intervals A_i form a solution of equation (3.1). This shows that the general essential solution of (3.1) is given by the most general a.-s.-s. of type $[I, <, \prec]$. This statement should not be considered as a solution of our problem; it is rather an interpretation of the problem in terms of auto-semi-similarities.

The developments of the previous sections allow us to go further towards the solution of the problem in the case when the type $[I, <, \prec]$ is real (*i. e.* when both orders in I are of real type).

Theorem 3.1. *Let the type $[I, <, \lessdot]$ be real, and let X be an arbitrary set of real numbers admitting an a.-s.-s. T of type $[I, <, \lessdot]$ relative to some partition $\{X_i\}_I$. Consider now an arbitrary function $\gamma(\xi)$ satisfying the following conditions 1^0 $\gamma(\xi)$ is defined for $\xi \in X$ and has for values types of order; 2^0 $\gamma(\xi)$ does not vanish identically on any interval X_i; 3^0 $\gamma(\xi_1)=\gamma(\xi_2)$ for all ξ_1, ξ_2 for which there exists an integer $m=0, \pm 1, \pm 2, \ldots$ such that $\xi_2=T^m(\xi_1)$. Then the types of order*

$$\alpha_i = \sum_{\xi \in X_i} \gamma(\xi), \quad i \in I, \tag{3.2}$$

give an essential solution of (3.1) of type $[I, <, \lessdot]$ and represent the most general essential solution of this type.

Proof. 1) We prove first that the α's given by (3.2) form an essential solution of (3.1). In fact we have

$$\sum_{i \in I^<} \alpha_i = \sum_{i \in I^<} \sum_{\xi \in X_i} \gamma(\xi) = \sum_{\xi \in X} \gamma(TT^{-1}\xi) = \sum_{\xi \in X} \gamma(T^{-1}\xi)$$
$$= \sum_{i \in I^{\lessdot}} \sum_{\xi \in T(X_i)} \gamma(T^{-1}\xi) = \sum_{i \in I^{\lessdot}} \sum_{\eta \in X_i} \gamma(\eta) = \sum_{i \in I^{\lessdot}} \alpha_i.$$

In these equalities we have used property 2^0 of $\gamma(\xi)$ by writing $\gamma(TT^{-1}\xi)=\gamma(T^{-1}\xi)$. Also, we used the fact that the transformation $\xi=T(\eta)$ *is a similarity* for $\eta \in X_i$.

2) Let $\{\alpha_i\}$ be an essential solution of (3.1). We prove that the α_i's are representable in the form (3.2).

As before, let us construct an ordered set A and an a.-s.-s. S of A of type $[I, <, \lessdot]$ relative to a partition $\{A_i\}_I$ with A_i of the type of order α_i. Consider for S the corresponding invariant partition as determined in Theorem 2.4. From Theorem 2.5 it follows that this invariant partition is of real type. Hence we may denote it by $\{A'_\xi\}_X$ with X being a set of real numbers. Since $\{A'_\xi\}_X$ is an invariant partition for S, there exists for each $\xi \in X$ a uniquely determined $\xi'=T(\xi) \in X$ such that $S(A'_\xi)=A'_{\xi'}$. Clearly T is a one-to-one correspondence of X onto X. Further, since $\{A'_\xi\}_X$ is a subdivision of $\{A_i\}_I$ we may consider the corresponding partition $\{X_i\}_I$ of X: $\xi \in X_i$ providing A'_ξ is contained in A_i. It is clear that T is an a.-s.-s. of X relative to $\{X_i\}_I$. The order relation induced in I by the partition $\{X_i\}_I$ is the same as the one induced by $\{A_i\}_I$, *i. e.* it is "$<$". The partition $\{T(X_i)\}_I$ can be obtained by considering $\{A'_\xi\}_X$ as a subdivision of $\{S(A_i)\}_I$; hence it induces in I the order relation "$\lessdot$". This proves that the a.-s.-s. T is of type $[I, <, \lessdot]$ relative to $\{X_i\}_I$. We now define $\gamma(\xi)=$ type of order of A'_ξ. Since $A_i = \sum_{\xi \in X_i} A'_\xi$, formula (3.2) follows immediately.

Concerning the properties of the function $\gamma(\xi)$, 1^0 is true by definition and 2^0 results from (3.2), already proved, since all the α_i's are $\neq 0$. Property 3^0 results from the fact that S is a *similarity* on each A'_ξ and that $S(A'_\xi)=A'_{T(\xi)}$. It follows that $\gamma(\xi)=\gamma(T(\xi))$, and by an immediate induction, $\gamma(\xi)=\gamma(T^m(\xi))$ for any integer m, which proves property 3^0.

Remark on Theorem 3.1. The theorem would solve our problem completely for the case of a real type $[I, <, \lessdot]$ if we knew all the sets X of real numbers admitting a.-s.-s.'s of this type, and if for any such set X we knew all its a.-s.-s.'s of this type. It would even be sufficient to know some such sets X and some of the corresponding a.-s.-s.'s if it could be proved that they allow a representation of all the solutions of (3.1). In the next section we solve our problem in this way for the simplest case of type $[I, <, \lessdot]$ with I composed of two elements 0 and 1, the order relations being $0<1$ and $1 \lessdot 0$.

4. The case of $\alpha_0+\alpha_1=\alpha_1+\alpha_0$. Theorem 3.1 gives the most general solution of this equation in the following form.

Let X be a set of real numbers, $X=X_0+X_1$ a partition of X, T an a.-s.-s. of X relative to this partition, so that T is a similarity on X_0 and X_1 and $T(X_1)<T(X_0)$. We can then write

$$X=X_0+X_1=T(X_1)+T(X_0). \tag{4.1}$$

Since the set of all real numbers is similar to the open interval $0<\xi<1$, we can further suppose

$$0<X<1. \tag{4.2}$$

If then $\gamma(\xi)$ satisfies conditions 1^0, 2^0, and 3^0 of Theorem 3.1, the types

$$\alpha_0=\sum_{\xi \in X_0}\gamma(\xi), \qquad \alpha_1=\sum_{\xi \in X_1}\gamma(\xi) \tag{4.3}$$

form a solution of our problem.

Rather than to determine all suitable sets X and all their a.-s.-s.'s T of the required type, we shall transform the representation (4.3) into the form given in Theorem I of the Introduction where (besides the exceptional cases (b)) the types α_0 and α_1 are represented by formulas only slightly different from (4.3) with X_0 and X_1 being open intervals of the real axis.

In order to achieve this transformation we introduce the translation $R(\xi)=\xi+1$ on the real axis and define

$$\widetilde{X}=\sum_{m=-\infty}^{\infty} R^m(X). \tag{4.4}$$

In view of (4.2), (4.4) represents a partition of $\widetilde{X}$ and for each $\xi\in\widetilde{X}$ there exists a unique integer m such that $\xi\in R^m(X)$ or such that $R^{-m}(\xi)\in X=X_0+X_1=T(X_1)+T(X_0)$.

Clearly the translation R restricted to $\widetilde{X}$ forms an auto-similarity of $\widetilde{X}$. We now define a transformation U as follows:

$$\textit{If}\quad \xi\in R^m(X_i),\qquad U(\xi)=R^{m+i}TR^{-m}(\xi). \tag{4.5}$$

It is clear that U is a one-valued transformation of $\widetilde{X}$ into itself. An immediate verification shows that U possesses an inverse U^{-1} given by

$$\textit{If}\quad \xi\in R^mT(X_i),\qquad U^{-1}(\xi)=R^{m-i}T^{-1}R^{-m}(\xi). \tag{4.6}$$

Hence U is a one-to-one auto-correspondence of $\widetilde{X}$. We prove now

(4.7) *U is an auto-similarity of $\widetilde{X}$.*

In fact, if $(\xi_1,\xi_2)\subset\widetilde{X}$, then $\xi_1=R^{m_1}(\xi_1^0)$, $\xi_2=R^{m_2}(\xi_2^0)$, $\xi_1^0\in X_{i_1}$, $\xi_2^0\in X_{i_2}$. Suppose $\xi_1<\xi_2$; then either $m_1<m_2$ or $m_1=m_2$ and $\xi_1^0<\xi_2^0$. In the first case $U(\xi_1)=R^{m_1+i_1}T(\xi_1^0)$ and $U(\xi_2)=R^{m_2+i_2}T(\xi_2^0)$ there may be doubt only if $m_1+i_1=m_2+i_2$, *i. e.* if $i_1=1$, $i_2=0$ and $m_1+1=m_2$. But then $T(\xi_1^0)\in T(X_1)$, $T(\xi_2^0)\in T(X_0)$ and $T(X_1)<T(X_0)$ which proves $U(\xi_1)<U(\xi_2)$. In the second case $m_1=m_2$ and $\xi_1^0<\xi_2^0$, hence $i_1\leqslant i_2$. The only doubt arises here if $i_1=i_2$. But then, since T is a similarity on X_i, we get $T(\xi_1^0)<T(\xi_2^0)$ and again, $U(\xi_1)<U(\xi_2)$.

We prove next

$$UR=RU. \tag{4.8}$$

In fact, if

$$\xi\in R^m(X_i),\qquad R(\xi)\in R^{m+1}(X_i),$$
$$U(R(\xi))=R^{m+1+i}TR^{-m-1}(R(\xi))=R^{m+1+i}TR^{-m}(\xi)=R(U(\xi)).$$

It follows from (4.8) that the group of similarities generated by R and U is an abelian group and all of its similarities have the form U^kR^l.

Consider now for $k>0$ the partition of X composed of all non-empty intersections $\bigcap_{m=0}^{k-1}T^{-m}(X_{i_m})$, $i_m=0,1$ (relative to this partition T^k is an a.-s.-s.). We prove

$$\textit{For}\ k=1,2,\ldots,\ \textit{if}\ \xi\in\bigcap_{m=0}^{k-1}T^{-m}(X_{i_m}),\quad U^k(\xi)=R^{i_0+i_1+\ldots+i_{k-1}}T^k(\xi). \tag{4.9}$$

This is obtained by an easy induction from $k-1$ to k by using (4.5). Similarly, by using (4.6), we obtain

$$\begin{gathered}\textit{For}\ k=-1,-2,\ldots,\ \textit{if}\ \xi\in\bigcap_{m=1}^{-k}T^m(X_{i_m}),\\ U^k(\xi)=R^{-i_1-i_2-\ldots-i_{-k}}T^k(\xi).\end{gathered} \tag{4.9'}$$

From (4.9) and (4.9′) we obtain

$$\begin{gathered}\textit{If}\quad \xi\in X\quad \textit{and}\quad U^kR^l(\xi)\in X,\quad \textit{then}\quad U^kR^l(\xi)=T^k(\xi)\\ \textit{and}\quad 0\leqslant k(k+l)\leqslant k^2.\end{gathered} \tag{4.10}$$

In fact, for $k>0$, ξ must belong to some intersection $\bigcap_{m=0}^{k-1}T^{-m}(X_{i_m})$ and in view of (4.9) we have $l=-(i_0+i_1+\ldots+i_{k-1})$, $0\leqslant k+l\leqslant k$, $0\leqslant k(k+l)\leqslant k^2$. For $k<0$ we get $l=i_1+\ldots+i_{-k}$, $k\leqslant k+l\leqslant 0$, $0\leqslant k(k+l)\leqslant k^2$.

We shall consider the class $\widetilde{\mathfrak{N}}$ of all proper initial intervals of $\widetilde{X}$ (*i. e.* of all those which are neither empty nor $=\widetilde{X}$). This class is an ordered set with the natural order relation $\subset$. We denote by Θ the initial interval formed by all $\xi\in\widetilde{X}$ with $\xi<0$. Obviously, R and U become auto-similarities on the class $\widetilde{\mathfrak{N}}$ and generate an abelian group of such auto-similarities. We have then

$$R(\Theta)-\Theta=X, \tag{4.11}$$

$$\begin{aligned}U^{-1}R(\Theta)-\Theta&=X_0, & R(\Theta)-U^{-1}R(\Theta)&=X_1,\\ U(\Theta)-\Theta&=T(X_1), & R(\Theta)-U(\Theta)&=T(X_0).\end{aligned} \tag{4.11'}$$

Corresponding to the interval X of $\widetilde{X}$ we have the subclass $\mathfrak{N}$, the interval of the ordered class $\widetilde{\mathfrak{N}}$, composed of all initial intervals Λ of $\widetilde{X}$ satisfying

$$\Theta\subset\Lambda\subsetneqq R(\Theta). \tag{4.12}$$

Similarly to (4.4) we have

$$\tilde{\mathfrak{N}} = \sum_{m=-\infty}^{+\infty} R^m(\mathfrak{N}). \tag{4.13}$$

Consequently, to each $A \in \tilde{\mathfrak{N}}$ there corresponds a unique integer m such that $R^{-m}(A) \in \mathfrak{N}$. Further, by use of property (4.10) we obtain

(4.14) *For $A \in \mathfrak{N}$ and for each integer k there exists a unique integer l such that $U^k R^l(A) \in \mathfrak{N}$; l satisfies $0 \leqslant k(k+l) \leqslant k^2$.*

We shall distinguish two principal cases. The first is the one where there exists an initial interval $A \in \tilde{\mathfrak{N}}$ and two integers k, l with $|k|+|l|>0$, so that $U^k R^l(A)=A$ — in other words, when there exists an invariant A for some $U^k R^l$. The second case is the one where there is no such A, k and l. We consider first the

Case I. *For some $U^k R^l$, $|k|+|l|>0$, there exists a $A \in \tilde{\mathfrak{N}}$ with $U^k R^l(A)=A$.*

Here we must have $k \neq 0$; otherwise, $k=0$, $R^l(A)=A$; hence $l=0$ which is contrary to the condition $|k|+|l|>0$. Further, we have $U^{-k}R^{-l}(A)=A$. Thus, if for some $k<0$ there exists an l and a A such that $U^k R^l(A)=A$, the same is true also for $-k$. For each k and each A there exists at most one l such that $U^k R^l(A)=A$; otherwise if there existed another one, l', we would have $R^{l'-l}(A)=A$ which is impossible.

Consider now the smallest positive k for which there exists an integer l and a $A \in \tilde{\mathfrak{N}}$ such that $U^k R^l(A)=A$. This value of k will be denoted as k_0. There may be many A's and l's satisfying $U^{k_0}R^l(A)=A$. If A is invariant for $U^{k_0}R^l$ then the same is also true of $R^{-m}(A)$: $U^{k_0}R^l(R^{-m}(A))=R^{-m}(U^{k_0}R^l(A))=R^{-m}(A)$. Therefore we can always suppose that the invariant A is in $\mathfrak{N}$. It follows from (4.14) that the corresponding integer l can take only one of the k_0+1 values $l=0,-1,\ldots,-k_0$.

Consider now for each of these integers l (for which there exists an invariant A) the class of all $A \supset \Theta$ satisfying $U^{k_0}R^l(A)=A$. Clearly the intersection of all these A's is again an initial interval of $\tilde{X}$ satisfying the same equation. It is the smallest A^l of all such A's. We take the smallest initial interval among the A^l; it corresponds to some l_0, $0 \leqslant l_0 \leqslant k_0$, and will be denoted by A_0; A_0 has the property

$$U^{k_0}R^{l_0}(A_0)=A_0, \qquad \Theta \subset A_0 \subsetneqq R(\Theta) \tag{4.15}$$

and there is no A with $\Theta \subset A \subsetneqq A_0$, $U^k R^l(A)=A$ where $0<k\leqslant k_0$.

We now prove two lemmas.

Lemma 1. *There exists a unique couple of integers (k_1,l_1) such that $0 \leqslant k_1 \leqslant k_0-1$, $k_0l_1-k_1l_0=1$ and such that all the initial intervals $U^k R^l(A_0)$ figure in their order in the sequence*

$$\{U^{mk_1}R^{ml_1}(A_0)\}, \quad -\infty<m<\infty, \qquad U^{(m-1)k_1}R^{(m-1)l_1}(A_0) \subsetneqq U^{mk_1}R^{ml_1}(A_0).$$

Proof. For any k we can find a k' with $0 \leqslant k' \leqslant k_0-1$ such that for some integer m, $k=mk_0+k'$. Hence $U^k R^l(A_0)= =U^k R^l U^{-mk_0}R^{-ml_0}(A_0)=U^{k'}R^{l-ml_0}(A_0)$. If we suppose that this initial interval lies in $\mathfrak{N}$, $l-ml_0$ can take only one value determined by k' (see (4.14)). Consequently there are at most k_0 such initial intervals in $\mathfrak{N}$ corresponding to the k' in the range $0 \leqslant k' \leqslant k_0-1$. On the other hand, to each k' in this range there corresponds an l' such that $U^{k'}R^{l'}(A_0) \in \mathfrak{N}$. Two such initial intervals must be distinct, otherwise for $0 \leqslant k'<k'' \leqslant k_0-1$ and the corresponding l' and l'' we would have $U^{k'}R^{l'}(A_0)=U^{k''}R^{l''}(A_0)$, $U^{k''-k'}R^{l''-l'}(A_0)=A_0$, which is contrary to the definition of k_0 since $0<k''-k' \leqslant k_0-1$. Hence there are exactly k_0 such initial intervals in $\mathfrak{N}$. The first of these is A_0 (by 4.15)). In $R(\mathfrak{N})$ there are also k_0 initial intervals $U^k R^l(A_0)$ which are obtained by applying the similarity R to those in $\mathfrak{N}$; $R(A_0)$ is clearly the smallest among those in $R(\mathfrak{N})$. Among all the $U^{k'}R^l(A_0)$ which are $\supsetneqq A_0$, there must be then a well determined smallest one; this one will be in $\mathfrak{N}$ if $k_0 \geqslant 2$, or $=R(A_0) \subset R(\mathfrak{N})$ if $k_0=1$. We denote the smallest one by $U^{k_1}R^{l_1}(A_0)$, $0 \leqslant k_1 \leqslant k_0-1$. We have, therefore, $A_0 \subsetneqq U^{k_1}R^{l_1}(A_0)$ and there is no $U^k R^l(A_0)$ lying strictly between these two. By applying the similarity $U^{(m-1)k_1}R^{(m-1)l_1}$, $-\infty<m<\infty$, we get $U^{(m-1)k_1}R^{(m-1)l_1}(A_0) \subsetneqq U^{mk_1}R^{ml_1}(A_0)$. No other $U^k R^l(A_0)$ can lie between these two initial intervals. It follows that $U^{(k_0-1)k_1}R^{(k_0-1)l_1}(A_0) \subsetneqq R(\Theta)$ and $U^{k_0k_1}R^{k_0l_1}(A_0)$ contains $R(\Theta)$ and is the first among $U^k R^l(A_0)$ which contains $R(\Theta)$. Hence $R(A_0)=U^{k_0k_1}R^{k_0l_1}(A_0)=R^{k_0l_1-k_1l_0}U^{k_0k_1}R^{k_1l_0}(A_0)=R^{k_0l_1-k_1l_0}(A_0)$ and thus $k_0l_1-k_1l_0=1$.

For any couple of integers (k,l) we can then write (in view of $k_0l_1-k_1l_0=1$) $k=(k_0l-kl_0)k_1+(kl_1-k_1l)k_0$, $l=(k_0l-kl_0)l_1+(kl_1-k_1l)l_0$ which gives $U^kR^l(\Lambda_0)=U^{mk_1}R^{ml_1}(\Lambda_0)$ with $m=k_0l-kl_0$. This finishes the proof of the lemma.

***Lemma* 2.** *With the notations of Lemma* 1, *the formula*

$$\tilde{X}=\sum_{m=-\infty}^{+\infty}[U^{mk_1}R^{ml_1}(\Lambda_0)-U^{(m-1)k_1}R^{(m-1)l_1}(\Lambda_0)] \tag{4.16}$$

determines a partition of $\tilde{X}$. *A corresponding partition of* $\tilde{\mathfrak{R}}$ *is given by the intervals of* $\tilde{\mathfrak{R}}$, $U^{(m-1)k_1}R^{(m-1)l_1}(\Lambda_0)\subsetneqq\Lambda\subset U^{mk_1}R^{ml_1}(\Lambda_0)$, $-\infty<m<\infty$. *If* $U^kR^l(\Theta)$ *belongs to the m-th interval of this partition, then those and only those* $U^{k'}R^{l'}(\Theta)$ *belong to the same interval for which there exists an integer* n *such that* $k'=k+nk_0$, $l'=l+nl_0$. *The index* m *of the interval of the partition of* $\tilde{\mathfrak{R}}$ *to which* $U^kR^l(\Theta)$ *belongs is given by* $m=k_0l-kl_0$.

Proof. From Lemma 1 it follows that all the intervals of (4.16) are mutually disjoint. On the other hand since $U^{-k_1}R^{-l_1}(\Lambda_0)$ is $\subsetneqq\Theta\subsetneqq R(\Theta)\subset U^{k_0k_1}R^{k_0l_1}(\Lambda_0)=R(\Lambda_0)$, it follows, by applying the similarity R^m, that $U^{(mk_0-1)k_1}R^{(mk_0-1)l_1}(\Lambda_0)=U^{-k_1}R^{m-l_1}(\Lambda_0)\subsetneqq R^m(\Theta)\subsetneqq R^{m+1}(\Theta)\subset U^{(m+1)k_0k_1}R^{(m+1)k_0l_1}(\Lambda_0)=R^{m+1}(\Lambda_0)$. Consequently each interval $R^m(X)=R^{m+1}(\Theta)-R^m(\Theta)$ is contained in the union of intervals of (4.16) for indices m' with $mk_0\leqslant m'\leqslant(m+1)k_0$. In view of (4.4) the intervals of (4.16) cover the whole of $\tilde{X}$ and hence form a partition of $\tilde{X}$. The partition of $\tilde{X}$ clearly induces the partition of $\tilde{\mathfrak{R}}$ given in the Lemma.

Consider now $U^kR^l(\Theta)$. There is a unique system of integers m, n solving the equations $k=mk_1+nk_0$, $l=ml_1+nl_0$. These integers are given by $m=k_0l-kl_0$, $n=kl_1-k_1l$. In view of $U^{-k_1}R^{-l_1}(\Lambda_0)\subsetneqq\Theta\subset\Lambda_0$, we obtain $U^{(m-1)k_1}R^{(m-1)l_1}(\Lambda_0)\subsetneqq U^kR^l(\Theta)\subset U^{mk_1}R^{ml_1}(\Lambda_0)$. Thus $U^kR^l(\Theta)$ belongs to the interval with index $m=k_0l-kl_0$ of the partition of $\tilde{\mathfrak{R}}$. On the other hand, $U^{k'}R^{l'}(\Theta)$ belongs to the same interval if and only if $m'=k_0l'-k'l_0=k_0l-kl_0$. This means $k_0(l'-l)=l_0(k'-k)$. Since k_0 and l_0 are relatively prime (in view of $k_0l_1-k_1l_0=1$), the last equation is equivalent to the existence of an integer n such that $k'-k=nk_0$ and $l'-l=nl_0$ which finishes the proof of the Lemma.

Case I is divided into two subcases depending on whether $k_0=1$ or $k_0>1$.

Subcase I'. $k_0=1$. It follows from (4.14) that $l_0=0$ or $=-1$. From Lemma 1 we obtain $k_1=0$, $l_1=1$, and $\Theta\subset\Lambda_0\subsetneqq R(\Theta)\subset R(\Lambda_0)$. We cannot have $\Theta=\Lambda_0$ since this would mean $\Theta=U(\Theta)$, or $\Theta=UR^{-1}(\Theta)$, which would be in contradiction to (4.11'). For the same reason, no one of the $U^kR^l(\Theta)=\Lambda_0$. Therefore we have two possibilities: $\Theta\subsetneqq\Lambda_0\subsetneqq U^{-1}R(\Theta)$ or $U^{-1}R(\Theta)\subsetneqq\Lambda_0\subsetneqq R(\Theta)$. In the first case $\Theta\subsetneqq\Lambda_0=UR^{l_0}(\Lambda_0)\subsetneqq UR^{l_0}U^{-1}R(\Theta)=R^{l_0+1}(\Theta)$, which gives $l_0=0$. In the second case $R(\Theta)\supsetneqq\Lambda_0=UR^{l_0}(\Lambda_0)\supsetneqq UR^{l_0}U^{-1}R(\Theta)=R^{l_0+1}(\Theta)$ which gives $l_0=-1$.

Consider the case $l_0=0$. Since $\Lambda_0\subsetneqq R(\Theta)\subsetneqq R(\Lambda_0)=U^{k_1}R^{l_1}(\Lambda_0)$ by Lemma 2, all the initial intervals $U^mR(\Theta)$ also satisfy $\Lambda_0\subsetneqq U^mR(\Theta)\subsetneqq R(\Lambda_0)$. Similarly, $R^{-1}(\Lambda_0)\subsetneqq U^m(\Theta)\subsetneqq\Lambda_0$ for all m. Further, from $\Theta\subsetneqq U(\Theta)$, it follows that $U^m(\Theta)\subsetneqq U^{m+1}(\Theta)$ and $U^mR(\Theta)\subsetneqq U^{m+1}R(\Theta)$ for $-\infty<m<\infty$. Consequently for $m=0,1,2,\ldots$ we obtain a strictly increasing sequence $U^m(\Theta)\subset\Lambda_0$ and a strictly decreasing sequence $U^{-m}R(\Theta)\supset\Lambda_0$. Therefore the union of the increasing sequence and intersection of the decreasing sequence are two initial intervals Λ' and Λ'' such that $\Lambda'\subset\Lambda_0\subset\Lambda''$ (in fact we have $\Lambda'=\Lambda_0$). Consider then the following partition of X_0 (compare (4.11')):

$$X_0=\sum_{m=0}^{\infty}[U^{m+1}(\Theta)-U^m(\Theta)]+(\Lambda''-\Lambda')+\sum_{m=-\infty}^{-1}[U^mR(\Theta)-U^{m-1}R(\Theta)].$$

By virtue of Theorem 3.1 we can then write

$$\alpha_0=\sum_{\xi\in X_0}\gamma(\xi)=\sum_{m=0}^{\infty}\sum_{\xi\in X'_0}\gamma(\xi)+\sum_{\xi\in\Lambda''-\Lambda'}\gamma(\xi)+\sum_{m=-\infty}^{-1}\sum_{\xi\in X''_m}\gamma(\xi). \tag{4.17}$$

We put here $X'_m=U^{m+1}(\Theta)-U^m(\Theta)$ and $X''_m=U^mR(\Theta)-U^{m-1}R(\Theta)$. The similarity $U^{m+1}R^{-1}$ transforms $R(\Theta)-U^{-1}R(\Theta)=X_1$ into X'_m. If follows from (4.10) that for $\eta\in X_1$, $U^{m+1}R^{-1}(\eta)=T^{m+1}(\eta)$. By using property 3° of $\gamma(\xi)$ (see Theorem 3.1) we obtain

$$\sum_{\xi\in X'_m}\gamma(\xi)=\sum_{\eta\in X_1}\gamma(U^{m+1}R^{-1}(\eta))=\sum_{\eta\in X_1}\gamma(T^{m+1}(\eta))=\sum_{\eta\in X_1}\gamma(\eta)=\alpha_1.$$

In the same way (we note that for $m\leqslant -1$, the similarity U^m transforms X_1 onto X''_m) we obtain $\sum\limits_{\xi\in X''_m}\gamma(\xi)=\sum\limits_{\eta\in X_1}\gamma(\eta)=\alpha_1$. From (4.17) we further obtain

$$\alpha_0=\alpha_1\omega+\sum_{\xi\in A''-A}\gamma(\xi)+\alpha_1\omega^*$$

which is a representation of the form (b) of the Introduction.

In the case $l_0=-1$, by a similar argument we obtain the partition of X_1

$$X_1=\sum_{m=1}^{\infty}[U^{-m-1}R^{m+1}(\Theta)-U^{-m}R^m(\Theta)]+(A''-A')+$$
$$+\sum_{m=-\infty}^{0}[U^{-m}R^{m+1}(\Theta)-U^{-m+1}R^m(\Theta)].$$

Here we arrive at the formula

$$\alpha_1=\alpha_0\omega+\sum_{\xi\in A''-A'}\gamma(\xi)+\alpha_0\omega^*$$

which is again a representation of type (b).

Subcase I''. $k_0\geqslant 2$. In the present case $U^{k_1}R^{l_1}(A_0)\in\mathfrak{N}$. By virtue of (4.14) we have $0\leqslant k_0+l_0\leqslant k_0$. In view of $k_0l_1-k_1l_0=1$, and $k_0\geqslant 2$, neither of the two equalities $k_0+l_0=0$ or $k_0+l_0=k_0$ is possible. Hence we have the strict inequality $0<k_0+l_0<k_0$.

Applying Lemma 2 to the initial intervals Θ, $U^{-1}R(\Theta)$ and $R(\Theta)$, we find that they lie in three intervals of $\mathfrak{N}$ of the form

$$U^{(m-1)k_1}R^{(m-1)l_1}(A_0)\subsetneqq A\subset U^{mk_1}R^{ml_1}(A_0),\quad\text{for}\quad m=0,\ k_0+l_0,\quad\text{and}\quad k_0.$$

Consequently we have the sequence of inclusions:

$$U^{mk_1}R^{ml_1}(\Theta)\subset U^{mk_1}R^{ml_1}(A_0)\subsetneqq U^{(m+1)k_1}R^{(m+1)l_1}(\Theta)\quad\text{for any integer } m,$$

in addition to the inclusions:

$$U^{(k_0+l_0-1)k_1}R^{(k_0+l_0-1)l_1}(A_0)\subsetneqq U^{-1}R(\Theta)\subset U^{(k_0+l_0)k_1}R^{(k_0+l_0)l_1}(A_0)$$
$$U^{(k_0-1)k_1}R^{(k_0-1)l_1}(A_0)\subsetneqq R(\Theta)\subset U^{k_0k_1}R^{k_0l_1}(A_0).$$

We introduce the notation:

$$Y'_m=U^{mk_1}R^{ml_1}(A_0)-U^{mk_1}R^{ml_1}(\Theta),\quad\text{for}\quad m\neq k_0+l_0,$$
$$Y'_{k_0+l_0}=U^{(k_0+l_0)k_1}R^{(k_0+l_0)l_1}(A_0)-U^{-1}R(\Theta),$$
$$Y''_m=U^{(m+1)k_1}R^{(m+1)l_1}(\Theta)-U^{mk_1}R^{ml_1}(A_0)\quad\text{for all } m \text{ except}$$
$$m=k_0+l_0-1\ \text{ or }\ =k_0-1,$$
$$Y''_{k_0+l_0-1}=U^{-1}R(\Theta)-U^{(k_0+l_0-1)k_1}R^{(k_0+l_0-1)l_1}(A_0),$$
$$Y''_{k_0-1}=R(\Theta)-U^{(k_0-1)k_1}R^{(k_0-1)l_1}(A_0).$$

With these notations we can write

$$X_0=U^{-1}R(\Theta)-\Theta=\sum_{m=0}^{k_0+l_0-1}[Y'_m+Y''_m],$$

$$X_1=R(\Theta)-U^{-1}R(\Theta)=\sum_{m=k_0+l_0}^{k_0-1}[Y'_m+Y''_m].$$

It is clear that $Y'_m=U^{mk_1}R^{ml_1}(Y'_1)$ for all m except $m=k_0+l_0$. For this exceptional value of m we have $Y'_{k_0+l_0}=U^{-1}R(Y'_0)$ which is immediately verified by using the relations $k_0l_1-k_1l_0=1$ and $U^{k_0}R^{l_0}(A_0)=A_0$. Similarly we have $Y''_m=U^{mk_1}R^{ml_1}(Y''_0)$ except for $m=k_0+l_0-1$, k_0-1. In these exceptional cases it is immediately seen that $Y''_{k_0+l_0-1}=U^{-k_1-1}R^{-l_1+1}(Y''_0)$ and $Y''_{k_0-1}=U^{-k_1}R^{-l_1+1}(Y''_0)$.

By an argument similar to the one used in Subcase I' we obtain now

$$\alpha_0=\sum_{m=0}^{k_0+l_0-1}\Big[\sum_{\xi\in Y'_m}\gamma(\xi)+\sum_{\xi\in Y''_m}\gamma(\xi)\Big]=\Big[\sum_{\xi\in Y'_0}\gamma(\xi)+\sum_{\xi\in Y''_0}\gamma(\xi)\Big](k_0+l_0),$$

$$\alpha_1=\sum_{m=k_0+l_0}^{k_0-1}\Big[\sum_{\xi\in Y'_m}\gamma(\xi)+\sum_{\xi\in Y''_m}\gamma(\xi)\Big]=\Big[\sum_{\xi\in Y'_0}\gamma(\xi)+\sum_{\xi\in Y''_0}\gamma(\xi)\Big](-l_0).$$

Hence we can write $\alpha_0=\delta(k_0+l_0)$, $\alpha_1=\delta(-l_0)$ which means that α_0 and α_1 are of the type (a) of the introduction. They can be represented in the form (1) of Theorem I of the Introduction by taking $\xi_0=\dfrac{k_0+l_0}{k_0}$, $\gamma'_0=\delta_t$, $\gamma'_1=0$, and $\gamma'(\xi)=0$ for all ξ which are not of the form $m+m_0\xi_0$ for any integers m and m_0, where $t=$gr. c. divisor of k_0+l_0 and k_0.

Case II. *There exist no invariant initial intervals for any* U^kR^l *when* k *and* l *are not both* 0.

In the present case we can prove the following lemma:

Lemma 3. *For any integers k and l, not both 0, the sequence of initial intervals $U^{mk}R^{ml}(\Theta)$, $m=0,1,2,\ldots$ is a strictly increasing or decreasing sequence depending on whether $\Theta\subset U^kR^l(\Theta)$ or $\Theta\supset U^kR^l(\Theta)$. In the first case $\bigcup_{m=0}^{\infty} U^{mk}R^{ml}(\Theta)=\widetilde{X}$; in the second $\bigcap_{m=0}^{\infty} U^{mk}R^{ml}(\Theta)=0$.*

Proof. Since in the present case we cannot have $\Theta=U^kR^l(\Theta)$ there are only two possibilities: either $\Theta\subsetneqq U^kR^l(\Theta)$ or $\Theta\supsetneqq U^kR^l(\Theta)$. In the first case, applying the similarity $U^{mk}R^{ml}$ we obtain

$$U^{mk}R^{ml}(\Theta)\subsetneqq U^{(m+1)k}R^{(m+1)l}(\Theta),\quad m=0,1,2,\ldots,$$

which shows that $U^{mk}R^{ml}(\Theta)$ form a strictly increasing sequence. If $\bigcup_{m=0}^{\infty} U^{mk}R^{ml}(\Theta)$ were $\neq\widetilde{X}$, it would be a proper initial interval Λ. We would then have

$$U^kR^l(\Lambda)=\bigcup_{m=0}^{\infty} U^{(m+1)k}R^{(m+1)l}(\Theta)=\bigcup_{m=0}^{\infty} U^{mk}R^{ml}(\Theta)=\Lambda,$$

and Λ would be an invariant initial interval for U^kR^l which is impossible.

In the second case we prove similarly that the sequence $\{U^{mk}R^{ml}(\Theta)\}$ is strictly decreasing and if $\bigcap_{m=0}^{\infty} U^{mk}R^{ml}(\Theta)$ were not $=0$ it would be an invariant initial interval for U^kR^l.

We now define the real number η_0. Consider, for any positive integer k, the unique integer $l\leqslant 0$ for which

$$\Theta\subset U^kR^l(\Theta)\subsetneqq R(\Theta). \tag{4.18}$$

Denote by I_k the interval of the real axis:

$$I_k\colon \frac{-l}{k}\leqslant\xi\leqslant\frac{-l+1}{k}. \tag{4.19}$$

We shall show that all the intervals I_k have a non-empty intersection. This is equivalent to the fact that any two intervals I_k and $I_{k'}$ have a non-empty intersection. This can be proved if we

show that for any k and k', the left end-point of I_k is smaller than the right end-point of $I_{k'}$:

$$\frac{-l}{k}<\frac{-l'+1}{k'}, \tag{4.20}$$

where l' is the integer corresponding to k'.

We prove (4.20) in the following way: From the first inclusion in (4.18) it follows, by Lemma 3, that $U^{mk}R^{ml}(\Theta)$ is an increasing sequence. Hence $\Theta\subset U^{k'k}R^{k'l}(\Theta)$. The second inclusion can be transformed into $U^kR^{l-1}(\Theta)\subsetneqq\Theta$. Hence the sequence $\{U^{mk}R^{m(l-1)}(\Theta)\}$ is strictly decreasing and $U^{k'k}R^{k'(l-1)}(\Theta)\subsetneqq\Theta$. The two inclusions so obtained can be written as follows:

$$R^{-k'l}(\Theta)\subset U^{k'k}(\Theta)\subsetneqq R^{k'(-l+1)}(\Theta).$$

Similarly we obtain

$$R^{-kl'}(\Theta)\subset U^{kk'}(\Theta)\subsetneqq R^{k(-l'+1)}(\Theta).$$

It follows that $R^{-k'l}(\Theta)\subsetneqq R^{k(-l'+1)}(\Theta)$ and hence $-k'l<k(-l'+1)$, which gives formula (4.20).

Since the interval I_k is of diameter $\frac{1}{k}$ the intersection of all I_k is composed of one point only which we shall denote by η_0.

We prove now the following property:

(4.21) $\quad 0<\eta_0<1,\quad$ *η_0 is an irrational number.*

In fact, from (4.18) and (4.14) it follows that $0\leqslant k(k+l)\leqslant k^2$ and consequently $0\leqslant\frac{k+l}{k}\leqslant 1$ and $0\leqslant\frac{-l}{k}\leqslant 1$. Since $\frac{-l}{k}\leqslant\eta_0\leqslant\frac{-l+1}{k}$ for all $k=1,2,\ldots$ it follows that $0\leqslant\eta_0\leqslant 1$.

Suppose now that η_0 is rational, $\eta_0=\frac{p}{q}$ with p and q integers, $p\geqslant 0$, $q\geqslant 1$. Consider then, for an arbitrary positive integer m, the integer $k=mq$ and the corresponding integer l such that (4.18) holds. We have then $\frac{-l}{mq}\leqslant\frac{p}{q}\leqslant\frac{-l+1}{mq}$. It follows that mp is either $=-l$ or $=-l+1$. This means that $\frac{p}{q}$ is either equal to the left or to the right end of the interval I_{mq}. Furthermore, $\frac{p}{q}$ must be on the same side of I_{mq} for all values of m; this follows from the strict inequality in (4.20) applied to $k=mq$ and $k'=m'q$.

Suppose first that $\frac{p}{q}$ is the left end of I_{mq}. Then $l=-mp$ and (4.18) takes the form

$$\Theta \subset U^{mq}R^{-mp}(\Theta) \subsetneqq R(\Theta) \quad \text{for all} \quad m=1,2,3,\ldots$$

which is in contradiction to Lemma 3.

If we suppose that $\frac{p}{q}$ is equal to the right end of I_{mq} we get $l=-mp+1$ and (4.18) gives

$$\Theta \subset U^{mq}R^{-mp+1}(\Theta) \subsetneqq R(\Theta)$$

$$R^{-1}(\Theta) \subset U^{mq}R^{-mp}(\Theta) \subsetneqq \Theta$$

which is again in contradiction to Lemma 3. Thus, property (4.21) is proved.

Consider now on the real axis the set Ω of all real numbers of the form $k\eta_0+l$ with k and l arbitrary integers. Since η_0 is irrational, each $\eta \in \Omega$ has a unique representation $\eta=k\eta_0+l$, and we can therefore define

(4.22) $$A(\eta)=U^kR^l(\Theta).$$

It is clear that $A(\eta)$ is a one-to-one correspondence between the set Ω and the set of all initial intervals $U^kR^l(\Theta)$.

We prove now

(4.23) *$A(\eta)$ is a similarity.*

To prove this take $\eta_1=k_1\eta_0+l_1$, $\eta_2=k_2\eta_0+l_2$ such that $\eta_1<\eta_2$. Putting $k=k_2-k_1$, we shall then consider three cases.

1) $k=0$. The inequality $\eta_1<\eta_2$ now gives $l_1<l_2$ and therefore

$$A(\eta_1)=U^{k_1}R^{l_1}(\Theta) \subsetneqq R^{l_2-l_1}U^{k_1}R^{l_1}(\Theta)=A(\eta_2).$$

2) $k>0$. Consider the corresponding l for which (4.18) holds. By the definition of η_0 we have $\frac{-l}{k}<\eta_0<\frac{-l+1}{k}$; this means that $0<k\eta_0+l<1$. Since $\eta_1<\eta_2$ gives $k\eta_0+(l_2-l_1)>0$, it follows that $l_2-l_1 \geqslant l$. Consequently, by (4.18), $\Theta \subset U^kR^l(\Theta) \subset U^kR^{l_2-l_1}(\Theta)$ and hence $U^{k_1}R^{l_1}(\Theta) \subset U^{k_2}R^{l_2}(\Theta)$. This inclusion can be replaced by $\subsetneqq$ since $k_2-k_1 \neq 0$. Thus we obtain again $A(\eta_1) \subsetneqq A(\eta_2)$.

3) $k<0$. Consider $k'=-k$ and the corresponding l' such that

$$\Theta \subset U^{-k}R^{l'}(\Theta) \subsetneqq R(\Theta).$$

We have now $\frac{l'}{k}<\eta_0<\frac{l'-1}{k}$, i. e. $-1<k\eta_0-l'<0$. Together with the inequality $k\eta_0+(l_2-l_1)>0$, this gives $l_2-l_1 \geqslant -l'+1$. The inclusion $U^{-k}R^{l'}(\Theta) \subsetneqq R(\Theta)$ now gives $\Theta \subsetneqq U^kR^{-l'+1}(\Theta) \subset U^kR^{l_2-l_1}(\Theta)$ and hence we again obtain $A(\eta_1) \subsetneqq A(\eta_2)$.

It is well known that the set Ω is everywhere dense on the real axis; hence every real number ξ is completely determined by the corresponding Dedekind cut of Ω. For each real number ξ we now define two initial intervals of $\widetilde{X}$, $A'(\xi)$ and $A''(\xi)$:

(4.24) $$A'(\xi)=\bigcup_{\substack{\eta \in \Omega \\ \eta<\xi}} A(\eta), \qquad A''(\xi)=\bigcap_{\substack{\eta \in \Omega \\ \eta>\xi}} A(\eta).$$

We have immediately

(4.25) *For* $\xi \in \Omega$, $A'(\xi) \subset A(\xi) \subset A''(\xi)$.

We prove further

(4.26) *If* $\xi_1<\xi_2$, $A'(\xi_1) \subset A''(\xi_1) \subsetneqq A'(\xi_2) \subset A''(\xi_2)$.

Here the first and third inclusions are obvious from definition (4.24). To prove the second we use the fact that Ω is everywhere dense on the real axis and choose two real numbers η_1 and η_2 in Ω such that $\xi_1<\eta_1<\eta_2<\xi_2$. Then by (4.23) and the definition (4.24) we get

$$A''(\xi_1) \subset A(\eta_1) \subsetneqq A(\eta_2) \subset A'(\xi_2)$$

which proves the second inclusion of (4.26).

Besides the translation R, $R(\xi)=\xi+1$, we consider on the real axis the translation P, $P(\xi)=\xi+\eta_0$. Clearly, $P^mR^n(\xi)=\xi+m\eta_0+n$ for any integers m and n. It is easy to show that

(4.27) *For* $\eta \in \Omega$, $A(P^mR^n(\eta))=U^mR^n(A(\eta))$.

In fact, if $\eta=k\eta_0+l$ then $P^mR^n(\eta)=(m+k)\eta_0+(n+l)$ and $A(P^mR^n(\eta))=U^{m+k}R^{n+l}(\Theta)=U^mR^n(A(\eta))$.

We have further

(4.28) *For any real* ξ, $A'(P^mR^n(\xi))=U^mR^n(A'(\xi))$,
$A''(P^mR^n(\xi))=U^mR^n(A''(\xi))$.

Clearly the set of all numbers $\eta \epsilon \Omega$ satisfying $\eta<\xi$ (or $\eta>\xi$) is transformed by the translation $P^m R^n$ onto the set of all numbers $\eta' \epsilon \Omega$ satisfying $\eta'<P^m R^n(\xi)$ (or $\eta'>P^m R^n(\xi)$). It follows by definition (4.24)

$$\Lambda'(P^m R^n(\xi)) = \bigcup_{\substack{\eta' \epsilon \Omega \\ \eta'<P^m R^n(\xi)}} \Lambda(\eta') = \bigcup_{\substack{\eta \epsilon \Omega \\ \eta<\xi}} \Lambda(P^m R^n(\eta)).$$

By using (4.27) we thus obtain the first equality of (4.28). Similarly we obtain the second equality.

We now prove

$$\widetilde{X} = \sum_{-\infty<\xi<\infty} [\Lambda''(\xi)-\Lambda'(\xi)]. \tag{4.29}$$

This equation means that all *non-empty* intervals $[\Lambda''(\xi)-\Lambda'(\xi)]$ of the set $\widetilde{X}$ form a partition of $\widetilde{X}$ with the set of indices ξ ordered in the natural order of real numbers.

We remark first that from (4.26) we have, for $\xi_1<\xi_2$ $[\Lambda''(\xi_1)-\Lambda'(\xi_1)]<[\Lambda''(\xi_2)-\Lambda'(\xi_2)]$ if both these intervals are non-empty (if one of them is empty the inequality has no meaning). It remains to be proved that every point $\zeta \epsilon \widetilde{X}$ belongs to some interval $[\Lambda''(\xi)-\Lambda'(\xi)]$.

In fact consider the decomposition $\Omega=\Omega'+\Omega''$ where Ω' is the set of all $\eta \epsilon \Omega$ such that $\zeta \notin \Lambda(\eta)$, and Ω'' is the set of all $\eta \epsilon \Omega$ for which $\zeta \epsilon \Lambda(\eta)$. It is clear from (4.23) that Ω' is an initial interval and Ω'' a final interval of Ω. Consequently they form a Dedekind cut in Ω and we have the following three possibilities: 1) there is a last element $\eta' \epsilon \Omega'$, 2) there is a first element $\eta'' \epsilon \Omega''$, 3) there exists a unique real number ξ such that $\Omega'<\xi<\Omega''$.

By applying (4.24) and (4.25) we obtain in each of these cases:

1) $\zeta \epsilon \Lambda''(\eta')-\Lambda(\eta') \subset \Lambda''(\eta')-\Lambda'(\eta')$,
2) $\zeta \epsilon \Lambda(\eta'')-\Lambda'(\eta'') \subset \Lambda''(\eta'')-\Lambda'(\eta'')$,
3) $\zeta \epsilon \Lambda''(\xi)-\Lambda'(\xi)$,

which shows that in every case ζ belongs to some interval of the right side of (4.29) which finishes the proof of this formula.

We now introduce the number $\xi_0=1-\eta_0 \epsilon \Omega$. We have

$$\Lambda(0)=\Theta, \quad \Lambda(\xi_0)=U^{-1}R(\Theta), \quad \Lambda(1)=R(\Theta). \tag{4.30}$$

By using (4.11′) and the decomposition (4.29) we can now write

$$X_0=\Lambda(\xi_0)-\Lambda(0)=[\Lambda''(0)-\Lambda(0)]+ \\ +\sum_{0<\xi<\xi_0}[\Lambda''(\xi)-\Lambda'(\xi)]+[\Lambda(\xi_0)-\Lambda'(\xi_0)], \tag{4.31}$$

$$X_1=\Lambda(1)-\Lambda(\xi_0)=[\Lambda''(\xi_0)-\Lambda(\xi_0)]+ \\ +\sum_{\xi_0<\xi<1}[\Lambda''(\xi)-\Lambda'(\xi)]+[\Lambda(1)-\Lambda'(1)]. \tag{4.31′}$$

From (4.3) we obtain

$$a_0=\sum_{\zeta \epsilon X_0}\gamma(\zeta)=\sum_{\zeta \epsilon \Lambda''(0)-\Lambda(0)}\gamma(\zeta)+\sum_{0<\xi<\xi_0}\sum_{\zeta \epsilon \Lambda''(\xi)-\Lambda'(\xi)}\gamma(\zeta)+\sum_{\zeta \epsilon \Lambda(\xi_0)-\Lambda'(\xi_0)}\gamma(\zeta), \tag{4.32}$$

$$a_1=\sum_{\zeta \epsilon X_1}\gamma(\zeta)=\sum_{\zeta \epsilon \Lambda''(\xi_0)-\Lambda(\xi_0)}\gamma(\zeta)+\sum_{\xi_0<\xi<1}\sum_{\zeta \epsilon \Lambda''(\xi)-\Lambda'(\xi)}\gamma(\zeta)+\sum_{\zeta \epsilon \Lambda(1)-\Lambda'(1)}\gamma(\zeta). \tag{4.32′}$$

We now introduce the following notations:

$$\gamma_0'=\sum_{\zeta \epsilon \Lambda''(\xi_0)-\Lambda(\xi_0)}\gamma(\zeta), \qquad \gamma_1'=\sum_{\zeta \epsilon \Lambda(\xi_0)-\Lambda'(\xi_0)}\gamma(\zeta)$$

$$\gamma'(\xi)=\sum_{\zeta \epsilon \Lambda''(\xi)-\Lambda'(\xi)}\gamma(\zeta) \qquad \text{for} \quad 0<\xi<1.$$

We shall prove the following properties.

(4.34) *For* $\eta \epsilon \Omega$, $0\leqslant\eta<1$, $\displaystyle\sum_{\zeta \epsilon \Lambda''(\eta)-\Lambda(\eta)}\gamma(\zeta)=\gamma_0'$,

(4.34′) *For* $\eta \epsilon \Omega$, $0<\eta\leqslant 1$, $\displaystyle\sum_{\zeta \epsilon \Lambda(\eta)-\Lambda'(\eta)}\gamma(\zeta)=\gamma_1'$,

(4.34″) *For* $\eta \epsilon \Omega$, $0<\eta<1$, $\gamma'(\eta)=\gamma_1'+\gamma_0'$.

We prove first (4.34). For $0\leqslant\eta<1$, $\Lambda''(\eta)-\Lambda(\eta)\subset X$ (see (4.31) and (4.31′)). Let $\eta=k\eta_0+l$. Then $\eta=P^{k+1}R^{l-1}(\xi_0)$. Hence by (4.27) and (4.28) $[\Lambda''(\eta)-\Lambda(\eta)]=U^{k+1}R^{l-1}[\Lambda''(\xi_0)-\Lambda(\xi_0)]$.

By an argument used in Subcase I′, applying (4.10) and property 3⁰ of Theorem 3.1, we obtain

$$\sum_{\zeta \epsilon \Lambda''(\eta)-\Lambda(\eta)}\gamma(\zeta)=\sum_{\zeta' \epsilon \Lambda''(\xi_0)-\Lambda(\xi_0)}\gamma(U^{k+1}R^{l-1}(\zeta'))$$
$$=\sum_{\zeta' \epsilon \Lambda''(\xi_0)-\Lambda(\xi_0)}\gamma(T^{k+1}(\zeta'))=\sum_{\zeta' \epsilon \Lambda''(\xi_0)-\Lambda(\xi_0)}\gamma(\zeta')=\gamma_0'.$$

The proof of (4.34′) is similar. Formula (4.34″) is a consequence of (4.34) and (4.34′) since

$$\gamma'(\eta)=\sum_{\zeta \epsilon \Lambda''(\eta)-\Lambda'(\eta)}\gamma(\zeta)=\sum_{\zeta \epsilon \Lambda(\eta)-\Lambda'(\eta)}\gamma(\zeta)+\sum_{\zeta \epsilon \Lambda''(\eta)-\Lambda(\eta)}\gamma(\zeta).$$

We prove further

(4.35) *For any real numbers* ξ_1 *and* ξ_2 *such that* $0<\xi_1<\xi_2<1$ *and* $\xi_2-\xi_1 \in \Omega$, $\gamma'(\xi_1)=\gamma'(\xi_2)$.

In fact, under the hypothesis of (4.35) $\xi_2-\xi_1=m\eta_0+n$ for some integers m and n and $A''(\xi_i)-A'(\xi_i)\subset X$ for $i=1,2$. Consequently $\xi_2=P^m R^n(\xi_1)$, $A''(\xi_2)-A'(\xi_2)=U^m R^n[A''(\xi_1)-A'(\xi_1)]$ and by the same argument as above we have

$$\gamma'(\xi_2)=\sum_{\zeta\in A''(\xi_2)-A'(\xi_2)}\gamma(\zeta)=\sum_{\zeta'\in A''(\xi_1)-A'(\xi_1)}\gamma(\zeta')=\gamma'(\xi_1).$$

By using (4.33), (4.34), and (4.34′) we obtain from (4.32) and (4.32′):

$$\alpha_0=\gamma'_0+\sum_{0<\xi<\xi_0}\gamma'(\xi)+\gamma'_1,\qquad \alpha_1=\gamma'_0+\sum_{\xi_0<\xi<1}\gamma(\xi)+\gamma'_1,$$

which is formula (1) of Theorem I of the Introduction.

Property 1^0 of this theorem is clearly given by (4.35) and property 2^0 by (4.34′). Thus the theorem is completely proved.

5. Concluding remarks.

Remark 1. *The multiplicity of representations in Theorem I.*

In general, the same couple of solutions α_0 and α_1 of the equation $\alpha_0+\alpha_1=\alpha_1+\alpha_0$ may have several representations of the same type as well as representations of different types. To some extent, an extreme case of such multiplicity of representations can be shown by the example $\alpha_0=\alpha_1=\alpha$ where α is the type of order of the half-open interval $0\leqslant\xi<1$. In fact we have here $\alpha=\alpha m$ with any natural number m which gives an infinite number of representations of type (a) of the Introduction. Further, $\alpha=\alpha\omega+\delta+\alpha\omega^*$ with $\delta=1$, or $\delta=\alpha+1$ which gives two different representations of type (b). Finally, we can write

$$\alpha=\gamma'_0+\sum_{0<\xi<\xi_0}\gamma'(\xi)+\gamma'_1=\gamma'_0+\sum_{\xi_0<\xi<1}\gamma'(\xi)+\gamma'_1$$

for any irrational ξ_0 by choosing

$$\gamma'_0=1\ \text{ or }\ =\alpha+1,\qquad \gamma'_1=0\ \text{ or }\ =\alpha,\qquad \gamma'(\eta)=\gamma'_1+\gamma'_0\ \text{ for }\ \eta\in\Omega,$$
$$\gamma'(\xi)=1\ \text{ for }\ \xi\notin\Omega.$$

This gives infinitely many representations of type (1) of Theorem I with different values of ξ_0; for the same value of ξ_0 it gives four different representations.

Remark 2. *Representability of solutions* α_0,α_1 *in different forms.* As mentioned in the Introduction, A. Tarski proved that if one of the solutions α_0,α_1 is enumerable or of dispersed type, then they may always be represented in one of the forms (a) or (b). This result can be obtained as an immediate corollary of Theorem I and of the following statement which can be proved without great difficulty:

Let α_0 *and* α_1 *be representable in the form* (1) *of Theorem I.*

1^0 *If* ξ_0 *is irrational and* $\gamma'(\xi)=\gamma$ *for all except an at most enumerable set of* ξ*'s, then* $\alpha_0=\alpha_1$.

2^0 *If* ξ_0 *is rational,* $\xi_0=\dfrac{p}{q}$, *with* p *and* q *rel. prime, then there exists a type of order* δ *such that* $\alpha_0=\delta p$, $\alpha_1=\delta(q-p)$.

As mentioned in the Introduction, A. Lindenbaum constructed solutions α_0 and α_1 which cannot be represented in either of the form (a) or (b), and hence are representable only in form (1).

Since Lindenbaum's example was not published it is of interest to indicate briefly how such solutions α_0,α_1 can be constructed. We choose them of form (1) with $\gamma'_0=\gamma'_1=0$ and ξ_0 irrational:

$$\alpha_0=\sum_{0<\xi<\xi_0}\gamma'(\xi),\qquad \alpha_1=\sum_{\xi_0<\xi<1}\gamma'(\xi).$$

The $\gamma'(\xi)$ are chosen in such a way that for $\xi_1-\xi_2\notin\Omega$ no proper interval of a set of type $\gamma'(\xi_1)$ can be similar to an interval of a set of type $\gamma'(\xi_2)$. For instance, we can choose the types of order $\gamma'(\xi)$ for each equivalence class of ξ's mod. Ω among the types of order $2^{\omega_\nu^*}$ (in Cantor's notation, ω_ν^* being the inverse type of the initial ordinal number ω_ν), for $0\leqslant\nu<\omega_\mu$, where $\aleph_\mu=\mathfrak{c}$.

The $\gamma'(\xi)$ being so fixed, consider a set A of type $\alpha_0+\alpha_1$ and the corresponding partitions

$$A=A_0+A_1,\quad A_0=\sum_{0<\xi<\xi_0}\Gamma(\xi),\quad A_1=\sum_{\xi_0<\xi<1}\Gamma(\xi),\ \text{with } \Gamma(\xi) \text{ of type } \gamma'(\xi).$$

We may also write

$$A=\sum_{0<\xi<1}\Gamma(\xi)\quad\text{with}\quad \Gamma(\xi_0)=0.$$

We first prove that if K' and K'' are two intervals of A and if a similarity S transforms K' onto K'', $S(K')=K''$, then either $K'\subset\Gamma(\xi')$ for some ξ' and $K''\subset\Gamma(\xi')$ with $\xi''-\xi'\in\Omega$, or $K'=K_0'+\sum_{\theta'<\xi<\tau'}\Gamma(\xi)+K_1'$ with $K_0'=$ final interval of $\Gamma(\theta')$, $K_1'=$ initial interval of $\Gamma(\tau')$, $0<\theta'<\tau'<1$, and similarly $K''=K_0''+\sum_{\theta''<\xi<\tau''}\Gamma(\xi)+K_1''$ and the similarity S must satisfy $S(K_0')=K_0''$, $S(K_1')=K_1''$, $S(\Gamma(\xi))=\Gamma(\sigma(\xi))$ for $\theta'<\xi<\tau$, where σ is a similarity of the closed interval $[\theta',\tau']$ onto $[\theta'',\tau'']$. Further, $\sigma(\xi)-\xi\in\Omega$ for all ξ in $\theta'<\xi<\tau'$. Since $\sigma(\xi)$ must be a continuous function in the open interval (θ',τ') the range of $\sigma(\xi)-\xi$ is connected, and since it is $\subset\Omega$ it must be reduced to one point $\eta\in\Omega$. Hence $\sigma(\xi)=\xi+\eta$ and σ is a translation by a number $\eta\in\Omega$.

If α_0 and α_1 were representable in form (a) we would have $\alpha_0=\delta m$, $\alpha_1=\delta n$ with m and n natural numbers. There would then exist a decomposition $A=\sum_{i=1}^{m+n}K^{(i)}$ in disjoint similar intervals. This would mean that there exists a partition of $[0,1]$ into $m+n$ intervals by numbers $0=\varkappa^{(0)}<\varkappa^{(1)}<\ldots<\varkappa^{(m+n)}=1$ so that

$$K^{(i)}=K_0^{(i)}+\sum_{\varkappa^{(i-1)}<\xi<\varkappa^{(i)}}\Gamma(\xi)+K_1^{(i)}.$$

Furthermore there would exist numbers $\eta_i\in\Omega$ such that $[\varkappa^{(i-1)},\varkappa^{(i)}]$ is translated by η_i onto $[\varkappa^{(i)},\varkappa^{(i+1)}]$. This gives $\eta_i==\varkappa^{(i)}-\varkappa^{(i-1)}=\varkappa^{(i+1)}-\varkappa^{(i)}$, so that $\eta_1=\eta_2=\ldots=\eta_{m+n-1}=\eta$ and $(m+n)\eta=1$. Since $\eta=k\eta_0+l$, we get $(m+n)k=0$ and $(m+n)l=1$, which is impossible since $(m+n)\geqslant 2$.

If α_0 and α_1 were representable in form (b), say $\alpha_1=\alpha_0\omega+\delta+\alpha_0\omega^*$, we would have a sequence of consecutive disjoint similar intervals $K^{(i)}$, $i=0,1,2,\ldots$ with $K^{(0)}=A_0$. This would lead to an increasing infinite sequence $0=\varkappa^{(0)}<\varkappa^{(1)}<\ldots<1$ with $\varkappa^{(i)}-\varkappa^{(i-1)}=\varkappa^{(i+1)}-\varkappa^{(i)}=\eta\in\Omega$ which is impossible.

The example could be constructed in a more general way without the simplifying assumption that $\gamma_0'=\gamma_1'=0$. We could have accepted non-zero types of order γ_0' and γ_1' (or one of them non-zero) restricted by the condition that a set of type γ_i', $i=0,1$, does not contain a proper interval similar to an interval of a set of type γ_{1-i}' nor with an interval of a set of type $\gamma'(\xi)$ for $\xi\notin\Omega$.

The solutions α_0,α_1 which we have constructed above possess the interesting property that not only are they representable only in the form (1) but their representation in this form is unique. For any of the other forms of representation we can construct solutions which are representable only in this form and whose representation in this form is unique. Here we should consider as proper representations of form (1) only those with ξ_0 irrational (since representation (1) with ξ_0 rational is equivalent to a representation of type (a)). To construct such solutions for type (a) we choose a non-enumerable type of order δ such that a set of this type has no two similar proper intervals (unless they are equal)[3]. Then the solutions $\alpha_0=\delta m$ and $\alpha_1=\delta n$ for any natural m and n do not admit of any other representation of form (a) nor of any representation of the two other forms. For the form (b) we take two types of order α_0 and δ, both $\neq 0$ and $\neq 1$ such that the corresponding sets have no similar proper intervals. Then α_0 and $\alpha_1=\alpha_0\omega+\delta+\alpha_0\omega^*$ admit of no other representation.

We shall make one further remark concerning the solutions of form (1). It is clear that if this form is a proper representation (with ξ_0 irrational) and if the type α_0 has a first element, then $\gamma_0'\neq 0$ and a set of type γ_0' has a first element. Similarly, if α_0 has a last element then $\gamma_1'\neq 0$ and has a last element. The converse of this remark is not true; α_0 may be without a first and/or last element without having a representation of type (1) with $\gamma_0'=0$ and/or $\gamma_1'=0$. For example, we can consider the above constructed solutions, taking $\gamma_0'\neq 0\neq\gamma_1'$ with γ_0' without a first element and γ_1' without last element. Since the representation of these solutions is unique, the converse is proven not true.

Remark 3. *Partially ordered sets.* The considerations of this paper may be extended to partially ordered (p.-ordered) sets. The notion of interval is defined here in the same way as for ordered sets. As a partition of a p.-ordered set A, we can in general consider any decomposition of A in mutually disjoint intervals. However, such a partition $\{A_i\}_I$ is not of much use for our purposes if it does not induce a partial order relation in I such that we may write $A=\sum_{i\in I}A_i$ with the usual definition of partial order in the union of p.-ordered sets under a p.-ordered set of indices. We have

[3]) Such types can be constructed.

this property if and only if for any two different intervals A_ι and $A_{\iota'}$ of the partition the relation $x<y$ for any $x \in A_\iota$ and any $y \in A_{\iota'}$ implies that $A_\iota < A_{\iota'}$. In this case we shall call $\{A_\iota\}_I$ a p.-ordered partition. If the partial order induced in I is an order relation we shall call it an ordered partition. The type of a p.-ordered partition is the partial order type induced in the set of indices. Hence a partition of real type is always ordered, but an enumerable p.-ordered partition may not be an ordered partition.

With these remarks in mind it is easy to extend the notions and theorems of the present paper to partially ordered sets. Since we consider mostly partitions of real type this extension will deal essentially only with ordered partitions of p.-ordered sets.

We add finally that following a communication by A. Tarski, the considerations and results of the present paper con be extended to types of general relations. The main additional tool used in this extension is a theorem proved by A. Tarski and Bjarni Jónson (as yet not published) that every relations-type admits of a unique ***ordered*** partition into indecomposable relations-types (a relations-type ξ is called indecomposable if $\xi=\alpha+\beta$ implies $\alpha=0$ or $\beta=0$).

Sur la représentation des ensembles ordonnés.

Par

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M. W. Sierpiński a prouvé [1]) le théorème suivant: ν étant un nombre ordinal quelconque, tout ensemble ordonné de puissance $\aleph_\nu$ est semblable à un ensemble de suites transfinies de type ω_ν, formées de nombres 0 et 1 et ordonnées d'après le principe de premières différences. Pour les continus ordonnés, M. J. Novák a établi [2]) le résultat suivant: Soit C un continu ordonné dont la séparabilité soit égale à $\mathfrak{m}(C)$. Il existe un nombre ordinal ϑ de puissance au plus égale à $\mathfrak{m}(C)$, tel que C est semblable à un ensemble de suites transfinies de type $\leqslant\vartheta$ formées de nombres 0 et 1 et ordonnées d'après le principe de premières différences. Il a posé le problème, s'il est possible de poser ϑ égal au nombre ordinal initial de puissance $\mathfrak{m}(C)$. Le théorème 1 du travail présent donne une réponse affirmative.

M. J. Novák a prouvé [3]) le théorème suivant: Soit P un continu ordonné; $\mathfrak{P}$ un système disjoint d'intervalles fermés et de sous-ensembles de P composés d'un seul point. Supposons que $\mathfrak{P}$ possède au moins deux éléments et qu'on ait $\cup\mathfrak{P}=P$. Dans ces hypothèses, $\mathfrak{P}$ est un continu ordonné.

Dans le théorème 2, j'ai établi ce résultat inverse: Tout continu ordonné est semblable à un système $\overline{M}$ d'intervalles fermés et de sous-ensembles composés d'un seul point d'un certain continu ordonné V_{ω_ν}, ce système vérifiant la relation $\cup\overline{M}=V_{\omega_\nu}$.

Enfin, M. Novák a défini [2]) un système $\mathfrak{P}$ d'intervalles fermés du continu ordonné C jouissant de certaines propriétés qu'il

[1]) W. Sierpiński, *Sur une propriété des ensembles ordonnés*, Fundamenta Mathematicae **34** (1949), p. 56.

[2]) J. Novák, *On Partition of an Ordered Continuum*, Fundamenta Mathematicae **39** (1952).

[3]) J. Novák, *On some Ordered Continua of Power $2^{\aleph_0}$ Containing a Dense Subset of Power $\aleph_1$*, Czechoslovak Mathematical Journal **76** (1951), 63-79.