par exemple

$$f = \left\{ t \sin \frac{\pi \log t}{\log \lambda_0} \right\} \qquad (\lambda_0 \text{ positif et } \neq 1).$$

Si (2), l'équation

$$x^2 = f \tag{3}$$

n'est pas résoluble dans Q. En effet, supposons, au contraire, qu'il existe un opérateur x satisfaisant à cette équation. Cet opérateur est réel ou purement imaginaire, c'est-à-dire il est de la forme f/g ou bien de la forme $i\,f/g$, où les fonctions f et g (appartenant à C) sont réelles. On a

$$T_{\lambda_0}(x^2) = (T_{\lambda_0} x)^2 = T_{\lambda_0} f = -\lambda_0^2 f = -\lambda_0^2 x^2,$$

d'où

$$T_{\lambda_0} x = \pm i \lambda_0 x.$$

Or, ce n'est pas possible, car on voit aussitôt que l'opérateur $T_{\lambda_0} x$ est réel ou imaginaire, suivant que x est réel ou imaginaire.

Cette contradiction prouve que l'équation (3) n'est pas résoluble.

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Singular integrals and periodic functions

by

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1. The purpose of this note is to extend to periodic functions some of the results about singular integrals known for the non-periodic case. We shall be more specific later and begin by recalling basic facts.

Let $x = (\xi_1, \ldots, \xi_k)$, $y = (\eta_1, \ldots, \eta_k), \ldots$ denote points in the k-dimensional Euclidean space E^k. By x we shall also denote the vector joining the origin $O = (0, \ldots, 0)$ with the point x. The length of the vector x will be denoted by $|x|$. If $x \neq 0$, by x' we shall mean the projection of x onto the unit sphere Σ having O for centre. Thus

$$x' = \frac{x}{|x|}, \qquad |x'| = 1.$$

We shall consider kernels $K(x)$ of the form

$$K(x) = \frac{\Omega(x')}{|x|^k} = \frac{\Omega(x')}{r^k} \qquad (r = |x|), \tag{1.1}$$

where Ω is a scalar (real or complex) function defined on Σ and satisfying the following conditions:

1° $\Omega(x')$ is continuous on Σ and its modulus of continuity $\omega(\delta)$ satisfies the Dini condition

$$\int_0^1 \frac{\omega(\delta)}{\delta} \, d\delta < \infty;$$

2° The integral of $\Omega(x')$ extended over Σ is zero.

Condition 1° is certainly satisfied if the function Ω satisfies a Lipschitz condition of positive order. It could be considerably relaxed in very important special cases, but for the problems discussed in this paper it is the most suitable one. On the other hand, condition 2° is absolutely essential, as explained in [2][1].

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[1]) Numbers in brackets refer to the bibliography at the end of the paper.

We shall denote by $K_\varepsilon(x)$, and call it the *truncated kernel*, the expression defined by the equations

$$K_\varepsilon(x)=\begin{cases} K(x) & \text{for } |x|\geqslant\varepsilon, \\ 0 & \text{elsewhere.}\end{cases}$$

Let $f(x)\epsilon L^p$, $p\geqslant 1$. The convolution

$$\tilde{f}_\varepsilon(x)=\int_{E^k} f(y)K_\varepsilon(x-y)\,dy=\int_{E^k} K_\varepsilon(y)f(x-y)\,dy$$

exists as an absolutely convergent integral for all x. We write here dy for $d\eta_1\ldots d\eta_k$. The limit of $\tilde{f}_\varepsilon(x)$ as $\varepsilon\to 0$, if it exists, will be denoted by $\tilde{f}(x)$ and called the *Hilbert transform*, associated with the kernel K, of $f(x)$. The classical Hilbert transform corresponds, except for a numerical constant, to $k=1$ and $K(x)=1/x$.

We state without proof a number of known results concerning these transforms[2]).

THEOREM A. *The function $\tilde{f}(x)$ exists almost everywhere if $f\epsilon L^p$, $p\geqslant 1$. More generally, if F is a totally additive function of set in E^k, the limit*

$$\lim_{\varepsilon\to 0}\int_{E^k} K_\varepsilon(x-y)F(dy) \tag{1.1a}$$

exists almost everywhere.

THEOREM B. *If $f(x)$ belongs to L^p, $1<p<\infty$, so does $\tilde{f}(x)$. Moreover,*

$$\|\tilde{f}\|_p\leqslant A_p\|f\|_p. \tag{1.2}$$

Here and hereafter constants like A_p are positive and depend not only on the parameters shown explicitly but also on the kernel K (*i. e.* on the function Ω). By A (or $B, C, \ldots$) we shall denote constants depending at most on the kernel K, and we may use identical notation for different constants. By $\|f\|_p$ we mean the norm

$$\Big(\int_{E^k}|f|^p\,dx\Big)^{1/p}.$$

Theorem B breaks down for $p=1$ and $p=\infty$. If $f\epsilon L$, the function $\tilde{f}$ need not be integrable, even locally. Nor does the boundedness of f imply that of $\tilde{f}$ (even if we assume the existence of $\tilde{f}$, which is not guaranteed here by Theorem A). For these extreme cases we have the following results:

[2]) Theorems A-F, as well as part 1° of Theorem G, are proved in [2]. Cases 2° and 3° of Theorem G are established by Mr Cotlar [3].

THEOREM C. *If $|f|(1+\log^+|f|)$ is integrable over E^k, then $\tilde{f}$ is integrable over every set S of finite measure and*

$$\int_S|\tilde{f}|\,dx\leqslant A_S\int_{E^k}|f|(1+\log^+|f|)\,dx+B_S. \tag{1.3}$$

THEOREM D. *If $f\epsilon L$,* then $|\tilde{f}|^\alpha$ *is integrable over every set S of finite measure if $0<\alpha<1$. Moreover,*

$$\int_S|\tilde{f}|^\alpha dx\leqslant\frac{A}{1-\alpha}|S|^{1-\alpha}\Big(\int_{E^k}|f|\,dx\Big)^\alpha. \tag{1.4}$$

THEOREM E. *If $f\epsilon L$, then the measure of the set E_y in which we have $|\tilde{f}(x)|\geqslant y>0$, satisfies the inequality*

$$|E_y|\leqslant\frac{A}{y}\|f\|_1. \tag{1.5}$$

THEOREM F. *Suppose that $|f(x)|\leqslant 1$ for all x, and let*

$$g(x)=\lim_{\varepsilon\to+0}\int_{E^k}\{K_\varepsilon(x-y)-K_1(x-y)\}f(y)\,dy. \tag{1.6}$$

Then, for any bounded set S,

$$\int_S\exp\lambda|g(x)|\,dx\leqslant A_S, \tag{1.7}$$

provided $0<\lambda\leqslant\lambda_0=\lambda_0(S,K)$. If f is, in addition, continuous, the last integral is finite for every $\lambda>0$.

THEOREM G. *Given any f, let*

$$\Phi(x)=\sup_{\varepsilon>0}\Big|\int_{E^k}K_\varepsilon(x-y)f(y)\,dy\Big|.$$

Then in the three cases

(i) $f\epsilon L^p$, $p>1$;

(ii) $|f|(1+\log^+|f|)\epsilon L$;

(iii) $f\epsilon L$,

the inequalities (1.2), (1.3), (1.4) *hold, respectively, with $\tilde{f}$ replaced by Φ.*

2. Let us now consider a system $e_1, e_2, \ldots, e_k$ of independent vectors in E^k and let $x_0=0, x_1, x_2, \ldots$ be the sequence of all lattice points generated by these vectors:

$$x_\nu=m_1e_1+m_2e_2+\ldots+m_kx_k, \tag{2.1}$$

where $m_1, m_2, \ldots, m_k$ are arbitrary integers. If we define a function $K^*(x)$ by the formula

$$K^*(x)=K(x)+\sum_{\nu=1}^{\infty}\{K(x-x_\nu)-K(-x_\nu)\}, \tag{2.2}$$

the series on the right converges absolutely and uniformly over any bounded set provided we discard the first few terms. (Of course we can also replace the terms in curly brackets by $K(x+x_\nu)-K(x_\nu)$). For if $x \in S$ and ν is large enough, then

$$|K(x-x_\nu)-K(-x_\nu)| \leqslant \frac{1}{|x_\nu|^k}\,\omega\left(\frac{A}{|x_\nu|}\right) \tag{2.3}$$

(see [2], p. 95), and condition 1° imposed on the kernel K implies that the terms on the right here form a convergent series.

It is also easy to see that the function K^* has periods $e_1, e_2, \ldots, e_k$. For let us assign to the kernel $K(x)$ the value 0 at the origin. We may then write the series (2.2) in the form $\sum_{\nu=0}^{\infty}\{K(x-x_\nu)-K(-x_\nu)\}$. Hence

$$K^*(x)-K^*(x+e_1)=\sum_{\nu=0}^{\infty}\{K(x-x_\nu)-K(x-x_\nu^*)\},$$

where $x_\nu^*=x_\nu-e_1$. The last series converges absolutely. Let us fix the values of $m_2, \ldots, m_k$ and consider the sum of the terms with variable m_1. This sum is zero due to the cancellation of "adjoining" terms and the fact that $K(x)$ tends to zero as $|x|$ tends to infinity. Hence the sum of the whole last series is zero. It follows that $K^*(x)$ has the period e_1, and so also periods $e_2, \ldots, e_k$.

Of course, the passage from $K(x)$ to $K^*(x)$ is a clasical one. If $k=1$, $e_1=2\pi$, $K(x)=1/x$, then

$$K^*(x)=\frac{1}{2}\cot\frac{1}{2}x.$$

If $k=2$, $K(z)=1/z^2$, then $K^*(z)$ is the classical $\wp$ function of Weierstrass, etc.

Let us now suppose for the sake of simplicity that the vectors $e_1, e_2, \ldots, e_k$ are all mutually orthogonal and of length 1. We may assume that they are situated on coordinate axes. Let $f(x)=f(\xi_1, \xi_2, \ldots, \xi_k)$ be a function of period 1 in each ξ_j and integrable over every bounded set. We shall consider the convolution

$$f^*(x)=\int_R f(y)K^*(x-y)\,dy=\int_R K^*(y)f(x-y)\,dy, \tag{2.4}$$

where R is the "fundamental" cube

$$|\xi_j| \leqslant \frac{1}{2} \qquad (j=1,2,\ldots,k) \tag{R}$$

and the integral is taken in the principal value sense. For such functions f^* we easily obtain results analogous to Theorems A-G.

In what follows, $f(x)=f(\xi_1, \xi_2, \ldots, \xi_k)$ will systematically denote a function of period 1 in each variable (for simplicity, we shall use the abbreviation "periodic function"), integrable over R. By $\mathfrak{M}_p[f]$ we shall denote the norm

$$\Big(\int_R |f|^p\,dx\Big)^{1/p}.$$

THEOREM 1. *The integral* (2.4) *exists for almost every value of x. The same holds for the integral*

$$\int_R K^*(x-y)F(dy) \tag{2.5}$$

for any totally additive function of set F.

THEOREM 2. *If $f \in L^p$, $1<p<\infty$, then f^* also belongs to L^p and*

$$\mathfrak{M}_p[f^*] \leqslant A_p \mathfrak{M}_p[f]. \tag{2.6}$$

THEOREM 3. *If $|f|\log^+|f|$ is integrable over R, then*

$$\int_R |f^*|\,dx \leqslant A\int_R |f|\log^+|f|\,dx+B. \tag{2.7}$$

THEOREM 4. *If f is merely integrable over R, then for every $0<\alpha<1$ and every set $S\subset R$,*

$$\int_S |f^*|^\alpha dx \leqslant |S|^{1-\alpha}\frac{A}{1-\alpha}\Big(\int_R |f|\,dx\Big)^\alpha. \tag{2.8}$$

THEOREM 5. *If f is integrable over R, then the measure of the set E_y of points $x\in R$ at which $|f^*|\geqslant y>0$, satisfies the inequality*

$$|E_y| \leqslant \frac{A}{y}\mathfrak{M}_1[f]. \tag{2.9}$$

THEOREM 6. *If $|f|\leqslant 1$, then*

$$\int_R \exp\lambda|f^*|\,dx \leqslant A, \tag{2.10}$$

provided λ is small enough, $0<\lambda\leqslant\lambda_0(K^*)$. If f is also continuous, then the last integral is finite for every $\lambda>0$.

Given a function f, let

$$\varphi(x)=\sup_{0<\varepsilon\leqslant 1/2}\Big|\int_R f(x-y)K_\varepsilon^*(y)\,dy\Big|,$$

where $K_\varepsilon^*(x)$ is the function equal to $K^*(x)$ except in the ε-neighbourhoods of the lattice points x_ν, in which it is equal to zero.

THEOREM 7. *According as*

(i) $$f \epsilon L^p, \qquad p>1,$$

(ii) $$|f| \log^+|f| \epsilon L,$$

(iii) $$f \epsilon L,$$

we have respectively the inequalities (2.6), (2.7), (2.8), *with* f^* *replaced by* φ.

3. These results can be easily obtained from Theorems A-G. For let R_1 denote the cube

$$(\mathrm{R}_1) \qquad |\xi_j| \leqslant 1 \qquad (j=1,2,\ldots,k),$$

and let $f_1(x)$ be the function equal to $f(x)$ in R_1 and to zero elsewhere. The difference K^*-K being bounded in R,

$$|K^*(x)-K(x)|<B \quad \text{for} \quad x \epsilon R,$$

and the function $f(x-y)$ being identical with $f_1(x-y)$ for x and y in R, we see immediately that the second integral (2.4) converges for $x \epsilon R$ if and only if the integral

$$(3.1) \qquad \int_R K(y) f_1(x-y)\,dy \qquad (x \epsilon R)$$

does, and that the difference between these two integrals is numerically less than

$$B\int_R |f(x-y)|\,dy = B\int_R |f(y)|\,dy.$$

On the other hand, if we replace R in (3.1) by E^k we obtain the function $\tilde{f}_1(x)$, and the error committed will not exceed

$$C\int_{E^k-R} |f_1(x-y)|\,dy \leqslant C\int_R |f_1(y)|\,dy = C\int_R |f(y)|\,dy.$$

Collecting results we see that, for x in R, the integral $f^*(x)$ exists if and only if $\tilde{f}_1(x)$ does, and that

$$(3.2) \qquad |f^*(x)-\tilde{f}_1(x)| \leqslant A\mathfrak{M}_1[f] \qquad (x \epsilon R).$$

In particular, $f^*(x)$ exists almost everywhere in R. A similar argument applies to the integral (2.5) and Theorem 1 is established.

Passing to Theorem 2 we observe that (3.2) leads to

$$\mathfrak{M}_p[f] \leqslant \mathfrak{M}_p[\tilde{f}_1] + A\mathfrak{M}_1[f] \leqslant \|\tilde{f}_1\|_p + A\mathfrak{M}_p[f] \leqslant A_p\|f_1\|_p + A\mathfrak{M}_p[f] \leqslant A_p\mathfrak{M}_p[f],$$

which completes the proof.

If $|f|\log^+|f|$ is integrable, (3.2) and Theorem C, with $S=R$, give

$$\int_R |f^*|\,dx \leqslant \int_R |\tilde{f}_1|\,dx + A\mathfrak{M}_1[f]$$
$$\leqslant A\int_{R_1} |f_1|\log^+|f_1|\,dx + A + A\mathfrak{M}_1[f] \leqslant A\int_R |f|\log^+|f|\,dx + B,$$

and (2.7) is established.

Passing to Theorem 4, let us integrate the inequality

$$|f^*|^\alpha \leqslant |\tilde{f}_1|^\alpha + A^\alpha \mathfrak{M}_1^\alpha[f],$$

which follows from (3.2), over S. We get

$$\int_S |f^*|^\alpha dx \leqslant \int_S |\tilde{f}_1|^\alpha dx + |S|A^\alpha\mathfrak{M}_1^\alpha[f]$$
$$\leqslant \frac{A}{1-\alpha}|S|^{1-\alpha}\Big(\int_{E^k}|f_1|\,dx\Big)^\alpha + |S|A^\alpha\mathfrak{M}_1^\alpha[f]$$
$$\leqslant \frac{A}{1-\alpha}|S|^{1-\alpha}\Big(\int_R |f|\,dx\Big)^\alpha + |S|A^\alpha\mathfrak{M}_1^\alpha[f] \leqslant \frac{A}{1-\alpha}|S|^{1-\alpha}\mathfrak{M}_1^\alpha[f],$$

since $|S|\leqslant 1$. This gives (2.8).

For the proof of Theorem 5 we may suppose that $\mathfrak{M}_1[f]=1$. Let us denote the constants A in (1.5) and (3.2) by A' and A'' respectively. Let us also temporarily assume that $y/2 \geqslant A''$. From (3.2) we see that the set of points $x \epsilon R$ at which $|f^*| \geqslant y$ is contained in the set of points x at which $|\tilde{f}| \geqslant y/2$ and so, by Theorem E, has a measure not exceeding

$$A'\Big(\frac{1}{2}y\Big)^{-1}\int_{E^k}|f_1|\,dx = A'2^{k+1}y^{-1}.$$

This gives (2.9), with $A=2^{k+1}A'$, provided $y \geqslant 2A''$. Increasing the constant A so that A/y exceeds 1 for $0<y\leqslant 2A''$ we shall have the inequality (2.9) trivially satisfied for such y's. Thus (2.9) holds for all positive y.

Suppose now that $|f|\leqslant 1$. The inequality (3.2) shows that $|f^*-\tilde{f}_1|\leqslant A$ in R. If we denote by $g_1(x)$ the function (1.6) corresponding to f_1, then clearly $|\tilde{f}_1-g_1|\leqslant A$ in R. Hence, by (3.2),

$$|f^*(x)-g_1(x)| \leqslant A \quad \text{for} \quad x \epsilon R.$$

Using this and (1.7) for $S=R$, we obtain

$$\int_R \exp\lambda|f^*(x)|\,dx \leqslant \int_R e^{\lambda A}e^{\lambda|g_1(x)|}\,dx \leqslant A \quad \text{for} \quad \lambda\leqslant\lambda_0,$$

which is (2.10). Similarly we obtain the result concerning continuous functions f.

The proof of Theorem 7 is analogous to the proofs of the preceding results provided we use Theorem G and the inequality

$$|\tilde{f}_\varepsilon(x)-f^*_\varepsilon(x)|\leqslant A\mathfrak{M}_1[f],$$

analogous to (3.2) and established similarly.

Remarks. 1° The best possible constant A_p in (1.2) satisfies inequalities

$$A_p\leqslant\frac{A}{p-1}\quad\text{for}\quad 1<p\leqslant 2,\qquad A_p\leqslant Ap\quad\text{for}\quad p\geqslant 2. \tag{3.3}$$

Moreover, if $\bar{A}_p$ is the constant corresponding to the kernel $K(-x)$, then

$$\bar{A}_p=A_{p'}\quad\text{for}\quad p'=\frac{p}{p-1}\qquad(1<p<\infty)$$

(see [2]). It follows from the proof of Theorem 2 that the same results hold for the constant A_p in (2.6).

That the constant A_p in part (i) of Theorem G satisfies inequality (3.3) is proved in [3]. Therefore this inequality also holds in part 1° of Theorem 7.

2° Using the second inequality (3.3) we can deduce (2.10) from (2.6) by a familiar argument. For, if $|f|\leqslant 1$, then

$$\int\limits_R \exp\lambda|f^*|\,dx<2\int\limits_R\cosh\lambda|f^*|\,dx=2\left(1+\sum_{\nu=1}^{\infty}\frac{\lambda^{2\nu}}{(2\nu)!}\int\limits_R|f^*|^{2\nu}dx\right)$$
$$\leqslant 2+2\sum_{\nu=1}^{\infty}\frac{(2A\lambda\nu)^{2\nu}}{(2\nu)!}\int\limits_R|f|^{2\nu}dx\leqslant 2+2\sum_{\nu=1}^{\infty}\frac{(2A\lambda\nu)^{2\nu}}{(2\nu)!},$$

and the last series is finite provided $\lambda A<e^{-1}$.

Since the second estimate (3.3) holds if we replace f^* by φ in (2.6), it follows that also in (2.10) we can replace f^* by φ.

3° Theorem D has an analogue for the functions (1.1a), with $\|\tilde{f}\|_1$ in (1.4) replaced by the total variation of F over E^k (see [2]). A corresponding result holds for the function (2.5).

4. We now pass to a different group of theorems. Let us consider a periodic function $f(x)=f(\xi_1,\xi_2,\ldots,\xi_k)$ and its Fourier series

$$f(x)\sim\sum c_{\mu_1,\mu_2,\ldots,\mu_k}e^{2\pi i(\mu_1\xi_1+\ldots+\mu_k\xi_k)}=\sum c_m e^{2\pi i(m,x)}, \tag{4.1}$$

where $m=(\mu_1,\ldots,\mu_k)$, and (m,x) is the scalar product $\mu_1\xi_1+\ldots+\mu_k\xi_k$ of m and x.

If $f\epsilon L^p$, $p>1$, or if only $|f|\log^+|f|$ is integrable, then the periodic function $f^*(x)$ is also integrable and we may consider its Fourier series

$$f^*(x)\sim\sum c^*_m e^{2\pi i(m,x)}. \tag{4.2}$$

Our next problem will be to consider relations between the series (4.1) and (4.2). For $k=1$ and $K(x)=1/x$, the series (4.2) is the conjugate of (4.1), and in the general case we shall also call (4.2) the *conjugate* of (4.1), corresponding to the kernel K.

We first compute the Fourier coefficients γ_m of $K^*(x)$,

$$\gamma_m=\int\limits_R K^*(y)e^{-2\pi i(m,y)}dy, \tag{4.2a}$$

where the integral is taken in the principal value sense. That these coefficients exist, is clear if we observe that in the neighbourhood of the origin $K^*(x)$ differs from $K(x)$ by a bounded function, that for $|x|$ small

$$K(x)e^{-2\pi i(m,x)}=K(x)\{1+O(|x|)\}=K(x)+O(|x|^{-k+1}),$$

and that the integral of K over R, in the principal value sense, exists owing to condition 2° imposed on K.

Next, we show that for $m\neq(0,0,\ldots,0)$ we have

$$\gamma_m=\int\limits_{E^k}K(y)e^{-2\pi i(m,y)}dy=\lim_{\varepsilon\to 0}\int\limits_{E^k}K_\varepsilon(y)e^{-2\pi i(m,y)}dy, \tag{4.3}$$

i. e. the Fourier coefficient of K^* is equal to the corresponding Fourier transform of K.

One remark is indispensable here. The function K_ε, being in L^2, has by the theorem of Plancherel a Fourier transform, also in L^2. This transform is, however, defined almost everywhere only, while here we insist on its existence at the lattice points m. We must therefore show that under the conditions imposed on K the last integral in (4.3), defined as

$$\lim_{\varrho\to\infty}\int\limits_{|y|\leqslant\varrho}K_\varepsilon(y)e^{-2\pi i(m,y)}dy=\lim_{\varrho\to\infty}\int\limits_{\varepsilon\leqslant|y|\leqslant\varrho}K(y)e^{-2\pi i(m,y)}dy, \tag{4.4}$$

exists for each m. This will follow, as we are going to show, from the existence — already established — of the γ_m. We may assume that $m\neq(0,0,\ldots,0)$, since in the remaining case the limit (4.4) clearly exists and is zero.

Let R_ν be the cube with centre x_ν and congruent to R; thus $R=R_0$. Let $\Gamma(\varepsilon)$ denote the sphere with centre at the origin and radius ε, $0<\varepsilon\leqslant 1/2$. Using (2.2) we have

$$\begin{aligned}&\int\limits_{R-\Gamma(\varepsilon)}K^*(y)e^{-2\pi i(m,y)}dy\\ &=\int\limits_{R-\Gamma(\varepsilon)}K(y)e^{-2\pi i(m,y)}dy+\sum_{\nu=1}^{\infty}\int\limits_{R-\Gamma(\varepsilon)}\{K(y-x_\nu)-K(-x_\nu)\}e^{-2\pi i(m,y)}dy.\end{aligned} \tag{4.5}$$

Owing to the convergence of the series of the right sides in (2.3) we see that if in the last sum in (4.5) we replace the domain of integration $R-\Gamma(\varepsilon)$ by R, we commit an error $O(\varepsilon)$. Suppose that in the new series we only retain terms with $|x_\nu|\leqslant\varrho$. The contribution of the omitted terms tends to zero as $\varrho\to\infty$, and the sum retained is

$$\sum_{0<|x_\nu|\leqslant\varrho}\int_R\{K(y-x_\nu)-K(-x_\nu)\}e^{-2\pi i(m,y)}dy=\sum_{0<|x_\nu|\leqslant\varrho}\int_{R_\nu}K(y)e^{-2\pi i(m,y)}dy.$$

Observing that the measure of the union of the sets R_ν with $|x_\nu|\leqslant\varrho,\ \nu\neq 0$, differs from the measure of the sphere $|y|\leqslant\varrho$ by $O(\varrho^{k-1})$, and that $K(y)=O(|y|^{-k})$ for $|y|\to\infty$, we can write (4.5) in the form

$$\int_{R-\Gamma(\varepsilon)}K^*(y)e^{-2\pi i(m,y)}dy=\int_{2\leqslant|y|\leqslant\varrho}K(y)e^{-2\pi i(m,y)}dy+O(\varepsilon)+o_\varrho(1),$$

where $o_\varrho(1)$ is a quantity tending to zero as $\varrho\to\infty$. This formula not only proves the existence of (4.4) but also the equations (4.3), provided $m\neq(0,0,\ldots,0)$.

If $m=(0,0,\ldots,0)$, the preceding argument shows that

$$\gamma_0=-\lim_{\varrho\to\infty}\sum_{0<|x_\nu|\leqslant\varrho}K(-x_\nu)=-\lim_{\varrho\to\infty}\sum_{0<|x_\nu|\leqslant\varrho}K(x_\nu).$$

The latter quantity need not be zero and the formula (4.3) fails then. However, γ_0 is zero, for example in the case when the sum of the $K(x_\nu)$ extended over the x_ν situated in the circle $|x_\nu|\leqslant\varrho$ is zero. It is clearly so in the classical case $n=1$, $K(x)=1/x$. If $k=2$ and

$$K(z)=\frac{e^{im\varphi}}{|z|^2}=\frac{z^m}{|z|^{m+2}}\qquad(\varphi=\arg z,\ m=\pm1,\pm2,\ldots),$$

we have $\gamma_0=0$ if m is not divisible by 4. In particular, $\gamma_0=0$ for $K(z)=1/z^2$.

Remarks. 1° An argument similar to the proof of (4.3) shows that if the integral of f over R is zero, *i. e.* if $c_0=0$, then

$$f^*(x)=\int_{E^k}f(y)K(x-y)\,dy=\lim_{\varepsilon\to0}\Big\{\lim_{\varrho\to\infty}\int_{\varepsilon\leqslant|y|\leqslant\varrho}K(y)f(x-y)\,dy\Big\},$$

the inner limit existing for all x and the outer one if and only if the integral (2.4) converges. Also

$$f^*(x)=\int_{E^k}f(y)K(x-y)\,dy=\lim_{\varepsilon\to0}\Big\{\lim_{\varrho\to\infty}\int_{\varepsilon\leqslant|y|\leqslant\varrho}f(y)K(x-y)\,dy\Big\}.$$

2° We might slightly modify the definition of K^* so that all the previous results hold and (4.3) is also valid for $m=(0,0,\ldots,0)$. For if we set

$$I_\nu=\int_{R_\nu}f(y)\,dy,$$

$$K^*(x)=K(x)+\sum_{\nu=1}^{\infty}\{K(x+x_\nu)-I_\nu\},$$

the new kernel K^* differs from the old one by a constant only and we now have $\gamma_0=0$.

5. THEOREM 8. *If $f^*(x)$ is integrable (in particular if $f\in L^p$, $p>1$) then the Fourier series* (4.2) *of f^* has coefficients*

$$c_m^*=c_m\gamma_m, \tag{5.1}$$

where γ_m is given by (4.2a) *or, for $m\neq(0,0,\ldots,0)$, by* (4.3).

Given any trigonometric series $\Sigma c_m e^{2\pi i(m,x)}$, we may call

$$\sum c_m\gamma_m e^{2\pi i(m,x)}$$

the *conjugate* of the former, corresponding to the kernel K^*. Thus the result may be stated that if the conjugate function f^* is integrable, the Fourier series of f^* is the conjugate of the Fourier series of f.

This theorem is very well known in the one-dimensional case (see [8], pp. 153, 163). It is obtained there either through complex methods or by considering a certain definition of integral more general than that of Lebesgue. While the first approach fails for general k, the second is applicable straightforwardly.

Let $f(x)=f(\xi_1,\ldots,\xi_k)$ be measurable and periodic. Let us consider any partition P of the cube R into a finite number of parallelpipedes with edges parallel to the axes. These parallelpipeds will be denoted by $\Delta_1,\Delta_2,\ldots,\Delta_N$ and their measures by $|\Delta_1|,\ldots,|\Delta_N|$. Let $x_j=(\xi_1^j,\xi_2^j,\ldots,\xi_k^j)$ be any point of Δ_j and let $t=(\tau_1,\tau_2,\ldots,\tau_k)$ be any vector with $0\leqslant\tau_j\leqslant1$. Let us consider the sum

$$\sum_{j=1}^{N}f(x_j+t)|\Delta_j|. \tag{5.2}$$

It will be denoted by S or $S(t)$, $S[f]$, $S(f,t)$; of course it also depends on P and the points x_j.

If S converges in measure to a limit I as the norm of the partition P (*i. e.* the largest diameter of the Δ_j) tends to 0, we shall say that the function $f(x)$ is *integrable B over R* and that

$$(B)\int_R f(x)\,dx=I.$$

This is an immediate extension of the definition familiar in the case $k=1$ (see [8], p. 151) and the same proof as there shows that, *if f is integrable L over R it is also integrable B and both integrals have the same value*. Therefore, Theorem 8 will be a corollary of the following result:

THEOREM 9. *Suppose that $f(x)$ is periodic and L integrable. Then the function $f^*(x)$, as well as the functions $f(x)e^{-2\pi i(m,x)}$, are B-integrable over R, and the Fourier coefficients of f^*, in the B-sense, satisfy the equation* (5.1).

We first observe that, if f is a trigonometric monomial $e^{2\pi i(m,x)}$, then the conjugate function is (see (2.4))

$$\int_R K^*(y)\,e^{2\pi i(m,x-y)}\,dy = e^{2\pi i(m,x)}\int_R K^*(y)\,e^{-2\pi i(m,y)}\,dy = \gamma_m e^{2\pi i(m,x)}.$$

Hence, for any trigonometric polynomial $\sum c_m e^{2\pi i(m,x)}$ the conjugate function is $\sum c_m\gamma_m e^{2\pi i(m,x)}$.

We shall now show that for any $f\epsilon L$ and for any lattice point m the function $f(x)e^{-2\pi i(m,x)}$ is integrable B over R and the integral is $c_m\gamma_m$. In view of the preceding remark we may assume that $c_m=0$. Let us set $f=f'+f''$, where f' is a trigonometric polynomial and $\mathfrak{M}_1[f'']$ is small. Without loss of generality we may assume that the coefficient of $e^{2\pi i(m,x)}$ in f' is zero. Thus $f^*=f'^*+f''^*$ and

$$S(f^*e^{-2\pi i(m,x)},t)=S(f'^*e^{-2\pi i(m,x)},t)+S(f''^*e^{-2\pi i(m,x)},t). \tag{5.3}$$

The first term on the right tends to zero as the norm of the partition approaches zero. The absolute value of the second term is

$$\Big|\sum_{j=1}^{N} f''^*(x_j+t)\,e^{-2\pi i(m,x_j)}|\Delta_j|\Big|,$$

and the sum between the signs of absolute value is the conjugate g^* of the function

$$g(t)=\sum f''(x_j+t)\,e^{-2\pi i(m,x_j)}|\Delta_j|.$$

By Theorem 4, with $\alpha=1/2$ and $S=R$, the quantity $\mathfrak{M}_{1/2}[g^*]$ does not exceed a fixed multiple of

$$\mathfrak{M}_1[g(t)]\leqslant\sum_{j=1}^{N}|\Delta_j|\int_R|f''(x_j+t)|\,dt=\int_R|f''(t)|\,dt,$$

and so is small. It follows that g^*, and so also the last term in (5.3), is small except for t's belonging to a set of small measure, no matter what is the partition P. Therefore the left side of (5.3) is small except for t's in a small set, provided the norm of P is small enough. This shows that $f^*(x)e^{-2\pi i(m,x)}$ is integrable B over R and the integral is zero. Thus the proof of Theorem 9 is completed.

6. We shall now give a few illustrations for the results obtained. Suppose that the function Ω in the numerator of the kernel K is a spherical harmonic Y_n of order n. In other words, $Y_n(x')|x|^n$ is a homogeneous polynomial P of degree n in the variables $\xi_1,\xi_2,\ldots,\xi_k$, satisfying in these variables Laplace's equation $\Delta P=0$. Thus

$$K(x)=\frac{Y_n(x')}{|x|^k}\qquad (n=1,2,\ldots). \tag{6.1}$$

In the simplest case $k=2$, we have $Y_n=e^{in\varphi}$.

It is well known that the Fourier transform of K in (6.1) is a numerical multiple of Y_n; more precisely,

$$(2\pi)^{-k/2}\int_{E^k}\frac{Y_n(y')}{|y|^k}\,e^{-2\pi i(x,y)}\,dy=i^n Y_n(x')\,2^{-k/2}\,\frac{\Gamma\left(\frac{1}{2}n\right)}{\Gamma\left(\frac{1}{2}n+\frac{1}{2}k\right)}\ {}^{3)}. \tag{6.2}$$

Combining this with previous results we have the following theorem:

THEOREM 10. *Let $P(x)=P_n(x)$ be a homogeneous polynomial of degree n in the variables $\xi_1,\xi_2,\ldots,\xi_k$, satisfying Laplace's equation $\Delta P=0$. Given any L-integrable function*

$$f(x)\sim\sum c_m e^{2\pi i(m,x)}, \tag{6.3}$$

consider the series

$$\sum_{m\neq 0} c_m P(m')\,e^{2\pi i(m,x)}=\sum_{m\neq 0} c_m\frac{P(m)}{|m|^n}\,e^{2\pi i(m,x)}. \tag{6.4}$$

Then,

(i) *If $f\epsilon L^p$, $p>1$, the series* (6.4) *is the Fourier series of a function f^* of the class L^p, and f^* satisfies* (2.6);

(ii) *If $|f|\log^+|f|$ is integrable,* (6.4) *is the Fourier series of an $f^*\epsilon L$ and satisfying* (2.7);

---

[3]) See Bochner [1]. A different and independent proof for the case $k=3$ was obtained at about the same time by Prof. Szegö, but never published. In the case $k=2$, $y_n=e^{in\varphi}$, $n>0$, the proof of the formula (6.2) is very simple and the formula itself apparently much older though we cannot give any reference. See also Giraud [4]. It may be added that Bochner sums the integral (6.2) near $y=\infty$ by Abel's method but since the integral converges the sum in both cases must be the same.

Developping the function Ω into a series of spherical harmonics, $\Omega(y')\sim\Sigma Y_n(y')$, and using the formulas (6.2) we formally obtain the Fourier transform of the kernel $K=\Omega/r^n$. It can be shown that this argument is justified under very general conditions on Ω. We shall return to this problem elsewhere.

(iii) *If f is merely integrable*, (6.4) *is the Fourier series, in the B-sense, of a function* f^* *satisfying* (2.8);

(iv) *If* $|f|\leqslant 1$ *the function* f^* *satisfies* (2.10).

The simplest cases here are

$$P(x)=P_1(x)=\xi_j, \tag{6.5}$$

or

$$P(x)=P_2(x)=\xi_i\,\xi_j \qquad (i\neq j). \tag{6.6}$$

In these cases part (i) of Theorem had been proved by Marcinkiewicz [5]. His method does not yield the remaining parts of Theorem 9 since it consists of repeated application of a result from the case $k=1$ and somewhat looses strength as k increases. His proof works for more general cases than (6.5) or (6.6). Of course, also the argument given above applies to general series (5.1a), provided we know that the multipliers γ_m are the Fourier coefficients of a suitable kernel K of our type. To the problem what properties of the γ_m guarantee that assertion we shall return elsewhere.

7. We shall now prove results concerning the behaviour of the conjugate series in the case the function f satisfies a Lipschitz (Hölder) condition

$$|f(x+h)-f(x)|\leqslant C\,|h|^\alpha,$$

with C independent of x and h. If this condition is satisfied we shall write

$$f\in\Lambda_\alpha.$$

If we disregard constant functions, only the case $0<\alpha\leqslant 1$ need be considered.

In the one-dimensional case there is a familiar result, due to Privalov, asserting that if f is in Λ_α, $0<\alpha<1$, so is f^* (see *e. g.* [8], p. 156). The result is false for $\alpha=1$ (see below). The theorem which follows is an extension of Privalov's result to the k-dimensional case:

THEOREM 11. *Suppose that* $f(x)$ *is periodic and of the class* Λ_α, $0<\alpha<1$, *and that* $\Omega\in\Lambda_\beta, \beta>\alpha$, *on* Σ. *Then* $f^*\in\Lambda_\alpha$.

Proof. Suppose that $|h|$ is small. We may assume that the integral of K^* over R is zero. Then, denoting by $\Gamma(x,r)$ the sphere with centre x and radius r, we may write f^* in the form

$$\begin{aligned} f^*(x) &= \int_R [f(x-t)-f(x)]K^*(t)\,dt \\ &= \int_{R-\Gamma(0,3|h|)} [f(x-t)-f(x)]K^*(t)\,dt+O(|h|^\alpha), \end{aligned} \tag{7.1}$$

since the integrand here is $O(|t|^\alpha)\cdot O(|t|^{-k})$, and the integral of the latter function over $\Gamma(0,3|h|)$ is $O(|h|^\alpha)$. Thus

$$\begin{aligned} f^*(x+h) &= \int_{R-\Gamma(0,3|h|)} [f(x+h-t)-f(x)]K^*(t)\,dt+O(|h|^\alpha) \\ &= \int_{R(-h)-\Gamma(-h,3|h|)} [f(x-t)-f(x)]K^*(t+h)\,dt+O(|h|^\alpha), \end{aligned} \tag{7.2}$$

where $R(-h)$ denotes the cube R translated by $-h$. A simple calculation shows that if we replace in the last integral the domain of integration by $R-\Gamma(0,3|h|)$ we commit an error $O(|h|)+O(|h|^\alpha)=O(|h|)^\alpha$; for in the neighbourhood of the boundary of R the integrand is $O(1)$, and in the shell $|h|\leqslant|t|\leqslant 3|h|$ the integrand is $O(|h|^\alpha)\cdot O(|h|^{-k})$.

Thus

$$f^*(x+h)-f^*(x)=O(|h|^\alpha)+\int_{R-\Gamma(0,3|h|)}[f(x-t)-f(x)][K^*(t+h)-K^*(t)]\,dt. \tag{7.3}$$

The first factor in the integrand here is $O(|t|^\alpha)$. In estimating the second factor, in which $|h|\leqslant|t|/3$, we use the series (2.2) and the formula

$$\frac{\Omega[(t+h)']}{|t+h|^k}-\frac{\Omega(t')}{|t|^k}=\frac{\Omega[(t+h)']-\Omega(t')}{|t+h|^k}+\Omega(t')\left[\frac{1}{|t+h|^k}-\frac{1}{|t|^k}\right].$$

The contribution of the term $K(t)$ on the right is

$$O\left(\frac{|h|^\beta}{|t|^{k+\beta}}\right)+O\left(\frac{|h|}{|t|^{k+1}}\right)=O\left(\frac{|h|^\beta}{|t|^{k+\beta}}\right),$$

and the contribution of the remaining terms is

$$O(|h|^\beta)+O(|h|)=O(|h|^\beta).$$

Thus, collecting results, we see that the last integral does not exceed

$$O(|h|^\beta)\int_{|t|\geqslant 3|h|}\frac{dt}{|t|^{k+\beta-\alpha}}+O(|h|^\beta)\int_R O(1)\,dt=O(|h|^\alpha)+O(|h|^\beta)=O(|h|^\alpha),$$

which shows that $f^*\in\Lambda_\alpha$.

The following result is an obvious corollary of Theorem 11:

THEOREM 12. *Suppose that the function* $f(x)$ *given by* (6.3) *is of the class* $\Lambda_\alpha, 0<\alpha<1$, *and that* $P(x)$ *is the same as in Theorem* 10. *Then the séries* (6.4) *is the Fourier series of a function in* Λ_α.

Let now $f(x)$ denote any, not necessarily periodic, function defined in a domain $D\subset E^k$. We shall say that the *function* $f(x)$ *satisfies condition* Λ_*, and write

$$f\in\Lambda_*,$$

if $f(x)$ is continuous and if

$$|f(x+h)+f(x-h)-2f(x)| \leqslant C|h|, \tag{7.4}$$

for any x and h such that $x, x \pm h$ are in D, with C independent of x and h.

Clearly, $\Lambda_1 \subset \Lambda_*$. If the left side in (7.4) is $o(|h|)$ as $|h| \to 0$, uniformly in $x \epsilon D$, we shall write $f \epsilon \lambda_*$.

In a number of problems the class Λ_* seems to be a more natural one to consider than Λ_1. For example, the theorem of Privalov quoted above does not hold for $a=1$, but it can be shown that if $f(x) \sim \sum c_m e^{2\pi i m x}$ is in Λ_* (in particular, if $f \epsilon \Lambda_1$) then the conjugate function

$$\sum c_m (-i \operatorname{sign} m) e^{2\pi i m x}$$

is also in Λ_* (see [7]). An analogous result could be established for the k-dimensional case, but the proof then is decidedly more difficult and unlike the proof of Theorem 11 is not an imitation of the argument in the one-dimensional case. For this reason we shall confine our attention here to a rather special result which is of interest on account of certain applications.

Let us suppose that f is periodic and of the class Λ_1 and that Ω is merely bounded. We then still have (7.1) and (7.2). For $f^*(x+h)+f^*(x-h)-2f^*(x)$ we get an expression analogous to the right side of (7.3) with $K^*(t+h)-K^*(t)$ replaced by $K^*(t+h)+K^*(t-h)-2K^*(t)$. Suppose that for $|t| \geqslant 2|h|$ we have an inequality

$$|K(t+h)+K(t-h)-2K(t)| \leqslant A \frac{|h|^\gamma}{|t|^{k+\gamma}} \qquad (\gamma > 1) \tag{7.5}$$

valid for some γ and A independent of t and h. Then, using formula (2.2), we find, as before, that

$$K^*(t+h)+K^*(t-h)-2K^*(t) = O\left(\frac{|h|^\gamma}{|t|^{k+\gamma}}\right) + O(|h|^\gamma)$$

and

$$f^*(x+h)+f^*(x-h)-2f^*(x)$$

$$= O(|h|) + O(|h|^\gamma) \int\limits_{|t| \geqslant 3|h|} \frac{dt}{|t|^{k+\gamma-1}} + O(|h|^\gamma) \int\limits_R O(1)\,dt = O(h),$$

so that $f^* \epsilon \Lambda_*$.

A similar argument shows that if f is *continuously differentiable* (by this we mean that all derivatives of f of order 1 exist and are continuous) then $f^* \epsilon \lambda_*$. For we may write $f = f_1 + f_2$, where f_1 is a finite polynomial and $|f_2(x+t)-f_2(x)| \leqslant \varepsilon |t|$ for $|t|$ small enough. Then clearly,

$$|f_2^*(x+h)+f_2^*(x-h)-2f_2^*(x)| \leqslant C\varepsilon |h|,$$

for $|h|$ small enough, and since f_1^* is a trigonometric polynomial and so satisfies condition λ_*, the function $f^* = f_1^* + f_2^*$ also satisfies condition λ_*.

Inequality (7.5) is certainly valid, with $\gamma = 2$, if Ω has bounded second derivatives with respect to the spherical coordinates. For assuming, as we may, that Ω is real-valued, by the Mean-Value Theorem

$$K(t+h)+K(t-h)-2K(t) = |h|^2 K''(t+\theta h) = O(|h|^2)$$

for $|t| \geqslant 2|h|$, K'' denoting here the second directional derivative and $-1 < \theta < 1$. Thus

THEOREM 13. *Suppose that K satisfies (7.5). Then the assumption $f \epsilon \Lambda_1$ implies $f^* \epsilon \Lambda_*$, and if f is continuously differentiable, f^* satisfies condition λ_*. The conclusions hold, in particular, if f^* is defined by the series (6.4) and $P(m)$ is the same as in Theorem 10.*

In the concluding section of this paper we shall give a few observations about functions $f \epsilon \lambda_*$.

8. In this section we prove a few results about discrete analogous of the Hilbert transform.

Let $X = (\ldots, x_{-1}, x_0, x_1, \ldots, x_n, \ldots)$ be any two-way infinite sequence of real or complex numbers. For any $p > 0$ we shall denote by $\|X\|_p$ the p-th norm of X:

$$\|X\|_p = \left(\sum |x_n|^p\right)^{1/p}.$$

The class of sequences X with $\|X\|_p$ finite will be denoted by l^p.

Let $\tilde{X} = (\ldots, \tilde{x}_{-1}, \tilde{x}_0, \tilde{x}_1, \ldots)$ denote the sequence

$$\tilde{x}_m = {\sum_n}' \frac{x_n}{m-n}, \tag{8.1}$$

the prime indicating that the term $n=m$ is omitted in summation. This is a discrete analogue of the Hilbert transform

$$\tilde{f}(x) = \int_{-\infty}^{+\infty} \frac{f(y)}{x-y}\,dy, \tag{8.2}$$

and it is very well known that

$$\|\tilde{X}\|_2 \leqslant A \|X\|_2,$$

a result which was extended by Riesz [6] to

$$\|\tilde{X}\|_p \leqslant A_p \|X\|_p, \qquad p > 1. \tag{8.3}$$

The best value of A_2 is π; for other p's the best value of A_p is unknown.

Riesz [6] deduced (8.3) from the inequality $\|\tilde{f}\|_p \leqslant A_p \|f\|_p$, valid for the function (8.2). His argument is applicable to a more general class of discrete transforms which we are going to introduce now.

Let us again consider the space E^k, and a kernel $K(x) = \Omega(x')/|x|^k$ with properties described in section 1. Let $e_1, e_2, \ldots, e_k$ be a system of k linearly independent vectors in E^k; thus, in particular, all the e_j are different from zero. Let $p_0 = 0, p_1, p_2, \ldots$ be the sequence of all lattice points in E^k generated by this system, *i. e.* the p's are of the form $\mu_1 e_1 + \mu_2 e_2 + \ldots + \mu_k e_k$, where the coefficients μ_j are arbitrary real integers. For any sequence $X = (x_0, x_1, x_2, \ldots)$ of real or complex numbers we define the transform $\tilde{X} = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \ldots)$ by the formulae

$$\tilde{x}_m = \sum_n{}' x_n K(p_m - p_n). \tag{8.4}$$

For such sequences $\tilde{X}$ we have the following result generalizing (8.3):

THEOREM 14. *If X is in l^p, $p > 1$, so is $\tilde{X}$, and*

$$\|\tilde{X}\|_p \leqslant A_p \|X\|_p,$$

where A_p depends only on p and the kernel K.

The series on the right are all absolutely convergent if $X \in l^p$, $p \geqslant 1$. For $p = 1$ this is immediate, and for $p > 1$ follows by an application of Hölder's inequality; for since $K(x) = O(|x|^{-k})$ as $|x| \to \infty$, the series $\sum_n{}' |K(p_m - p_n)|^q$ is finite for every $q > 1$.

For the sake of simplicity we assume that the vectors $e_1, e_2, \ldots, e_k$ are all mutually orthogonal, of length 1, and situated on the coordinate axes. The proof in the general case remains essentially the same. Let R_m denote the cube with centre p_m and edges of length 1, parallel to the axes. By R'_m we shall denote the concentric and similarly situated cube with edges $1/2$. Given a sequence $X = (x_0, x_1, \ldots, x_m, \ldots)$ let $f(x)$ denote the function taking the value x_m at the points of R'_m $(m = 0, 1, 2, \ldots)$ and equal to zero elsewhere in E^k. The function f is in L^p if and only if X is in l^p, and the ratio $\|X\|_p / \|f\|_p$ depends on k and p only. Hence, on account of Theorem B of Section 1,

$$\sum_m \int_{R'_m} |\tilde{f}(x)|^p dx = \int_{\cup R'_m} |\tilde{f}(x)|^p dx \leqslant \|\tilde{f}\|_p^p \leqslant A_p^p \|f\|_p^p \leqslant A_p^p \sum_m |x_m|^p. \tag{8.6}$$

For $x \in R'_m$ we may write

$$\tilde{f}(x) = \sum_{n \neq m} x_n \int_{R'_n} K(x - y)\, dy + x_m \int_{R'_m} K(x - y)\, dy. \tag{8.7}$$

Let $\omega(\delta)$ be the modulus of continuity of Ω on Σ. Without loss of generality we may assume that $\omega(\delta) \geqslant \delta$, since otherwise in the inequalities that follow we replace $\omega(\delta)$ by $\omega_1(\delta) = \mathrm{Max}(\delta, \omega(\delta))$. We easily verify that, for $x \in R'_m$ and $y \in R'_n$

$$|K(x - y) - K(p_m - p_n)| \leqslant A \frac{\omega\left(\dfrac{1}{|p_m - p_n|}\right)}{|p_m - p_n|^k},$$

and (8.7) may be written

$$\tilde{f}(x) = 2^{-k} \sum_{n \neq m} x_n K(p_m - p_n) + x_m \int_{R'_m} K(x - y)\, dy + O\left\{\sum_{n \neq m} \frac{\omega\left(\dfrac{1}{|p_n - p_m|}\right)}{|p_n - p_m|^k}\right\}$$

$$= 2^{-k} \tilde{x}_m + x_m \tilde{\varphi}_m(x) + O\left\{\sum_{n \neq m} |x_n|\, |p_n - p_m|^{-k} \omega\left(\frac{1}{|p_m - p_n|}\right)\right\},$$

where φ_m is the characteristic function of the cube R'_m. Thus $|\tilde{x}_m|^p$ does not exceed a fixed multiple of the sum of the three expressions

$$|\tilde{f}(x)|^p, \qquad |x_m|^p |\tilde{\varphi}_m(x)|^p, \qquad \sum_{n \neq m} |x_n|\, |p_m - p_n|^{-k} \omega\left(\frac{1}{|p_m - p_n|}\right).$$

Let us integrate these expressions over the cube R'_m and sum the results over all m. It is enough to show that all the three sums are majorized by a fixed multiple of $\Sigma |x_n|^p$.

This is certainly the case for the sum involving $|\tilde{f}|^p$ (see (8.6)). Since

$$\int_{R'_m} |\tilde{\varphi}_m(x)|^p dx \leqslant \|\tilde{\varphi}_m\|_p^p \leqslant A_p^p \|\varphi_m\|_p^p \leqslant A_p^p,$$

also the second sum satisfies the condition. Finally, setting

$$a_m = |p_m|^{-k} \omega\left(\frac{1}{|p_m|}\right) \qquad \text{for} \quad m > 0, \ a_0 = 0,$$

and observing that the Dini condition imposed on ω implies that $\Sigma a_m = a$ is finite, we may write the following inequalities, in which p' is the exponent conjugate to p:

$$\sum_m \Big\{\sum_n |x_n| a_{m-n}\Big\}^p = \sum_m \Big\{\sum_n |x_n| a_{m-n}^{1/p} a_{m-n}^{1/p'}\Big\}^p$$

$$\leqslant \sum_m \Big\{\sum_n |x_n|^p a_{m-n}\Big\} \Big\{\sum_n a_{m-n}\Big\}^{p/p'} = a^{p/p'} \sum_m \Big\{\sum_n |x_n|^p a_{m-n}\Big\}$$

$$= a^{1+p/p'} \sum_n |x_n|^p = a^p \sum_n |x_n|^p.$$

This completes the proof of Theorem 14.

Of course, Theorem 14 can be restated in the language of bilinear (or quadratic) forms:

$$\left|\sum x_m y_n K(p_m - p_n)\right| \leqslant A_p \|X\|_p \|Y\|_{p'}, \qquad p>1.$$

9. Let us now change our notation slightly and let m, and similarly n, denote the general lattice point in E^k, *i. e.* $m = \mu_1 e_1 + \ldots + \mu_k e_k$, where the μ's are arbitrary integers and $e_1, \ldots, e_k$ are unit vectors mutually orthogonal.

THEOREM 15. *The series*

$$\sum_m{}' K(m) e^{2\pi i(m,x)} \tag{9.1}$$

is the Fourier series of a bounded function $\chi(x)$. *The number*

$$M = \operatorname{ess\,sup} |\chi(x)|$$

is the norm of the linear transformation

$$\tilde{x}_m = \sum_n{}' K(m-n) x_n, \tag{9.2}$$

considered as a transformation from l^2 *into* l^2.

We already know that the transformation (9.2) is bounded and from this fact we shall be able to deduce the boundedness of the function (9.1). Since $\Sigma |K(m)|^2$ is finite, (9.1) is in any case the Fourier series of a function $\chi \epsilon L^2$. Similarly, if $\sum |x_m|^2$ converges,

$$\sum x_m e^{2\pi i(m,x)} \sim \psi(x) \epsilon L^2.$$

By (9.2), $\sum \tilde{x}_m e^{2\pi i(m,x)}$ is the Fourier series of the integrable function

$$\varphi(x) = \psi(x)\chi(x). \tag{9.3}$$

Since $\sum |\tilde{x}_m|^2$ is finite, the function φ is even quadratically integrable and we can write

$$\sum |\tilde{x}_m|^2 = \int_R |\varphi|^2 dx = \int_R |\psi|^2 |\chi|^2 dx \leqslant M^2 \int_R |\psi|^2 dx = M^2 \sum |x_m|^2,$$

so that the number $M (\leqslant \infty)$ is not less than the norm of the transformation (9.2). Moreover, one immediately sees that M is actually equal to the norm of the transformation. Since the transformation is bounded, the theorem follows.

An interesting illustration is provided in the two-dimensional case by the transformation

$$\tilde{x} = \sum_n{}' \frac{x_n}{(m-n)^2}, \tag{9.4}$$

where m and n denote complex integers. This seems to be the most natural extension of the classical Hilbert transformation (8.1) to the two-dimensional case. The norm of (9.4) is the upper bound of the modulus of the function given by the Fourier series

$$\sum_{\mu,\nu}{}' \frac{e^{2\pi i(\mu x + \nu y)}}{(\mu + i\nu)^2}.$$

The latter function occurs already in the work of Kronecker on elliptic functions and is expressible in terms of elliptic theta functions.

10. We conclude by a few remarks concerning *smooth functions*.

Suppose a function $f(x)$ is determined in the neighbourhood of a point $x_0 \epsilon E^k$. We say that f is *smooth* at x_0, if

$$f(x_0+h) + f(x_0-h) - 2f(x_0) = o(h) \quad \text{as} \quad |h| \to 0. \tag{10.1}$$

If f is smooth *and continuous* at every point of an open set D, f will be called *smooth in* D. The latter notion has close connection with condition λ_* introduced in Section 7, the only difference being that the notion of smoothness in D does not presuppose the uniformity of the "o" in (10.1) with respect to $x_0 \epsilon D$.

Clearly, if f is differentiable at x_0 (*i. e.*, if it has a total differential at x_0), then f is smooth at x_0, but the converse need not be true. Thus smooth functions may be considered as a generalization of differentiable functions. Similarly, functions of the class λ_* may be considered as a generalization of continuously differentiable functions (*i. e.*, functions with continuous first partial derivatives).

The notion of smoothness of functions is familiar in the simplest case $k=1$ (see [7]), and the definition (10.1) seems to be a natural extension of that special case to general k. In what follows we shall prove a few simple results concerning smooth functions. We shall not presuppose any longer that the functions f considered are periodic, and the results themselves will have little connection with the previous discussion.

(a) *If f is smooth in D and real-valued, and if f has a maximum (or minimum) at x_0, then f is differentiable at x_0 and the partial derivatives of f at x_0 with respect to the coordinates are zero.*

This is immediate since (10.1) can be written

$$\{f(x_0+h) - f(x_0)\} + \{f(x_0-h) - f(x_0)\} = o(|h|),$$

and since for $|h|$ small enough both terms in curly brackets are of the same sign, we get $f(x_0+h) - f(x_0) = o(|h|)$, which is the desired result.

(b) *If f is smooth in D and real-valued, then the set S of the points of differentiability of f is dense in D; indeed, it is of the power of the continuum in every sphere K totally contained in D.*

We may suppose that the closure of K is in D. Let $g(x)$ be a real-valued and continuously differentiable function vanishing on the boundary of K, positive inside K and taking a large value at the centre of K. Then the sum $h=f+g$ certainly has a maximum at a point $x_0 \epsilon K$, and so is differentiable at x_0. Hence, also f is differentiable at x_0, which shows that S is dense in D.

Let now $l(x)=a_1\xi_1+a_2\xi_2+\ldots+a_k\xi_k$ be a real-valued linear function with coefficients $a_1, a_2, \ldots, a_k$ numerically small but otherwise quite arbitrary. The function $h(x)=f(x)+g(x)+l(x)$ will then still have a maximum at a point $x_0(\xi_1,\ldots,\xi_k)\epsilon K$, and so will be differentiable at that point. Moreover, the first partial derivatives of $h(x)$ at $x_0(\xi_1,\ldots,\xi_k)$ will be zero, and so the first partial derivatives of $f(x)+g(x)$ at that point will be $-a_1,\ldots,-a_k$. Thus the point $x_0(a_1,\ldots,a_k)$ varies with the system $(a_1,\ldots,a_k)$. It follows that the set of points $x_0(a_1,\ldots,a_k)$ is of the power of the continuum, $h=f+g+l$ is differentiable in a subset of K of the power of the continuum, and the same holds for f.

(c) *The partial derivatives* $u_{\xi_i}(\xi_1,\ldots,\xi_k)$ *of the potential*

$$u(\xi_1,\ldots,\xi_k)=u(x)=\int_{E^k} f(y)\frac{dy}{|x-y|^{k-2}} \qquad (k>2) \tag{10.2}$$

corresponding to a continuous density f, satisfy condition λ_* in every finite sphere.

Without loss of generality we may suppose that $f(y)$ vanishes for $|y|$ large. It is a classical fact that under the assumption of continuity of f the partial derivatives u_{ξ_i} exist everywhere, are continuous and given by the formulae

$$u_{\xi_i}(x)=-(k-2)\int_{E^k} f(y)\frac{\xi_i-\eta_i}{|x-y|^k}\,dy. \tag{10.3}$$

It is also very well known that the second partial derivatives of u need not exist at individual points, and statement (c) is a substitute for the existence of these derivatives.

It is enough to give a sketch of proof since the whole argument follows familiar lines. On the right of the last formula we have a convolution of f with the kernel $K(x)=-(k-2)\,\xi_i/|x|^k$. In the integrals

$$\int_{E^k} f(y)K(x-y)\,dy, \qquad \int_{E^k} f(y)K(x\pm h-y)\,dy,$$

we consider separately the parts extended over the sphere $|y-x|\leqslant 2|h|$ and over the remainder of the space E^k. Since f is bounded, and

$K(z)=O(|z|^{-k+1})$ for small $|z|$, the parts extended over the sphere are all $O(|h|)$. Since

$$\Big|\int_{|x-y|\geqslant 2|h|} f(y)\{K(x+h-y)+K(x-h-y)-2K(x-y)\}\,dy\Big| < O(|h|^2)\int_{|x-y|\geqslant 2|h|} |f(y)|\,|x-y|^{-k-1}\,dy=O(|h|),$$

collecting results we see that $u_{\xi_i}(x+h)+u_{\xi_i}(x-h)-2u_{\xi_i}(x)=O(|h|)$.

So far we have only used the boundedness of f and showed that then $u_{\xi_i}(x)$ satisfies condition Λ_* in every sphere. Since in the formula (10.3) we may replace $f(y)$ on the right by $f(y)-f(x)$, the condition Λ_* refines to λ_* if f is continuous.

Of course, (c) also holds for $k=2$ if we replace (10.2) by the logarithmic potential. The result in this case was pointed to us by W. H. Oliver, and clearly the proof for $k>2$ is essentially the same.

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