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On the degree of regularity of surfaces formed by the asymptotic integrals of differential equations

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Consider the system of differential equations

(0.1) $$d\eta_i/dt = F^{(i)}(\eta_1,\ldots,\eta_n) \qquad (i=1,2,\ldots,n)$$

and assume that the origin of the coordinate system $\Theta_n=(0,\ldots,0)$ is its singular point, *i. e.* that $F^{(i)}(0,\ldots,0)=0$ $(i=1,2,\ldots,n)$. Further, let us assume that all the characteristic roots of the matrix

$$C=(c_{ij}), \qquad c_{ij}=F^{(i)}_{\eta_j}(0,\ldots,0) \qquad (i,j=1,2,\ldots,n)$$

have the real parts different from zero and that there exists at least one pair of roots whose real parts are of opposite signs. Let us denote by S the set made up by the trajectories of the system (0.1) tending to the singular point Θ_n for $t\to\infty$.

I. G. Petrovskiĭ showed in 1934 [6], under fairly general assumptions regarding the regularity of functions $F^{(i)}(i=1,2,\ldots,n)$, that the set S is a continuous manifold. In 1940, M. Martin proved the analyticity of the surface S, assuming that the functions $F^{(i)}(i=1,2,\ldots,n)$ are analytic and putting certain additional restrictions on the matrix C.

In the present paper we prove that the manifold S is a surface of class C^p if the functions $F^{(i)}$ $(i=1,2,\ldots,n)$ are of class C^p $(p\geqslant 1)$ (cf. theorem 1, § 1). Besides, we give an effective construction of the sequence of functions uniformly convergent to the function whose graph is the surface S. The derivatives up to the pth order of the functions of the above sequence are uniformly convergent to the corresponding derivatives of the limit function.

The proof of theorem 1 is given in § 3 (theorem 1 bis) in a special coordinate system (coordinates $z_1,z_2,\ldots,z_n$). The passage from the coordinates $\eta_1,\ldots,\eta_n$ to $z_1,\ldots,z_n$ is given in § 2.

*) This result was communicated to the VIII Congress of Polish Mathematicians (September, 1953). It is also mentioned in the paper [7].

§ 1. Definition 1. We say that *the real matrix* $C=(C_{ij})$ $(i,j=1,2,\ldots,n)$ *is of type* T_q $(0<q<n)$ if the numbers of the characteristic roots whose real parts are negative or positive are respectively q and $n-q$.

Consider the system of differential equations

$$d\eta_i/dt=\sum_{j=1}^{n} c_{ij}\eta_j \qquad (i=1,2,\ldots,n)$$

which, by setting

(1.1) $$H=(\eta_1,\ldots,\eta_n), \qquad C=(c_{ij})$$

we may also write in the vectorial form

(1.2) $$dH/dt=CH.$$

It is known (cf. [5], chapter III, p. 207) that if the matrix C is of type T_q, the set made up by the trajectories of the system (1.2) tending to $\Theta_n=(0,\ldots,0)$ for $t\to 0$ is a certain q-dimensional hyperplane passing through the origin of the coordinate system Θ_n. We shall call it the negative characteristic hyperplane and denote it by N. Analogously we define the positive characteristic hyperplane P as hyperplane of the trajectories of the system (1.2) tending to Θ_n for $t\to-\infty$.

Let a system of differential equations

(1.3) $$d\eta_i/dt=F^{(i)}(\eta_1,\ldots,\eta_n) \qquad (i=1,2,\ldots,n)$$

be given. By introducing the vector-function $F(H)=\big(F^{(1)}(H),\ldots,F^{(n)}(H)\big)$ (cf. (1.1)) we write the system (1.3) in the form

(1.4) $$dH/dt=F(H).$$

THEOREM 1. *Let us assume that the function $F(H)$ is of class C^p $(p\geqslant 1)$ in the neighbourhood of Θ, $F(\Theta_n)=0$ $(i=1,2,\ldots,n)$, and that the matrix $C=(c_{ij})$, $c_{ij}=F^{(i)}_{\eta_j}(\Theta_n)$ $(i,j=1,2,\ldots,n)$ is of the type T_q.*

Under these hypotheses there exists a neighbourhood of the point Θ_n (a homeomorph of the n-dimensional open sphere) such that all the trajectories of the system (1.4) contained in this neighbourhood and tending to Θ_n for $t\to\infty$ make up a surface S of class C^p slight[1]*) with respect to the hyperplane N and tangent to it at Θ_n (interior point of S).*

Remark 1 (cf. [7], remarque 4). By the transformation $\tau=-t$ we obtain from theorem 1 an analogous theorem for the set R made up by the trajectories of the system (1.4) tending to Θ_n for $t\to-\infty$. From

[1]) This means that every hyperplane parallel to the positive characteristic hyperplane cuts the surface S at most at one point.

that follows the existence of a transformation of class C^p $(p \geqslant 1)$ which transforms the system (1.4) into a new system in such a manner that the surfaces S and R are transformed into plane surfaces, S^* and R^*, lying in the plane $\xi_{q+1} = \ldots = \xi_n = 0$ and in the plane $\xi_1 = \ldots = \xi_q = 0$ respectively. This transformation enables us to reduce the investigation of the asymptotic behaviour of the integrals tending to Θ_n to the investigation of a system of the form

$$d\xi_i/dt = G^{(i)}(\xi_1, \ldots, \xi_q, 0, \ldots, 0) \qquad (i = 1, 2, \ldots, q)$$

which has a knot at the point Θ_n.

Remark 2. M. Martin [4] has shown the analycity of the manifolds S and R under the conditions of analycity of the function $F(H)$ and under a certain additional assumptions [2]) regarding the matrix C. From our theorem it follows, without these additional assumptions regarding the matrix C, that the surfaces S and R are of class C^∞.

§ 2. Definition 2. Let $D = (d_{ij})$ $(i, j = 1, 2, \ldots, n)$ be an arbitrary real matrix and $W = (w_1, \ldots, w_n)$ an arbitrary real vector. In the present paper by the symbols $\|D\|$ and $\|W\|$ we shall denote the square root of the sum of the squares of all elements, *i. e.*

$$\|D\| = \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^2\Big)^{1/2}, \qquad \|W\| = \Big(\sum_{i=1}^{n} w_i^2\Big)^{1/2}.$$

In the proofs of the theorems we shall make use of the following inequalities, which are immediately implied by the above definition:

$$\|D_1 + D_2\| \leqslant \|D_1\| + \|D_2\|, \qquad \|D_1 \cdot D_2\| \leqslant \|D_1\| \cdot \|D_2\|,$$

$$\Big\|\int_{t_0}^{t} W(\tau)\,d\tau\Big\| \leqslant \int_{t_0}^{t} \|W(\tau)\|\,d\tau \quad \text{for} \quad t \geqslant t_0$$

under the hypothesis that $\int_{t_0}^{t} \|W(\tau)\|\,d\tau$ exists.

Let $Z = (z_1, \ldots, z_n)$ be a vector in the n-dimensional space and q a positive integer $0 < q < n$. Let us write

$$Z = (X, Y), \qquad X = (z_1, \ldots, z_q), \qquad Y = (z_{q+1}, \ldots, z_n).$$

We denote by $\Theta_n, \Theta_q, \Theta_{n-q}$ the origins of the coordinate systems Z, X and Y respectively.

Consider the system of differential equations written in the vectorial form

$$\text{(2.1)} \qquad dX/dt = AX + L^{(1)}(Z), \qquad dY/dt = BY + L^{(2)}(Z),$$

[2]) That is besides the assumption that the matrix C is ot type T_q.

in which A, B are constant real square matrices and $L^{(i)}(Z)$ $(i = 1, 2)$ are vector-functions.

HYPOTHESIS K_p. *We say that the system* (2.1) *satisfies the hypothesis* K_p $(p \geqslant 1)$ *if the following four conditions are fulfilled:*

W_1) $L^{(1)}(\Theta_n) = \Theta_q, \quad L^{(2)}(\Theta_n) = \Theta_{n-q},$

W_2) *there exists a number* $\lambda > 0$ *such that*

$$\|e^{At}\| \leqslant q e^{-\lambda t}, \qquad \|e^{-Bt}\| \leqslant (n-q)\, e^{-\lambda t} \quad \textit{for} \quad t \geqslant 0,$$

W_3) $UAU \leqslant -\lambda \|U\|^2$ *for every vector* $U = (u_1, \ldots, u_q)$,

W_4) *there exist constant matrices* $G^{(i)}$ $(i = 1, 2)$ *and vector-functions* $\mathcal{L}^{(i)}(Z)$ $(i = 1, 2)$ *of class* C^p *in the neighbourhood of* Θ_n *such that*

$$\text{(2.2)} \qquad L^{(1)}(Z) = G^{(1)}X + \mathcal{L}^{(1)}(Z), \qquad L^{(2)}(Z) = G^{(2)}Y + \mathcal{L}^{(2)}(Z),$$

$$\text{(2.3)} \qquad \|G^{(i)}\| \leqslant 4^{-1}\lambda n^{-3/2} \qquad (i = 1, 2),$$

$$\text{(2.4)} \qquad \mathcal{L}^{(1)}_{z_j}(\Theta_n) = \Theta_q, \qquad \mathcal{L}^{(2)}_{z_j}(\Theta_n) = \Theta_{n-q} \qquad (j = 1, 2, \ldots, n).$$

Remark 3. From the condition W_4) of hypothesis K_p follows the existence of a positive number R such that in the closed sphere with centre at Θ_n and radius R the functions $L^{(i)}(Z)$ are of class C^p and satisfy Lipschitz's condition

$$\text{(2.5)} \qquad \|L^{(i)}(Z) - L^{(i)}(\hat{Z})\| \leqslant 2^{-1}\lambda n^{-3/2}\|Z - \hat{Z}\| \qquad (i = 1, 2).$$

Indeed since $G^{(1)}, G^{(2)}$ are constant matrices and the functions $\mathcal{L}^{(i)}(Z)$ $(i = 1, 2)$ are of class C^p in the neighbourhood of Θ_n, it follows from (2.2) that $L^{(i)}(Z)$ are of class C^p in the neighbourhood of Θ_n. Inequalities (2.5) are obtained from the relations (2.2)-(2.4) on the basis of the mean value theorem applied to the functions $\mathcal{L}^{(i)}(Z)$ $(i = 1, 2)$, which are of class C^1 in the neighbourhood of Θ_n $(p \geqslant 1)$.

Definition 3. We shall denote by F_R the common part of the n-dimensional open sphere $V(\Theta_n, R)$ with centre at Θ_n and radius R and a cylinder defined by the inequality $\|X\| < R/2q$.

Definition 4. We shall denote by S the set formed by the trajectories of the system (2.1) which are contained in F_R and tend to Θ_n for $t \to \infty$.

In § 3 we shall prove the following theorem:

THEOREM 1 bis. *If the system* (2.1) *satisfies the hypothesis* K_p, *then the set* S *is a surface of class* C^p $(p \geqslant 1)$ *slight with respect to the hyperplane* $Y = \Theta_{n-q}$ *and tangent to it at the point* Θ_n *(interior point of* S*)*.

To reduce the proof of theorem 1 to the proof of theorem 1 bis we make use of the following lemma:

LEMMA 1. *If the functions* $F^{(i)}(H)=F^{(i)}(\eta_1,\ldots,\eta_n)$ $(i=1,2,\ldots,n)$ *are of class* C^p $(p\geqslant 1)$ *in the neighbourhood of* $\Theta_n=(0,\ldots,0)$, $F^{(i)}(\Theta_n)=0$ $(i=1,2,\ldots,n)$ *and the matrix* $C=(c_{ij})$, $c_{ij}=F^{(i)}_{\eta_j}(\Theta_n)$ $(i,j=1,2,\ldots,n)$ *is of type* T_q, *then there exists a real non-singular constant matrix* M, *such that the linear transformation* $Z=MH$ *transforms the system* (1.4) *into the system* (2.1), *which satisfies the hypothesis* K_p.

The proof of lemma 1, which is in fact based only on the classical theorem of Jordan on reducing a matrix to the canonical form, will be given in § 4 at the end of the paper.

From lemma 1 it follows that the system (1.4), satisfying the assumptions of theorem 1, may be transformed by means of a suitable non-singular transformation $Z=MH$ with constant real coefficients into the system (2.1), which satisfies the hypothesis K_p. Since the negative characteristic hyperplane of the system

$$dX/dt=[A+G^{(1)}]X,\qquad dY/dt=[B+G^{(2)}]Y,$$

which satisfies the inequalities W$_2$) and (2.3) is the hyperplane $Y=\Theta_{n-q}$, therefore, in view of the properties of the transformation $Z=MH$, it is evident that theorem 1 bis implies theorem 1.

§ 3. The proof of theorem 1 bis, to which this paragraph is devoted, will be preceded by some lemmas.

LEMMA 2. *Suppose that the inequalities* W$_2$) *with respect to the system of differential equations*

$$dX/dt=AX+L^{(1)}(Z),\qquad dY/dt=BY+L^{(2)}(Z) \tag{3.1}$$

are satisfied and the functions $L^{(i)}(Z)$ $(i=1,2)$ *are continuous for* $\|Z\|\leqslant R$. *Then each integral* $Z(t)=\big(X(t),Y(t)\big)$ *of the system* (3.1) *such that*

$$\|Z(t)\|\leqslant R \quad \text{for} \quad t\geqslant t_0,\qquad X(t_0)=\Xi,\qquad \Xi=(\xi_1,\ldots,\xi_q), \tag{3.2}$$

satisfies the system of integral equations

$$\begin{aligned} X(t)&=\exp[A(t-t_0)]\Xi+\int_{t_0}^{t}\exp[A(t-\tau)]L^{(1)}[Z(\tau)]d\tau,\\ Y(t)&=-\int_{t}^{\infty}\exp[-B(\tau-t)]L^{(2)}[Z(\tau)]d\tau. \end{aligned} \tag{3.3}$$

Proof. Every integral of the system (3.1) satisfying the second of the conditions (3.2) satisfies, for suitably chosen V_0, the system of integral equations

$$\begin{aligned} X(t)&=\exp[A(t-t_0)]\Xi+\int_{t_0}^{t}\exp[A(t-\tau)]L^{(1)}[Z(\tau)]d\tau,\\ Y(t)&=\exp[B(t-t_0)]V(t),\quad\text{where} \end{aligned} \tag{3.4}$$

$$V(t)=V_0+\int_{t_0}^{t}\exp[-B(\tau-t_0)]L^{(2)}[Z(\tau)]d\tau.$$

It follows that $Z(t)$ satisfies the first of the equations (3.3). To show that the second is also satisfied let us observe that by $\|V(t)\|\leqslant\|\exp[-B(t-t_0)]\|\cdot\|Y(t)\|$, inequality W$_2$) and the first of the conditions (3.2) we obtain the inequality $\|V(t)\|\leqslant R(n-q)e^{-\lambda(t-t_0)}$, and hence $V(t)\to\Theta_{n-q}$ for $t\to\infty$. It follows that

$$V_0=-\int_{t_0}^{\infty}\exp[-B(\tau-t_0)]L^{(2)}[Z(\tau)]d\tau$$

which proves, together with (3.4), that $Z(t)$ satisfies also the second of the equations (3.3).

LEMMA 3. *Suppose that the system* (3.1) *satisfies the hypothesis* K_p. *Let* R *be a positive number such that the inequalities* (2.5) (cf. remark 3) *hold in the closed sphere with centre at* Θ_n *and radius* R. *Let us denote by* ω *the* $q+1$ *dimensional set of points* $(t,\Xi)=(t,\xi_1,\ldots,\xi_q)$ *defined by the inequalities*

$$\|\Xi\|<r,\qquad t_0\leqslant t<\infty\qquad (r=R/2q). \tag{ω}$$

We assert that there exists a function $Z(t,\Xi)=\big(X(t,\Xi),Y(t,\Xi)\big)$ *which has the following properties* P$_1$) *and* P$_2$):

P$_1$) $Z(t,\Xi)$ *is in the set* ω *the limit of uniformly convergent sequence of successive approximations* ${}^{\nu}Z(t,\Xi)=\big({}^{\nu}X(t,\Xi),{}^{\nu}Y(t,\Xi)\big)$ *given by the relations*

$${}^{1}X(t,\Xi)=\exp[A(t-t_0)]\Xi,\qquad {}^{1}Y(t,\Xi)=\Theta_{n-q}, \tag{3.5}$$

$$\begin{aligned} {}^{\nu+1}X(t,\Xi)&=\exp[A(t-t_0)]\Xi+\int_{t_0}^{t}\exp[A(t-\tau)]L^{(1)}[{}^{\nu}Z(\tau,\Xi)]d\tau,\\ &\qquad\qquad (\nu=1,2,\ldots)\\ {}^{\nu+1}Y(t,\Xi)&=-\int_{t}^{\infty}\exp[-B(\tau-t)]L^{(2)}[{}^{\nu}Z(\tau,\Xi)]d\tau. \end{aligned} \tag{3.6}$$

For every fixed $\Xi\,(\|\Xi\|<r)$, $Z(t,\Xi)$ *is an integral of the system* (3.1) *and satisfies the conditions*

$$\|Z(t,\Xi)\|\leqslant R \quad \text{for} \quad t\geqslant t_0,\quad Z(t,\Xi)\to\Theta_n \quad\text{for}\quad t\to\infty,\quad X(t_0,\Xi)=\Xi. \tag{3.7}$$

If $\Xi=\Theta_q$ *then*

$$Z(t,\Theta_q)=\Theta_n\quad \text{for}\quad t\geqslant t_0. \tag{3.8}$$

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P$_2$) *For every fixed t, $t \geqslant t_0$ the function $Z(t,\Xi)$ is of the class C^p with respect to $\Xi = (\xi_1, \ldots, \xi_q)$ for $\|\Xi\| < r$, where all the derivatives of order $1,2,\ldots,p$ of $Z(t,\Xi)$ are limits of uniformly converging sequences of the corresponding derivatives of ${}^\nu Z(t,\Xi)$.*

Proof. For the sake of clarity, we divide the proof of lemma 3 into four stages.

I. We assert that the functions of the sequence ${}^\nu Z(t,\Xi) = ({}^\nu X(t,\Xi), {}^\nu Y(t,\Xi))$ formally determined by the relations (3.5) and (3.6), are defined and continuous in the set ω for $1 \leqslant \nu < \infty$ and have the following properties $(\nu = 1,2,\ldots)$:

$$\|{}^{\nu+1}Z(t,\Xi) - {}^\nu Z(t,\Xi)\| \leqslant qr2^{-\nu} \quad \text{in the set } \omega, \tag{3.9}$$

$$\|{}^\nu Z(t,\Xi)\| \leqslant [1+2^{-1}+\ldots+2^{-(\nu-1)}]qr < R \quad \text{in the set } \omega, \tag{3.10}$$

$$d({}^{\nu+1}X)/dt = A\cdot{}^{\nu+1}X + L^{(1)}({}^\nu Z), \qquad d({}^{\nu+1}Y)/dt = B\cdot{}^{\nu+1}Y + L^{(2)}({}^\nu Z), \tag{3.11}$$

$${}^\nu X(t_0,\Xi) = \Xi, \tag{3.12}$$

$${}^\nu Z(t,\Xi) \to \Theta_n \quad \text{for } t \to \infty, \tag{3.13}$$

$${}^\nu Z(t,\Theta_q) = \Theta_n \quad \text{for } t \geqslant t_0. \tag{3.14}$$

Remark 4. From the assumptions W$_2$) of hypothesis K_p it follows at once that the relations (3.6) define the function ${}^{\nu+1}Z(t,\Xi)$, which is continuous in the set ω provided ${}^\nu Z(t,\Xi)$ is a continuous function such that $\|{}^\nu Z(t,\Xi)\| \leqslant R$ for $(t,\Xi) \in \omega$.

We make use of remark 4 in proving that the sequence of functions formally defined by (3.5) and (3.6) is infinite.

By (3.5) and inequality W$_2$) we obtain, in view of $r = R/2q$, $\|Z(t,\Xi)\| = \|{}^1X\| \leqslant q\exp[-\lambda(t-t_0)]\|\Xi\| \leqslant qr < R$ for $t \geqslant t_0$.

Thus the relations (3.10), (3.13), (3.14) hold for $\nu = 1$ and by remark 4 the relations (3.6) define the function ${}^2Z(t,\Xi) = ({}^2X(t,\Xi), {}^2Y(t,\Xi))$, whence, by assumptions W$_2$), W$_1$) and (2.5) (cf. remark 3), we obtain

$$\begin{aligned}\|{}^2Z - {}^1Z\| &\leqslant \|{}^2X - {}^1X\| + \|{}^2Y - {}^1Y\| \\ &\leqslant q\int_{t_0}^{t} \exp[-\lambda(t-\tau)]\|L^{(1)}[{}^1Z(\tau,\Xi)]\|d\tau + \\ &\quad + (n-q)\int_t^\infty \exp[-\lambda(\tau-t)]\|L^{(2)}[{}^1Z(\tau,\Xi)]\|d\tau \\ &\leqslant \frac{q}{\lambda}\cdot\frac{\lambda}{2n}qr + \frac{n-q}{\lambda}\cdot\frac{\lambda}{2n}qr = \frac{qr}{2},\end{aligned}$$

which proves that (3.9) holds for $\nu = 1$.

Let

$${}^{\nu+1}m = \sup_{t_0 \leqslant t < \infty} \|{}^{\nu+1}Z(t,\Xi) - {}^\nu Z(t,\Xi)\|. \tag{3.15}$$

We have

$${}^{\nu+1}m \leqslant {}^\nu m/2 \qquad (\nu = 1,2,\ldots) \tag{3.16}$$

under the assumption that ${}^{\nu-1}Z, {}^\nu Z$ are continuous and that $\|{}^{\nu-1}Z\| \leqslant R$, $\|{}^\nu Z\| \leqslant R$. Indeed, from (3.6) under the assumption that $\|{}^{\nu-1}Z\| \leqslant R$, $\|{}^\nu Z\| \leqslant R$ we obtain, by assumption (2.5) and remark 4, the inequalities

$$\|{}^{\nu+1}X - {}^\nu X\| \leqslant q\int_{t_0}^{t} \exp[-\lambda(t-\tau)](\lambda/2n)\|{}^\nu Z - {}^{\nu-1}Z\|d\tau \leqslant {}^\nu mq/2n,$$

$$\|{}^{\nu+1}Y - {}^\nu Y\| \leqslant (n-q)\int_t^\infty \exp[-\lambda(\tau-t)](\lambda/2n)\|{}^\nu Z - {}^{\nu-1}Z\|d\tau \leqslant {}^\nu m(n-q)/2n$$

from which it follows that $\|{}^{\nu+1}Z - {}^\nu Z\| \leqslant {}^\nu m/2$ for $t \geqslant t_0$. From this and by the definition (3.15) we immediately obtain the relation (3.16), which was to be shown.

Suppose that the functions ${}^\nu Z(t,\Xi)$ $(\nu = 1,2,\ldots,p)$ are continuous in the set ω and the inequalities (3.9), (3.10), (3.14) hold for $\nu = 1,2,\ldots,p-1$. We shall show that the relations (3.6) define the functions ${}^{p+1}X, {}^{p+1}Y$ and that the inequalities (3.9), (3.10), (3.14) hold for $\nu = p$. Indeed,

$$\begin{aligned}\|{}^pZ\| &\leqslant \|{}^pZ - {}^{p-1}Z\| + \|{}^{p-1}Z\| \\ &\leqslant 2^{-(p-1)}qr + [1+2^{-1}+\ldots+2^{-(p-2)}]qr = [1+2^{-1}+\ldots+2^{-(p-1)}]qr\end{aligned}$$

It follows by $r = R/2q$ (cf. the definition of the set ω) that $\|{}^pZ\| \leqslant R$, and thus, by remark 4, the formulas (3.6) determine the functions ${}^{p+1}X, {}^{p+1}Y$, and since by the induction hypothesis we have also $\|{}^{p-1}Z\| \leqslant R$, we obtain from inequality (3.16)

$${}^{p+1}m \leqslant {}^pm/2. \tag{3.17}$$

Since (3.9) holds for $\nu = p-1$, we obtain from (3.17) and definition (3.15) the inequality

$$\|{}^{p+1}Z - {}^pZ\| \leqslant 2^{-p}qr \quad \text{for } t \geqslant t_0,$$

and from (3.6), the induction hypothesis (3.14) and W$_1$) it follows that relation (3.14) holds also for $\nu = p$.

Thus we have proved the existence of the sequence ${}^\nu Z(t,\Xi) = ({}^\nu X(t,\Xi), {}^\nu Y(t,\Xi))$ satisfying the relations (3.5), (3.6), (3.9), (3.10), (3.14) and the relations (3.11), (3.12), which are simple consequences of the relations (3.5) and (3.6). For the inductive proof of relation (3.13), which, as has been shown, is true for $\nu = 1$, let us suppose that

$${}^{p-1}Z(t,\Xi) \to \Theta_n \quad \text{for } t \to \infty. \tag{3.18}$$

Since $\|^{p-1}Z(t,\Xi)\| \leqslant R$, we obtain from the assumptions W_1), W_2) and (2.5) the inequalities

$$\|^pX\| \leqslant q\exp[-\lambda(t-t_0)]r + q(\lambda/2n)\int_{t_0}^{t}\exp[-\lambda(t-\tau)]\,\|^{p-1}Z(\tau,\Xi)\|\,d\tau,$$

$$\|^pY\| \leqslant (n-q)(\lambda/2n)\int_{t}^{\infty}\exp[-\lambda(\tau-t)]\,\|^{p-1}Z(\tau,\Xi)\|\,d\tau,$$

from which, by $\lambda>0$ and assumption (3.18), follows

$$\|^pZ\| \leqslant \|^pX\| + \|^pY\| \to 0 \quad \text{for} \quad t\to\infty$$

on the basis of l'Hospital's theorem applied to the integrals

$$J_1 = \int_{t_0}^{t}\exp[-\lambda(t-\tau)]\,\|^{p-1}Z(\tau,\Xi)\|\,d\tau = e^{-\lambda t}\int_{t_0}^{t}e^{\lambda\tau}\|^{p-1}Z(\tau,\Xi)\|\,d\tau,$$

$$J_2 = \int_{t}^{\infty}\exp[-\lambda(\tau-t)]\,\|^{p-1}Z(\tau,\Xi)\|\,d\tau = e^{\lambda t}\int_{t}^{\infty}e^{-\lambda\tau}\|^{p-1}Z(\tau,\Xi)\|\,d\tau.$$

The sequence $\{^{\nu}Z(t,\Xi)\}$, which we have defined, satisfies also the relations (3.13).

Now we pass to the second stage of the proof of lemma 1.

II. We assert that there exists a vectorial function $Z(t,\Xi) = (X(t,\Xi), Y(t,\Xi))$ which is continuous in the set ω and satisfies the following conditions:

(3.19) $\quad {}^{\nu}Z(t,\Xi) \rightrightarrows Z(t,\Xi)$ for $\nu\to\infty$ in the set ω,

(3.20) $\quad X(t_0,\Xi) = \Xi$,

(3.21) $\quad \|Z(t,\Xi)\| \leqslant R$ in the set ω,

(3.22) $\quad Z(t,\Theta_q) = \Theta_n$ for $t\geqslant t_0$,

(3.23) $\quad Z(t,\Xi)\to\Theta_n$ for every fixed Ξ when $t\to\infty$ $(\|\Xi\|<r)$,

(3.24) $\quad d({}^{\nu}X)/dt \rightrightarrows dX/dt, \quad d({}^{\nu}Y)/dt \rightrightarrows dY/dt$ in the set ω.

The existence and the continuity of the function $Z(t,\Xi)$ satisfying the relation (3.19) follows from the uniform convergency of the series $^1Z(t,\Xi) + \sum_{\nu=1}^{n}[{}^{\nu+1}Z(t,\Xi) - {}^{\nu}Z(t,\Xi)]$ (cf. (3.9)). The relations (3.20)-(3.22) follow from (3.19) by (3.12), (3.10), and (3.14) respectively. To prove (3.23) let us choose an arbitrary $\delta>0$ and find k and T_{Ξ} so large that (cf. (3.19), (3.13))

$$\|Z(t,\Xi) - {}^kZ(t,\Xi)\| \leqslant \delta/2 \quad \text{in the set } \omega,$$

$$\|^kZ(t,\Xi)\| \leqslant \delta/2 \quad \text{for } t\geqslant T_{\Xi} \quad (\Xi \text{ being fixed}).$$

From the above inequalities it follows at once that

$$\|Z(t,\Xi)\| \leqslant \delta \quad \text{for} \quad t\geqslant T_{\Xi},$$

which shows that relation (3.23) holds true.

From (3.11), (3.19) and the uniform continuity of the function $L^{(i)}(Z)$ follows the uniform convergency of the sequences $\{d({}^{\nu}X)/dt\}, \{d({}^{\nu}Y)/dt\}$, from which by applying once more relation (3.19) we obtain the relations (3.24).

III. From (3.11), (3.19), (3.24) we obtain

$$dX/dt = AX + L^{(1)}(Z), \qquad dY/dt = BY + L^{(2)}(Z),$$

which, together with the relations (3.20), (3.21), (3.23), proves that for every fixed $\Xi(\|\Xi\|<r)$, $Z(t,\Xi) = (X(t,\Xi), Y(t,\Xi))$ is an integral of the system (3.1) satisfying the conditions (3.7). It follows, on the basis of I and II (cf. in particular (3.19), (3.22)) that the property P_1) holds true.

IV. In the proof[3]) of the property P_2) we make use of the following theorem of the theory functions of real variable (theorem A) giving only an outline of its proof.

THEOREM A. *If $\Xi = (\xi_1, \xi_2, \ldots, \xi_q)$,*

(3.25) $\quad {}^{\nu}W(\Xi) \rightrightarrows W(\Xi)$ *for* $\nu\to\infty$ *in the sphere* $\|\Xi\|<r$

and all the derivatives of function ${}^{\nu}W(\Xi)$ $(\nu=1,2,\ldots)$ *up to the order* p *with respect to the variables* $\xi_1, \xi_2, \ldots, \xi_q$ *are bounded in common and have a common module of continuity*[4]) *in the sphere* $\|\Xi\|<r$, *then the limit function* $W(\Xi)$ *is of class* C^p *in the open sphere* $\|\Xi\|<r$ *and all the derivatives up to the order* p *of function* $W(\Xi)$ *are the limits of uniformly convergent sequences of the corresponding derivatives of functions* ${}^{\nu}W(\Xi)$.

Proof. Denote by ${}^{\nu}F(\Xi)$ the vector made up of all derivatives of the function ${}^{\nu}W(\Xi)$ up to the order p. By Arzela's theorem (cf. [1], p. 132) and the assumptions of theorem A it follows that from every sequence $\{^{\alpha_\nu}F(\Xi)\}$ chosen from the sequence $\{^{\nu}F(\Xi)\}$ one can choose a sequence $\{^{\beta_\nu}F(\Xi)\}$ which is uniformly convergent in the sphere $\|\Xi\|<r$. Thanks to the choice of the sequence $\{\alpha_\nu\}$ being arbitrary one can easily show

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[3]) This proof has been simplified thanks to a certain remark of A. Pliś.

[4]) Let $f(\Xi)$ be a real valued function defined in the set Ω. Let us set $\sigma(\delta,f) = \sup|f(\Xi) - f(\hat{\Xi})|$ for all $\Xi, \hat{\Xi}\in\Omega$ such that $\varrho(\Xi,\hat{\Xi})<\delta$. The function $\sigma(\delta,f)$ is called the module of continuity of the function $f(\Xi)$ if $\sigma(\delta)\to 0$ when $\delta\to 0$. We say that the functions f belonging to a certain family F are equi-continuous in the set Ω, or that they have a common module of continuity, if there exists a function $\sigma(\delta)$ defined for $\delta>0$ such that $\lim\sigma(\delta)=0$ for $\delta\to 0$ and $\sigma(\delta,f)\leqslant\sigma(\delta)$ for every function f belonging to the family F.

(cf. [3], v. II, p. 127) on the basis of relation (3.25) and the definition of the function ${}^{\nu}F(\Xi)$ that all the derivatives up to the order p of the sequence of functions ${}^{\nu}W(\Xi)$ are uniformly convergent to the corresponding derivatives of the limit-function $W(\Xi)$. Thus the function $W(\Xi)$ is (cf. the assumptions of theorem A) of class C^p in the sphere $\|\Xi\|<r$.

Definition 5. We shall denote by $D^k W(\Xi)$ an arbitrary (fixed) derivative of the order k of the function $W(\Xi)$ with respect to the variables $\xi_1, \xi_2, \ldots, \xi_q$.

From theorem A, definition 5 and relation (3.19) it follows at once that in order to prove the property P_2) it is sufficient to show that the sequence $\{{}^{\nu}Z(t,\Xi)\}$ has the following property P_2^*):

Property P_2^*). *There exist finite constants $a_1, a_2, \ldots, a_p$ and increasing functions $\sigma_1(u), \sigma_2(u), \ldots, \sigma_p(u)$ such that for every $(t,\Xi)\epsilon\omega, (t,\hat{\Xi})\epsilon\omega$ the following inequalities hold*:

$$\|D^{k}\cdot{}^{\nu}Z(t,\Xi)\|\leqslant a_k \quad \textit{for} \quad 1\leqslant\nu<\infty \qquad (k=1,2,\ldots,p), \tag{3.26}$$

$$\begin{gathered}\|D^{k}\cdot{}^{\nu}Z(t,\Xi)-D^{k}\cdot{}^{\nu}Z(t,\hat{\Xi})\|\leqslant\sigma_k(\|\Xi-\hat{\Xi}\|) \quad \textit{for} \quad \nu=1,2,\ldots,\\ \sigma_k(u)\to 0 \quad \text{for} \quad u\to 0 \qquad (k=1,2,\ldots,p).\end{gathered} \tag{3.27}$$

Proof. First we show by induction that for every $(t,\Xi)\epsilon\omega$, $(t,\hat{\Xi})\epsilon\omega$ we have

$$\|{}^{\nu}Z(t,\Xi)-{}^{\nu}Z(t,\hat{\Xi})\|\leqslant 2q\|\Xi-\hat{\Xi}\| \qquad (\nu=1,2,\ldots). \tag{3.28}$$

Indeed, from (3.5) and inequality W_2) we obtain

$$\|{}^{1}Z(t,\Xi)-{}^{1}Z(t,\hat{\Xi})\|\leqslant q\|\Xi-\hat{\Xi}\|,$$

while (3.6) and (2.5) and inequality W_2) imply the inequalities

$$\begin{gathered}\|{}^{k+1}X(t,\Xi)-{}^{k+1}X(t,\hat{\Xi})\|\\ \leqslant q\|\Xi-\hat{\Xi}\|+q\int_{t_0}^{t}\exp[-\lambda(t-\tau)](\lambda/2n)\|{}^{k}Z(\tau,\Xi)-{}^{k}Z(\tau,\hat{\Xi})\|\,d\tau,\\ \|{}^{k+1}Y(t,\Xi)-{}^{k+1}Y(t,\hat{\Xi})\|\\ \leqslant(n-q)\int_{t}^{\infty}\exp[-\lambda(\tau-t)](\lambda/2n)\|{}^{k}Z(\tau,\Xi)-{}^{k}Z(\tau,\hat{\Xi})\|\,d\tau,\end{gathered}$$

from which, under the assumption that (3.28) holds for $\nu=k$, we obtain the inequality

$$\begin{aligned}\|{}^{k+1}Z(t,\Xi)-{}^{k+1}Z(t,\hat{\Xi})\|\leqslant q\|\Xi-\hat{\Xi}\|+(q/\lambda)(\lambda/2n)2q\|\Xi-\hat{\Xi}\|+\\ +[(n-q)/\lambda](\lambda/2n)2q\|\Xi-\hat{\Xi}\|=2q\|\Xi-\hat{\Xi}\|\end{aligned}$$

which shows that (3.28) holds for $\nu=k+1$.

We shall denote[5]) by $L_{(Z)}^{(i)}(Z)$ the jacobian matrix of the vector-function $L^{(i)}(Z)$ $(i=1,2)$ with respect to the variables $Z=(z_1,\ldots,z_n)$.

From assumption (2.5) we obtain

$$\|L_{(Z)}^{(i)}(Z)\|\leqslant\lambda/2n \quad \text{for} \quad \|Z\|\leqslant R \qquad (i=1,2) \tag{3.29}$$

and from the uniform continuity of the function $L_{(Z)}^{(i)}(Z)$ in the bounded and closed set $\|Z\|\leqslant R$ follows the existence of an increasing function $\delta(r)$ such that

$$\begin{gathered}\|L_{(Z)}^{(i)}(Z_1)-L_{(Z)}^{(i)}(Z_2)\|\leqslant\delta(\|Z_1-Z_2\|),\\ \delta(r)\to 0 \quad \text{for} \quad r\to 0 \qquad (i=1,2).\end{gathered} \tag{3.30}$$

From (3.28), (3.10), (3.30) and by the monotonicity of $\delta(r)$ we obtain the inequalities $(\nu=1,2,\ldots)$

$$\|L_{(Z)}^{(i)}[{}^{\nu}Z(t,\Xi)]-L_{(Z)}^{(i)}[{}^{\nu}Z(t,\hat{\Xi})]\|\leqslant\delta(2q\|\Xi-\hat{\Xi}\|) \qquad (i=1,2), \tag{3.31}$$

from which, by (3.29), it follows that for two vectors $W=(w_1,\ldots,w_n)$ and $\hat{W}=[\hat{w}_1,\ldots,\hat{w}_n)$ such that $\|W\|\leqslant\beta$, $\|\hat{W}\|\leqslant\beta$ the inequalities

$$\begin{gathered}\|\{L_{(Z)}^{(i)}[{}^{\nu}Z(t,\Xi)]\}W-\{L_{(Z)}^{(i)}[{}^{\nu}Z(t,\hat{\Xi})]\}\hat{W}\|\\ \leqslant(\lambda/2n)\|W-\hat{W}\|+\beta\delta(2q\|\Xi-\hat{\Xi}\|) \qquad (i=1,2;\ \nu=1,2,\ldots)\end{gathered} \tag{3.32}$$

hold.

Now we shall prove the relations (3.26) and (3.27) for $k=1$, showing by induction for every $(t,\Xi)\epsilon\omega$, $(t,\hat{\Xi})\epsilon\omega$ the inequalities

$$\|D^{1}\cdot{}^{\nu}Z(t,\Xi)\|\leqslant 2q \qquad (\nu=1,2,\ldots), \tag{3.33}$$

$$\|D^{1}\cdot{}^{\nu}Z(t,\Xi)-D^{1}\cdot{}^{\nu}Z(t,\hat{\Xi})\|\leqslant 4qn\lambda^{-1}\delta(2q\|\Xi-\hat{\Xi}\|) \qquad (\nu=1,2,\ldots). \tag{3.34}$$

The relations (3.33) and (3.34) hold for $\nu=1$ since from (3.5) and W_2) we obtain

$$\|D^{1}\cdot{}^{1}Z(t,\Xi)\|\leqslant q, \qquad \|D^{1}\cdot{}^{1}Z(t,\Xi)-D^{1}\cdot{}^{1}Z(t,\hat{\Xi})\|=0.$$

Assume that (3.33) and (3.34) hold for $\nu=1,2,\ldots,m$. To prove that they also hold for $\nu=m+1$, let us observe first that

$$D^{1}\cdot\{L^{(i)}[{}^{m}Z(\tau,\Xi)]\}=\{L_{(Z)}^{(i)}[{}^{m}Z(\tau,\Xi)]\}D^{1}\cdot{}^{m}Z(\tau,\Xi) \qquad (i=1,2). \tag{3.35}$$

From (3.29) and the hypothesis of induction (3.33) it follows that

$$\|\{L_{(Z)}^{(i)}[{}^{m}Z(\tau,\Xi)]\}D^{1}\cdot{}^{m}Z(\tau,\Xi)\|\leqslant 2q\lambda/2n \qquad (i=1,2) \tag{3.36}$$

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[5]) Since $p\geqslant 1$, the functions $L^{(i)}(Z)$ $(i=1,2)$, which are by hypothesis of the class C^p in the set $\|Z\|\leqslant R$, are of the class C^1 in this set.

and by the hypothesis of induction (3.34) and inequality (3.32), in which we set $\nu=m$, $W=D^{1\cdot m}Z(t,\Xi)$, $\hat{W}=D^{1\cdot m}Z(t,\hat{\Xi})$, $\beta=2q$, we obtain

$$(3.37)\quad \big\|\{L^{(i)}_{(Z)}[{}^mZ(\tau,\Xi)]\}D^{1\cdot m}Z(\tau,\Xi)-\{L^{(i)}_{(Z)}[{}^mZ(\tau,\hat{\Xi})]\}D^{1\cdot m}Z(\tau,\hat{\Xi})\big\|$$
$$\leqslant(\lambda/2n)(4qn/\lambda)\,\delta(2q\|\Xi-\hat{\Xi}\|)+2q\delta(2q\|\Xi-\hat{\Xi}\|)=4q\delta(2q\|\Xi-\hat{\Xi}\|).$$

From (3.6), (3.35)-(3.37) and W_2) it follows (cf. [2], v. I, p. 233) that

$$(3.38)\quad \begin{aligned} D^{1\cdot m+1}X(t,\Xi)&=\exp[A(t-t_0)]D^1\Xi+\\ &+\int_{t_0}^{t}\exp[A(t-\tau)]\{L^{(1)}_{(Z)}[{}^mZ(\tau,\Xi)]\}D^{1\cdot m}Z(\tau,\Xi)\,d\tau,\\ D^{1\cdot m+1}Y(t,\Xi)&=-\int_{t}^{\infty}\exp[-B(\tau-t)]\{L^{(2)}_{(Z)}[{}^mZ(\tau,\Xi)]\}D^{1\cdot m}Z(\tau,\Xi)\,d\tau \end{aligned}$$

and

$$(3.39)\quad \begin{aligned} \|D^{1\cdot m+1}X(t,\Xi)\|&\leqslant q+(q/\lambda)2q\lambda/2n,\\ \|D^{1\cdot m+1}Y(t,\Xi)\|&\leqslant[(n-q)/\lambda]2q\lambda/2n \end{aligned}$$

and

$$(3.40)\quad \begin{aligned} \|D^{1\cdot m+1}X(t,\Xi)-D^{1\cdot m+1}X(t,\hat{\Xi})\|&\leqslant(q/\lambda)4q\delta(2q\|\Xi-\hat{\Xi}\|),\\ \|D^{1\cdot m+1}Y(t,\Xi)-D^{1\cdot m+1}Y(t,\hat{\Xi})\|&\leqslant[(n-q)/\lambda]4q\delta(2q\|\Xi-\hat{\Xi}\|). \end{aligned}$$

From (3.39) we immediately obtain relation (3.33) for $\nu=m+1$ and from (3.40) relation (3.34) for $\nu=m+1$.

Assume that the relations (3.26) and (3.27), which are satisfied, as has been shown, for $k=1$, hold for $k=1,2,\dots,h-1$, where $2\leqslant h\leqslant p$. The proof of the property P_2^*) will be finished if we show that they hold for $k=h$.

It is obvious that if the function ${}^\nu Z(t,\Xi)$ has derivatives up to the order h with respect to the variables $\xi_1,\dots,\xi_q$, then

$$(3.41)\quad D^{h\cdot}\{L^{(i)}[{}^\nu Z(t,\Xi)]\}={}^\nu F^{(i)}(t,\Xi)+\{L^{(i)}_{(Z)}[{}^\nu Z(t,\Xi)]\}D^{h\cdot\nu}Z(t,\Xi)\quad(i=1,2)$$

where the functions ${}^\nu F^{(i)}(t,\Xi)$ $(i=1,2)$ are aggregates of the product of derivatives up to order $h-1$ of the functions ${}^\nu z_j(t,\Xi)$ by the derivatives up to the order h of functions $L^{(i)}(Z)$ in which ${}^\nu Z(t,\Xi)$ is substituted for Z. These factors are, by our assumptions, bounded by constants independent of ν and have a common module of continuity independent of ν. It follows that the functions ${}^\nu F^{(i)}(t,\Xi)$ $(i=1,2)$ are bounded by constants independent of ν

$$(3.42)\quad \|{}^\nu F^{(i)}(t,\Xi)\|\leqslant M\quad\text{in the set }\omega\text{ for }\nu=1,2,\dots\quad(i=1,2)$$

and the existence of an increasing function $\delta^*(r)$ such that

$$(3.43)\quad \begin{gathered} \|{}^\nu F^{(i)}(t,\Xi)-{}^\nu F^{(i)}(t,\hat{\Xi})\|\leqslant\delta^*(\|\Xi-\hat{\Xi}\|),\\ 1\leqslant\nu<\infty\qquad(i=1,2),\qquad \delta^*(r)\to0\quad\text{for}\quad r\to0. \end{gathered}$$

We shall show by induction that for all positive integers ν the inequalities

$$(3.44)\quad \|D^{h\cdot\nu}Z(t,\Xi)\|\leqslant 2nM/\lambda,$$

$$(3.45)\quad \begin{aligned} &\|D^{h\cdot\nu}Z(t,\Xi)-D^{h\cdot\nu}Z(t,\hat{\Xi})\|\\ &\quad\leqslant(2n/\lambda)[\delta^*(\|\Xi-\hat{\Xi}\|)+(2nM/\lambda)\,\delta(2q\|\Xi-\hat{\Xi}\|)] \end{aligned}$$

hold, from which follow at once the relations (3.26) and (3.27) for $k=h$.

For the proof we observe that from (3.5) we obtain the relations

$$\|D^{h\cdot1}Z(t,\Xi)\|=0,\qquad \|D^{h\cdot1}Z(t,\Xi)-D^{h\cdot1}Z(t,\hat{\Xi})\|=0,$$

which show that (3.44) and (3.45) hold for $\nu=1$. Assume that the inequalities (3.44) and (3.45) hold for every $\nu\leqslant m$ (the hypothesis of induction). From (3.29) and the hypothesis of induction (3.44) it follows that

$$(3.46)\quad \begin{gathered} \big\|\{L^{(i)}_{(Z)}[{}^\nu Z(t,\Xi)]\}D^{h\cdot\nu}Z(t,\Xi)\big\|\leqslant(\lambda/2n)(2nM/\lambda)=M,\\ \nu=1,2,\dots,m,\qquad i=1,2 \end{gathered}$$

and by the hypothesis of induction (3.45) and the inequality (3.32), in which we set $W=D^{h\cdot\nu}Z(t,\Xi)$, $\hat{W}=D^{h\cdot\nu}Z(t,\hat{\Xi})$, $\beta=2nM/\lambda$, we obtain for $\nu=1,2,\dots,m$, $i=1,2$

$$(3.47)\quad \begin{aligned} &\|\{L^{(i)}_{(Z)}[{}^\nu Z(t,\Xi)]\}D^{h\cdot\nu}Z(t,\Xi)-\{L^{(i)}_{(Z)}[{}^\nu Z(t,\hat{\Xi})]\}D^{h\cdot\nu}Z(t,\hat{\Xi})\|\\ &\leqslant(\lambda/2n)(2n/\lambda)[\delta^*(\|\Xi-\hat{\Xi}\|)+(2nM/\lambda)\,\delta(2q\|\Xi-\hat{\Xi}\|)]+\\ &\qquad\qquad+(2nM/\lambda)\,\delta(2q\|\Xi-\hat{\Xi}\|)\\ &=\delta^*(\|\Xi-\hat{\Xi}\|)+(4nM/\lambda)\,\delta(2q\|\Xi-\hat{\Xi}\|). \end{aligned}$$

From (3.6), (3.41), (3.42), (3.46), (3.43), (3.47), it follows that (cf. [2], v. I, p. 233)

$$(3.48)\quad \begin{aligned} D^{h\cdot m+1}X&=\int_{t_0}^{t}\exp[A(t-\tau)]\{{}^\nu F^{(1)}(\tau,\Xi)+L^{(1)}_{(Z)}[{}^mZ(\tau,\Xi)]D^{h\cdot m}Z(\tau,\Xi)\}\,d\tau,\\ D^{h\cdot m+1}Y&=-\int_{t}^{\infty}\exp[-B(\tau-t)]\{{}^\nu F^{(2)}(\tau,\Xi)+\\ &\qquad+L^{(2)}_{(Z)}[{}^mZ(\tau,\Xi)]D^{h\cdot m}Z(\tau,\Xi)\}\,d\tau; \end{aligned}$$

moreover, the inequalities

(3.49) $$\|D^{h.m+1}X\| \leqslant 2qM/\lambda, \qquad \|D^{h.m+1}Y\| \leqslant 2(n-q)M/\lambda,$$

(3.50) $$\begin{gathered}\|D^{h.m+1}X(t,\Xi) - D^{h.m+1}X(t,\hat{\Xi})\| \\ \leqslant (q/\lambda)\{2\delta^*(\|\Xi-\hat{\Xi}\|) + (4nM/\lambda)\,\delta(2q\|\Xi-\hat{\Xi}\|)\}, \\ \|D^{h.m+1}Y(t,\Xi) - D^{h.m+1}Y(t,\hat{\Xi})\| \\ \leqslant [(n-q)/\lambda]\{2\delta^*(\|\Xi-\hat{\Xi}\|) + (4nM/\lambda)\,\delta(2q\|\Xi-\hat{\Xi}\|)\}\end{gathered}$$

hold.

Since (3.44) and (3.45) for $\nu=m+1$ follow from (3.49) and (3.50) respectively, therefore the inductive proof of the relations (3.44) and (3.45), and thus also of the property P_2^*), is completed.

LEMMA 4. *The system (3.1), satisfying the hypothesis K_p, has precisely one integral $Z(t,\Xi)=(X(t,\Xi),Y(t,\Xi))$ satisfying the conditions (3.7). For every fixed t ($t\geqslant t_0$) the function $Z(t,\Xi)$ is of class C^p with respect to $\Xi=(\xi_1,\xi_2,\ldots,\xi_q)$ for $\|\Xi\|<r$.*

Proof. The existence of the integral $Z(t,\Xi)$, which has the properties mentioned in lemma 4, follows from lemma 3. Denote by $Z^*(t,\Xi)=(X^*(t,\Xi),\ Y^*(t,\Xi))$ any integral of the system (3.1) satisfying the conditions (3.7). From lemma 2 and inequalities W_2) and (2.5) we obtain

(3.51) $$\begin{gathered}\|X(t,\Xi)-X^*(t,\Xi)\| \\ \leqslant q\int_{t_0}^{t}\exp[-\lambda(t-\tau)](\lambda/2n)\|Z(\tau,\Xi)-Z^*(\tau,\Xi)\|\,d\tau, \\ \|Y(t,\Xi)-Y^*(t,\Xi)\| \\ \leqslant (n-q)\int_{t}^{\infty}\exp[-\lambda(\tau-t)](\lambda/2n)\|Z(\tau,\Xi)-Z^*(\tau,\Xi)\|\,d\tau.\end{gathered}$$

Let us set $m=\sup\limits_{t_0\leqslant\tau<\infty}\|Z(\tau,\Xi)-Z^*(\tau,\Xi)\|$. From (3.51) we obtain

$$\|Z(t,\Xi)-Z^*(t,\Xi)\| \leqslant (q/\lambda)(\lambda/2n)m + [(n-q)/\lambda](\lambda/2n)m = m/2,$$

whence, by the definition of m, it follows that

$$Z(\tau,\Xi)=Z^*(\tau,\Xi) \quad \text{for} \quad t_0\leqslant\tau<\infty.$$

This inequality implies the unicity of the integral of the system (3.1) satisfying the conditions (3.7). Thus the proof of lemma 4 is finished.

Proof of theorem 1 bis. Let $Z(t,\Xi)$ be an integral of the system (3.1) satisfying the conditions (3.7). By lemma 4 there exists precisely one such integral, and function $Z(t_0,\Xi)$ and therefore also the function $Y(t_0,\Xi)$ are of class C^p for $\|\Xi\|<R/2q$.

It follows that the function

(3.52) $$\Psi(X)=Y(t_0,X)$$

of the independent variable X is of class C^p for $\|X\|<R/2q$. By W_3) one can prove that $Z(t,\Xi)\in F_R$ for $t\geqslant t_0$ (see def. 3). Since the right sides of the system (3.1) are independent of the time t, it follows from the definition of the function $\Psi(X)$ that $Y=\Psi(X)$, $\|X\|<R/2q$ is the equation of the set S (cf. definition 4, § 2). To prove theorem 1 bis it is sufficient to show the contact of the surface S with the hyperplane $Y=\Theta_{n-q}$ at the point Θ_n. Since the surface S is of class $C^p (p\geqslant 1)$, it must be shown that $\Psi_{x_i}(\Theta_q)=\Theta_{n-q}$ for $i=1,2,\ldots,q$, or, by (3.52) that

(3.53) $$Y_{x_i}(t_0,\Theta_q)=\Theta_{n-q} \qquad (i=1,2,\ldots,q).$$

Let us observe that from lemma 2 it follows that

(3.54) $$Y(t,\Xi) = -\int_t^\infty \exp[-B(\tau-t)]L^{(2)}[Z(\tau,\Xi)]\,d\tau.$$

Making use of (2.2), (2.4), (3.8) it is easy to show that

(3.55) $$\left\{\frac{\partial}{\partial\xi_i}L^{(2)}[Z(t,\Xi)]\right\}_{\Xi=\Theta_q} = \{L^{(2)}_{(Z)}[Z(t,\Theta_q)]\}Z_{\xi_i}(t,\Theta_q) = G^{(2)}Y_{\xi_i}(t,\Theta_q).$$

By (3.55) we obtain from (3.54) (cf. (3.29), (3.21), (3.33), P_2), W_2))

(3.56) $$Y_{\xi_i}(t,\Theta_q) = -\int_t^\infty \exp[-B(\tau-t)]G^{(2)}Y_{\xi_i}(\tau,\Theta_q)\,d\tau.$$

From (3.56), W_2), (2.3) it follows that

(3.57) $$\|Y_{\xi_i}(t,\Theta_q)\| \leqslant (n-q)(\lambda/4n)\int_t^\infty \exp[-\lambda(\tau-t)]\|Y_{\xi_i}(\tau,\Theta_q)\|\,d\tau \quad \text{for } t\geqslant t_0.$$

Let us set $m_i=\sup\limits_{t_0\leqslant\tau<\infty}\|Y_{\xi_i}(\tau,\Theta_q)\|$. Now we obtain from (3.57)

$$\|Y_{\xi_i}(t,\Theta_q)\| \leqslant [(n-q)\lambda/4n](m_i/\lambda) < m_i/4 \quad \text{for} \quad t\geqslant t_0,$$

whence, by the definition of m_i, follow the relations

$$Y_{\xi_i}(t,\Theta_q)=\Theta_{n-q} \quad \text{for} \quad t\geqslant t_0 \qquad (i=1,2,\ldots,q),$$

and thus in particular also the relations (3.53), which were to be shown.

§ 4. To complete our considerations it suffices to prove lemma 1 only (cf. § 2). To begin with we shall recall some theorems of the theory of matrices and linear transformations.

Let $C=(c_{ij})$ $(i,j=1,2,\ldots,n)$ be an arbitrary real square matrix and $\lambda_1,\lambda_2,\ldots,\lambda_n$ the full sequence of all its characteristic roots. We assume that there exists a positive integer q such that $0<q<n$ and

$$\operatorname{Re}\lambda_1\leqslant\ldots\leqslant\operatorname{Re}\lambda_q<\operatorname{Re}\lambda_{q+1}\leqslant\ldots\leqslant\operatorname{Re}\lambda_n.$$

It is known (cf. [8], p. 295-296) that for every $\varepsilon>0$ there exists a non-singular matrix M such that the matrix

$$J=MCM^{-1}=\begin{pmatrix}J_1 & 0\\ 0 & J_2\end{pmatrix}$$

has Jordan's canonical form such that

1) $\lambda_1,\lambda_2,\ldots,\lambda_q$ is the sequence of all characteristic roots of the matrix J_1,

2) $\lambda_{q+1},\lambda_{q+2},\ldots,\lambda_n$ is the sequence of all characteristic roots of the matrix J_2,

3) the matrices J_1 and J_2 have the form K

$$K=\begin{bmatrix}N_1 & L_1 & & & & & & & \\ & N_2 & L_2 & & & & & & \\ & & \ddots & & & & & & \\ & & & N_{r-1} & L_{r-1} & & & & \\ & & & & N_r & & & & \\ & & & & & a_{2r+1} & \varepsilon_{2r+1} & & \\ & & & & & & \ddots & & \\ & & & & & & & a_{s-1} & \varepsilon_{s-1}\\ & & & & & & & & a_s\end{bmatrix},$$

where

$$N_j=\begin{bmatrix}a_j & -\beta_j\\ \beta_j & a_j\end{bmatrix},\quad L_j=\begin{bmatrix}\varepsilon_j & 0\\ 0 & \varepsilon_j\end{bmatrix}\quad (j=1,2,\ldots,r),\quad |\varepsilon_k|<\varepsilon$$

for every index k, a_k $(k=1,\ldots,s)$ and β_k $(k=1,\ldots,r)$, denote respectively the real and the imaginary parts of the roots of the matrix C, $\beta_k\neq 0$ and the vacant places are occupied by zeros.

Denote by N the matrix

$$N=\begin{bmatrix}N_1 & & & & \\ & \ddots & & & \\ & & N_r & & \\ & & & a_{2r+1} & \\ & & & & \ddots \\ & & & & & a_s\end{bmatrix},\quad \text{where}\quad N_j=\begin{bmatrix}a_j & -\beta_j\\ \beta_j & a_j\end{bmatrix}\quad (j=1,\ldots,r).$$

Obviously

$$\|K-N\|\leqslant s^{1/2}\varepsilon \tag{4.1}$$

and on account of the form of the matrix e^{Nt}

$$e^{Nt}=\begin{bmatrix}S_1 & & & & \\ & \ddots & & & \\ & & S_r & & \\ & & & e^{a_{2r+1}t} & \\ & & & & \ddots \\ & & & & & e^{a_st}\end{bmatrix},\quad \text{where}\quad S_j=\begin{bmatrix}e^{a_jt}\cos\beta_jt & -e^{a_jt}\sin\beta_jt\\ e^{a_jt}\sin\beta_jt & e^{a_jt}\cos\beta_jt\end{bmatrix}$$

$(j=1,2,\ldots,r)$, the inequality

$$\|e^{Nt}\|\leqslant 2\sum_{p=1}^{r}e^{a_pt}+\sum_{p=2r+1}^{s}e^{a_pt} \tag{4.2}$$

holds.

The above considerations lead to the following lemma:

Lemma 5. *Let C be a matrix of the type T_q* (cf. def. 1, § 1). *Then there exists a number $\lambda>0$ (dependent on the matrix C) such that for every $\varepsilon>0$ one can find a real non-singular matrix M, and the q matrices A, G, and $n-q$ matrices B,H, satisfying the following relations:*

$$MCM^{-1}=\begin{bmatrix}J_1 & 0\\ 0 & J_2\end{bmatrix}\qquad J_1=A+G,\qquad J_2=B+H, \tag{4.3}$$

$$\|G\|\leqslant q^{1/2}\varepsilon,\qquad \|H\|\leqslant(n-q)^{1/2}\varepsilon, \tag{4.4}$$

$$\begin{gathered}\|e^{At}\|\leqslant qe^{-\lambda t},\qquad \|e^{-Bt}\|\leqslant(n-q)e^{-\lambda t}\quad \textit{for}\quad t\geqslant 0,\\ UAU\leqslant-\lambda\|U\|^2\quad \textit{for every vector}\quad U=(u_1,\ldots,u_q).\end{gathered} \tag{4.5}$$

Proof. Denote by $\lambda_1,\lambda_2,\ldots,\lambda_n$ the sequence of all characteristic roots of the matrix C of type T_q and assume that (cf. def. 1, § 1)

$$\operatorname{Re}\lambda_1\leqslant\operatorname{Re}\lambda_2\leqslant\ldots\leqslant\operatorname{Re}\lambda_q<0<\operatorname{Re}\lambda_{q+1}\leqslant\ldots\leqslant\operatorname{Re}\lambda_n. \tag{4.6}$$

Let λ be an arbitrary number satisfying the inequality

$$0<\lambda<|\operatorname{Re}\lambda_i|\qquad (i=1,2,\ldots,n). \tag{4.7}$$

The existence of matrices A, B of type N and of the matrices G, H, M satisfying the relations (4.3) and (4.4) follows immediately from the considerations preceding lemma 5. $\lambda_1,\lambda_2,\ldots,\lambda_q$ and $\lambda_{q+1},\lambda_{q+2},\ldots,\lambda_n$ are sequences of all the characteristic roots of the matrices A and B respectively. The inequalities (4.5) hold by (4.6), (4.7), the inequalities of the form (4.2) and by the identity

$$UNU = a_1(u_1^2+u_2^2)+\ldots+a_r(u_{2r-1}^2+u_{2r}^2)+a_{2r+1}u_{2r+1}^2+\ldots+a_s u_s^2$$

satisfied by the matrices A and B of the type N.

Lemma 6. *Assume that the matrix C of the coefficients of the system of differential equations*

$$dH/dt = CH+\Phi(H) \tag{4.8}$$

is of type T_q and that $\Phi(H)$ is of class C^p in the neighbourhood of $\Theta_n (p\geqslant 1)$

$$\Phi(\Theta_n)=\Theta_n, \tag{4.9}$$

$$\Phi_{\eta_j}(\Theta_n)=\Theta_n \qquad (j=1,2,\ldots,n). \tag{4.10}$$

Then there exists a real non-singular linear transformation $Z=MH$, which transforms the system (4.8) *into the system*

$$dX/dt = AX+L^{(1)}(Z), \qquad dY/dt=BY+L^{(2)}(Z), \tag{4.11}$$

where $Z=(X,Y)$, satisfying the hypothesis K_p.

Proof. Let M be a matrix chosen according to lemma 5 for the matrix C and the number $\varepsilon=\lambda/4n^2$. It follows that the linear transformation $Z=MH$ transforms the system (4.8) into the system (4.11), which satisfies the inequalities W_2) and the relations

$$L^{(1)}(Z)=G^{(1)}X+Ł^{(1)}(Z), \qquad L^{(2)}(Z)=G^{(2)}Y+Ł^{(2)}(Z) \tag{4.12}$$

where $G^{(i)}$ are constant matrices satisfying the inequalities (2.3), and the function

$$\big(Ł^{(1)}(Z),Ł^{(2)}(Z)\big)=M\Phi(M^{-1}Z) \tag{4.13}$$

is of class C^p in the neighbourhood of Θ_n[6]). By (4.10) we obtain from the formula (4.13)

$$Ł^{(1)}_{z_j}(\Theta_n)=\Theta_q \qquad Ł^{(2)}_{z_j}(\Theta_n)=\Theta_{n-q} \qquad (j=1,2,\ldots,n)$$

and since from the relations (4.12), (4.13), (4.9) it follows that $L^{(1)}(\Theta_n)=\Theta_q$, $L^{(2)}(\Theta_n)=\Phi_{n-q}$ therefore the proof of lemma 6 is complete.

Remark 5. Lemma 1 is the immediate consequence of lemma 6. Indeed from the assumptions of lemma 1 it follows that the function $\Phi(H)$ defined by the relation $\Phi(H)=F(H)-CH$, $F(H)=\big(F^{(1)}(H),\ldots,F^{(n)}(H)\big)$ is of class C^p in the neighbourhood of Θ_n and $\Phi(\Theta_n)=\Theta_n$, $\Phi_{\eta_j}(\Theta_n)=\Theta_n$ $(j=1,2,\ldots,n)$.

[6]) This follows from (4.13) since $\Phi(H)$ is of class C^p in the neighbourhood of Θ_n and M is a constant non-singular matrix.

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