Lemme 1. *Pour tout* $I = I_{i_0 i_1 \dots i_\lambda \dots (\lambda<\alpha)} \in \mathfrak{P}$ *nous avons* $\alpha \leqslant \delta(I)$.

Puisque $\alpha \leqslant \delta(I) < \omega_\nu$, il résulte de ce lemme que l'indice α de l'intervalle arbitraire $I_{i_0 i_1 \dots i_\lambda \dots (\lambda<\alpha)}$ du système $\mathfrak{P}$ présente la propriété $\alpha < \omega_\nu$.

Lemme 2. *Pour que deux intervalles différents* $I_{i_0 i_1 \dots i_\lambda \dots (\lambda<\alpha)}$, $I_{j_0 j_1 \dots j_\lambda \dots (\lambda<\beta)}$ *du système* $\mathfrak{P}$ *présentent la propriété* $I_{i_0 i_1 \dots i_\lambda \dots (\lambda<\alpha)} \subset I_{j_0 j_1 \dots j_\lambda \dots (\lambda<\beta)}$, *il faut et il suffit que* $\alpha > \beta$, $i_\lambda = j_\lambda$ *pour tout* $\lambda < \beta$.

Lemme 3. *Pour que deux intervalles* $I_{i_0 i_1 \dots i_\lambda \dots (\lambda<\alpha)}$, $I_{j_0 j_1 \dots j_\lambda \dots (\lambda<\beta)}$, *du système* $\mathfrak{P}$ *aient un seul point commun, il faut et il suffit qu'il existe un nombre ordinal* $\delta < \min(\alpha, \beta)$ *tel que* $i_\lambda = j_\lambda$ *pour tout* $\lambda < \delta$, $i_\delta + j_\delta = 1$, $i_\lambda \neq i_\delta$ *pour tout* λ *vérifiant la relation* $\delta < \lambda < \alpha$, $j_\lambda \neq j_\delta$ *pour tout* λ *vérifiant la relation* $\delta < \lambda < \beta$.

D'après les lemmes 2 et 3, dans tout couple d'intervalles du système $\mathfrak{P}$ ceux-ci sont disjoint ou n'ont qu'un seul point commun ou encore l'un d'eux est sous-ensemble de l'autre. Par conséquent, le système $\mathfrak{P}$ vérifie l'axiome 1. Les axiomes 2-4 sont vérifiés d'après la définition du système $\mathfrak{P}$; alors, $\mathfrak{P}$ est une partition. L'ordre de l'intervalle $I = I_{i_0 i_1 \dots i_\lambda \dots (\lambda<\alpha)}$ est égal à son indice α, car tout intervalle du système $\mathfrak{P}$ contenant l'intervalle I et différent de I peut être écrit sous la forme $I_{j_0 j_1 \dots j_\lambda \dots (\lambda<\beta)}$, où $\beta < \alpha$ et $i_\lambda = j_\lambda$ pour tout $\lambda < \beta$ d'après le lemme 2. Alors, l'ordre de la partition $\alpha(\mathfrak{P})$ est au plus égal à ω_ν d'après le lemme 1.

Nous avons construit une partition du continu M^* d'ordre $\leqslant \omega_\nu$; la similitude de M et M^* entraîne facilement l'existence de la partition du continu M d'ordre $\leqslant \omega_\nu$.

On the Concepts of Completeness and Interpretation of Formal Systems.

By

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Introduction.

1. When in the history of mathematics new methods of proof or new concepts were introduced, *e. g.* the duality principle in geometry, complex or continuous variables, doubts often arose about these methods; that is, doubts whether the methods had the properties which were expected of them. To resolve these objections required two pieces of work: stating what conditions the new methods had to satisfy, and deciding whether they did. These two problems have often been called the problem of giving foundations for the new branch of mathematics.

Now, the properties which one needs in a mathematical system, naturally depend on the applications for which the system is intended, and often one system has many interesting applications, either in mathematics or outside it as a scientific theory. It is therefore natural that the problem of foundations should have been formulated in quite different ways.

2. Hilbert decided that the problem should be formulated as the *consistency problem*, that is, the new system should be formalized so that the idea of „provability in the system" is made precise, and it should be established (by suitable methods) that not all formulae of the system are provable. This decision is readily motivated by the use of a formal system as a scientific theory (see *e. g.* [1], § 6); Hilbert's special interest in this formulation is perhaps due to the fact that his first logical investigations concerned Euclidean geometry, which is traditionally applied as a theory of a certain branch of physics.

The intuitionists and finitists, who have written mostly on analysis and set theory, demand a particular interpretation of logical constants and quantifiers; roughly speaking, it should be very closely related to the one customary for finite decidable sets.

Frege, Russell and others developed the theory of classes to provide foundations for (pretty diverse) logical systems: and by this they meant that a model could be constructed for the system by terms of the theory of classes (for the notion of "models" and developments in this spirit see [2]).

3. We shall not list here applications of formal systems where these various formulations are specially appropriate. It is enough for us that there are mathematicians who operate more freely with a given system once foundations of their favourite sort have been provided for the system.

4. *Our main problem is to discuss a schema for interpreting a given formal system which covers the formulations just described as special cases* [1]). Quite roughly: the simple idea at the back of the schema is that we wish to study a given formal system $\mathfrak{F}$ by getting "information" about a familiar system F from proofs in $\mathfrak{F}$. The information consists in this: with a formula $\mathfrak{A}$ of $\mathfrak{F}$ is associated a class of formulae $(A_1, A_2, \ldots)$ of F so that if $\mathfrak{A}$ is provable in $\mathfrak{F}$ all (or one of the) formulae $A_1, A_2, \ldots$ can be proved in F, and if $\mathfrak{A}$ is disprovable in $\mathfrak{F}$ this is not so. No condition is imposed if $\mathfrak{A}$ is neither provable nor disprovable in $\mathfrak{F}$.

5. A special case is a complete interpretation when $\mathfrak{A}$ can be proved in $\mathfrak{F}$ if and only if all (or one of the) formulae $A_1, A_2, \ldots$ can be proved in F. This case is relevant to discussing the axiomatic characterization of concepts and completeness (*e. g.* in the sense of [3]). This is a generalization of the idea of completeness used in Gödel's proof [4] of the completeness of the predicate calculus, where however the concept of completeness was not defined for general formal systems (or, more precisely, for interpretations of general formal systems since in this sense completeness is not a property of the system, but of its intended interpretation).

[1]) Our schema is pretty general, and, by suitable specialization, allows one to make precise many demands concerning interpretation. We do not take the question seriously whether it covers all the things which might in some circumstances be described as interpretations.

Remark. Both the concept of models and the general concept of interpretation involve the relation between two systems. Consistency, decidability, and completeness in the sense of Hilbert-Post are (internal) properties of the system considered. This fact gives a clue to the applications of formal systems in which the latter concepts may be expected to be appropriate criteria: in applications as a scientific theory the test that a system is *right* is that it gives the right classification of the relevant empirical propositions, and has nothing to do with its relation to other formal systems; or, if a system is complete in the sense that every formula is either provable or disprovable, and is extended by new rules of inference, the same formulae (of the original system) will be provable in the original and extended system, provided the latter is consistent; again the new rules of inference are justified by an internal property of the (extended) system.

Interpretation.

6. By a *formal system* we mean a set of rules stating what symbols are to be used, what sequences of symbols are to be called formulae, and what sequences of formulae are to be called proofs. Throughout we suppose that numbers have been given to sequences of symbols and that we have a primitive recursive formula $\mathfrak{F}(n)$ which holds if and only if n is the number of a formula, thus the formulae of the system considered are recursively enumerable. If also proofs of a system are recursively enumerable we call the system *formalized.*

Let $\mathfrak{F}$ be a consistent formal system containing a symbol for negation; its formulae are denoted by Gothic capitals.

Let F be a formal system, and denote its formulae by Roman capitals; we suppose that from its formulae have been selected a class of true formulae and a class of false formulae, which are exclusive, but do not necessarily cover all formulae of F.

Remark. We use the words "true", "false", because

(i) what we need is a classification of formulae of F, and we shall sometimes understand by "true" ("false"): provable in F (not provable in F), or: consistent over F (inconsistent over F), or: when F contains the symbol of negation, provable in F (disprovable in F where a formula A is called *disprovable* if $\neg A$ is provable);

(ii) we often use free variable F, where the analogue to "$A(b)$ is disprovable" with free variable b, is a term $\mathfrak{n}$ so that the formula $\neg A(\mathfrak{n})$ is provable; the analogue is not that "the formula $\neg A(b)$ is provable", nor that "the formula $A(b)$ is not provable" since $A(b)$ may simply be undecided (in the system F).

Definition. Let $g(n,a)$ be a computable function of the variables n,a. If $g(n,a)$ is the number of a formula A_n of F in some numbering of expressions in F whenever a is the number of a formula $\mathfrak{A}$ of $\mathfrak{F}$ in some numbering of expressions in $\mathfrak{F}$; then $g(n,a)$ is called a *disjunctive interpretation* of $\mathfrak{F}$ by F, provided

(i) when $\mathfrak{A}$ is provable in $\mathfrak{F}$, there is an A_n which is true,

(ii) when $\mathfrak{A}$ is disprovable in $\mathfrak{F}$, each A_n is false;

and $g(n,a)$ is called a *conjunctive interpretation* of $\mathfrak{F}$ by F, provided

(i) when $\mathfrak{A}$ is provable in $\mathfrak{F}$, each A_n is true,

(ii) when $\mathfrak{A}$ is disprovable in $\mathfrak{F}$, there is a false A_n.

Remark 1. When an interpretation is actually set up, $g(n,a)$ will generally be defined by some simple recursive scheme. We do not restrict here the type of scheme so that a theorem of the form: such and such a kind of interpretation is impossible, should be strong.

Remark 2. We shall usually strengthen conditions (i) and (ii), and require a computable function $D(a,p)$ so that in the disjunctive interpretation $A_{D(a,p)}$ is true when p is the number of a proof of $\mathfrak{A}$ in $\mathfrak{F}$ (and $C(a,p)$ so that in the conjunctive interpretation $A_{C(a,p)}$ is false when p is the number of a proof of $\neg\mathfrak{A}$): this may be described by saying that from a proof of $\mathfrak{A}$ in $\mathfrak{F}$ we find a true A_n.

Remark 3. It will sometimes be convenient to say that "proofs of $\mathfrak{A}$ in $\mathfrak{F}$ give information about the class of formulae $A_1, A_2, \ldots$ of F", the class being defined by the condition $(En)[m = g(n,a)]$. In language fashionable in the twenties a disjunctive interpretation would be described by saying "$\mathfrak{A}$ is a *Partialurteil* concerning the propositions $A_1, A_2, \ldots$" and a conjunctive one by "A_n are particular cases of $\mathfrak{A}$".

7. There are some (trivial) general results about interpretations:

Theorem 1. *If $\mathfrak{F}$ and F contain the predicate calculus, we can get a conjunctive interpretation of $\mathfrak{F}$ by F from a disjunctive one, and conversely.*

Without loss of generality we bind the free variables in formulae of $\mathfrak{F}$ and F by universal quantifiers.

Let the formulae $B_1, B_2, \ldots$ be the formulae associated with $\neg\mathfrak{A}$ in the given disjunctive interpretation. Then associate with $\mathfrak{A}$ the formulae: $\neg B_1$ *and* $\neg B_2$ *and* ... This is then a conjunctive interpretation.

Remark. The result does not always hold when F is a free variable formal system, *e. g.* it is shown in appendix II of [1] that arithmetic with quantifiers but even without induction has no conjunctive interpretation by decidable free variable formulae if any verifiable formula of arithmetic may be used as an axiom — although it has (several) disjunctive interpretations.

The converse is similarly trivial.

Theorem 2[2]. *If $\mathfrak{F}$ is formalized (and consistent!) it has interpretations by numerical formulae without variables.*

Let $\mathfrak{n}(a)$ be the number of the formula $\neg\mathfrak{A}$, and let $\mathrm{Prov}(m,n)$ be the recursive relation which holds if and only if m is the number of a proof in $\mathfrak{F}$ of the formula with number n.

Then the sequence of formulae

$$\mathrm{Prov}(1,a) \quad \text{or} \quad \mathrm{Prov}(2,a) \quad \text{or} \quad \ldots,$$

i. e. "the formula $\mathfrak{A}$ is provable in $\mathfrak{F}$" is a disjunctive interpretation of $\mathfrak{A}$, and hence, by Theorem 1, the sequence

$$\neg\mathrm{Prov}[1,\mathfrak{n}(a)], \quad \text{and} \quad \neg\mathrm{Prov}[2,\mathfrak{n}(a)], \quad \text{and} \quad \ldots,$$

i. e. "the formula $\mathfrak{A}$ is consistent over $\mathfrak{F}$" is a conjunctive interpretation.

Remark. The latter interpretation may also be written as an interpretation of the formula $\mathfrak{A}$ by the single free variable formula $\neg\mathrm{Prov}[n,\mathfrak{n}(a)]$ (instead of a sequence) with the free variable n.

8. Theorem 2 shows that a consistency proof of a formalized system leads to an interpretation in our sense, by a very familiar system, namely arithmetic without variables. By an equally slight argument it is seen that *models* are also interpretations.

Let $\mathfrak{F}$ be an axiom system of the predicate calculus, to which, possibly, other inference rules are added. If the predicate symbols

[2]). Already proved in [1], § 16.

of $\mathfrak{F}$ are $B_1, \ldots, B_k$ then a model of $\mathfrak{F}$ in F is a set of predicates $B_1^*, \ldots, B_k^*$ of F, B_i^* having the same number of free places as B_i, and a predicate $D^*(a)$ of F (the domain) so that

When B^* is substituted for B in a formula $\mathfrak{A}$ of $\mathfrak{F}$ to yield the formula $\mathfrak{A}^*$ of F, and $\mathfrak{A}$ is provable in $\mathfrak{F}$ then $\mathfrak{A}^*$ can be proved in F provided its variables are required to satisfy D^* (are restricted to the domain D^*).

Remark. If $\mathfrak{A}$ is the formula

$$(x)(Ey)(z)R(x,y,z),$$

R quantifier free, $\mathfrak{A}^*$ is $(x)(Ey)(z)R^*(x,y,z)$, and the restriction of the variables to D^* is expressed by

$$(x)(Ey)(z)[D^*(x) \,\&\, D^*(z) \cdot\!\rightarrow\!\cdot D^*(y) \,\&\, R^*(x,y,z)];$$

we denote this formula by A.

It is clear that if A of F is associated with $\mathfrak{A}$ of $\mathfrak{F}$ we have an interpretation which is both disjunctive and conjunctive because a single formula of F is associated with each formula of $\mathfrak{F}$.

Remark 1. When $\mathfrak{F}$ is an axiom system of the predicate calculus we get a model for the whole system provided we have a model (in the sense above) for the axioms of $\mathfrak{F}$. This is not generally so if $\mathfrak{F}$ contains other rules of inference, such as the induction schema.

Remark 2. Models in which a predicate symbol of $\mathfrak{F}$ is replaced by a single predicate of F, are used when one looks on the predicate symbols of $\mathfrak{F}$ as being (implicitly) defined by its axioms. If one thinks of them as variables, a conjunctive interpretation is used of all models of $\mathfrak{F}$ in F: predicates $B_{i1}^*, \ldots, B_{ik}^*$, $i=1,2,\ldots$, with the right number of free places are ordered, and one associates with $\mathfrak{A}$ the conjunction $A_1, A_2, \ldots$ For the pure predicate calculus an interesting interpretation is got when F is the system consisting of all ω-consistent extensions of Z_μ.

Remark 3. By a *finitist* (or *free variable*) *model* we mean a model in a free variable formal system F, whose predicates are decidable; where $\mathfrak{A}^*$ has a computable *Erfüllung* in F (see [1], § 5).

Remark 4. Recently, somewhat more general definitions of models have been given by Kemeny [2] and Henkin [3], which are also interpretations in our sense provided, *e. g.* in [3], p. 83, 84, the "models", "frames", "assignments" are defined in a formal system.

We recalled above several simple facts about models because they constitute the best known method which goes by the name of "interpreting" a given system $\mathfrak{F}$, and because it has been thought that they provide a satisfactory general schema for making precise the vague ideas about interpretation. We are using a broader concept of interpretation (where the structures of the formulae $\mathfrak{A}$ and their associates A are not necessarily so similar as with models) briefly for these reasons:

(1) Models are suitable when the system $\mathfrak{F}$ has logical categories of "individuals", "predicates", etc., which is not necessary for an application of $\mathfrak{F}$ (say, as a scientific theory).

(2) Even if $\mathfrak{F}$ has these categories, there is no general reason for associating like categories in $\mathfrak{F}$ and F; for instance, in the version of *quantum theory* by Schrödinger, observables, which are presumably called "individuals" in the empirical propositions of the subject, are associated with differential equations which are not called individuals in analysis.

(3) Many systems $\mathfrak{F}$ in use have models only in rather sophisticated systems F and have no finitist models, though they have a finitist interpretation in our sense.

(4) Lastly, models are only suitable if one wishes to interpret predicate symbols, and not if one wishes to interpret logical connectives or quantifiers [3]).

Finitist Interpretations.

If F is a free variable formal system whose predicates are decidable we call an *interpretation* of $\mathfrak{F}$ by F *finitist* (see [1], where finitist interpretations of number theory are discussed at length). By Theorem 2 above any consistent formalized $\mathfrak{F}$ has a finitist interpretation so that the problem of finitist interpretations is interesting only for formal systems $\mathfrak{F}$ whose proofs cannot be re-

[3]) One very important use of models is in independence proofs. It should be observed that other interpretations in our sense also lead to independence proofs, usually by the application of a diagonal argument like Gödel's classical case.

cursively enumerated, *e. g.* ω-consistent extensions of arithmetic, [1], § 18.

Some natural questions concerning these interpretations were not decided in [1]: it seems worth while to restate here the main results of interpretations of number theory which were found in [1], and then fill some gaps.

9. If $\mathfrak{F}$ is the system of arithmetic consisting of a set of verifiable free variable formulae of number theory as axioms and the predicate calculus as rules of inference, from Herbrand's theorem we get in [1], § 21, an interpretation by the system F_0 consisting of the same axioms and the elementary calculus with free variables as rules of inference.

10. For the same $\mathfrak{F}$ we get a disjunctive interpretation by a system F_1, which contains free function variables, in [1], § 24; with a formula $(x)(Ey)(z)R(x,y,z)$, R quantifier free, of $\mathfrak{F}$, are associated formulae $R[a,\mathfrak{y}_1,f(\mathfrak{y}_1)], R[a,\mathfrak{y}_2,f(\mathfrak{y}_2)], \ldots$ where $\mathfrak{y}$ are terms of the predicate calculus made up of the variable a, function symbols of $\mathfrak{F}$, and the function variable f (= functionals of the predicate calculus): it follows from the first ε-theorem in [5] that if $(x)(Ey)(z)R(x,y,z)$ has been proved in $\mathfrak{F}$ we can find from the proof a term $\mathfrak{y}$ so that $R[a,\mathfrak{y},f(\mathfrak{y})]$ can be proved in F_1; and it is shown in § 24 of [1] that to any functional $\mathfrak{y}$ of the predicate calculus we can find a number a_0 and a function f_0 so that $R[a,\mathfrak{y}_0,f_0(\mathfrak{y}_0)]$ is false provided $(x)(Ey)(z)R(x,y,z)$ has been disproved in $\mathfrak{F}$. This shows that the sequence $R[a,\mathfrak{y}_2,f(\mathfrak{y}_1)], R[a,\mathfrak{y}_2,f(\mathfrak{y}_2)], \ldots$ constitutes a disjunctive interpretation of the formula $(x)(Ey)(z)R(x,y,z)$ of $\mathfrak{F}$ by the system F_1. This interpretation can be expressed by: there is a counter example of the predicate calculus to any *Erfüllung* of $\neg(x)(Ey)(z)R(x,y,z)$. (The generalization to more complicated formulae of $\mathfrak{F}$ is clear.)

11. For full number theory with induction, Z_μ of [5], only one interpretation is given, in [1], § 39, which is the analogue to the interpretation described above in § 10. Now the functionals used are ordinal recursive ones of finite order instead of the simpler functionals of the predicate calculus.

Remark. All these interpretations remain correct when an arbitrary set of verifiable free variable formulae of $\mathfrak{F}$, not necessarily provable in $\mathfrak{F}$, are added to $\mathfrak{F}$ as axioms (ω-consistent extensions) and if we mean by "true" in F: verifiable. The method of getting these interpretations is easily adapted to the extensions (c) of [1], § 18, where any verifiable free variable formula $A[a,f_0(a)]$ with a computable term $f_0(a)$ may be added as an axiom to $\mathfrak{F}$; but there can be no one interpretation for all such extensions satisfying the condition that $g(n,a)$ of § 6 above is computable because computable functions cannot be enumerated by a computable function.

12. *We wish to discuss to what extent the use of function variables is necessary.*

The fact that we used functionals for the interpretation of full number theory, § 11, seemed rather interesting, but is not. Consider again the formula $(x)(Ey)(z)R(x,y,z)$, and observe that one can enumerate functions of one variable which are zero except for a finite number of arguments by means of a primitive recursive function $\eta(x;b)$. If η is substituted for the function variable $f(b)$ in $\mathfrak{y}$, of § 10, the latter reduces to a function $\overline{\mathfrak{y}}(x;a)$. Since the value of an ordinal recursive functional with argument f depends only on the values of f for a finite number of arguments of f, $R[a,\mathfrak{y},f(\mathfrak{y})]$ will be verifiable provided $R\{a,\overline{\mathfrak{y}}(x;a),\eta(x;\overline{\mathfrak{y}}(x;a)]\}$ is verifiable (with the two free variables x and a), and, of course, conversely. Therefore we get an interpretation of $(x)(Ey)(z)R(x,y,z)$ by the sequence:

$$R\{a,\overline{\mathfrak{y}}_1(x;a),\eta[x;\overline{\mathfrak{y}}_1(x;a)]\}$$

or

$$R\{a,\overline{\mathfrak{y}}_2(x;a),\eta[x;\overline{\mathfrak{y}}_2(x;a)]\},\ldots,$$

which contains no function variables (cf. §§ 31-33 of [1]).

13. An interpretation without function variables seemed desirable for this (technical) reason: in [1] it was easy to check condition (ii) for disjunctive interpretations when F was without function variables, as in § 9 above: from any A_n (written with its variables: $A_n(a_1,\ldots,a_k)$), one could prove $\mathfrak{A}$ by the predicate calculus, hence also the formula

$$\neg\mathfrak{A}\to(Ex_1)\ldots(Ex_k)\neg A_n(x_1\ldots x_k),$$

so that from a proof of $\neg\mathfrak{A}$ we get a proof of

$$(Ex_1)\ldots(Ex_k)\neg A_n(x_1\ldots x_k),$$

whence, by Hilbert's first ε-theorem in [5], we get terms $\mathfrak{n}_1,\dots,\mathfrak{n}_k$ so that $A_n(\mathfrak{n}_1,\dots,\mathfrak{n}_k)$ is false. On the other hand, checking this condition for the interpretations of § 10 and particularly of § 11 was a tough piece of work; for simplicity we describe the proof for three variables only: we replaced $R[a,\mathfrak{y},f(\mathfrak{y})]$ with an ordinal recursive $\mathfrak{y}$ by

$$R\{a,\overline{\mathfrak{y}}(x;\,a),\ \eta[x;\,\overline{\mathfrak{y}}(x;\,a)]\},$$

and proved from the latter in Z_μ the formula

$$(x)(Ey)(z)R(x,y,z)$$

and hence the implication

$$\neg(x)(Ey)(z)R(x,y,z)\rightarrow(Ex)(Ea)\neg R\{a,\overline{\mathfrak{y}}(x;\,a),\ \eta[x;\,\overline{\mathfrak{y}}(x;a)]\};$$

(the possibility of proving this implication is established effectively in §§ 31-33 of [1]), thus from a proof of

$$\neg(x)(Ey)(z)R(x,y,z)$$

in Z_μ we get a proof, still in Z_μ, of

(13.1) $$(Ex)(Ea)\neg R\{a,\overline{\mathfrak{y}}(x;\,a),\ \eta[x;\,\overline{\mathfrak{y}}(x;a)]\}.$$

Now we applied Hilbert's substitution method, as developed by Ackermann in [6], to the proof of (13.1), and thereby we found numbers $\mathfrak{x}$ and $\mathfrak{a}$ so that

$$R\{\mathfrak{a},\overline{\mathfrak{y}}(\mathfrak{x};\,\mathfrak{a}),\ \eta[\mathfrak{x};\,\overline{\mathfrak{y}}(\mathfrak{x};\,\mathfrak{a})]\}$$

is false: *i. e.* $\mathfrak{a}$ and the function $\eta[\mathfrak{x};\,b]$ with the variable b constitute a counter example to $R[a,\mathfrak{y},f(\mathfrak{y})]$.

Remark 1. The method of proof is described here rather more simply than in [1].

Remark 2. The argument described above proves rather more: we can find a counter example to $R[a,\mathfrak{y},f(\mathfrak{y})]$, even if $\neg(x)(Ey)(z)R(x,y,z)$ is *not provable in* Z_μ, but if one of the formulae associated with it in our interpretation is verifiable, say $\neg R[\mathfrak{x}_1,g(\mathfrak{x}_1),\mathfrak{z}]$ where $g(b)$ is a free function variable, $\mathfrak{x}_1$ and $\mathfrak{z}$ are functionals.

We take for $g(b)$ the function $\eta(x;b)$ and prove the implication

(13.2) $$(x)\neg R\{\overline{\mathfrak{x}}_1(x),\eta[x;\,\overline{\mathfrak{x}}_1(x)],\overline{\mathfrak{z}}(x)\}\rightarrow$$
$$\rightarrow(Ex)(Ea)\neg R\{a,\overline{\mathfrak{y}}(x;\,a),\eta[x;\,\overline{\mathfrak{y}}(x;\,a)]\}$$

(i) in the predicate calculus if the functionals $\mathfrak{x}_1$ and $\mathfrak{z}$ belong to the predicate calculus,

(ii) in Z_μ if they are ordinal recursive functionals; then we apply the relevant version of the substitution method to the proof of (13.2), and if the premis is verifiable we find numbers $\mathfrak{x}$ and $\mathfrak{a}$, as required.

Remark 3. It is rather misleading to contrast the proofs of conditions (ii) in the interpretation of § 9 and § 10 without recalling that the proof of condition (i) given in [5] is much harder in the former than in the latter.

The substitution method is much more complicated for full number theory than for the predicate calculus. It therefore seems of interest to see if there is a disjunctive interpretation of full number theory such that the formula $\mathfrak{A}$ can be proved from any one of its associates A_n by the predicate calculus only. (The interest derives particularly from remark 2 above.)

Remark 4. Another consequence of interest would follow from such an interpretation: we could normalize proofs of $\mathfrak{A}$ into first a proof of an A_n, so to speak the proper number theoretical part of the proof, and then quantifiers are introduced by the operations of the predicate calculus of first order. We shall see in Theorem 3 that this is false for proofs of $\mathfrak{A}$ in Z_μ. However such a splitting up of proofs is possible if we step from A_n to $\mathfrak{A}$ by the predicate calculus of second order. For the proof of this fact consider again a formula $\mathfrak{A}$ of the form $(x)(Ey)(z)R(x,y,z)$ where A_n is $R[a,\mathfrak{y}_n,f(\mathfrak{y}_n)]$; we use the predicate calculus of second order to which the symbol $\mathfrak{y}_n$, but no axioms for it have been added. From A_n we prove (already in the predicate calculus of first order) $(Ey)R[a,y,f(y)]$, and hence, by the predicate calculus of second order,

(13.3) $$(f)(Ey)R[a,y,f(y)].$$

By the predicate calculus of second order (*e. g.* [5], p. 488),

$$(y)(Ez)\mathfrak{A}(y,z)\rightarrow(Ef)(y)\mathfrak{A}[y,f(y)]$$

so that, setting $\neg R(a,b,c)$ for $\mathfrak{A}(b,c)$, we get

(13.4) $$(f)(Ey)R[a,y,f(y)]\rightarrow(Ey)(z)R(a,y,z).$$

(13.4) together with (13.3) yields $(Ey)(z)R(a,y,z)$, and hence

$$(x)(Ey)(z)R(x,y,z).$$

14. Theorem 3. *There is no disjunctive interpretation of Z_μ by a formal system F satisfying the following conditions:*

its variables are free individual variables,

its predicates are decidable,

its function symbols computable,

and from each A_n associated with $\mathfrak{A}$, the latter can be proved by the predicate calculus of first order.

Remark. We have not attempted to state the theorem in its best possible form; we know, *e. g.*, that there can be no conjunctive interpretation, and by examining proofs of the predicate calculus it is clear that if $\mathfrak{A}$ can be proved from A_n it can also be proved from a formula A_n' which contains no function symbols and predicate symbols other than those of $\mathfrak{A}$ and where A_n is provable in the predicate calculus from A_n'; since the function symbols and predicate symbols of Z_μ are computable so are those of A_n'.

For the proof it is convenient to consider an *Herbrand interpretation* of formulae $(x)(Ey)(z)A(x,y,z)$ of Z_μ, by which we mean a sequence of disjunctions

$$A[a\mathfrak{y}_1(a)a_1] \vee A[a\mathfrak{y}_2(a,a_1)a_2] \vee \dots \vee A[a\mathfrak{y}_n(a,a_1,\dots,a_{n-1})a_n]$$

where the a are individual variables, $\mathfrak{y}_i(a,a_1,\dots,a_{i-1})$, $1<i\leqslant n$ are computable free variable terms whose only variables are a, a_j, $1\leqslant j<i$.

Lemma. *There is no Herbrand interpretation of number theory Z_μ.*

Consider the formula

$$(x)(Ey)(z)[z>y\to P(y,x)=P(z,x)] \tag{14.1}$$

where $P(a,b)$ is the primitive recursive function defined as follows:

(i) prov (a,b) is the primitive recursive function which $=0$ if a is the number of a proof in Z_μ of the formula with number b, $=1$ otherwise. Here, and below, some numbering of expressions in Z_μ, *e. g.* of [5], is assumed to be chosen.

(ii) $\mathrm{prov}_1(a,b)=\prod_{n=1}^{a}\mathrm{prov}(n,b)$, *i. e.* $=0$ if some integer $\leqslant a$ is the number of a proof in Z_μ of the formula with number b. $\mathrm{prov}_1(a,b)$ is also primitive recursive.

(iii) $P(a,b)=0$ if and only if the following conditions are satisfied:

b is the number of the conjunction $A_1\& A_2\&\dots\& A_n$ where the formula A_i has the number a_i, and there exists a sequence $b^{(1)},b^{(2)},\dots,b^{(m)}$, $m\leqslant n$, where $b^{(1)}$ is the number of the first proof of A_1, and $b^{(i+1)}$ is the first number $>b^{(i)}$ which is the number of a proof of A_{i+1}, and:

$$a=b^{(m)}, \quad \text{or} \quad a\geqslant b^{(n)}.$$

Note that the only numbers $\leqslant b^{(i)}$ for which $P(a,b)=0$ are a which are equal to some $b^{(j)}$, $1\leqslant j\leqslant i$.

Note also that if A_i can be proved (in Z_μ) there are arbitrarily long proofs of A_i so that if the conjunction $A_1\& A_2\&\dots\& A_m$, $m\leqslant n$, can be proved in Z_μ there exists a sequence $b^{(1)},\dots,b^{(m)}$ satisfying the conditions above. The sequence is defined u n i q u e l y.

$P(a,b)$ is primitive recursive since it can be defined by applying quantifiers restricted to f i n i t e sets to a primitive recursive formula. We need the following definitions:

$\lambda(n,x)$: the exponent of p_n in the factorization of x into powers of primes, p_n being the n^{th} prime;

$n(b)$: the number of conjuncts of the formula with number b;

$\nu(r,b)$: for $r\leqslant n(b)$, the number of the r^{th} conjunct A_r.

Then $P(a,b)=0$ if and only if either

$$a\neq 0\,\&\,(En)\{n\leqslant n(b)\,\&\,a=\mu_x(Ep)(q)(r)\{q<r\leqslant n\to\cdot\lambda(q,p)<\lambda(r,p)\,\&\,\mathrm{prov}[\lambda(r,p),\nu(r,b)]=0\,\&\,x=\lambda(n,p)\}\}$$

or

$$(Ep)(q)(r)\{q<r\leqslant n(b)\to\cdot\lambda(q,p)<\lambda(r,p)\,\&\,\mathrm{prov}[\lambda(r,p),\nu(r,b)]=0\,\&\,a\geqslant\lambda[n(b),p]\}.$$

Then,

$$(x)(Ey)(z)[z>y\to P(y,x)=P(z,x)] \tag{14.1}$$

can be proved in Z_μ, since for large y $P(y,b)=0$ if the formula $A_1\&\dots\& A_n$ (with number b) can be proved in Z_μ, and $P(y,b)=1$ if it cannot be proved in Z_μ. But (14.1) has no Herbrand interpretation, that is, to any disjunction

$$\begin{aligned} &a_1>\mathfrak{y}_1(a)\to P[\mathfrak{y}_1(a),a]=P(a_1,a)\cdot\vee\dots\vee\\ &a_n>\mathfrak{y}_n(a,a_1,\dots,a_{n-1})\to P[\mathfrak{y}_n(a,a_1,\dots,a_{n-1})a]=P(a_n,a)\end{aligned} \tag{14.2}$$

where the $\mathfrak{y}$ are computable terms, we find numbers $\mathfrak{a},\mathfrak{a}_1,\dots,\mathfrak{a}_n$ which make (14.2) false.

Remark. To show that the first disjunct of (14.2) is false requires that (14.1) is not satisfied by a computable function $\mathfrak{y}_1(a)$. This is already achieved by the formula

$$(x)(Ey)(z)[z>y \to \mathrm{prov}_1(y,x)=\mathrm{prov}_1(z,x)] \tag{14.3}$$

as shown in appendix I of [1]. But since $\mathrm{prov}_1(a,b)$ is monotone decreasing for each fixed b, (14.3) has the Herbrand interpretation:

$$a_1 > \mathfrak{y}_1(a) \to \mathrm{prov}_1[\mathfrak{y}_1(a),a] = \mathrm{prov}_1(a_1,a)\cdot \vee,$$
$$a_2 > a_1 \to \mathrm{prov}_1(a_1,a) = \mathrm{prov}(a_2,a)$$

since if for some $a=\mathfrak{a}$, $a_1=\mathfrak{a}_1$, the first disjunct is false, $\mathrm{prov}_1(\mathfrak{a}_1,\mathfrak{a})=0$, and hence $a_2>\mathfrak{a}_1 \to \mathrm{prov}(a_2,\mathfrak{a})=0$.

The reason why we have to replace (14.3) by (14.1) is that instead of the monotone convergent sequences $\mathrm{prov}_1(n,x)$ (of 0 and 1), we want convergent sequences $P(n,x)$ (of 0 and 1) which for infinitely many x have many long stretches of l though $P(n,x)\to 0$: and there must be no computable estimate, depending on x and the lengths of the first r stretches, which bounds the length of the $(r+1)^{\text{th}}$ stretch.

We give (metamathematical) instructions for introducing abbreviations $p(i,b)$ for certain terms of Z_μ. These instructions are intended to be used as follows: we suppose a set of computable terms $\mathfrak{y}_i$, $1\leqslant i\leqslant n$, (of (14.2)) to be given, and we denote representatives of them in Z_μ by $\bar{\mathfrak{y}}_i$; then the terms which we denote by $p(1,b),\ldots,p(n,b)$ are to be formed according to our rules; in other words, the recursive definition is to be applied n times only, and (for our given set of $\mathfrak{y}_i$, $1\leqslant i\leqslant n$) these terms can be written out in full. When in the sequel we speak of the *number of an expression* which contains the symbols $p(i,b)$ we always mean the number of the expression (of Z_μ) which is got when $p(i,b)$ is replaced by the definiendum.

Denote

$$\mu_z\{\mathrm{prov}\{z,\nu[1,s(b,b)]\}=0\}$$

by $p(1,b)$, and

$$\mu_z\{(Ep)(q)(r)[q<r\leqslant i+1 \to\cdot \lambda(q,p)<\lambda(r,p)\ \&$$
$$\mathrm{prov}\{\lambda(r,p),\nu[r,s(b,b)]\}=0\ \&\ z=\lambda(i+1,p)]\vee$$
$$[P\{\bar{\mathfrak{y}}_{i+1}[s(b,b),p(1,b),\ldots,p(i,b)],s(b,b)\}=0\ \&$$
$$z=\bar{\mathfrak{y}}_{i+1}[s(b,b),p(1,b),\ldots,p(i,b)]+1]\}$$

by $p(i+1,b)$.

Remark. Observe that in the numbering of proofs in Z_μ, if $p(j,b)$ is the number of a proof, $p(j,b)+1$ is not, since all proofs have even numbers. The need for the somewhat complicated definition of $p(i+1,b)$ will appear in (iii) below.

Now consider the conjunction

$$\begin{aligned}
&\mathrm{prov}_1\{\bar{\mathfrak{y}}_1[s(b,b)],\nu[1,s(b,b)]\}=1\ \&\\
&\mathrm{prov}_1\{\bar{\mathfrak{y}}_2[s(b,b),p(1,b)],\nu[2,s(b,b)]\}=1\ \&\\
&\quad\vdots\\
&\mathrm{prov}_1\{\bar{\mathfrak{y}}_n[s(b,b),p(1,b),\ldots,p(n-1,b)],\nu[n,s(b,b)]\}=1,
\end{aligned}\tag{14.4}$$

where b is a free variable, $\bar{\mathfrak{y}}_i$ are terms of Z_μ representing the computable functions $\mathfrak{y}_i$, and $s(a,b)$ is the primitive recursive function whose value is the number of the expression got when the variable in the expression with number b is replaced by the numeral a. Let $\mathfrak{p}$ be the number of (14.4) so that $s(\mathfrak{p},\mathfrak{p})$ is the number of the formula $\mathfrak{S}$ got when b in (14.4) is replaced by $\mathfrak{p}$.

We shall show that (14.2) is false if we replace a by $s(\mathfrak{p},\mathfrak{p})$, a_i by $p(i,\mathfrak{p})$, $1\leqslant i\leqslant n$.

(i) Observe that $\mathfrak{S}$ is a correct decidable formula of Z_μ. The first conjunct is correct since otherwise

$$\mathrm{prov}_1\{\bar{\mathfrak{y}}_1[s(\mathfrak{p},\mathfrak{p})],\nu[1,s(\mathfrak{p},\mathfrak{p})]\}=0, \tag{14.5}$$

that is some integer $q_1\leqslant\bar{\mathfrak{y}}_1[s(\mathfrak{p},\mathfrak{p})]$, would be the number of a proof in Z_μ of the formula with number $\nu[1,s(\mathfrak{p},\mathfrak{p})]$, *i. e.* of

$$\mathrm{prov}_1\{\bar{\mathfrak{y}}[s(\mathfrak{p},\mathfrak{p})],\nu[1,s(\mathfrak{p},\mathfrak{p})]\}=1 \tag{14.6}$$

itself; since both (14.5) and (14.6) are decidable in Z_μ, one of them can be proved in Z_μ, and by the consistency of Z_μ it must be (14.6). The number of its shortest proof is the value of the expression $p(1,\mathfrak{p})$: $>\bar{\mathfrak{y}}_1[s(\mathfrak{p},\mathfrak{p})]$. Since the expression which we denote by $p(1,\mathfrak{p})$ can be evaluated in Z_μ, also the second conjunct of $\mathfrak{S}$ is a decidable formula of Z_μ, and it is correct by the argument above. Again

$$p(2,\mathfrak{p})>\bar{\mathfrak{y}}_2[s(\mathfrak{p},\mathfrak{p}),p(1,\mathfrak{p})].$$

Generally, suppose the first i conjuncts of $\mathfrak{S}$ have been proved, and the terms $p(j,\mathfrak{p})$, $j<i$, have been evaluated. Then $p(i,\mathfrak{p})$ can also be evaluated, and the $(i+1)^{\text{th}}$ conjunct can be proved in Z_μ.

Observe first that under our conditions there is a sequence $b^{(j)}$, $j\leqslant i$, with $b=s(\mathfrak{p},\mathfrak{p})$, which satisfies the conditions in the definition of $P(a,b)$.

Observe also that the i^{th} conjunct of $\mathfrak{S}$ states that

$$\bar{\mathfrak{y}}_i[s(\mathfrak{p},\mathfrak{p}),p(1,\mathfrak{p}),\ldots,p(i-1,\mathfrak{p})]$$

is less than the least number of a proof of the formula with number $\nu[i,s(\mathfrak{p},\mathfrak{p})]$, and is therefore certainly less than $b^{(i)}$.

Now, if

$$P\{\bar{\mathfrak{y}}_i[s(\mathfrak{p},\mathfrak{p}),p(1,\mathfrak{p}),\ldots,p(i-1,\mathfrak{p})],s(\mathfrak{p},\mathfrak{p})\}\neq 0, \tag{14.7}$$

$p(i,\mathfrak{p})=b^{(i)}$ by the least number definition of $p(i,b)$ above. If (14.7) is false

$$p(i,\mathfrak{p})=\bar{\mathfrak{y}}_i[s(\mathfrak{p},\mathfrak{p}),p(1,\mathfrak{p}),\ldots,p(i-1,\mathfrak{p})]+1.$$

Since $P(a,b)$ is primitive recursive, the condition (14.7) can be decided in Z_μ, and hence $p(i,\mathfrak{p})$ can be evaluated. Note that in either case

$$p(i,\mathfrak{p})>\bar{\mathfrak{y}}_i[s(\mathfrak{p},\mathfrak{p}),p(1,\mathfrak{p}),\ldots,p(i-1,\mathfrak{p})].$$

Thus the $(i+1)^{\text{th}}$ conjunct of $\mathfrak{S}$ is a decidable formula of Z_μ, and by the argument above it is correct.

Hence $\mathfrak{S}$ itself is correct.

(ii) $P\{\bar{\mathfrak{y}}_1[s(\mathfrak{p},\mathfrak{p})],s(\mathfrak{p},\mathfrak{p})\}=1,\quad P[p(1,\mathfrak{p}),s(\mathfrak{p},\mathfrak{p})]=0.$

Since even the first conjunct of $\mathfrak{S}$ cannot be proved by a proof with number $\leqslant\bar{\mathfrak{y}}_1[s(\mathfrak{p},\mathfrak{p})]$,

$$P\{\bar{\mathfrak{y}}_1[s(\mathfrak{p},\mathfrak{p})],s(\mathfrak{p},\mathfrak{p})\}=1.$$

Since $p(1,\mathfrak{p})=b^{(1)}$,

$$P[p(1,\mathfrak{p}),s(\mathfrak{p},\mathfrak{p})]=0.$$

(iii) $P[p(i+1,\mathfrak{p}),s(\mathfrak{p},\mathfrak{p})]=0$ if and only if

$$P\{\bar{\mathfrak{y}}_{i+1}[s(\mathfrak{p},\mathfrak{p}),p(1,\mathfrak{p}),\ldots,p(i,\mathfrak{p})],s(\mathfrak{p},\mathfrak{p})\}\neq 0. \tag{14.8}$$

If (14.8) holds, by the definition of $p(i+1,b)$, $p(i+1,\mathfrak{p})=b^{(i+1)}$.
If (14.8) is false,

$$\bar{\mathfrak{y}}_{i+1}[s(\mathfrak{p},\mathfrak{p}),p(1,\mathfrak{p}),\ldots,p(i,\mathfrak{p})]=b^{(j)},\quad \text{for some } j\leqslant i,$$

and then

$$p(i+1,\mathfrak{p})=b^{(j)}+1.$$

Thus $p(i+1,\mathfrak{p})<b^{(n)}$ and is not the number of a proof at all so that

$$P[p(i+1,\mathfrak{p}),s(\mathfrak{p},\mathfrak{p})]=1.$$

Now consider (14.2).

Let $a=s(\mathfrak{p},\mathfrak{p})$, $a_1=p(1,\mathfrak{p})$:

$$p(1,\mathfrak{p})>\bar{\mathfrak{y}}_1[s(\mathfrak{p},\mathfrak{p})],$$

and by (ii)

$$P\{\bar{\mathfrak{y}}_1[s(\mathfrak{p},\mathfrak{p})],s(\mathfrak{p},\mathfrak{p})\}\neq P[p(1,\mathfrak{p}),s(\mathfrak{p},\mathfrak{p})].$$

Generally, let $a_j=p(j,\mathfrak{p})$, $1\leqslant j\leqslant n$. Then

$$p(i,\mathfrak{p})>\bar{\mathfrak{y}}_i[s(\mathfrak{p},\mathfrak{p}),p(1,\mathfrak{p}),\ldots,p(i-1,\mathfrak{p})]$$

by (i), and

$$P\{\bar{\mathfrak{y}}_i[s(\mathfrak{p},\mathfrak{p}),p(1,\mathfrak{p}),\ldots,p(i-1,\mathfrak{p})],s(\mathfrak{p},\mathfrak{p})\}\neq P[p(i,\mathfrak{p}),s(\mathfrak{p},\mathfrak{p})]$$

by (iii).

Thus, putting

$$a=s(\mathfrak{p},\mathfrak{p}),\quad a_i=p(i,\mathfrak{p}),\quad 1\leqslant i\leqslant n,$$

yields a counter example to the whole disjunction (14.2).

This proves the Lemma.

To prove the theorem, consider a sequence of free variable formulae $A_1(a_1,\ldots,a_{n_1}),\ldots,A_k(a_1,\ldots,a_{n_k})$ associated with $\mathfrak{A}$. If $\mathfrak{A}$ could be proved by the predicate calculus from each A, say A_k,

$$\mathfrak{A}\vee(Ey_1),\ldots,(Ey_{n_k})\rightarrow A(y_1,\ldots,y_{n_k}) \tag{14.9}$$

could be proved in the predicate calculus, and by Herbrand's theorem we should get a Herbrand interpretation of (14.9).

Take for $\mathfrak{A}$ the formula (14.1). By condition (i) of § 6 one A_k should be verifiable since $\mathfrak{A}$ is provable in Z_μ, so that from an Herbrand interpretation of (14.9) we should get one for (14.1) itself, which is false.

Completeness.

15. The definitions by Hilbert and Post of the concept of completeness of a formal system $\mathfrak{F}$ make it an internal property of the system: in one form $\mathfrak{F}$ is called *complete* if any formula of $\mathfrak{F}$ can either be proved or disproved in $\mathfrak{F}$ (provided $\mathfrak{F}$ contains a symbol of negation), and in the more general form $\mathfrak{F}$ is called *complete* if the system $\mathfrak{F}$ becomes inconsistent when a formula of $\mathfrak{F}$ which is not provable in $\mathfrak{F}$ is added to $\mathfrak{F}$ as an axiom; this definition is suitable either if $\mathfrak{F}$ does not contain the symbol of negation or, roughly speaking, contains free variables (like the propositional calculus).

These definitions are a rather sophisticated development of a much vaguer notion of completeness, which is concerned with the complete characterization of some notion by a formal system. Thus the propositional calculus was called complete because, on the one hand, the idea of truth of any formula of this calculus could be defined by truth tables, and, on the other hand, it could be shown that all identically true formulae and only such formulae could be proved in the (usual) propositional calculus. Similarly the predicate calculus was intended to characterize the notion of logical truth or validity, in the (vague) sense that those and only those formulae (of the first order predicate calculus) are valid which are "true in any domain": and Gödel's proof which he called *proof of the completeness of the predicate calculus*, consisted in showing that

(i) if a formula is provable in the predicate calculus it is valid in number theory Z_μ extended by verifiable free variable formulae, and

(ii) if it is not so provable, it is false in a certain ω-consistent extension of Z_μ. (It is well known that the predicate calculus is not complete in the sense of Hilbert-Post.) On the other hand number theory was intended to characterize the notion of verifiability, where the formula $A(b)$ is called *verifiable* if $A(\mathfrak{n})$ is correct for every recursive term $\mathfrak{n}$, and Gödel showed in [7] that this was not achieved by any formalized system in the sense that in any consistent such system there is a primitive recursive formula $A(b)$ so that $A(\mathfrak{n})$ is correct for each $\mathfrak{n}$ without $A(b)$ being provable in the system. This was called a *proof of incompleteness*.

In each case a formula of the system considered was associated with a class of formulae (of the same system or other systems) and the formula was intended to be provable if and only if each formula of the class was correct (in some prescribed sense). From this point of view completeness of a system is not an internal property of the system. (We observe that also Henkin on completeness in the theory of types associates completeness of a formal system with a certain *intended class of interpretations* in [3], p. 81.)

Remark. To stress this fact we shall not speak of the completeness of a formal system, but of the completeness of a certain interpretation of the formal system. The reason why one often speaks of the completeness of the first order predicate calculus is, no doubt, that many logicians intended this formal system to have just that interpretation whose completeness Gödel established: in other words, completeness of the intended interpretation.

16. From this point of view it is natural to call a disjunctive interpretation *complete* if the formula $\mathfrak{A}$ of $\mathfrak{F}$ can be proved in $\mathfrak{F}$ when some A_n is true, and to call a conjunctive interpretation *complete* if the formula $\mathfrak{A}$ of $\mathfrak{F}$ can be proved in $\mathfrak{F}$ when each A_n is true.

Remark 1. We might equally well say in the conjunctive case, if $\mathfrak{A}$ is not provable in $\mathfrak{F}$, we find an $\mathfrak{n}$ so that $A_\mathfrak{n}$ is not true.

Remark 2. If F is a subsystem of $\mathfrak{F}$, and "true" means: x provable in F, a sufficient condition for completeness of a disjunctive interpretation is that $\mathfrak{A}$ can be proved from A_n in $\mathfrak{F}$, for each n.

17. Trivially, if $\mathfrak{F}$ is consistent and complete in the sense of Hilbert-Post, the interpretation of $\mathfrak{A}$: "the system $\mathfrak{F}$ is consistent when the formula $\mathfrak{A}$ is added to the axioms of $\mathfrak{F}$" is complete in our sense. If each formula of $\mathfrak{F}$ is either provable or disprovable both interpretations of Theorem 2 are complete.

Concerning the interpretations of number theory in [1].

(a) Consider first the interpretation of a formalized system $\mathfrak{F}$ by F_0 mentioned in § 9. Denote the formulae of F_0 associated with the formula $\mathfrak{A}$ of $\mathfrak{F}$ by $A_1, A_2, A_3, \ldots$, and recall that

(i) F_0 is a subsystem of $\mathfrak{F}$,

and

(ii) each A_n implies $\mathfrak{A}$ in $\mathfrak{F}$.

Thus if some A_n has been proved in F_0, it can also be proved in $\mathfrak{F}$, and $\mathfrak{A}$ can be proved in $\mathfrak{F}$. Thus the interpretation satisfies the condition imposed on complete disjunctive interpretations in § 16. (Note that the argument just given is a special case of remark 2 in § 16.) By "true in F_0" we mean here: provable in F_0.

Our argument applies also if by $\mathfrak{F}$ we mean a formal system of arithmetic consisting of the predicate calculus as rules of inference, and any verifiable free variable formula of $\mathfrak{F}$ as an axiom (*i. e.* the so-called *ω-consistent extensions* of $\mathfrak{F}$); and by F_0 we mean the system consisting of the elementary calculus with free variables as rules of inference, and any verifiable formula of F_0 as an axiom. Here we mean by "true in F_0": verifiable in F_0.

(b) Next consider the interpretation of $\mathfrak{F}$ by F_1 in § 10 above. Denote the formulae of F_1 now associated with $\mathfrak{A}$ by $A_1^*, A_2^*, A_3^*, \ldots$ It is shown in [5], p. 150-151, that A_i implies A_i (of (a) above) in the elementary calculus with free function and individual variables, and if A_i^* can be proved in F_1, A_i can be proved in F_0. Now the argument given above shows that the interpretation of $\mathfrak{F}$ by F_1 is complete.

(c) Finally, consider the interpretation of full number theory, as described in § 11 and § 12: by full number theory $\mathfrak{Z}$ we mean the system Z_μ to which are added symbols for ordinal recursive functions of finite order, and all its ω-consistent extensions. There, with the formula $\mathfrak{A}$

$$(x_1)(Ey_1),\ldots,(x_n)(Ey_n)A(x_1,\ldots,x_n y_1,\ldots,y_n)$$

is associated the sequence

$$A[af_2(\mathfrak{y}_1^{(r)}),\ldots,f_n(\mathfrak{y}_1^{(r)},\ldots,\mathfrak{y}_{n-1}^{(r)})\mathfrak{y}_1^{(r)},\ldots,\mathfrak{y}_n^{(r)}] \qquad (\S\ 11)$$

or the sequence $A^{(r)}$, *i. e.*

$$A\{a\eta_2[x;\bar{\mathfrak{y}}_1^{(r)}(x;a),\ldots,\eta_n[x;\bar{\mathfrak{y}}_1^{(r)}(x;a),\ldots,\bar{\mathfrak{y}}_{n-1}^{(r)}(x;a)]\bar{\mathfrak{y}}_1^{(r)}(x;a),\ldots,\bar{\mathfrak{y}}_n^{(r)}(x;a)\} \qquad (\S\ 12)$$

where the $\mathfrak{y}$ are ordinal recursive functionals, and the $\bar{\mathfrak{y}}(x;a)$ are ordinal recursive functions of finite order. Also, by §§ 31-33 of [1], for each r, $A^{(r)}$ implies $\mathfrak{A}$ in the full number theory described. Since $A^{(r)}$ is a free variable formula of the (non formalized) number theory $\mathfrak{Z}$ considered, $A^{(r)}$ is an axiom of $\mathfrak{Z}$ provided it is verifiable, so that, if $A^{(r)}$ is verifiable, $\mathfrak{A}$ can be proved in $\mathfrak{Z}$. Thus, if we call $A^{(r)}$ "true" when it is verifiable, our interpretation of $\mathfrak{Z}$ is complete.

It is probable that the interpretation in question is complete if the number theory considered is the system Z_μ itself, and if we call $A^{(r)}$ "true" when it is provable in the (formalized) system of [1], § 38.

18. When one considered interpretations by models it was natural to define completeness by models. Again it seems, however suitable models may be for formulating completeness in certain cases (*e. g.* validity of logical formulae), they are not suitable for a general definition of completeness, the reasons being much the same as those of § 7. In addition, when we consider the problem of finding a complete axiomatic characterization for a notion (*i. e.* a *class* of formulae of some given system as illustrated in § 15) models are not suitable because the formulae of the class may not have the form of models of some one formula; *e. g.* the notion of counter example in [1].

19. The obvious problem of whether every (consistent) system has a complete disjunctive and conjunctive interpretation is trivial unless the system F is suitably restricted.

Remark. For any formalized system the disjunctive interpretation of Theorem 2 is complete.

There are some interesting unsolved problems in this connection, but generally the interesting problem is to decide whether an intended interpretation of a given formal system is complete.

20. We conclude with a curious isolated result on complete conjunctive interpretations of certain consistent formalized systems of numbers theory by models defined in ω-consistent extensions of Z_μ. The method applies also to ω-inconsistent systems of number theory.

Remark. The interest, if any, of these models is that they are defined very simply from Gödel's arithmetization (cf. first sentence of the footnote in [5], p. 206), and, like the Skolem model in cases discussed in [8], they are non standard models, which have been mentioned in the literature recently. Also, the method applies apparently only if the logical connectives are just & and $\vee$, in particular, if the symbol of negation has been eliminated.

We describe these models without much formal detail, and, to fix ideas, we consider a certain number theory Z_1 which is got from the system Z, described in [5], p. 324, as follows:

(i) The function symbols $+1$, $+$, $\times$ and the constant 0 of Z are eliminated by predicate symbols: $N(a,b)$, a is successor of b; $S(a,b,c)$, c is the sum of a and b; $P(a,b,c)$, c is the product of a and b; $Z(a)$, a is zero; also the axioms of Z are modified appropriately. Universal and existential quantifiers and ε-terms and the ε-formula, [5], p. 13, are used, but no formulae with free variables.

Remark 1. By [6] the system Z_1 is consistent.

Remark 2. The elimination of function symbols and free variables is usual with models (cf. [4]).

(ii) We suppose expressions of Z_1 have been numbered, and since the terms of Z_1 can be recursively enumerated there is a computable function $j_1(n)$ so that the value of $j_1(n)$ is the number of the n^{th} term of Z_1; actually $j_1(n)$ can be defined by a primitive recursive definition. Further we have a substitution function $s_1(n,m,p,a)$ which is the number of the expression of Z_1 got by substituting in the expression $\mathfrak{A}$ of Z_1 with number a the terms with numbers $j_1(n)$, $j_1(m)$, $j_1(p)$ for the free variables b_1, b_2, b_3 in $\mathfrak{A}$.

Remark. We substitute the n^{th}, m^{th}, p^{th} term and not the numerals n, m, p, as in the substitution function in Theorem 3; also they are not the terms with numbers n, m, p.

(iii) We denote by $\mathrm{Prov}_1(a,p,q)$ the primitive recursive formula which holds if and only if either $a=0$ or a is not the number of a formula of Z_1, and p is the number of a proof in Z_1 of the formula $\mathfrak{Q}$ with number q, or if a is the number of a formula of Z_1, p is the number of a proof by Z_1 from $\mathfrak{A}$ of $\mathfrak{Q}$. The value of the primitive recursive function $\mathfrak{e}_1(a)$ is the number of the formula $\neg\mathfrak{A}$, and, finally, $t_1(n)$ is the expression $\beta\{\mu_x \mathrm{Prov}_1[0,x,\mathfrak{e}(n)]\}$, where $\beta(0)=1$, $\beta(n+1)=0$.

Remark. $t_1(n)$ is not computable, but can be "evaluated" in the informal system consisting of ω-consistent extensions of Z_μ: $t_1(n)=0$ if $\mathfrak{N}$, the formula with number n, is inconsistent over Z_1, $t_1(n)=1$ if $\mathfrak{N}$ is consistent, since Z_1 does not contain formulae with free variables.

(iv) *Positive formulae* of Z_1 are prenex formulae of Z_1 which neither contain the ε-symbol so that its quantifiers are universal and existential variables, nor the symbol of negation. Any such formula can be written as

$$(20.1)\qquad (x_{11})\dots(x_{1n_1})(Ey_{11})\dots(Ey_{1m_1})\dots(x_{n1})\dots(x_{nn_n})(Ey_{n1})\dots(Ey_{nm_n})$$
$$[(B_{11}\vee\dots\vee B_{1k_1})\,\&\,\dots(B_{p1}\vee\dots\vee B_{pk_p})]$$

where B are unnegated prime formulae, *i. e.* the predicate symbols of 20 (i), $=$, $<$ whose arguments are the variables x_{ij}, $1\leqslant i\leqslant n$, $1\leqslant j\leqslant n_i$, y_{rs}, $1\leqslant r\leqslant n$, $1\leqslant s\leqslant m_r$.

Remark. Any formula of Z_1 without the ε-symbol is equivalent over Z_1 to a positive formula of Z_1: first write it in prenex form (20.1), when the B may also be negated prime formulae; then replace $\neg a=b$ by $a<b\vee b<a$; $\neg a<b$ by $a=b\vee b<a$; $\neg N(a,b)$ by $(Ez)[N(z,b)\,\&\,\neg z=a]$; similarly with the other predicates of 20 (i); lastly transform the resulting formula into its prenex form.

Theorem 4. *Given a positive formula $\mathfrak{A}$ of Z_1, denote by A_n the formula obtained when the predicate*

$$(m)\neg\mathrm{Prov}_1\{nt_1(n),m,\mathfrak{e}[s(x_1,x_2,x_3,f)]\}$$

of Z_μ is substituted for the predicate $F(x_1,x_2,x_3)$ in $\mathfrak{A}$, where the formula $F(b_1,b_2,b_3)$ has the number f in our numbering, and associate the resulting sequence with $\mathfrak{A}$: this is then a complete conjunctive interpretation of (positive formulae of) Z_1 by $\mathfrak{Z}$, the system of ω-consistent extensions of Z_μ.

Remark 1. We use predicates with three arguments because no predicate of Z_1 has more.

Remark 2. "True" means: provable in some ω-consistent extension of Z_μ. We do not attempt to give a detailed proof of the theorem.

Remark 3. As in remark 1, § 8, we have to check for each positive formula of Z_1 that our predicates constitute a model, and not only for the axioms of Z_1 because Z_1 is not an axiom system of the predicate calculus.

Lemma 1. *If the formula $\mathfrak{A}$, say* (20.1), *is proved in Z_1, each of our models A_n is a true formula (of $\mathfrak{Z}$).*

Observe that $\neg\mathfrak{A}$ cannot be proved in Z_1 from any formula $\mathfrak{N}$ which is consistent over Z_1.

Eliminate from $\mathfrak{A}$ the existential quantifiers by ε-expressions, as described *e. g.* in [1], § 24, denoting the resulting formula by $\widetilde{\mathfrak{A}}$. The quantifiers y_{ij} are replaced by ε-terms $\mathfrak{y}_{ij}(x_{11},\dots,x_{in_i})$, and there are primitive recursive functions $y_{ij}(a_{11},\dots,a_{in_i})$ so that when $\mathfrak{x}_{11},\dots,\mathfrak{x}_{in_i}$ are the terms with numbers $j_1(a_{11}),\dots,j_1(a_{in_i})$, $\mathfrak{y}_{ij}(\mathfrak{x}_{11},\dots,\mathfrak{x}_{in_i})$ has the number $y_{ij}(a_{11},\dots,a_{in_i})$.

Since $\mathfrak{A}$ can be proved in Z_1, so can $\widetilde{\mathfrak{A}}$, and hence each of the disjunctions $\widetilde{B}_{i1}\vee\dots\vee\widetilde{B}_{ik_i}$ whatever terms of Z_1 are substituted for the all variables $x_{11},\dots,x_{nn_n}$. Hence by the consistency of the system Z_1, $\neg\widetilde{B}_{i1}\,\&\dots\&\neg\widetilde{B}_{ik_i}$ cannot be proved in Z_1 from any formula $\mathfrak{N}$ which is consistent over Z_1, so that either $\neg\widetilde{B}_{i1}$, or $\neg\widetilde{B}_{i2}$ or $\dots\neg\widetilde{B}_{ik_i}$ cannot be proved from $\mathfrak{N}$ in Z_1.

This proves the lemma.

Remark 1. For any n the model is formalized by a free variable formula. In particular the model has a primitive recursive *Erfüllung* $y_{11}(a_{11}, \ldots, a_{1n_1}), \ldots, y_{nm_n}(a_{11}, \ldots, c_{nn_n})$ though even a proved formula (20.1) in general has no computable *Erfüllung*. The reason is that the formal *Erfüllung* of $\mathfrak{A}$ by ε-terms provides a primitive recursive *Erfüllung* for the model by the order of these ε-terms in our numbering.

Note that a rather similar phenomenon is found with the Skolem model, *e. g.* [8], end of § 1.

Remark 2. Our definition of a model breaks down if applied to the predicate calculus: thus the formula $\mathfrak{F}$:

$$(x)[\neg F_1(x) \vee \neg F_2(x)]$$

is certainly consistent over the predicate calculus, but neither $\neg F_1(a)$ nor $\neg F_2(a)$ can be proved from $\mathfrak{F}$ by the predicate calculus. Suppose f_i is the number of the formula $F_i(b)$, $s_2(m,n)$ the number of the formula got by substituting the m^{th} term of the predicate calculus for the free variable b in the formula $\mathfrak{N}$ of the predicate calculus with number n, $\mathrm{e}_2(n)$ is the number of $\neg\mathfrak{N}$, and $\mathrm{Prov}_2(m,n)$ holds if and only if m is the number of a proof of $\mathfrak{N}$ from $\mathfrak{F}$ by the predicate calculus. Then the analogue to our lemma would be to substitute $(m) \neg \mathrm{Prov}_2\{m, \mathrm{e}_2[s(\cdot, f_i)]\}$ for $F_i(\cdot)$ so that $\mathfrak{F}$ becomes

$$(x)\Big\{(Em)\,\mathrm{Prov}_2\{m, \mathrm{e}_2[s_2(x,f_1)]\} \vee (Em)\,\mathrm{Prov}_2\{m, \mathrm{e}_2[s_2(x,f_2)]\}\Big\}$$

which is false.

Similarly the predicates

$$(Em)\,\mathrm{Prov}_2[m, s(\cdot, f)]$$

do not provide a model: consider $(x)[F_1(x) \vee F_2(x)]$.

Lemma 2. *If* $\mathfrak{A}$, (20.1), *cannot be proved in* Z_1, *there is an* n *so that* A_n *can be disproved in some* ω*-consistent extension of* Z_μ.

Remark. The argument is rather uninformative and does not compute the number n, as is to be expected from a completeness proof of this sort.

If $\mathfrak{A}$ cannot be proved in Z_1, one of the disjuncts $\widetilde{B}_{t1} \vee \ldots \vee \widetilde{B}_{tk_t}$ cannot be proved in Z_1, and the formula $\neg\widetilde{B}_{t1} \,\&\, \ldots \,\&\, \neg\widetilde{B}_{tk_t}$ is consistent over Z_1; let its number be $\mathfrak{n}$. Then from it can be proved in Z_1 each $\neg\widetilde{B}_{tj}$, $1 \leqslant j \leqslant k_t$, and hence the formula $A_{\mathfrak{n}}$ is falsce (*i. e.* can be disproved in the extension of Z_μ by the free variable formula $\neg\mathrm{Prov}_1[0, m, \mathrm{e}(\mathfrak{n})]$.

The two lemmata together prove the theorem.

Remark. The proof applies readily to systems which are extensions of Z_1 by consistent, but not necessarily ω-consistent formulae of Z_1.

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