A generalization of a Zygmund-Bernstein theorem

by J. Albrycht (Poznań)

R. Bellman [1], using an interpolation formula of B. Civin [3], has proved the following

Theorem 1. *If*

$$s_n(t)=\sum_{k=-l}^{l} a_k e^{i\lambda_k t} \qquad (\ldots\lambda_{k+1}>\lambda_k>\ldots>\lambda_0>\ldots>\lambda_{-k}>\lambda_{-k-1}>\ldots)$$

then, $|\lambda_l|=|\lambda_{-l}|$,

$$\mathfrak{M}_p\{s_n'(t)\}\leqslant\lambda_l\mathfrak{M}_p\{s_n(t)\} \quad (p\geqslant 1),$$

where $\mathfrak{M}_p\{f(t)\}=\lim_{T\to\infty}\left(\frac{1}{T}\int_{-T}^{T}|f(t)|^p\,dt\right)^{1/p}$.

This theorem is an extension of Zygmund-Bernstein's theorem [4] to the case of general trigonometric polynomials with non-integral exponents.

The purpose of this Note is to give a further extension of Theorem 1 to a wider class of functions than L^p, $p\geqslant 1$. This extension is based on two lemmata, the first of which is an interpolation formula of Civin and the second a generalization of Bellman's lemma.

Lemma 1. *If*

$$f(t)=\int_{-R}^{R} e^{i\xi t}\,ds(\xi), \qquad g(t)=\int_{-R}^{R}\xi e^{i\xi t}\,ds(\xi),$$

where $s(\xi)$ *is a complex-valued function of bounded variation on* $(-R,+R)$, *then*

$$g(t)=\sum_{k=-\infty}^{\infty}c_k f\left(\frac{k\pi}{R}-\frac{\pi}{2R}+t\right),$$

where

$$\xi e^{\frac{i\pi\xi}{2R}}=\sum_{k=-\infty}^{\infty}c_k e^{\frac{i\pi\xi}{R}k} \quad \text{and} \quad \sum_{k=-\infty}^{\infty}|c_k|<R<\infty.$$

Let M, N be a pair of functions complementary in the sense of Birnbaum-Orlicz (see paper [2], p. 8). We shall denote by $\mathfrak{N}$ the class of

all complex-valued functions $y(t)$, $-\infty<t<+\infty$, measurable in every finite interval, and such that

$$\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}N[|y(t)|]\,dt<\infty.$$

Now we denote by $\overline{\mathfrak{M}}$ the class of all complex-valued functions $x(t)$, $-\infty<t<+\infty$, measurable in every finite interval, such that the product $x(t)y(t)$ for every $y\in\mathfrak{N}$ is integrable in every finite interval and

$$\overline{\lim_{T\to\infty}}\frac{1}{2T}\left|\int_{-T}^{T}x(t)y(t)\,dt\right|<\infty.$$

For elements of this class we define the pseudonorm by formula

$$\|x\|=\sup_E\left\{\overline{\lim_{T\to\infty}}\frac{1}{2T}\left|\int_{-T}^{T}x(t)y(t)\,dt\right|\right\},$$

where E denotes the set of elements which belong to $\mathfrak{N}$ and satisfy the condition

$$\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}N[|y(t)|]\,dt\leqslant 1.$$

The class $\overline{\mathfrak{M}}/Q$, where Q denotes a set of elements belonging to $\overline{\mathfrak{M}}$ for which the pseudonorm is zero, we call the Marcinkiewicz-Orlicz space; it is linear, metric, and complete [1]).

Lemma 2. *If* f, *and* g *are elements of* $\overline{\mathfrak{M}}$ *and have the same meaning as in Lemma* 1, *then*

$$\|g\|\leqslant\Big(\sum_{k=-\infty}^{\infty}|c_k|\Big)\|f\|.$$

This lemma follows immediately from the inequality

$$\frac{1}{2T}\left|\int_{-T}^{T}\sum_{k=-l}^{l}c_k f\left(\frac{k\pi}{R}-\frac{\pi}{2R}+t\right)y(t)\,dt\right| \leqslant\sum_{k=-l}^{l}|c_k|\cdot\frac{1}{2T}\left|\int_{-T}^{T}f\left(\frac{k\pi}{R}-\frac{\pi}{2R}+t\right)y(t)\,dt\right|,$$

[1]) This facts and other properties of the Marcinkiewicz-Orlicz space will be proved in a separate Note which be devoted to the investigation of that space.

because for $y \in E$ it leads to the inequality

$$\overline{\lim_{T\to\infty}} \frac{1}{2T} \left| \int_{-T}^{T} \sum_{k=-l}^{l} c_k f\left(\frac{k\pi}{R} - \frac{\pi}{2R} + t\right) y(t)\,dt \right|$$
$$\leqslant \sum_{k=-l}^{l} |c_k| \overline{\lim_{T\to\infty}} \frac{1}{2T} \left| \int_{-T}^{T} f\left(\frac{k\pi}{R} - \frac{\pi}{2R} + t\right) y(t)\,dt \right|$$
$$\leqslant \Big(\sum_{k=l}^{l} |c_k|\Big) \|f\| \leqslant \Big(\sum_{k=-\infty}^{\infty} |c_k|\Big) \|f\|,$$

which proves the Lemma.

THEOREM 2. *If*

(1) $$s_n(t) = \sum_{k=-l}^{l} a_k e^{i\lambda_k t} \qquad (\ldots \lambda_{k+1} > \lambda_k > \ldots > \lambda_0 > \ldots > \lambda_{-k} > \lambda_{-k-1} > \ldots)$$

then, $|\lambda_l| = |\lambda_{-l}|$, $\|s_n'(t)\| \leqslant \lambda_l \|s_n(t)\|$.

We can represent the trigonometric sum (1) in the following form:

$$s_n(t) = \int_{-\lambda_l}^{\lambda_l} e^{i\xi t} ds(\xi) = f(t),$$

where

$$s(\xi) = \sum_{k<\xi} a_k, \qquad \int_{-\lambda_l}^{\lambda_l} |ds(\xi)| \leqslant \sum_{x=-l}^{l} |a_k| < \infty.$$

We also have

$$g(t) = f'(t) = i \int_{-\lambda_l}^{\lambda_l} \xi e^{i\xi t} ds(\xi).$$

Therefore we see that the theorem follows, as in [1], from the above lemmata and a remark that the functions f and g are elements of the class $\mathfrak{M}$ since they are trigonometric polynomials.

References

[1] R. Bellman, *A generalization of Zygmund-Bernstein theorem*, Duke Math. Jour. 10 (1943), p. 649-651.

[2] Z. W. Birnbaum, W. Orlicz, *Über die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen*, Studia Mathematica 3 (1931), p. 1-68.

[3] P. Civin, *Inequalities for trigonometric integrals*, Duke Math. Jour. 8 (1941), p. 658-665.

[4] A. Zygmund, *A remark on conjugate series*, Proceedings of the London Math. Soc. (2), vol. 34 (1932), p. 392-400.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Démonstration du théorème de Osgood-Carathéodory par la méthode des points extrémaux

par W. KLEINER (Kraków)

1. Introduction. Soit D un domaine plan simplement connexe contenant le point à l'infini et dont la frontière C ne se réduit pas à un seul point. Soit

(1) $$\eta^{(n)} = \{\eta_0^{(n)}, \eta_1^{(n)}, \ldots, \eta_n^{(n)}\}$$

un système de points extrémaux de C du rang n, c'est-à-dire un système de $n+1$ points de C, tel que le produit

$$V(\eta^{(n)}) = \prod_{0\leqslant i<k\leqslant n} |\eta_i^{(n)} - \eta_k^{(n)}|$$

soit le plus grand. On peut supposer que

$$\prod_{k=1}^{n} |\eta_0^{(n)} - \eta_k^{(n)}| \leqslant \prod_{\substack{k=0\\k\neq i}}^{n} |\eta_i^{(n)} - \eta_k^{(n)}|, \qquad i=1,2,\ldots,n.$$

Posons

(2) $$f_n(z) = e^{i\Theta_n} \sqrt[n]{\prod_{k=1}^{n} \frac{z-\eta_k^{(n)}}{\eta_0^{(n)}-\eta_k^{(n)}}}, \qquad n=1,2,\ldots,$$

où les nombres réels Θ_n et les radicaux sont choisis de manière que $f_n(a) > 0$ pour un point fixe $a \in D$.

F. Leja a démontré[1]) que la suite (2) converge dans D vers une limite $f(z)$, et que $w = f(z)$ donne la représentation conforme du domaine D sur le „cercle" $K\{|w|>1\}$, de manière que $f(\infty) = \infty$.

En partant de la formule (2), nous allons démontrer le théorème suivant:

THÉORÈME DE OSGOOD-CARATHÉODORY. *Lorsque C est une courbe de Jordan, la fonction $f(z)$ est prolongeable continûment au domaine fermé $\bar{D} = D + C$, et elle établit une correspondance biunivoque et bicontinue entre $\bar{D}$ et $\bar{K}$.*

[1]) Voir F. Leja, *Sur les suites de polynômes, les ensembles fermés et la fonction de Green*, Ann. Soc. Pol. de Math. 12 (1933), p. 57-71; *Sur une suite de polynômes et la représentation conforme d'un domaine plan quelconque sur le cercle*, Ann. Soc. Pol. de Math. 14 (1935), p. 116-134.